Charles University in Prague Faculty of Mathematics and Physics

DOCTORAL THESIS



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Applications of descriptive set theory in mathematical analysis

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I declare that I carried out this doctoral thesis independently, and only with the cited sources, literature and other professional sources.

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Abstrakt: Charakterizujeme různé typy σ -pórovitosti pomocí nekonečné hry a vítězných strategií. Použijeme modifikaci této hry k důkazu některých nových i známých vepisovacích vět pro σ -ideály σ -pórovitého typu v lokálně kompaktních metrických prostorech. Ukážeme existenci uzavřené množiny, která je σ - $(1 - \varepsilon)$ -symetricky pórovitá pro každé $0 < \varepsilon < 1$, ale není σ -1-symetricky pórovitá.

Dále ukážeme, že množina unitárních reprezentací konečné abelovské grupy Γ na nekonečnědimensionálním separabilním komplexním Hilbertově prostoru H, které jsou realizovatelné akcí, je residuální v Rep (Γ, H) .

Klíčová slova: Determinovanost, nekonečné hry, pórovitost, unitární reprezentace grupy, akce grupy.

Title: Applications of descriptive set theory in mathematical analysis

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Abstract: We characterize various types of σ -porosity via an infinite game in terms of winning strategies. We use a modification of the game to prove and reprove some new and older inscribing theorems for σ -ideals of σ -porous type in locally compact metric spaces. We show that there exists a closed set which is σ - $(1 - \varepsilon)$ -symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ -1-symmetrically porous.

Next, we prove that the realizable by an action unitary representations of a finite abelian group Γ on an infinite-dimensional complex Hilbert space H form a comeager set in $\text{Rep}(\Gamma, H)$.

Keywords: Determinacy, infinite games, porosity, unitary group representations, group actions.

Contents

Introduction										
1	Infinite games and σ -porosity									
	1.1	Introd	uction	7						
	1.2	cterization of σ -porosity via an infinite game $\ldots \ldots \ldots \ldots \ldots \ldots$	10							
		1.2.1	Introduction.	11						
		1.2.2	Proof of the main theorem.	12						
	1.3	act non- σ -porous subsets of non- σ -porous sets	18							
		1.3.1	Introduction	18						
		1.3.2	Main result	19						
		1.3.3	Applications to concrete porosities	30						
2	Unitary representations of finite abelian groups realizable by an action									
	2.1	Introd	uction	34						
	2.2	Proof	of the main theorem	42						

Introduction

This thesis consists of this introduction and two chapters. The first chapter concerns the notion of σ -porosity and it contains an introductory section and two sections based each on one of these papers:

- Characterization of σ-porosity via an infinite game. Fund. Math. 216 (2012), no. 2, 109-118.
- Infinite games and σ-porosity. Submitted.
 (with M. Zelený)

The second chapter concerns unitary group representations realizable by an action and it is based on the following paper:

• Unitary representations of finite abelian groups realizable by an action. Submitted.

The aim of this introduction is to give the basic background to the studied topics, to briefly outline the content of the individual sections and to introduce the main results of the thesis.

There are many reasonable notions of 'small sets' in mathematical analysis. For example, meager subsets of a topological space are usually considered to be small in the sense of Baire categories and Lebesgue measure zero subsets of \mathbb{R}^n are usually considered to be small in the sense of Lebesgue measure. Such notions of smallness are frequently used to show that every point of a given space has a certain property, except points from an exceptional set of 'singular points' which is 'small'. In both chapters of this thesis, we are interested in some kind of smallness. In the first chapter, we investigate so called σ -porosity which is a useful notion of smallness for subsets of metric spaces. In the second chapter, we answer the question whether one particular subset of the space of all unitary representations of a finite abelian group on a given infinite-dimensional separable complex Hilbert space is meager or comeager.

One can find many similar definitions of porosity and σ -porosity throughout the literature. Their common idea is that a subset M of a metric space is porous if there exist 'big pores in M' (i.e. open sets disjoint with M) arbitrarily near to every point $x \in M$. Here we give one of the most frequently used definitions.

Definition. Let (X, d) be a metric space. Let $M \subset X$, $x \in X$, and R > 0. Then we define

$$\begin{split} \gamma(x,R,M) &= \sup\{r > 0 \colon \text{there exists } z \in B(x,R) \\ &\quad \text{such that } B(z,r) \cap M = \emptyset\}, \\ p(x,M) &= \limsup_{R \to 0+} \frac{\gamma(x,R,M)}{R}. \end{split}$$

We say that $M \subset X$ is porous at $x \in X$ if p(x, M) > 0. We say that M is porous if it is porous at each $x \in M$. We say that M is σ -porous if it is a countable union of porous sets.

This porosity is sometimes called 'ordinary porosity' to avoid confusion since there are also many other variants of porosity such as symmetric porosity in \mathbb{R} , left and right porosity in \mathbb{R} or strong porosity. In many cases, it is convenient to use a more general definition of a so called 'porosity-like relation' which is a relation between points and subsets of a given metric space with three basic properties which are common for virtually all known porosities.

The term 'porosity' was introduced in 1967 by E. P. Dolženko in [5] and it is possible to say that Dolženko started a systematic investigation of σ -porosity. In [5], he studied boundary behavior of complex functions and proved that certain exceptional sets of his interest are σ -porous. He observed that in an Euclidean space, σ -porous sets are meager and have Lebesgue measure zero. He also stated without proof that there is a meager set which has Lebesgue measure zero but which is not σ -porous (but it was first proved only later by L. Zajíček in [20] and all known proofs of this basic fact are relatively difficult). From the last two facts, it follows that σ -porosity of certain exceptional sets turns out to be a very interesting property since it is a strictly stronger property than being small in the sense of both Baire category and measure. Moreover, it is frequently easier to prove σ porosity of a given set than to provide two different proofs of its smallness, one on category and one on the measure.

Since then the porosity has been widely used, especially in the differentiation theory and in Banach space theory. The first such case is a paper of C. L. Belna, M. J. Evans and P. D. Humke from 1978 [1]. They proved that for every measurable and symmetrically differentiable function $f: \mathbb{R} \to \mathbb{R}$, the set of points $x \in \mathbb{R}$ at which f'(x) does not exist (i.e. the set of exceptional points with respect to differentiation) is σ -porous. In Banach space theory, σ -porosity was used for the first time in 1984 by D. Preiss and L. Zajíček in [15]. They showed that if f is a real-valued continuous convex function on a real Banach space with a separable dual, then the set of points at which f is not Fréchet differentiable is σ -porous. Another example is a paper of J. Lindenstrauss and D. Preiss from 2003 [12]. Among other things, they investigated a relationship between σ -porous and so called Γ -null subsets of Banach spaces. They showed that in some Banach spaces (namely in C(K) with K countable compact, in the Tsirelson space and in all subspaces of c_0 , every σ -porous set is Γ -null. They used this fact to prove that every countable collection of real-valued Lipschitz functions on one of these spaces has a common point of Fréchet differentiability. Some results in the differentiation theory of real functions of a real variable concerning porosity and σ -porosity are presented in the monograph [19] written by B. S. Thomson. Many results concerning porosity and σ -porosity in Banach space theory can be found in the monograph [13] written by J. Lindenstrauss, D. Preiss and J. Tišer. An interested reader can also consult two extensive topical surveys [21], [23] about porosity and σ -porosity written by L. Zajíček. All of the mentioned examples show that the structural properties of the σ -ideal of σ -porous sets deserve further investigation which is the motivation for the results presented in the first chapter of this thesis.

Let us look at the content of the first chapter of the thesis a little closer. We consider the following question.

Question. Let A be an analytic subset of a metric space X and let \mathcal{I} be a σ -ideal of subsets of X. Suppose that $A \notin \mathcal{I}$. Does there exist a closed set $F \subset A$ which is not in \mathcal{I} ?

This question was posed by L. Zajíček in [21] (for a Borel set A) for a σ -ideal \mathcal{I} of σ porous subsets of X. An affirmative answer was given independently by J. Pelant (for any topologically complete metric space X) and M. Zelený (for any compact metric space X). Their results are demonstrated in a joint paper [25] which combines the original idea of J. Pelant (giving an explicit construction of the set F) and techniques developed by M. Zelený. The case of some other types of porosity (including the ordinary one in a locally compact metric space X but also $\langle g \rangle$ -porosity in a locally compact metric space X or symmetrical porosity in \mathbb{R}) was solved (also affirmatively) by M. Zelený and L. Zajíček in [26]. They offer a less complicated method of construction of F using porosity-like relations mentioned earlier. Their nonconstructive proof uses tools from descriptive set theory. However, the authors admitted that their method cannot be easily applied to strong porosity and so Question for strong porosity still remained open (even in a compact metric space X).

Our main tasks are:

- (a) to find a suitable characterization of σ -P-porous subsets of X where P is a porositylike relation on a metric space X,
- (b) to apply this characterization to answer Question for the σ -ideal \mathcal{I} of σ -P-porous subsets of X where P and X are as much general as possible.

The first section of the first chapter contains the necessary introduction. Here, we recall the definition of a porosity-like relation. We also remind the terminology concerning infinite games as well as some known results which will be used later.

The second section is based on the paper [3]. This paper was inspired by the infinite game of Farah and Zapletal from [8] which characterizes σ -porous subsets of the Cantor space $\{0,1\}^{\mathbb{N}}$ and by its generalization by D. Rojas-Rebolledo who found similar games which characterize σ -porous, resp. σ -strong porous subsets of any zero-dimensional compact metric space (see [16]). (In both papers [8] and [16], the relevant infinite games were used to answer Question affirmatively in the corresponding settings.) Given a metric space X and an arbitrary porosity-like relation P on X, we associate an infinite game G(A) with any subset A of X. We characterize σ -P-porous subsets of X via this infinite game by proving the following theorem.

Theorem. The second player has a winning strategy in the game G(A) if and only if A is σ -P-porous.

The third section is based on the paper [4]. Here, we generalize the results of [8], [16] and [26] concerning Question in two directions. We give an affirmative answer to Question

in spaces which are more general than those considered in [8] and [16] and also for σ -ideals of σ -porous type which are not included in [26]. Since the precise formulation of the main result is a little bit technical, let us formulate it only in an informal way for now.

Let X be a compact metric space and let Q be a porosity-like relation on X satisfying some additional conditions. Then every analytic non- σ -Q-porous subset A of X contains a compact non- σ -Q-porous subset.

We apply this result to concrete porosities and obtain an (affirmative) answer to several different variants of Question. Namely, we deal with ordinary porosity, strong porosity, strong porosity and 1-symmetrical porosity. As it is described earlier, the first result has been already known but the method used in our work (based on an infinite game) aspires to be more elegant and easier than the known proofs. The other results are new. Finally, we apply this theorem to show that there exists a closed set in \mathbb{R} which is $\sigma - (1 - \varepsilon)$ -symmetrically porous for every $\varepsilon \in (0, 1)$ but which is not σ -1-symmetrically porous. This answers a question posed by M. J. Evans and P. D. Humke in [7].

The second chapter of the thesis is based on the paper [2]. Here, we investigate the topological space of all unitary representations of a finite abelian group on a given infinitedimensional separable complex Hilbert space. More specifically, we try to find out 'how many' of these representations are realizable by an action where the phrase 'how many' refers to Baire categories.

If H is a separable infinite-dimensional complex Hilbert space, we endow $\mathcal{U}(H)$, the unitary group of H, with the strong topology, i.e. the topology generated by the maps $T \in \mathcal{U}(H) \mapsto T(x) \in H, x \in H$. The group $\mathcal{U}(H)$ endowed with this topology is a Polish group. If Γ is a countable group, we denote by $\operatorname{Rep}(\Gamma, H)$ the set of all group homomorphisms from Γ to the unitary group $\mathcal{U}(H)$. Every element of $\operatorname{Rep}(\Gamma, H)$ is called a unitary representation of Γ on H. The set $\operatorname{Rep}(\Gamma, H)$ is a closed subspace of the Polish space $\mathcal{U}(H)^{\Gamma}$ and so it is also a Polish space.

Now, if (X, μ) is a standard probability space (i.e. a standard Borel space X together with a non-atomic probability Borel measure μ on X) and $L_0^2(X, \mu)$ is the Hilbert space of all complex-valued square-integrable (with respect to μ) functions on X with zero integral, every action a of the group Γ on X which preserves the measure μ induces in a canonical way so called Koopman unitary representation κ_0^a of Γ on $L_0^2(X, \mu)$ associated with the action a. This representation is defined by the formula

$$\kappa_0^a(\gamma)(f)(x) = f(a(\gamma^{-1}, x)), \quad \gamma \in \Gamma, \ f \in L^2_0(X, \mu), \ x \in X.$$

Finally, we say that a unitary representation π of a countable group Γ on a separable infinite-dimensional complex Hilbert space H is realizable by an action if there is a standard probability space (X, μ) and a measure preserving action a of Γ on (X, μ) such that π is unitarily equivalent to the Koopman representation κ_0^a of Γ on $L_0^2(X, \mu)$ associated with the action a.

Some interesting facts concerning the notion of realizability by an action can be found in [10, Appendix H, (F)]. Among other things, it is stated there that the set of realizable by an action representations is either meager or comeager in $\operatorname{Rep}(\Gamma, H)$ since it is invariant under conjugacy by elements of the unitary group $\mathcal{U}(H)$. And it is shown there that if Γ is torsion-free abelian then the set is meager. However, there are no other examples. Instead, the following question is formulated there.

Question. Let Γ be an infinite countable group and let H be an infinite-dimensional separable complex Hilbert space. Is the set of realizable by an action $\pi \in \operatorname{Rep}(\Gamma, H)$ meager in $\operatorname{Rep}(\Gamma, H)$?

Although the author of this thesis tried to answer this question for some infinite countable groups that are non-abelian or which have a torsion element he did not succeed. However, we consider the corresponding question for an arbitrary finite abelian group Γ and show that in this case, the realizable by an action representations form a comeager subset of $\text{Rep}(\Gamma, H)$.

1. Infinite games and σ -porosity

This chapter is based on the papers [3] and [4].

Section 1.1 contains some preliminaries which are common for both these papers.

Section 1.2 is based on [3]. It contains two subsections. Subsection 1.2.1 corresponds to Section 1 (Introduction) of [3] but it is slightly shortened here, such that we do not repeat some facts stated already in Section 1.1. Section 2 (Preliminaries) of [3] is entirely left out here, for the same reason. Subsection 1.2.2 is almost identical with Section 3 (Proof of the main theorem) of [3].

Section 1.3 is based on [4]. It contains three subsections. Similarly as above, subsection 1.3.1 corresponds to Section 1 (Introduction) of [4] but is shortened and Section 2 (Preliminaries) of [4] is entirely left out here. Subsections 1.3.2 and 1.3.3 are almost identical with Sections 3 (Main result) and 4 (Applications to concrete porosities) of [4], respectively.

1.1 Introduction

In this section, we recall the definition of an abstract porosity-like relation. We remind that σ -porosity is a local property. We also recall that every metrizable topological space has a σ -discrete open basis. Finally, we remind the notation concerning infinite games used in descriptive set theory as well as the well known and very difficult Martin's theorem.

Let (X, d) be a metric space. An open ball with center $x \in X$ and radius r > 0 is denoted by B(x, r). Since an open ball (considered as a set) does not uniquely determine its center and radius, we will identify every open ball with the pair (center, radius) throughout this chapter. Therefore two different open balls (i.e., two different pairs (center, radius)) can still determine the same subset of X. Now, for p > 0 and an open ball B with center $x \in X$ and radius r > 0, we can define $p \star B$ as an open ball with center x and radius pr. The closed ball with center $x \in X$ and radius r > 0 is denoted by $\overline{B}(x,r)$. We employ the same identification of closed balls with the pairs (center, radius) as for open balls. If $A \subset X$ is nonempty and r > 0 then $B(A, r) = \{x \in X : \operatorname{dist}(x, A) < r\}$. We also set $B(\emptyset, r) = \emptyset$. Finally, for a nonempty subset B of X, we set

diam
$$B = \sup\{d(a, b) \colon a, b \in B\}.$$

Definition 1.1.1 (porosity-like relation). Let X be a metric space and let $P \subseteq X \times 2^X$ be a relation between points of X and subsets of X. Then P is called a *point-set relation* on X. The symbol P(x, A) where $x \in X$ and $A \subseteq X$ means that $(x, A) \in P$.

A point-set relation P on X is called a *porosity-like relation* if the following conditions hold for every $A \subseteq X$ and $x \in X$:

(P1) if $B \subseteq A$ and P(x, A) then P(x, B),

(P2) we have P(x, A) if and only if there exists r > 0 such that $P(x, A \cap B(x, r))$,

(P3) we have P(x, A) if and only if $P(x, \overline{A})$.

If P is a porosity-like relation on X, $A \subseteq X$ and $x \in X$, we say that

- A is P-porous at x if P(x, A),
- A is *P*-porous if it is *P*-porous at every point $x \in A$,
- A is σ -P-porous if it is a countable union of P-porous sets.

Definition 1.1.2 ((ordinary) porosity). Let X be a metric space. Let $A \subseteq X$, $x \in X$ and R > 0. Let us define

$$\gamma(x, R, A) = \sup\{r > 0: \text{ there exists } z \in B(x, R)$$

such that $B(z, r) \cap A = \emptyset\},$
$$p(x, A) = \limsup_{R \to 0+} \frac{\gamma(x, R, A)}{R}.$$

We say that

- A is porous at x if p(x, A) > 0,
- A is *porous* if it is porous at every point $x \in A$,
- A is σ -porous if it is a countable union of porous sets.

Remark 1.1.3. To be exact, if we speak about ordinary porosity as a particular case of a porosity-like relation, we mean the following. Let us define the point-set relation P on a metric space X by posing $(x, A) \in P$ if and only if A is porous at x. Then P is a porosity-like relation on X, as can be easily verified.

We will need the following theorem.

Theorem 1.1.4 ([22, Lemma 3]). Let X be a metric space, P be a porosity-like relation on X and $A \subseteq X$. Then A is σ -P-porous if and only if for every $x \in A$ there exists r > 0such that $B(x, r) \cap A$ is σ -P-porous.

Definition 1.1.5 (σ -discrete system). Let X be a topological space. A system \mathcal{V} of subsets of X is said to be

- discrete if for every $x \in X$ there exists a neighborhood of x which intersects at most one set from the system \mathcal{V} ,
- σ -discrete if it is a countable union of discrete systems.

We will use the existence of a σ -discrete basis of open sets in a metric space. This is guaranteed by the following well known theorem.

Theorem 1.1.6. Let X be a metrizable topological space. Then X has an open basis which is σ -discrete.

It is also necessary to remind some basic definitions which concern infinite games. Let M be a nonempty set and $n \in \mathbb{N}$. We denote by M^n the set of all sequences $s = (s_0, s_1, \ldots, s_{n-1})$ of length n from M. We also set $M^0 = \{\emptyset\}$ where \emptyset is the *empty sequence* (of length 0). We denote by $M^{<\mathbb{N}}$ (resp. $M^{\mathbb{N}\cup\{0\}}$) the set of all finite (resp. infinite) sequences from M. This means that

$$M^{<\mathbb{N}} = \bigcup_{n=0}^{\infty} M^n.$$

The length of a finite sequence s is denoted by length (s). If $s \in M^{<\mathbb{N}}$ and $n \in \mathbb{N} \cup \{0\}$ are such that $n \leq \text{length}(s)$ then $s|n = (s_0, s_1, \ldots, s_{n-1}) \in M^n$. If $s, t \in M^{<\mathbb{N}}$ then we say that s is an *initial segment* of t and t is an *extension* of s if there exists $n \in \mathbb{N} \cup \{0\}$ such that $n \leq \text{length}(t)$ and s = t|n. If $s = (s_0, s_1, \ldots, s_{n-1}) \in M^n$ and $t = (t_0, t_1, \ldots, t_{m-1}) \in M^m$, then the *concatenation* of s and t is the sequence $s^{\wedge}t = (s_0, s_1, \ldots, s_{n-1}, t_0, t_1, \ldots, t_{m-1}) \in$ M^{n+m} . In the obvious way, we also understand the infinite concatenation $s_1^{\wedge}s_2^{\wedge}s_3^{\wedge} \ldots$ of a sequence $(s_n)_{n=1}^{\infty}$ of elements of $M^{<\mathbb{N}}$. If $x = (x_j)_{j=0}^{\infty} \in M^{\mathbb{N} \cup \{0\}}$ and $n \in \mathbb{N} \cup \{0\}$ then $x|n = (x_0, x_1, \ldots, x_{n-1}) \in M^n$. If $s \in M^{<\mathbb{N}}$ and $x \in M^{\mathbb{N} \cup \{0\}}$ then we say that s is an *initial* segment of x and x is an extension of s if s = x|n for some $n \in \mathbb{N} \cup \{0\}$.

A subset $T \subseteq M^{<\mathbb{N}}$ is called a *tree* on M if for every $t \in T$ and every initial segment s of t, we have $s \in T$. A sequence $x \in M^{\mathbb{N} \cup \{0\}}$ is called an *infinite branch* of T if $x | n \in T$ for every $n \in \mathbb{N} \cup \{0\}$. The *body* of T is the set of all infinite branches of T and is denoted by [T]. This means that

$$[T] = \{ x \in M^{\mathbb{N} \cup \{0\}} \colon x | n \in T \text{ for every } n \in \mathbb{N} \cup \{0\} \}.$$

A tree T is called *pruned* if every $s \in T$ has a proper extension in T, i.e. for every $s \in T$ there exists $t \in T$ such that t is an extension of s and $t \neq s$.

Let M be a nonempty set and $X \subseteq M^{\mathbb{N} \cup \{0\}}$. We associate X (which is called a *payoff* set then) with the following game:

I a_0 a_2 a_4 II a_1 a_3 a_5

Player I plays $a_0 \in M$, then player II plays $a_1 \in M$, I plays $a_2 \in M$, etc. Player I wins if $(a_n)_{n=0}^{\infty} \in X$, II wins in the opposite case. We denote this game by G(M, X).

A strategy for player I in the game G(M, X) is a tree $\rho \subseteq M^{<\mathbb{N}}$ on M such that

- ρ is nonempty,
- if $i \in \mathbb{N} \cup \{0\}$ and $(a_0, a_1, \dots, a_{2i}) \in \rho$ then $(a_0, a_1, \dots, a_{2i}, a_{2i+1}) \in \rho$ for every $a_{2i+1} \in M$,

• if $i \in \mathbb{N} \cup \{0\}$ and $(a_0, a_1, \dots, a_{2i-1}) \in \rho$ then there exists a unique $a_{2i} \in M$ such that $(a_0, a_1, \dots, a_{2i-1}, a_{2i}) \in \rho$.

If we say that player I follows the strategy ρ , we mean the following. Player I starts with the unique $a_0 \in M$ such that $(a_0) \in \rho$. If II replies by $a_1 \in M$ then $(a_0, a_1) \in \rho$ and I plays the unique $a_2 \in M$ such that $(a_0, a_1, a_2) \in \rho$, etc.

A strategy for player I is winning in the game G(M, X) if for every run $(a_n)_{n=0}^{\infty} \in M^{\mathbb{N} \cup \{0\}}$ of the game, in which I follows the strategy, we have $(a_n)_{n=0}^{\infty} \in X$ (and so I wins the run).

A (winning) strategy for II is defined in an analogous way.

The game G(M, X) is determined if one of the players has a winning strategy.

In the game G(M, X), both players play arbitrary elements of a given nonempty set M. In many cases, it is more convenient to let them obey some rules which are represented by a nonempty pruned tree $T \subseteq M^{<\mathbb{N}}$ (which determines so called *legal positions*). Let $X \subseteq [T]$ (X is called a *payoff set* again), then we define the game G(T, X) as follows:

Again, I plays $a_0 \in M$, II plays $a_1 \in M$, etc. But both players have now to choose their moves such that $(a_0, a_1, \ldots, a_n) \in T$ for every $n \in \mathbb{N} \cup \{0\}$. Player I wins if $(a_n)_{n=0}^{\infty} \in X$, II wins in the opposite case. The notions of (winning) strategy and determinacy are defined analogously as before. However, the game G(T, X) is only a particular case of the previous game. Indeed, it is easy to see that if we denote

 $X' = \{x \in M^{\mathbb{N} \cup \{0\}} : \text{(there exists } n \in \mathbb{N} \text{ such that } x | n \notin T \}$

and the smallest such n is even) or $(x \in [T] \cap X)$,

then I (resp. II) has a winning strategy in the game G(T, X) if and only if I (resp. II) has a winning strategy in the game G(M, X').

Now, we can formulate the well known (and very deep) Martin's theorem. Its proof can be found in [9, Theorem 20.5]. In this Theorem, we consider the discrete topology on the nonempty set M, the product topology on $M^{\mathbb{N}\cup\{0\}}$ and the derived topology on $[T] \subseteq M^{\mathbb{N}\cup\{0\}}$ where T is a nonempty pruned tree on M.

Theorem 1.1.7 ([14]). Let T be a nonempty pruned tree on a nonempty set M and let $X \subseteq [T]$ be a Borel set. Then the game G(T, X) is determined.

1.2 Characterization of σ -porosity via an infinite game

This section is based on the paper [3]. It contains two subsections. Subsection 1.2.1 corresponds to Section 1 (Introduction) of [3] but it is slightly shortened here, such that we do

not repeat some facts stated already in Section 1.1. Section 2 (Preliminaries) of [3] is entirely left out here, for the same reason. Subsection 1.2.2 is almost identical with Section 3 (Proof of the main theorem) of [3]. The only changes are made in the hierarchy of sections and subsections and cross-referencing between them.

1.2.1 Introduction.

The connection between σ -porosity and infinite games was first shown by M. Zelený in [24]. He defined an infinite game which is very similar to the well known Banach-Mazur game and using this game, he characterized both sets which can be covered by countably many closed uniformly porous sets and σ -very porous sets. He also found a sufficient condition for σ -porosity in the terminology of games.

For this work, very inspirational was the infinite game H(A) of Farah and Zapletal (see [8, Example 4.20]). Let us endow the Cantor space $\{0,1\}^{\mathbb{N}}$ with the metric $d(x,y) = \frac{1}{k}$ where k is the least such that $x(k) \neq y(k)$. For $n \in \mathbb{N}$ and $t \in \{0,1\}^n$, let $U_t = \{y \in \{0,1\}^{\mathbb{N}} : y \text{ is an extension of } t\}$. The Farah-Zapletal game associated with a set $A \subseteq \{0,1\}^{\mathbb{N}}$ is defined as follows:

Lasker	(S_1^1)		(S_2^1, S_2^2)		$\left(S_{3}^{1},S_{3}^{2},S_{3}^{3} ight)$		
							•
Steinitz		x_1		x_2		x_3	

On the first move, Lasker plays a system S_1^1 (possibly empty) consisting of some sets of the form U_t where $t \in \{0, 1\}$. Then, Steinitz plays $x_1 \in \{0, 1\}$. On the second move, Lasker plays two systems S_2^1 and S_2^2 , both of them consisting of some sets of the form U_t where $t \in \{0, 1\}^4$. Again, Steinitz plays $x_2 \in \{0, 1\}$. On the *n*th move, Lasker plays systems $S_n^1, S_n^2, \ldots, S_n^n$ consisting of some sets of the form U_t where $t \in \{0, 1\}^n$ and Steinitz plays $x_n \in \{0, 1\}$. After a run of this game, we obtain a point $x = (x_n)_{n=1}^{\infty} \in \{0, 1\}^{\mathbb{N}}$ constructed by Steinitz and a σ -porous set

$$C = \bigcup_{k=1}^{\infty} \left(\left\{ y \in \{0,1\}^{\mathbb{N}} \colon \{0,1\}^{\mathbb{N}} \setminus \bigcup_{n=k}^{\infty} \bigcup_{n=k} S_n^k \text{ is porous at } y \right\} \setminus \bigcup_{n=k}^{\infty} \bigcup_{n=k} S_n^k \right)$$

constructed by Lasker. Steinitz wins if $x \in A \setminus C$, Lasker wins in the opposite case. There is a claim [8, Claim 4.21] saying that Lasker has a winning strategy in the game H(A) if and only if the set A is σ -porous.

Later, D. Rojas-Rebolledo generalized the ideas from [8] and managed to find a similar game which characterizes σ -porosity and also σ -strong porosity in any zero-dimensional compact metric space (see [16]).

Let (X, d) be a nonempty metric space and let P be a porosity-like relation on X. In this subsection, we associate an infinite game G(A) (inspired by the game from [8]) with any subset A of X. This is a game between Boulder and Sisyfos (by using these names, we

follow the original terminology of J. Zapletal) where Boulder has a similar role to Steinitz in the game above and Sisyfos corresponds with Lasker. The game is defined as follows:

Boulder
$$B_1$$
 B_2 B_3
Sisyfos (S_1^1) (S_2^1, S_2^2) (S_3^1, S_3^2, S_3^3)

On the first move, Boulder plays a nonempty open set $B_1 \subseteq X$ such that diam $B_1 < \infty$ and Sisyfos plays an open set $S_1^1 \subseteq B_1$. On the second move, Boulder plays a nonempty open set B_2 such that $B_2 \subseteq B_1$ and diam $B_2 \leq \frac{1}{2}$ diam B_1 and Sisyfos plays open sets $S_2^1 \subseteq B_2$ and $S_2^2 \subseteq B_2$. On the *n*th move, n > 1, Boulder plays a nonempty open set B_n such that $B_n \subseteq B_{n-1}$ and diam $B_n \leq \frac{1}{2}$ diam B_{n-1} and Sisyfos plays open sets $S_n^1 \subseteq B_n, S_n^2 \subseteq B_n, \ldots, S_n^n \subseteq B_n$. Sisyfos wins the run if at least one of the following two conditions is satisfied:

- (i) $\bigcap_{n=1}^{\infty} B_n \cap A = \emptyset,$
- (ii) $\bigcap_{n=1}^{\infty} B_n = \{x\}$ and there exists $m \in \mathbb{N}$ such that $x \in X \setminus \bigcup_{n=m}^{\infty} S_n^m$ and $P\left(x, X \setminus \bigcup_{n=m}^{\infty} S_n^m\right)$.

Boulder wins in the opposite case.

In Section 1.2.2, we characterize σ -P-porous sets in X via this game by proving the following theorem.

Theorem 1.2.1. Sisyfos has a winning strategy in the game G(A) if and only if A is a σ -P-porous set.

Since virtually all types of porosities can be considered as porosity-like relations (namely ordinary porosity, symmetric porosity in \mathbb{R} , strong porosity, right and left porosity in \mathbb{R}), this is a more general result than in [8] and [16] in the assumptions both on the metric space X and on the porosity-like relation P.

1.2.2 Proof of the main theorem.

In this subsection, we prove Theorem 1.2.1. Let us fix a nonempty metric space (X, d), a porosity-like relation P on X and $A \subseteq X$ throughout this subsection.

We say that a finite (also empty) sequence (B_1, B_2, \ldots, B_i) of nonempty open sets in X is good if $B_{n+1} \subseteq B_n$, diam $B_1 < \infty$ and diam $B_{n+1} \leq \frac{1}{2}$ diam B_n , $n = 1, 2, \ldots, i-1$. So a finite sequence of nonempty open sets in X is good if and only if Boulder can play the set B_n on his *n*th move, $n = 1, 2, \ldots, i$ (this is clearly independent of Sisyfos' moves). If $T = (B_1, B_2, \ldots, B_i)$ is a good sequence of nonempty open sets, we say that a run of the game G(A) is T-compatible if Boulder played the sets B_1, B_2, \ldots, B_i in sequence on his first i moves.

If Boulder played the sets B_n , $n \in \mathbb{N}$, in a run of the game G(A) and $\bigcap_{n=1}^{\infty} B_n = \{x\}$ then x is called an outcome of the game. If Sisyfos wins the game by satisfying (ii) for some $m \in \mathbb{N}$, then every such m is called a witness of Sisyfos' victory.

Let ρ be a strategy for Sisyfos in the game G(A). For $m \in \mathbb{N} \cup \{0\}$ and a good sequence $T = (B_1, B_2, \ldots, B_i)$, we denote by M_m^T the set of all

$$x \in \begin{cases} A & \text{if } i = 0, \\ A \cap B_i & \text{if } i > 0 \end{cases}$$

such that in every run V of the game G(A) such that

- the outcome of V is x,
- V is T-compatible,
- Sisyfos followed the strategy ρ ,

all the witnesses of Sisyfos' victory (if there exist any) are greater than m. The set M_m^T depends also on the strategy ρ . This will not cause any difficulties since if we speak about this set later, the strategy ρ is fixed.

Let Boulder and Sisyfos play a run of the game G(A). Let

$$V = (B_1, \mathcal{S}_1, B_2, \mathcal{S}_2, \ldots),$$

$$\mathcal{S}_n = (S_n^1, S_n^2, \ldots, S_n^n), \ n \in \mathbb{N},$$

where Boulder played the set B_n and Sisyfos played the sets $S_n^1, S_n^2, \ldots, S_n^n$ on the *n*th move of the run. Then we will refer to the run itself by V and if we speak about the set B_n or about the set S_n^m , $m \in \{1, 2, \ldots, n\}$, $n \in \mathbb{N}$, we just use the symbols $B_n(V)$ and $S_n^m(V)$, respectively.

First of all, we prove the following two lemmata. Lemma 1.2.2 is well known at least for ordinary porosity.

Lemma 1.2.2. Let \mathcal{V} be a σ -discrete system of σ -P-porous sets in X. Then $\bigcup \mathcal{V}$ is also σ -P-porous.

Proof. Let $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where \mathcal{V}_n is a discrete system for every $n \in \mathbb{N}$. Let us fix $n \in \mathbb{N}$ and $x \in X$. There exists r > 0 such that B(x,r) intersects at most one set from the system \mathcal{V}_n . Therefore $B(x,r) \cap \bigcup \mathcal{V}_n$ is a σ -P-porous set. By Theorem 1.1.4, the set $\bigcup \mathcal{V}_n$ is σ -P-porous. Finally,

$$\bigcup \mathcal{V} = \bigcup_{n=1}^{\infty} \bigcup \mathcal{V}_n$$

is σ -P-porous as well.

Lemma 1.2.3. Let ρ be a strategy for Sisyfos in the game G(A). Let

$$T_0 = (B_1, B_2, \dots, B_i)$$

be a good sequence of nonempty open sets and let $m \in \mathbb{N} \cup \{0\}$. Then there exist a P-porous set $N_m^{T_0}$ and a σ -discrete system \mathcal{E} of subsets of X such that

- (a) $M_m^{T_0} = N_m^{T_0} \cup \bigcup \mathcal{E},$
- (b) for every $E \in \mathcal{E}$, there exists a finite sequence T of nonempty open sets in X such that $T_0^{\wedge}T$ is good and $E \subseteq M_{m+1}^{T_0^{\wedge}T}$.

Proof. Whenever we speak about a run of the game G(A) in this proof, we suppose that Sisyfos followed the strategy ρ in the run. Let us denote

 $Z = \bigcup \left\{ S_n^{m+1}(V) : n \ge m+1, V \text{ is a } T_0 \text{-compatible run of the game } G(A) \right\}.$

For every $x \in Z$, let us fix $n(x) \ge m+1$ and a T_0 -compatible run V(x) of the game G(A) such that x lies in the open set $S_{n(x)}^{m+1}(V(x))$. For $x \in Z$, let us denote

$$T(x) = (B_{i+1}(V(x)), B_{i+2}(V(x)), \dots, B_{n(x)}(V(x))).$$

Now, whenever $y \in S_{n(x)}^{m+1}(V(x))$ for some $x \in Z$ and V' is a $T_0^{\wedge}T(x)$ -compatible run giving y as its outcome then V' coincides with V(x) in its first n(x) moves, in particular $S_{n(x)}^{m+1}(V') = S_{n(x)}^{m+1}(V(x))$, and so $y \notin X \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V')$ and m+1 is not a witness of Sisyfos' victory in the run V'. Thus, if $y \in S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0}$ then also $y \in M_{m+1}^{T_0^{\wedge}T(x)}$, so we have $S_{n(x)}^{m+1}(V(x)) \cap M_m^{T_0} \subseteq M_{m+1}^{T_0^{\wedge}T(x)}$. (1.1)

Now, if \mathcal{B} is a σ -discrete basis of open sets in X (whose existence is guaranteed by Theorem 1.1.6) then the system

$$\mathcal{E}' = \left\{ G \in \mathcal{B} \colon G \subseteq S_{n(x)}^{m+1}(V(x)) \text{ for some } x \in Z \right\}$$

is a σ -discrete covering of Z. We define

$$\mathcal{E} = \left\{ M_{m+1}^{T_0} \right\} \cup \left\{ G \cap M_m^{T_0} \colon G \in \mathcal{E}' \right\}$$

and

$$N_m^{T_0} = M_m^{T_0} \setminus \left(Z \cup M_{m+1}^{T_0} \right).$$

The system \mathcal{E} is obviously σ -discrete and $M_m^{T_0} = N_m^{T_0} \cup \bigcup \mathcal{E}$. Moreover, if $E \in \mathcal{E}$ then either $E = M_{m+1}^{T_0} = M_{m+1}^{T_0 \wedge \emptyset}$ or $E = G \cap M_m^{T_0}$ for some $G \in \mathcal{E}'$. In the latter case, there exists $x \in Z$ such that

$$G \subseteq S_{n(x)}^{m+1}\left(V\left(x\right)\right)$$

and so

$$E \subseteq S_{n(x)}^{m+1}\left(V\left(x\right)\right) \cap M_{m}^{T_{0}} \subseteq M_{m+1}^{T_{0} \wedge T(x)}$$

where the last inclusion is due to (1.1).

It only remains to show that the set $N_m^{T_0}$ is *P*-porous. Let us choose $x \in N_m^{T_0}$ arbitrarily. Then $x \in M_m^{T_0} \setminus M_{m+1}^{T_0}$ and so there exists a T_0 -compatible run *V* of the game *G*(*A*) giving *x* as its outcome such that m + 1 is a witness of Sisyfos' victory in the run *V*, in particular

$$P\left(x, X \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V)\right).$$

But we have

$$N_m^{T_0} \subseteq X \setminus Z \subseteq X \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V)$$

and so it follows from (P1) (see Definition 1.1.1) that $P(x, N_m^{T_0})$.

Proof of Theorem 1.2.1. First, let us assume that $A = \bigcup_{n=1}^{\infty} A_n$ where A_n is a *P*-porous set for every $n \in \mathbb{N}$. On his *n*th move, let Sisyfos play $S_n^j = \emptyset$ for j < n and $S_n^n = B_n \setminus \overline{A_n}$. Let Boulder and Sisyfos play a run of the game G(A) such that Sisyfos follows the described strategy. We may assume that $\bigcap_{n=1}^{\infty} B_n = \{x\}$ and $x \in A$ because otherwise Sisyfos wins by (i). Then there exists $m \in \mathbb{N}$ such that $x \in A_m$. We have

$$X \setminus \bigcup_{n=m}^{m} S_n^m = \overline{A_m} \cup (X \setminus B_m)$$
(1.2)

and therefore

$$x \in A_m \subseteq X \setminus \bigcup_{n=m}^{\infty} S_n^m$$

Further, *P*-porosity of A_m implies that $P(x, A_m)$. But this is equivalent to $P(x, \overline{A_m})$ by (P3) (see Definition 1.1.1) and by (P2), this is equivalent to $P(x, \overline{A_m} \cup (X \setminus B_m))$ since $x \in B_m$. So we have $P\left(x, X \setminus \bigcup_{n=m}^{\infty} S_n^m\right)$ by (1.2). Therefore, Sisyfos wins by (ii) with *m* as a witness and the described strategy is winning.

Now, let us assume that Sisyfos has a winning strategy ρ in the game G(A). Let us denote $E_0 = A$. By Lemma 1.2.3, we have

$$A = E_0 = M_0^{\emptyset} = N_0^{\emptyset} \cup \bigcup \mathcal{E}$$
(1.3)

where N_0^{\emptyset} is *P*-porous and \mathcal{E} is a σ -discrete system of subsets of *X* such that for every $E_1 \in \mathcal{E}$, there exists a good sequence $T(E_1)$ such that $E_1 \subseteq M_1^{T(E_1)}$. Now, for every $E_1 \in \mathcal{E}$ we have

$$E_1 \subseteq M_1^{T(E_1)} = N_1^{T(E_1)} \cup \bigcup \mathcal{E}^{E_1}$$
(1.4)

where $N_1^{T(E_1)}$ is *P*-porous and \mathcal{E}^{E_1} is a σ -discrete system of subsets of *X* such that for every $E_2 \in \mathcal{E}^{E_1}$, there exists a finite sequence $T(E_1, E_2)$ of nonempty open sets such that $T(E_1) \wedge T(E_1, E_2)$ is good and $E_2 \subseteq M_2^{T(E_1) \wedge T(E_1, E_2)}$. Suppose that for some $k \in \mathbb{N}$, we already have $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \ldots, E_k \in \mathcal{E}^{E_1, E_2, \ldots, E_{k-1}}$ and finite sequences $T(E_1)$, $T(E_1, E_2), \ldots, T(E_1, E_2, \ldots, E_k)$ such that

$$H := T(E_1)^{\wedge} T(E_1, E_2)^{\wedge} \dots^{\wedge} T(E_1, E_2, \dots, E_k)$$

is good and $E_k \subseteq M_k^H$. Then we have

$$E_k \subseteq M_k^H = N_k^H \cup \bigcup \mathcal{E}^{E_1, E_2, \dots, E_k}$$

where N_k^H is *P*-porous and $\mathcal{E}^{E_1, E_2, \dots, E_k}$ is a σ -discrete system of subsets of *X* such that for every $E_{k+1} \in \mathcal{E}^{E_1, E_2, \dots, E_k}$, there exists a finite sequence $T(E_1, E_2, \dots, E_{k+1})$ of nonempty open sets such that the sequence

$$H^{\wedge}T(E_1, E_2, \dots, E_{k+1}) = T(E_1)^{\wedge}T(E_1, E_2)^{\wedge} \dots^{\wedge}T(E_1, E_2, \dots, E_{k+1})$$

is good and

$$E_{k+1} \subseteq M_{k+1}^{T(E_1)^{\wedge} T(E_1, E_2)^{\wedge} \dots^{\wedge} T(E_1, E_2, \dots, E_{k+1})}$$

By iterating this process, we get a system of P-porous sets

$$\mathcal{U} = \left\{ N_k^{T(E_1)^{\wedge}T(E_1, E_2)^{\wedge} \dots^{\wedge}T(E_1, E_2, \dots, E_k)} \cap \bigcap_{i=0}^k E_i : k \in \mathbb{N} \cup \{0\}, E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \dots, E_k \in \mathcal{E}^{E_1, E_2, \dots, E_{k-1}} \right\}$$

We show that $A \subseteq \bigcup \mathcal{U}$. Suppose that this is not true and so there exists $x \in A \setminus \bigcup \mathcal{U}$. By (1.3), there exists $E_1 \in \mathcal{E}$ such that $x \in E_1 \subseteq M_1^{T(E_1)}$. By (1.4), there exists $E_2 \in \mathcal{E}^{E_1}$ such that $x \in E_2 \subseteq M_2^{T(E_1)^{\wedge}T(E_1,E_2)}$. Next, there exists $E_3 \in \mathcal{E}^{E_1,E_2}$ such that $x \in E_3 \subseteq M_3^{T(E_1)^{\wedge}T(E_1,E_2)^{\wedge}T(E_1,E_2,E_3)}$. Continuing in this way, we get that there exists a sequence $(E_k)_{k=1}^{\infty}$ where $E_1 \in \mathcal{E}$ and $E_k \in \mathcal{E}^{E_1,E_2,\dots,E_{k-1}}$ for k > 1 such that

$$x \in E_k \subseteq M_k^{T(E_1)^{\wedge}T(E_1,E_2)^{\wedge}\dots^{\wedge}T(E_1,E_2,\dots,E_k)}$$

for every $k \in \mathbb{N}$. Therefore Boulder can play a run of the game G(A) in the following way. He plays all the sets from $T(E_1)$ in sequence on his first moves, then all the sets from $T(E_1, E_2)$ and so on. (If there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that all the sequences $T(E_1, E_2, \ldots, E_k), k > k_0$, are empty then the sequence

$$T(E_1)^{\wedge}T(E_1, E_2)^{\wedge}\ldots = T(E_1)^{\wedge}T(E_1, E_2)^{\wedge}\ldots^{\wedge}T(E_1, E_2, \ldots, E_{k_0})$$

is finite. Then Boulder can finish the run arbitrarily such that the outcome of the run is x.) After such a run, x is its outcome and any $m \in \mathbb{N}$ cannot be a witness of Sisyfos' victory since $x \in M_m^{T(E_1)^{\wedge}T(E_1,E_2)^{\wedge}...^{\wedge}T(E_1,E_2,...,E_m)}$ for every $m \in \mathbb{N}$. This is a contradiction with the assumption that the strategy ρ is winning for Sisyfos.

By (P1), it suffices to show that $\bigcup \mathcal{U}$ is a σ -*P*-porous set. For $k \in \mathbb{N} \cup \{0\}$ and $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \ldots, E_k \in \mathcal{E}^{E_1, E_2, \ldots, E_{k-1}}$, let us denote

$$Q(E_1, E_2, \dots, E_k) = N_k^{T(E_1)^{\wedge} T(E_1, E_2)^{\wedge} \dots^{\wedge} T(E_1, E_2, \dots, E_k)} \cap \bigcap_{i=0}^k E_i.$$

Then we have $\bigcup \mathcal{U} = \bigcup_{k=0}^{\infty} \bigcup \mathcal{U}_k$ where

$$\mathcal{U}_{k} = \left\{ Q(E_{1}, E_{2}, \dots, E_{k}) \colon E_{1} \in \mathcal{E}, E_{2} \in \mathcal{E}^{E_{1}}, \dots, E_{k} \in \mathcal{E}^{E_{1}, E_{2}, \dots, E_{k-1}} \right\}.$$

It is obviously sufficient to prove that $\bigcup \mathcal{U}_k$ is a σ -*P*-porous set for every $k \in \mathbb{N} \cup \{0\}$. For k = 0 we know that $\bigcup \mathcal{U}_0 = N_0^{\emptyset}$ which is a *P*-porous set. Now suppose that k > 0. To finish the proof, it suffices to prove the following claim and use it for j = 1.

Claim 1.2.4. For every $j \in \{1, 2, ..., k\}$ and for every $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, ..., E_{j-1} \in \mathcal{E}^{E_1, E_2, ..., E_{j-2}}$, the set

$$\bigcup_{E_j \in \mathcal{E}^{E_1, E_2, \dots, E_{j-1}}} \bigcup_{E_{j+1} \in \mathcal{E}^{E_1, E_2, \dots, E_j}} \dots \bigcup_{E_k \in \mathcal{E}^{E_1, E_2, \dots, E_{k-1}}} Q(E_1, E_2, \dots, E_k)$$

is σ -*P*-porous.

Proof. For j = k and for every $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, \ldots, E_{k-1} \in \mathcal{E}^{E_1, E_2, \ldots, E_{k-2}}$, the set

$$\bigcup_{E_k \in \mathcal{E}^{E_1, E_2, \dots, E_{k-1}}} Q(E_1, E_2, \dots, E_k)$$

is a union of a σ -discrete system (since $\mathcal{E}^{E_1, E_2, \dots, E_{k-1}}$ is σ -discrete) of *P*-porous sets (since $N_k^{T(E_1)^{\wedge}T(E_1, E_2)^{\wedge} \dots^{\wedge}T(E_1, E_2, \dots, E_k)}$ is *P*-porous). By Lemma 1.2.2, this set is σ -*P*-porous.

Let us assume that the assertion holds for j + 1 where $j \in \{1, 2, ..., k - 1\}$ and let $E_1 \in \mathcal{E}, E_2 \in \mathcal{E}^{E_1}, ..., E_{j-1} \in \mathcal{E}^{E_1, E_2, ..., E_{j-2}}$ be given. Then

$$\bigcup_{E_j \in \mathcal{E}^{E_1, E_2, \dots, E_{j-1}}} \left(\bigcup_{E_{j+1} \in \mathcal{E}^{E_1, E_2, \dots, E_j}} \dots \bigcup_{E_k \in \mathcal{E}^{E_1, E_2, \dots, E_{k-1}}} Q(E_1, E_2, \dots, E_k) \right)$$

is a union of σ -discrete system (since $\mathcal{E}^{E_1, E_2, \dots, E_{j-1}}$ is σ -discrete) of σ -*P*-porous sets (the assumption for j + 1). By Lemma 1.2.2, it is also σ -*P*-porous.

1.3 Compact non- σ -porous subsets of non- σ -porous sets

This section is based on the paper [4]. It contains three subsections. Subsection 1.3.1 corresponds to Section 1 (Introduction) of [4] but is shortened here, such that we do not repeat some facts stated already in previous sections. Section 2 (Preliminaries) of [4] is entirely left out here, for the same reason. Subsections 1.3.2 and 1.3.3 are almost identical with Sections 3 (Main result) and 4 (Applications to concrete porosities) of [4], respectively. The only changes are made in the hierarchy of sections and subsections, cross-referencing between them, refining one argument in the proof of Lemma 1.3.3 and correcting one typo.

1.3.1 Introduction

Here we are interested in structural properties of σ -ideals of σ -porous type. More precisely, the main question we will consider is the following one.

Question. Let A be an analytic subset of a metric space X and \mathcal{I} be a σ -ideal of subsets of X. Suppose that $A \notin \mathcal{I}$. Does there exist a closed set $F \subset A$ which is not in \mathcal{I} ?

This question was posed by L. Zajíček in [21] (for a Borel set A) for a σ -ideal \mathcal{I} of σ -porous subsets of X. An affirmative answer was given independently by J. Pelant (for any topologically complete metric space X) and M. Zelený (for any compact metric space X). Their results are demonstrated in a joint paper [25] which combines the original idea of J. Pelant (giving an explicit construction of the set F) and techniques developed by M. Zelený. The case of some other types of porosity (including the ordinary one in a locally compact metric space X but also $\langle g \rangle$ -porosity in a locally compact metric space X or symmetrical porosity in \mathbb{R}) was solved (also affirmatively) by M. Zelený and L. Zajíček in [26]. They offer a less complicated method of construction of F using so called 'porosity-like' relations. Their non-constructive proof uses tools from Descriptive Set Theory. However, the authors admitted that their method cannot be easily applied to strong porosity and so Question for strong porosity still remained open (even in a compact metric space X).

Later on, J. Zapletal characterized σ -porous sets in $2^{\mathbb{N}}$ via an infinite game (which was already reminded in Section 1.2). This game was used to reprove the positive answer to Question in this particular case ([8, Example 4.20]). The only attempt to answer Question for strong porosity (and ordinary porosity once again) was made by D. Rojas-Rebolledo, who generalized in [16] the ideas from [8]. He managed to give an affirmative answer to Question in any zero-dimensional compact metric space X.

Our aim is to generalize results of [8, 16, 26] in two directions. We give an affirmative answer to Question in spaces which are more general than those considered in [8, 16] and also for σ -ideals of σ -porous type which are not included in [26]. The main result of this section is Theorem 1.3.11. The complete formulation is a little bit technical so let us formulate the result in an informal way. Let X be a compact metric space and Q be a porosity-like relation on X satisfying some additional conditions. Then every analytic non- σ -Q-porous subset A of X contains a compact non- σ -Q-porous subset.

To prove this we proceed as follows. We modify the infinite game G(A) from Section 1.2 (now for a subset A of a compact metric space) to a more complicated form such that this modified game still characterizes σ -Q-porosity. Now consider non- σ -Q-porous analytic subset A of X. By the result of Farah and Zapletal ([8, Theorem 4.16]), we may assume that A is non- σ -Q-porous and Borel. Then we show that our game with A is determined using Martin Determinacy Theorem. The set A is non- σ -Q-porous and thus the second player does not have a winning strategy. By determinacy the first player has to have a winning strategy. Using a winning strategy of the first player we find a compact subset K of A such that the first player still has a winning strategy in the game played with K. This means that the second player does not have a winning strategy and so K is not σ -Q-porous, which will complete the proof.

In Subsection 1.3.3, we apply the last result to concrete porosities and obtain an (affirmative) answer to several different variants of Question. Namely, we deal with ordinary porosity, strong porosity, strong right porosity, and 1-symmetrical porosity. As it is described earlier, the first result has been already known but the method used in our work (based on an infinite game) aspires to be more elegant and easier than the known proofs. The other results are new. Finally, we show that there exists a closed set in \mathbb{R} which is σ -(1 - ε)-symmetrically porous for every $\varepsilon \in (0, 1)$ but which is not σ -1-symmetrically porous. This answers a question posed by M. J. Evans and P. D. Humke in [7].

1.3.2 Main result

The class \mathfrak{Q}

Now we define the class of porosity-like relations for which we prove affirmative answer to our Question in compact metric spaces. This class includes many interesting cases. The definition is technical but verification of conditions in concrete cases is straightforward.

Definition 1.3.1. Let (X, d) be a nonempty metric space. We say that a point-set relation Q on X belongs to the class $\mathfrak{Q}(X)$ if there are point-set relations Q^s and $Q_r^{s,q}$ on $X, s \in \mathbb{N}$, $r > 0, q \in (0, 1)$, such that the following conditions are satisfied:

- $(\mathrm{R1}) \ Q^s = \bigcap_{0 < q < 1} \bigcap_{R > 0} \bigcup_{0 < r < R} Q_r^{s,q} \text{ and } Q = \bigcup_{s \in \mathbb{N}} Q^s,$
- (R2) if $Q_r^{s,q}(x,A)$ and $0 < w < \frac{q}{2s}$ then $Q_r^{s,q-2sw}(x,B(A,rw))$,
- (R3) if $B \subset A$ and $Q_r^{s,q}(x, A)$ then $Q_r^{s,q}(x, B)$,
- (R4) we have $Q_r^{s,q}(x, A)$ if and only if $Q_r^{s,q}(x, A \cap B(x, 2r))$,
- (R5) the set $\{(x,r) \in X \times (0,\infty) : Q_r^{s,q}(x,A)\}$ is open in $X \times (0,\infty)$.

Convention 1.3.2. Throughout this subsection we will work with a fixed compact metric space K with a fixed point-set relation $Q \in \mathfrak{Q}(K)$. The corresponding point-set relations $Q_r^{s,q}, Q^s$ witnessing $Q \in \mathfrak{Q}(K)$ are fixed as well. We also fix a set $A \subset K$.

Lemma 1.3.3. Let $s \in \mathbb{N}$ and r > 0. Then we have:

(M) if $0 < q_1 < q_2 < 1$ and $Q_r^{s,q_2}(x, A)$, then $Q_r^{s,q_1}(x, \overline{A})$, in particular $Q_r^{s,q_1}(x, A)$,

(P) Q^s is a porosity-like relation; consequently, Q is a porosity like-relation.

Proof. (M) By (R2) applied to $w = \frac{q_2-q_1}{2s}$, we get $Q_r^{s,q_1}\left(x, B\left(A, \frac{r(q_2-q_1)}{2s}\right)\right)$. By (R3), we have $Q_r^{s,q_1}(x, \overline{A})$. Using (R3) we also get $Q_r^{s,q_1}(x, A)$.

We verify (P1)–(P3) from Definition 1.1.1 for $P = Q^s$ to prove (P).

(P1) This property is an immediate consequence of (R1) and (R3)

(P2) Suppose that $Q^s(x, A \cap B(x, r_0))$ for some $r_0 > 0$. By (R1), there exist sequences $(q_k)_{k=1}^{\infty}$ of real numbers from (0, 1) and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that $\lim_{k\to\infty} q_k = 1$, $\lim_{k\to\infty} r_k = 0$, and $Q_{r_k}^{s,q_k}(x, A \cap B(x, r_0))$ for every $k \in \mathbb{N}$. There exists $k_0 \in \mathbb{N}$ such that $2r_k \leq r_0$ for every $k \geq k_0$. Then $Q_{r_k}^{s,q_k}(x, A \cap B(x, 2r_k))$ for $k \geq k_0$ by (R3) and so $Q_{r_k}^{s,q_k}(x, A)$ for $k \geq k_0$ by (R4). Using (R1) and (M), we get $Q^s(x, A)$. The opposite implication follows by (P1).

(P3) Suppose that $Q^s(x, A)$. Choose $q \in (0, 1)$ and R > 0. By (R1), there exists $0 < \tilde{r} < R$ such that $Q^{s,q}_{\tilde{r}}(x, A)$. By (M) we have $Q^{s,q'}_{\tilde{r}}(x, \overline{A})$ for every 0 < q' < q. Using (R1) we get $Q^s(x, \overline{A})$. The opposite implication follows by (P1).

The fact that Q is a porosity-like relation follows directly from (R1).

Boulder-Sisyfos game

For the rest of this subsection, let us fix sequences

$$(R_n)_{n=1}^{\infty} \text{ and } (a_n)_{n=1}^{\infty}$$
 (1.5)

of real numbers from $(0, \infty)$ such that for every $n \in \mathbb{N}$ we have

$$R_{n+1} \le 2^{-(n+2)} R_n \tag{1.6}$$

and

$$\lim_{n \to \infty} \frac{a_n}{R_{n+2}} = 0. \tag{1.7}$$

Let D_n , $n \in \mathbb{N}$, be a finite a_n -net in K, i.e., a finite subset of K such that $K = \bigcup \{B(y, a_n) \colon y \in D_n\}$. Let $\mathcal{M}_n = \{B(y, a_n) \colon y \in D_n\}$.

We define a game H(A) for two players, who will be called Boulder and Sisyfos. These names were used by J. Zapletal in the original version of his game. The game is played as follows:

Boulder	B_1		B_2		B_3		
Sisyfos		(S_1^1)		(S_2^1, S_2^2)		(S_3^1, S_3^2, S_3^3)	•

On the first move, Boulder plays an open ball $B_1 \subset K$ with radius R_1 and Sisyfos plays an open set $S_1^1 \subset B_1$ where S_1^1 is a union (possibly empty) of some balls from the system \mathcal{M}_1 introduced in the previous paragraph. On the second move, Boulder plays an open ball B_2 with center in $\frac{1}{2} \star B_1$ and radius R_2 and Sisyfos plays two open sets S_2^1 and S_2^2 such that $S_2^1 \cup S_2^2 \subset B_2$ where S_2^j is a union of some balls from \mathcal{M}_2 , j = 1, 2. On the *n*th move, n > 1, Boulder plays an open ball B_n with center in $(1 - 2^{-(n-1)}) \star B_{n-1}$ and radius R_n and Sisyfos replies by playing open sets $S_n^1, S_n^2, \ldots, S_n^n$ such that $\bigcup_{j=1}^n S_n^j \subset B_n$ where S_n^j is a union of some balls from \mathcal{M}_n , $j = 1, 2, \ldots, n$.

We use the above notation in the next lemma.

Lemma 1.3.4. For every $n \in \mathbb{N}$, we have $B_{n+1} \subset \left(1 - \frac{1}{2^{n+1}}\right) \star B_n$.

Proof. Suppose that x_n is the center of B_n , x_{n+1} is the center of B_{n+1} , and $z \in B_{n+1}$. Then we have

$$d(z, x_n) \le d(z, x_{n+1}) + d(x_{n+1}, x_n) < R_{n+1} + (1 - 2^{-n}) R_n$$

$$\le (2^{-(n+2)} + 1 - 2^{-n}) R_n = (1 - 3 \cdot 2^{-(n+2)}) R_n < (1 - 2^{-(n+1)}) R_n.$$

By (1.6), we have $\lim_{n\to\infty} \dim B_n = 0$. Using this fact, Lemma 1.3.4, and the compactness of K, when a run of the game is over, we get a unique point x lying in the intersection of the balls B_n , $n \in \mathbb{N}$, played by Boulder. We call this point an *outcome* of the run. Sisyfos wins if at least one of the following conditions is satisfied:

- (a) $x \notin A$,
- (b) there exists $m \in \mathbb{N}$ such that one can find $s \in \mathbb{N}$, sequences $(n_k)_{k=1}^{\infty}$ of integers from $\{m, m+1, \ldots\}, (q_k)_{k=1}^{\infty}$ of real numbers from (0, 1), and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that
 - $x \in K \setminus \bigcup_{n=m}^{\infty} S_n^m$,
 - $\lim_{k\to\infty} n_k = \infty$,
 - $\lim_{k\to\infty} q_k = 1$,
 - $r_k \leq 2^{-(n_k+3)} R_{n_k}, k \in \mathbb{N},$
 - $Q_{r_k}^{s,q_k}(x, K \setminus S_{n_k}^m), k \in \mathbb{N}.$

Boulder wins in the opposite case. If condition (b) is satisfied for some $m \in \mathbb{N}$, then m is called a *witness of Sisyfos' victory*.

At first sight, condition (b) looks very complicated. For a better understanding, we can observe that it is stronger than the assertion that $Q^s(x, K \setminus \bigcup_{n=m}^{\infty} S_n^m)$ by (R1), (R3), and (M).

Characterization of σ -Q-porosity via the infinite game

In this subsubsection we show that the notion of σ -Q-porosity, where Q is the fixed porositylike relation belonging to the class \mathfrak{Q} , can be characterized by existence of a winning strategy for Sisysfos in our game. To this end we will need a couple of auxiliary notions.

We say that a finite (also empty) sequence of open balls (B_1, B_2, \ldots, B_i) is good if B_{n+1} is centered at $(1-2^{-n}) \star B_n$, $n = 1, \ldots, i-1$, and the radius of B_n equals R_n , $n = 1, \ldots, i$. That is, a finite sequence of open balls is good if the rules of the game H(A) allow Boulder to play the ball B_n on his *n*th move, $n = 1, 2, \ldots, i$.

For $n, m \in \mathbb{N}$ we define

$$d_n^m = \begin{cases} 1 - 2^{-n+m-1} & \text{if } m \le n, \\ \frac{1}{4} & \text{if } m > n. \end{cases}$$

Let σ be a strategy for Sisyfos in the game H(A). If $k \in \mathbb{N} \cup \{0\}$ and $l \in \mathbb{N}$ then we say that a good sequence of open balls (B_1, B_2, \ldots, B_i) is (k, l)-good (with respect to the strategy σ) if there exists a run of the game H(A) such that the following conditions hold:

- Sisyfos followed the strategy σ ,
- Boulder played the ball B_n on his *n*th move, n = 1, 2, ..., i,
- if k < i then the following conditions are satisfied for every positive $n \in \{k, k + 1, \dots, i-1\}$:
 - (H1) if $[l > n \text{ or } (l \le n \text{ and } S_n^l \cap (d_n^l \star B_n) = \emptyset)]$ then the center of B_{n+1} lies in $d_n^{l+1} \star B_n$,
 - (H2) if $[l \leq n \text{ and } S_n^l \cap (d_n^l \star B_n) \neq \emptyset]$ then the center of B_{n+1} lies in $d_n^l \star B_n$.

Let Boulder and Sisyfos play a run of the game H(A). Let $V = (B_1, S_1, B_2, S_2, ...)$, and $S_n = (S_n^1, S_n^2, ..., S_n^n)$, $n \in \mathbb{N}$, where Boulder played the ball B_n and Sisyfos played the sets $S_n^1, S_n^2, ..., S_n^n$ on the *n*th move of the run. Then we will refer to the run itself by V and if we talk about the ball B_n or about the set S_n^m , we just use the symbols $B_n(V)$ and $S_n^m(V)$, respectively.

We say that a run V of the game H(A) is (k, l)-good if Sisyfos followed the strategy σ and the sequence $(B_1(V), B_2(V), \ldots, B_j(V))$ is (k, l)-good for every $j \in \mathbb{N}$.

It is easy to see that if a finite sequence of open balls is (k, l+1)-good then it is also (k, l)-good since $d_n^{l+2} \star B_n \subset d_n^{l+1} \star B_n \subset d_n^l \star B_n$. It follows that if $l_1 > l_2$ and a finite sequence of open balls (a run of the game H(A), respectively) is (k, l_1) -good then it is also (k, l_2) -good.

If $T = (B_1, B_2, \ldots, B_i)$ is a good sequence of open balls, we say that a run V of the game H(A) is T-compatible if $B_n(V) = B_n$ for every $n \in \{1, 2, \ldots, i\}$.

For $m \in \mathbb{N} \cup \{0\}$ and a good sequence of open balls $T = (B_1, B_2, \dots, B_i)$, we denote by $M_m(T)$ the set of all

$$x \in \begin{cases} A & \text{if } T = \emptyset, \text{ i.e., } i = 0, \\ A \cap \left(\frac{1}{4} \star B_i\right) & \text{if } i > 0 \end{cases}$$

such that in every T-compatible (i, m + 1)-good run of the game H(A) giving x as its outcome, all the witnesses of Sisyfos' victory (if there exist any) are greater than m. The set $M_m(T)$ also depends on the set A and on the strategy σ but these will be always fixed.

Lemma 1.3.5. Let σ be a strategy for Sisyfos in the game H(A). Let $T_0 = (B_1, B_2, \ldots, B_i)$ be a good sequence of open balls and $m \in \mathbb{N} \cup \{0\}$. Then there exist a Q-porous set $N_m(T_0)$ and an at most countable collection \mathcal{T} of finite sequences of open balls such that the concatenation $T_0^{\wedge}T$ is (i, m + 1)-good for every $T \in \mathcal{T}$ and

$$M_m(T_0) \subset N_m(T_0) \cup \bigcup \{M_{m+1}(T_0^{\wedge}T) \colon T \in \mathcal{T}\}.$$

Proof. Define $N_m(T_0)$ as the set of all $x \in M_m(T_0)$ such that

- (I) there exists a T_0 -compatible (i, m + 2)-good run of the game H(A) giving x as its outcome such that m + 1 is a witness of Sisyfos' victory,
- (II) for every T_0 -compatible (i, m + 2)-good run V of the game H(A) and for every $n \ge \max\{i, m + 1\}$, we have $x \notin S_n^{m+1}(V) \cap (d_n^{m+1} \star B_n(V))$.

Suppose that $x \in M_m(T_0) \setminus (M_{m+1}(T_0) \cup N_m(T_0))$. By definition of $M_{m+1}(T_0)$ there exists a T_0 -compatible (i, m + 2)-good run with the outcome x and with a witness less or equal m + 1. Since $x \in M_m(T_0)$ and the run is also (i, m + 1)-good, the witness is equal m + 1. Thus condition (I) holds for x. Therefore condition (II) cannot be true by the definition of $N_m(T_0)$, and so there exist a T_0 -compatible (i, m + 2)-good run V(x) of the game H(A)and $n(x) \ge \max\{i, m + 1\}$ such that

$$x \in S_{n(x)}^{m+1}(V(x)) \cap \left(d_{n(x)}^{m+1} \star B_{n(x)}(V(x))\right).$$

Denote $B_j(x) = B(x, R_j)$ for j > n(x) where R_j were fixed in (1.5). Find N(x) > n(x) such that $B_{N(x)}(x) \subset S_{n(x)}^{m+1}(V(x))$ and denote

$$T(x) = (B_{i+1}(V(x)), \dots, B_{n(x)}(V(x)))^{\wedge} (B_{n(x)+1}(x), \dots, B_{N(x)}(x)).$$

Then the sequence $T_0^{\wedge}T(x)$ is (i, m+1)-good. Indeed, the sequence

 $T_0^{\wedge} (B_{i+1}(V(x)), \dots, B_{n(x)}(V(x)))$

is even (i, m+2)-good (as it is an initial segment of Boulder's moves in the (i, m+2)-good run V(x)) and the fact that

$$S_{n(x)}^{m+1}(V(x)) \cap \left(d_{n(x)}^{m+1} \star B_{n(x)}(V(x))\right) \neq \emptyset$$

allows Boulder to use condition (H2) and play the ball with center $x \in d_{n(x)}^{m+1} \star B_{n(x)}(V(x))$ on his (n(x) + 1)st move. Since $B_{N(x)}(x) \subset S_{n(x)}^{m+1}(V(x))$, we see that m + 1 cannot become a witness of Sisyfos' victory in any $T_0^{\wedge}T(x)$ -compatible run of the game H(A). Therefore we have

$$M_m(T_0) \cap \left(\frac{1}{4} \star B_{N(x)}(x)\right) \subset M_{m+1}(T_0^{\wedge}T(x)),$$

and so $x \in M_{m+1}(T_0^{\wedge}T(x))$. By Lindelöf's property, there exists an at most countable set

$$\{x_j \colon j \in \mathbb{N}\} \subset M_m(T_0) \setminus (M_{m+1}(T_0) \cup N_m(T_0))$$

such that $M_m(T_0) \setminus (M_{m+1}(T_0) \cup N_m(T_0))$ is covered by the system $\{\frac{1}{4} \star B_{N(x_j)}(x_j) : j \in \mathbb{N}\}$ of open sets and so it is also covered by the countable system $\{M_{m+1}(T_0 \wedge T(x_j)) : j \in \mathbb{N}\}$. Now, we can define $\mathcal{T} = \{\emptyset\} \cup \{T(x_j) : j \in \mathbb{N}\}$. Then we obviously have

$$M_m(T_0) \subset N_m(T_0) \cup \bigcup \{ M_{m+1}(T_0^{\wedge}T) \colon T \in \mathcal{T} \}.$$

It remains to show that $N_m(T_0)$ is Q-porous. Suppose that $x \in N_m(T_0)$ and V is a T_0 -compatible (i, m+2)-good run of the game H(A) such that x is its outcome and m+1 is a witness of Sisyfos' victory. By condition (b) (see p. 21), this means that there exist $s \in \mathbb{N}$ and sequences $(n_k)_{k=1}^{\infty}$ of integers from $\{m+1, m+2, \ldots\}, (q_k)_{k=1}^{\infty}$ of real numbers from (0, 1), and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that

- $x \in K \setminus \bigcup_{n=m+1}^{\infty} S_n^{m+1}(V),$
- $\lim_{k\to\infty} n_k = \infty$,
- $\lim_{k\to\infty} q_k = 1$,
- $r_k \le 2^{-(n_k+3)} R_{n_k}, k \in \mathbb{N},$
- $Q_{r_k}^{s,q_k}(x, K \setminus S_{n_k}^{m+1}(V)), k \in \mathbb{N}.$

We may assume that $n_k \ge \max\{i, m+2\}$ for every $k \in \mathbb{N}$. We know that the center of $B_{n+1}(V)$ lies in $d_n^{m+2} \star B_n(V)$ for every positive $n \ge i$ by conditions (H1) and (H2). Let us fix $k \in \mathbb{N}$. By condition (R4), we have

$$Q_{r_k}^{s,q_k}\left(x,\left(K\setminus S_{n_k}^{m+1}(V)\right)\cap B(x,2r_k)\right).$$
(1.8)

Since

$$\left(K \setminus S_{n_k}^{m+1}(V)\right) \cap B(x, 2r_k) = \left(K \setminus \left(S_{n_k}^{m+1}(V) \cap B(x, 2r_k)\right)\right) \cap B(x, 2r_k),$$

using (1.8) and (R4) again we get

$$Q_{r_k}^{s,q_k}\left(x, K \setminus \left(S_{n_k}^{m+1}(V) \cap B(x, 2r_k)\right)\right).$$

$$(1.9)$$

By condition (II), we have

$$N_m(T_0) \subset K \setminus \left(S_{n_k}^{m+1}(V) \cap \left(d_{n_k}^{m+1} \star B_{n_k}(V)\right)\right).$$
(1.10)

Now, let y_{n_k} be the center of $B_{n_k}(V)$, y_{n_k+1} be the center of $B_{n_k+1}(V)$, and let us take $z \in B(x, 2r_k)$. Then we have

$$d(z, y_{n_k}) \leq d(z, x) + d(x, y_{n_k+1}) + d(y_{n_k+1}, y_{n_k}) < 2r_k + R_{n_k+1} + d_{n_k}^{m+2} R_{n_k}$$

$$\leq 2^{-(n_k+2)} R_{n_k} + 2^{-(n_k+2)} R_{n_k} + d_{n_k}^{m+2} R_{n_k} = \left(2^{-(n_k+1)} + 1 - 2^{-n_k+m+1}\right) R_{n_k}$$

$$\leq \left(1 - 2^{-n_k+m}\right) R_{n_k} = d_{n_k}^{m+1} R_{n_k}.$$

Therefore we have $B(x, 2r_k) \subset d_{n_k}^{m+1} \star B_{n_k}(V)$, and so

$$K \setminus \left(S_{n_k}^{m+1}(V) \cap \left(d_{n_k}^{m+1} \star B_{n_k}(V) \right) \right) \subset K \setminus \left(S_{n_k}^{m+1}(V) \cap B(x, 2r_k) \right).$$
(1.11)

Finally, we have $Q_{r_k}^{s,q_k}(x, N_m(T_0))$ by (1.9), (1.10), (1.11), and (R3). Therefore also

$$Q^{s}(x, N_{m}(T_{0}))$$

by (R1) and (M), and we have $Q(x, N_m(T_0))$.

Theorem 1.3.6. Sisyfos (i.e., the second player) has a winning strategy in the game H(A) if and only if the set A is σ -Q-porous.

Proof. Suppose first that $A = \bigcup_{n=1}^{\infty} A_n$ such that A_n is *Q*-porous for every $n \in \mathbb{N}$. We define a strategy for Sisyfos as follows. For $n \in \mathbb{N}$ and $m \in \{1, 2, \ldots, n\}$, Sisyfos plays S_n^m as the union of all open balls $B \in \mathcal{M}_n$ for which $B \subset B_n \setminus A_m$, where B_n is the *n*th move of Boulder.

We show that this strategy is winning. Let Boulder and Sisyfos play a run of the game H(A) such that Sisyfos follows this strategy. Let x be an outcome of this run. If $x \notin A$ then Sisyfos satisfies condition (a) (see p. 21) and wins. If $x \in A$ then there exists $m \in \mathbb{N}$ such that $x \in A_m$. Then we have $x \notin \bigcup_{n=m}^{\infty} S_n^m$. Further, since $Q(x, A_m)$, there exists $s \in \mathbb{N}$ such that $Q^s(x, A_m)$, and so we know by condition (R1) that there exist sequences $(q_k)_{k=1}^{\infty}$ of real numbers from (0, 1) and $(r_k)_{k=1}^{\infty}$ of real numbers from $(0, \infty)$ such that

- $\lim_{k\to\infty} q_k = 1$,
- $\lim_{k\to\infty} r_k = 0$,
- $Q_{r_k}^{s,q_k}(x,A_m), k \in \mathbb{N}.$

There also exists $n_0 \ge m$ such that

$$s\frac{2^{n+6}a_n}{R_{n+1}} \le \inf\{q_k \colon k \in \mathbb{N}\}\tag{1.12}$$

for $n \ge n_0$ (where the numbers R_{n+1} and a_n were fixed in (1.5)) since the expression on the right hand side is strictly positive and the expression on the left hand side tends to zero which follows from (1.7) and the estimate (derived from (1.6))

$$0 < s \frac{2^{n+6}a_n}{R_{n+1}} \le s \frac{8a_n}{R_{n+2}}.$$
(1.13)

We may assume that $r_k \leq 2^{-(n_0+3)}R_{n_0}$ for every $k \in \mathbb{N}$. Let us choose $k \in \mathbb{N}$ arbitrarily and define n_k as the greatest integer such that

$$r_k \le 2^{-(n_k+3)} R_{n_k}. \tag{1.14}$$

Obviously, we have $n_k \ge n_0$ and $\lim_{k\to\infty} n_k = \infty$. Since (1.14) does not hold for $n_k + 1$ instead of n_k , we get

$$r_k > 2^{-(n_k+4)} R_{n_k+1} \ge s \frac{4a_{n_k}}{q_k} \tag{1.15}$$

using the estimate (1.12) for $n = n_k$ in the second inequality. It follows that $\frac{q_k}{2s} > \frac{2a_{n_k}}{r_k} > 0$. By condition (R2) applied to $w = \frac{2a_{n_k}}{r_k}$, we have

$$Q_{r_k}^{s,q_k-s\frac{4a_{n_k}}{r_k}}\left(x, B\left(A_m, 2a_{n_k}\right)\right).$$
(1.16)

Let us denote $\tilde{q}_k = q_k - s \frac{4a_{n_k}}{r_k}$. Using the first inequality from the estimate (1.15), we get

$$0 \le s \frac{4a_{n_k}}{r_k} \le s \frac{2^{n_k+6}a_{n_k}}{R_{n_k+1}}.$$
(1.17)

By (1.7), (1.13), and (1.17), we have

$$\lim_{k \to \infty} s \frac{4a_{n_k}}{r_k} = 0$$

and so

$$\lim_{k \to \infty} \tilde{q}_k = \lim_{k \to \infty} q_k - \lim_{k \to \infty} s \frac{4a_{n_k}}{r_k} = 1.$$

To verify condition (b), it suffices to show that $Q_{r_k}^{s,\tilde{q}_k}(x, K \setminus S_{n_k}^m)$, $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$ and suppose that $z \in B(x, 2r_k) \setminus B(A_m, 2a_{n_k})$. Then

$$B(z, 2a_{n_k}) \subset K \setminus A_m \tag{1.18}$$

by the definition of $B(A_m, 2a_{n_k})$. Denote the center of the ball B_{n_k} played by Boulder by x_{n_k} . If we use

- Lemma 1.3.4 and the fact that $x \in B_{n_k+1}$ (in the second inequality of (1.19)),
- an immediate consequence of (1.15) saying that $a_{n_k} \leq r_k$ (in the third inequality of (1.19)),
- estimate (1.14) (in the fourth inequality of (1.19)),

then we have for arbitrary $y \in B(z, 2a_{n_k})$ the following:

$$d(y, x_{n_k}) \le d(y, z) + d(z, x) + d(x, x_{n_k}) < 2a_{n_k} + 2r_k + (1 - 2^{-(n_k+1)})R_{n_k} \le 4r_k + (1 - 2^{-(n_k+1)})R_{n_k} \le 2^{-(n_k+1)}R_{n_k} + (1 - 2^{-(n_k+1)})R_{n_k} = R_{n_k}.$$
(1.19)

This gives us the inclusion

$$B(z, 2a_{n_k}) \subset B_{n_k}.\tag{1.20}$$

By putting (1.18) and (1.20) together, we get $B(z, 2a_{n_k}) \subset B_{n_k} \setminus A_m$. By the definition of D_{n_k} , there exists $y \in D_{n_k}$ such that $d(z, y) < a_{n_k}$. Then the ball $B(y, a_{n_k})$, which is in the system \mathcal{M}_{n_k} and which contains z, is also a subset of $B_{n_k} \setminus A_m$. By definition of the strategy for Sisyfos, it follows that $z \in B(y, a_{n_k}) \subset S_{n_k}^m$. So we have $B(x, 2r_k) \setminus B(A_m, 2a_{n_k}) \subset S_{n_k}^m$ and thus

$$B(x,2r_k) \setminus S_{n_k}^m \subset B(A_m,2a_{n_k}).$$
(1.21)

By (1.16), (1.21), and (R3), we get $Q_{r_k}^{s,\tilde{q}_k}(x, B(x, 2r_k) \setminus S_{n_k}^m)$. By (R4), this gives $Q_{r_k}^{s,\tilde{q}_k}(x, K \setminus S_{n_k}^m)$ as we wanted.

Now, let us assume that Sisyfos has a winning strategy σ in the game H(A) and that he follows this strategy in every run of the game H(A). We have $A = M_0(\emptyset)$ and, by Lemma 1.3.5, it follows

$$A = M_0(\emptyset) \subset N_0(\emptyset) \cup \bigcup \{M_1(T_1) \colon T_1 \in \mathcal{T}\}, \qquad (1.22)$$

where $N_0(\emptyset)$ is Q-porous and \mathcal{T} is an at most countable collection of (0, 1)-good sequences of open balls. Now, for every $T_1 \in \mathcal{T}$ we have

$$M_1(T_1) \subset N_1(T_1) \cup \bigcup \{ M_2(T_1^{\wedge} T_2) \colon T_2 \in \mathcal{T}(T_1) \},$$
(1.23)

where $N_1(T_1)$ is Q-porous and $\mathcal{T}(T_1)$ is an at most countable collection of finite sequences of open balls such that $T_1^{\wedge}T_2$ is $(\text{length}(T_1), 2)$ -good for every $T_2 \in \mathcal{T}(T_1)$. By iterating this process, we get a countable system of Q-porous sets

$$\mathcal{U} = \{ N_k(T_1, T_2, \dots, T_k) \colon k \in \mathbb{N} \cup \{0\}, T_1 \in \mathcal{T}, T_2 \in \mathcal{T}(T_1), \dots, T_k \in \mathcal{T}(T_1, \dots, T_{k-1}) \}$$

such that for every $k \in \mathbb{N} \cup \{0\}$ and $T_1 \in \mathcal{T}, T_2 \in \mathcal{T}(T_1), \ldots, T_k \in \mathcal{T}(T_1, T_2, \ldots, T_{k-1})$, the sequence $T_1^{\wedge}T_2^{\wedge} \ldots^{\wedge}T_k$ is $(\operatorname{length}(T_1^{\wedge}T_2^{\wedge} \ldots^{\wedge}T_{k-1}), k)$ -good. It suffices to show that $A \subset \bigcup \mathcal{U}$. Suppose that this is not true and so there exists $x \in A \setminus \bigcup \mathcal{U}$. By (1.22), there exists $T_1 \in \mathcal{T}$ such that $x \in M_1(T_1)$. By (1.23), there exists $T_2 \in \mathcal{T}(T_1)$ such that $x \in M_2(T_1^{\wedge}T_2)$. In this way, we get that there exists a sequence $(T_k)_{k=1}^{\infty}$ where $T_1 \in \mathcal{T}$ and $T_k \in \mathcal{T}(T_1, T_2, \ldots, T_{k-1})$ for k > 1 such that $x \in M_k(T_1^{\wedge}T_2^{\wedge} \ldots^{\wedge}T_k)$ for every $k \in \mathbb{N}$.

We use the sequence $(T_k)_{k=1}^{\infty}$ to construct a special run of the game H(A). Set $S = T_1^{\wedge}T_2^{\wedge}\ldots$ The sequence S is either finite or infinite. In the first case there exists $k_0 \in \mathbb{N} \cup \{0\}$ such that $S = T_1^{\wedge}T_2^{\wedge}\ldots^{\wedge}T_{k_0}$ and $T_k = \emptyset$ for every $k > k_0$. Then Boulder plays balls from S and then he continues by playing open balls centered at x. Sisyfos follows his winning strategy σ . The outcome of such a run is x. Moreover, since $x \in M_{k_0}(S)$, we have

 $x \in \frac{1}{4} \star B_{\text{length}(S)}$. It follows that the run is (length(S), m+1)-good for every $m \in \mathbb{N}$. If the sequence S is infinite, then Boulder plays open balls following the sequence S and Sisyfos follows his winning strategy σ .

In both cases the point x is the outcome of the run and any $m \in \mathbb{N}$ cannot be a witness of Sisyfos' victory since $x \in M_m(T_1^{\wedge}T_2^{\wedge} \dots^{\wedge}T_m)$ and the run is $(\text{length}(T_1^{\wedge}T_2^{\wedge} \dots^{\wedge}T_m), m + 1)$ -good for every $m \in \mathbb{N}$. This is a contradiction since the strategy σ is winning for Sisyfos.

Lemma 1.3.7. If the set A is Borel then the game H(A) is Borel.

Proof. Denote by \mathcal{B} and \mathcal{G} the family of all open balls in K and the family of all open subsets of K respectively. Denote the tree of all legal positions of the game H(A) by \mathfrak{T} . Let $[\mathfrak{T}]$ denote the set of all infinite branches of \mathfrak{T} . Then the payoff set P for the game H(A) is the set of all $V \in [\mathfrak{T}]$ of the form $V = (B_1, (S_1^1), B_2, (S_2^1, S_2^2), \ldots)$ such that Sisyfos satisfied neither (a) nor (b) in the run V. Then $[\mathfrak{T}]$ is a subset of $\prod_{n=1}^{\infty} (\mathcal{B} \times (\mathcal{G})^n)$, which will be considered as a topological space with the product topology, where each factor is equipped with the discrete topology as usual.

We define mappings $f : [\mathfrak{T}] \to K$ and $h_n^j : [\mathfrak{T}] \to \mathcal{G}, n \in \mathbb{N}, j \in \{1, 2, \dots, n\}$, by

- $\{f(V)\} = \bigcap_{n=1}^{\infty} B_n(V)$, i.e., f(V) is the outcome of V,
- $h_n^j(V) = S_n^j(V)$.

It is easy to check that the mappings f and h_n^j are continuous. Next, we define

$$W_m = \{ V \in [\mathfrak{T}] : m \text{ is a witness of Sisyfos' victory in the run } V \}.$$
(1.24)

Then we have

$$P = f^{-1}(A) \setminus \bigcup_{m=1}^{\infty} W_m$$

The set $f^{-1}(A)$ is a continuous preimage of a Borel set and so it is Borel. To finish the proof, it remains to show that W_m is a Borel set for every $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$. After taking into consideration (R5) and (M), we have $V \in W_m$ if and only if

- (i) $f(V) \in K \setminus \bigcup_{n=m}^{\infty} h_n^m(V)$ and
- (ii) there exists $s \in \mathbb{N}$ such that for every $k \in \mathbb{N}$ there exist $n_k \geq \max\{m, k\}, q_k \in (1 \frac{1}{k}, 1) \cap \mathbb{Q}$, and $r_k \in (0, 2^{-(n_k+3)}R_{n_k}] \cap \mathbb{Q}$ such that $Q_{r_k}^{s, q_k}(f(V), K \setminus h_{n_k}^m(V))$.

Further, we have $V \in [\mathfrak{T}]$ satisfies (i) if and only if

$$V \in \bigcap_{\substack{n=m \ G \ \text{is a union} \\ \text{of some balls} \\ \text{from } \mathcal{M}_n}}^{\infty} \left((h_n^m)^{-1} \left(\{G\} \right) \cap f^{-1}(K \setminus G) \right).$$

The set \mathcal{M}_n is finite, so it is easy to see that the set on the right hand side is closed in $[\mathfrak{T}]$. Finally, we have $Q_{r_k}^{s,q_k}(f(V), K \setminus h_{n_k}^m(V))$ if and only if

$$V \in \bigcup_{\substack{G \text{ is a union} \\ \text{ of some balls} \\ \text{ from } \mathcal{M}_{n_k}}} \left(\left(h_{n_k}^m \right)^{-1} \left(\{G\} \right) \cap f^{-1} \left(\{y \in K \colon Q_{r_k}^{s,q_k}(y, K \setminus G) \} \right) \right)$$

The set $\{y \in K : Q_{r_k}^{s,q_k}(y, K \setminus G)\}$ is open by (R5). Thus a straightforward verification gives that W_m is Borel and we are done.

We will need the following result of J. Zapletal. To state it we need another notion of abstract porosity.

Definition 1.3.8 ([8]). Let X be a Polish space and \mathcal{U} be a countable collection of its Borel subsets. An *abstract porosity* is a mapping por from the family of all subsets of \mathcal{U} to the family of all Borel subsets of X such that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{U}$ implies $\operatorname{por}(\mathcal{A}) \subset \operatorname{por}(\mathcal{B})$. The *porosity* σ -*ideal* associated with the porosity por is σ -generated by sets $\operatorname{por}(\mathcal{A}) \setminus \bigcup \mathcal{A}$, as \mathcal{A} runs through all subsets of \mathcal{U} .

Theorem 1.3.9. [8, Theorem 4.16] Let X be a Polish space and \mathcal{I} be a porosity σ -ideal of subsets of X and $A \subset X$ be analytic. If $A \notin \mathcal{I}$, then there exists a Borel set $B \subset A$ with $B \notin \mathcal{I}$.

Lemma 1.3.10. The σ -ideal \mathcal{I} of all σ -Q-porous subsets of K forms a porosity σ -ideal.

Proof. Let \mathcal{U} be a countable open basis of the space K. We define the mapping por by

$$\operatorname{por}(\mathcal{A}) = \{ x \in K \colon Q(x, K \setminus \bigcup \mathcal{A}) \}, \qquad \mathcal{A} \subset \mathcal{U}.$$

Using (R1), (R5), and (M), we get that $por(\mathcal{A})$ is Borel for every $\mathcal{A} \subset \mathcal{U}$. The monotonicity of por is obvious. The verification that \mathcal{I} is σ -generated by sets of the form $por(\mathcal{A}) \setminus \bigcup \mathcal{A}$, $\mathcal{A} \subset \mathcal{U}$, is straightforward.

Theorem 1.3.11. Let (K, d) be a nonempty compact metric space, $Q \in \mathfrak{Q}(K)$, and let $A \subset K$ be an analytic set which is not σ -Q-porous. Then there exists a compact set $F \subset A$ which is not σ -Q-porous.

Proof. Using Lemma 1.3.10 and Theorem 1.3.9 we may assume that A is Borel. Sisyfos does not have a winning strategy in the game H(A) by Theorem 1.3.6. But by Lemma 1.3.7 and Martin Determinacy Theorem ([14]), the game is determined and so Boulder has a winning strategy μ . We consider μ as a subset of \mathfrak{T} (cf. [9, 20.A]) and denote by $[\mu]$ the set of all infinite branches of μ . The fact that Sisyfos has only finitely many possible choices on each of his moves of the game H(A) easily implies that $[\mu]$ is compact in the topology derived from the topological space $\prod_{n=1}^{\infty} (\mathcal{B} \times (\mathcal{G})^n)$. Each run $V \in [\mu]$ is a run of the game H(A) won by Boulder. Let $f: [\mathfrak{T}] \to K$ be the mapping from the proof of Theorem 1.3.7, that is the mapping assigning to $V \in [\mathfrak{T}]$ its outcome. Recall that the mapping f is continuous. Define $F = f([\mu])$. Then F is compact and a subset of A by condition (a) because the strategy μ is winning for Boulder.

It remains to show that F is not σ -Q-porous. Since satisfaction of condition (b) does not depend on the set which the game is played with, it is obvious that μ is a winning strategy for Boulder also in the game H(F). Therefore Sisyfos does not have a winning strategy in the game H(F) and using Theorem 1.3.6 again, we get the conclusion. \Box

1.3.3 Applications to concrete porosities.

Using Theorem 1.3.11 we prove inscribing theorems for σ -porosity, σ -strong porosity, σ strong right porosity, and σ -1-symmetrical porosity. It will be clear that Theorem 1.3.11 can be applied to many other types of porosity. First of all we recall definitions of the mentioned porosities.

Let (X, d) be a metric space. Let $M \subset X$, $x \in X$, and R > 0. Then we define

$$\theta(x, R, M) = \sup\{r > 0: \text{ there exists an open ball } B(z, r)$$

such that $d(x, z) < R$ and $B(z, r) \cap M = \emptyset\},$
$$p(x, M) = \limsup_{R \to 0+} \frac{\theta(x, R, M)}{R}.$$

We say that $M \subset X$ is

- porous at $x \in X$ if p(x, M) > 0,
- strongly porous at $x \in X$ if $p(x, M) \ge 1$.

Let $M \subset \mathbb{R}$, $x \in \mathbb{R}$, and R > 0. Then we define

$$\begin{aligned} \theta^+(x,R,M) &= \sup\{r > 0; \text{ there exists an open ball } B(z,r), z > x, \\ &\qquad \text{ such that } |x-z| < R, \text{ and } B(z,r) \cap M = \emptyset\}, \\ p^+(x,M) &= \limsup_{R \to 0+} \frac{\theta^+(x,R,M)}{R}, \\ \theta^s(x,R,M) &= \sup\{r > 0; \text{ there exists an open ball } B(z,r), \\ &\qquad \text{ such that } |x-z| < R, \text{ and } (B(z,r) \cup B(2x-z,r)) \cap M = \emptyset\}, \end{aligned}$$

$$p^{s}(x,M) = \limsup_{R \to 0+} \frac{\theta^{s}(x,R,M)}{R}.$$

Let c > 0. We say that $M \subset X$ is

- right porous at $x \in \mathbb{R}$ if $p^+(x, M) > 0$,
- strongly right porous at $x \in \mathbb{R}$ if $p^+(x, M) \ge 1$,
- c-symmetrically porous at $x \in \mathbb{R}$ if $p^s(x, M) \ge c$.

Theorem 1.3.12 (cf. [25, Theorem 3.1]). Let (X, d) be a locally compact metric space. Let $A \subset X$ be a non- σ -porous analytic set. Then there exists a non- σ -porous compact set $F \subset A$.

Proof. First, suppose that the space (X, d) is compact. Let $s \in \mathbb{N}, q \in (0, 1)$, and r > 0. We define a point-set relation $Q_r^{s,q}$ on X by

$$Q_r^{s,q}(x,M) \Leftrightarrow \text{there exists a ball } B(y,\tilde{r}) \text{ such that } x \in \left(B(y,r) \setminus \overline{B}(y,\frac{1}{2}r)\right) \cap B(y,\frac{s\tilde{r}}{q})$$

and $B(y,\tilde{r}) \cap M = \emptyset$.

We set

$$Q^s = \bigcap_{q \in (0,1)} \bigcap_{R>0} \bigcup_{0 < r < R} Q_r^{s,q}$$
 and $Q = \bigcup_{s \in \mathbb{N}} Q^s$.

To show that $Q \in \mathfrak{Q}(X)$, we need to verify that the relations $Q_r^{s,q}$, $s \in \mathbb{N}$, r > 0, $q \in (0,1)$, satisfy conditions (R1)–(R5). Let us verify only (R2) and (R4), the other conditions are easy to check.

(R2) Let $s \in \mathbb{N}$, r > 0, $q \in (0,1)$, $M \subset X$, $x \in X$, $0 < w < \frac{q}{2s}$, and suppose that $Q_r^{s,q}(x, M)$. There exists an open ball $B(y, \tilde{r})$ such that

$$x \in \left(B(y,r) \setminus \overline{B}(y,\frac{1}{2}r)\right) \cap B(y,\frac{s\tilde{r}}{q})$$
 and $B(y,\tilde{r}) \cap M = \emptyset$.

So we have

$$\frac{s\tilde{r}}{q} > d(x,y) > \frac{r}{2} \tag{1.25}$$

and so $\tilde{r} - rw > r\left(\frac{q}{2s} - w\right) > 0$. Clearly, $B(y, \tilde{r} - rw) \cap B(M, rw) = \emptyset$ and by (1.25) we have

$$s\frac{\tilde{r}-rw}{q-2sw} > s\frac{\tilde{r}(1-\frac{2sw}{q})}{q-2sw} = \frac{s\tilde{r}}{q} > d(x,y).$$

Thus $x \in B(y, s\frac{\tilde{r}-rw}{q-2sw})$ and we can conclude that $Q_r^{s,q-2sw}(x, B(M, rw))$.

(R4) Let $s \in \mathbb{N}$, r > 0, $q \in (0, 1)$, $M \subset X$, and $x \in X$ be such that $Q_r^{s,q}(x, M \cap B(x, 2r))$. Then there exists an open ball $B(y, \tilde{r})$ such that

$$x \in \left(B(y,r) \setminus \overline{B}(y,\frac{1}{2}r)\right) \cap B(y,\frac{s\tilde{r}}{q})$$
 and $B(y,\tilde{r}) \cap M \cap B(x,2r) = \emptyset$.

First, let us assume that $\tilde{r} \leq r$. If $z \in B(y, \tilde{r})$ then

$$d(z,x) \le d(z,y) + d(y,x) < \tilde{r} + r \le 2r.$$

So we have $B(y, \tilde{r}) \subset B(x, 2r)$ and therefore $B(y, \tilde{r}) \cap M = B(y, \tilde{r}) \cap M \cap B(x, 2r) = \emptyset$. It follows that $Q_r^{s,q}(x, M)$. Now, let us assume that $\tilde{r} > r$. Then we have

$$B(y,r) \cap M = B(y,r) \cap M \cap B(x,2r) \subset B(y,\tilde{r}) \cap M \cap B(x,2r) = \emptyset$$

and the open ball B(y,r) witnesses that $Q_r^{s,q}(x,M)$. The opposite implication in (R4) is obvious.

It is also straightforward to verify that $M \subset X$ is porous at $x \in X$ if and only if M is Q-porous at x. Therefore, A is not σ -Q-porous and by Theorem 1.3.11, there exists a non- σ -Q-porous (and thus also non- σ -porous) compact set $F \subset A$.

Now, suppose that (X, d) is an arbitrary locally compact metric space. Since A is a non- σ -porous subset of X, there exists $x \in X$ such that $A \cap B(x, r)$ is a non- σ -porous subset of X for every r > 0 by Theorem 1.1.4. Let us take $r_0 > 0$ such that $\overline{B(x, r_0)}$ is compact and denote $A' = A \cap B(x, r_0)$. Since porosity is a local property, every $M \subset B(x, r_0)$ is σ -porous in X if and only if M is σ -porous in the compact metric space $\overline{B(x, r_0)}$. Therefore, A' is non- σ -porous in $\overline{B(x, r_0)}$. Due to the previous part of the proof, there exists a non- σ -porous (in $\overline{B(x, r_0)}$ and therefore also in X) compact set $F \subset A' \subset A$.

Theorem 1.3.13. Let (X, d) be a locally compact metric space. Let $A \subset X$ be a non- σ -strongly porous analytic set. Then there exists a non- σ -strongly porous compact set $F \subset A$.

Proof. Similarly as in the previous proof we may assume that X is compact. Let $q \in (0, 1)$ and r > 0. We define a point-set relation Q_r^q on X by

$$Q_r^q(x, M) \Leftrightarrow \text{there exists a ball } B(y, \tilde{r}) \text{ such that } x \in \left(B(y, r) \setminus \overline{B}(y, \frac{1}{2}r)\right) \cap B(y, \frac{\tilde{r}}{q})$$

and $B(y, \tilde{r}) \cap M = \emptyset$.

We set

$$Q = \bigcap_{q \in (0,1)} \bigcap_{R>0} \bigcup_{0 < r < R} Q_r^q.$$

One can easily check that $Q \in \mathfrak{Q}(X)$. Then $M \subset X$ is σ -strongly porous if and only if A is σ -Q-porous. Applying Theorem 1.3.11, we get the conclusion.

Theorem 1.3.14. Let $A \subset \mathbb{R}$ be a non- σ -strongly right porous analytic set. Then there exists a non- σ -strongly right porous compact set $F \subset A$.

Proof. Without any loss of generality, we may assume that $A \subset (0, 1)$. Let $q \in (0, 1)$ and r > 0. We define a point-set relation Q_r^q on [0, 1] by

$$Q_r^q(x, M) \Leftrightarrow \text{there exist } y \in \mathbb{R}, \tilde{r} > 0 \text{ such that } y > x, x \in \left(B(y, r) \setminus \overline{B}(y, \frac{1}{2}r)\right) \cap B(y, \frac{\tilde{r}}{q})$$

and $B(y, \tilde{r}) \cap M = \emptyset$.

We set

$$Q = \bigcap_{q \in (0,1)} \bigcap_{R>0} \bigcup_{0 < r < R} Q_r^q.$$

One can easily check that $Q \in \mathfrak{Q}([0,1])$. Then $M \subset (0,1)$ is σ -strongly right porous if and only if M is σ -Q-porous. Applying Theorem 1.3.11 we get the conclusion.

Remark 1.3.15. Theorem 1.3.14 has been already used in [11].

Theorem 1.3.16. Let $A \subset \mathbb{R}$ be a non- σ -1-symmetrically porous analytic set. Then there exists a non- σ -1-symmetrically porous compact set $F \subset A$.

Proof. Without any loss of generality, we may assume that $A \subset (0, 1)$. Let $q \in (0, 1)$ and r > 0. We define a point-set relation Q_r^q on [0, 1] by

$$Q_r^q(x,M) \Leftrightarrow \text{there exist } y \in \mathbb{R}, \tilde{r} > 0 \text{ such that } x \in \left(B(y,r) \setminus \overline{B}(y,\frac{1}{2}r)\right) \cap B(y,\frac{\tilde{r}}{q})$$

and $\left(B(y,\tilde{r}) \cup B(2x-y,\tilde{r})\right) \cap M = \emptyset.$

We set

$$Q = \bigcap_{q \in (0,1)} \bigcap_{R>0} \bigcup_{0 < r < R} Q_r^q.$$

We can easily verify that $Q \in \mathfrak{Q}([0,1])$ and that $M \subset (0,1)$ is σ -1-symmetrically porous if and only if M is σ -Q-porous. The rest of the proof follows from Theorem 1.3.11. \Box

Finally, we apply Theorem 1.3.16 to answer a question posed by M. J. Evans and P. D. Humke in [7]. This is the following question.

Question 1.3.17. Does there exist an F_{σ} set in [0,1] which is $\sigma \cdot (1 - \varepsilon)$ -symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ -1-symmetrically porous?

We answer this question affirmatively by proving the next theorem.

Theorem 1.3.18. There exists a closed set $F \subset [0,1]$ which is $\sigma \cdot (1 - \varepsilon)$ -symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ -1-symmetrically porous.

Proof. There exists a Borel set $A \subset (0, 1)$ which is $\sigma \cdot (1 - \varepsilon)$ -symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ -1-symmetrically porous ([6]). By Theorem 1.3.16, there exists a compact non- σ -1-symmetrically porous set $F \subset A$. Since F is a subset of A, it is still $\sigma \cdot (1 - \varepsilon)$ -symmetrically porous for every $0 < \varepsilon < 1$.

2. Unitary representations of finite abelian groups realizable by an action

This chapter is based on the paper [2]. It contains two sections. The introductory Section 2.1 is more detailed than Section 1 (Introduction) of [2] since it contains more detailed explanations of some basic facts concerning the relevant notions. Section 2.2 differs from Section 2 (Proof of the main theorem) of [2] only in the proof of Claim 2.2.2, where we give a little more detailed explanation of one particular argument.

2.1 Introduction

In this chapter, we investigate the topological space of unitary representations of a finite abelian group on a given infinite-dimensional separable complex Hilbert space. In this introductory section, we remind the terminology as well as some well known basic facts concerning the relevant notions and some results shown in [10]. Throughout this chapter, whenever we speak about a Hilbert space, we always mean an infinite-dimensional separable complex Hilbert space.

If H is a Hilbert space, let $\mathcal{U}(H)$ denote the unitary group of H consisting of all unitary operators on H. The strong topology on $\mathcal{U}(H)$ is the topology generated by the family of maps $U \in \mathcal{U}(H) \mapsto U(x) \in H$, $x \in H$, i.e. it is the smallest topology such that all these maps are continuous. The weak topology on $\mathcal{U}(H)$ is generated by the family of maps $U \in \mathcal{U}(H) \mapsto \langle U(x), y \rangle \in \mathbb{C}, x, y \in H$. In the following lemma, we remind some very well known basic facts concerning these topologies.

Lemma 2.1.1. Let $\mathcal{U}(H)$ be endowed with the strong topology.

- (i) The (strong) topology on $\mathcal{U}(H)$ coincide with the weak topology.
- (ii) The mapping $(U, V) \in \mathcal{U}(H)^2 \mapsto UV \in \mathcal{U}(H)$ is continuous.
- (iii) The mapping $U \in \mathcal{U}(H) \mapsto U^{-1} \in \mathcal{U}(H)$ is continuous.
- (iv) The unitary group $\mathcal{U}(H)$ endowed with the strong topology is a Polish group.

Proof. (i) The fact that the weak topology is weaker than the strong topology is clear by the definitions. To show the opposite, let $\{U_i : i \in I\}$ be a net of unitary operators converging to some $U \in \mathcal{U}(H)$ in the weak topology and let $x \in H$. Then we have

$$||U_i(x) - U(x)||^2 = ||U_i(x)||^2 + ||U(x)||^2 - 2\operatorname{Re}\langle U_i(x), U(x)\rangle$$

= 2 (||x||² - Re\langle U_i(x), U(x)\rangle),

and the expression on the right hand side tends to zero since

$$\lim \langle U_i(x), U(x) \rangle = \langle U(x), U(x) \rangle = ||U(x)||^2 = ||x||^2.$$

This proves that U_i converge to U also in the strong topology and we are done.

(ii) Let $\{(U_i, V_i) : i \in I\}$ be a net of pairs of unitary operators converging to some $(U, V) \in \mathcal{U}(H)^2$ and let $x \in H$. Then we have

$$||U_iV_i(x) - UV(x)|| \le ||U_i(V_i - V)(x)|| + ||(U_i - U)(V(x))||$$
$$\le ||(V_i - V)(x)|| + ||(U_i - U)(V(x))||,$$

and the expression on the right hand side tends to zero since $\lim V_i(x) = V(x)$ and $\lim U_i(V(x)) = U(V(x))$.

(iii) Let $\{U_i : i \in I\}$ be a net of unitary operators converging to some $U \in \mathcal{U}(H)$ and let $x \in H$. Then we have

$$||U_i^{-1}(x) - U^{-1}(x)|| = ||x - U_i U^{-1} x||,$$

and the expression on the right hand side tends to zero since

$$\lim U_i(U^{-1}(x)) = U(U^{-1}(x)) = x.$$

(iv) By (ii) and (iii), we already know that $\mathcal{U}(H)$ is a topological group. It remains to show that it is a Polish space. To do this, let $D \subseteq H$ be a countable dense subset of Hwhich is closed under rational linear combinations. Let $U \in \mathcal{U}(H)$ and let $\{U_i : i \in I\}$ be a net of unitary operators such that $\lim U_i(d) = U(d)$ for every $d \in D$. Then for every $x \in H$ and $d \in D$, we have

$$||U_i(x) - U(x)|| \le ||U_i(x) - U_i(d)|| + ||U_i(d) - U(d)|| + ||U(d) - U(x)||$$
$$= 2||x - d|| + ||U_i(d) - U(d)||.$$

For every $x \in H$, we can find $d \in D$ arbitrarily close to x, and so it follows from the previous estimate that $\lim U_i(x) = U(x)$ for every $x \in H$. This shows that the strong topology on $\mathcal{U}(H)$ is generated by the family of maps $U \in \mathcal{U}(H) \mapsto U(d) \in H$, $d \in D$. Now, it easily follows that the mapping $F: U \in \mathcal{U}(H) \mapsto (U(d))_{d \in D} \in H^D$ is a topological embedding of $\mathcal{U}(H)$ to the Polish space H^D , and it only remains to prove that the range of F is a G_δ subset of H^D . Let $\{B_n: n \in \mathbb{N}\}$ be an open basis of H. Then we have

$$F(\mathcal{U}(H)) = \{(y(d))_{d \in D} \in H^D : (y(d))_{d \in D} \text{ can be extended to a linear operator on } H$$

such that it preserves the norm and its range is dense in H

$$= \{ (y(d))_{d \in D} \in H^D : \forall p, q \in \mathbb{Q} \ \forall d_1, d_2 \in D \ y(pd_1 + qd_2) = py(d_1) + qy(d_2), \\ \forall y \in D \ ||y(d)|| = ||d||, \ \forall n \in \mathbb{N} \ \exists d \in D \ y(d) \in B_n \},$$

and so $F(\mathcal{U}(H))$ is the intersection of the closed sets

$$\bigcap_{\substack{p,q \in \mathbb{Q} \\ d_1, d_2 \in D}} \{ (y(d))_{d \in D} \in H^D \colon y(pd_1 + qd_2) = py(d_1) + qy(d_2) \}$$

and

$$\bigcap_{d' \in D} \{ (y(d))_{d \in D} \in H^D \colon ||y(d')|| = ||d'|| \}$$

and of the G_{δ} set

$$\bigcap_{n \in \mathbb{N}} \bigcup_{d' \in D} \{ (y(d))_{d \in D} \in H^D \colon y(d') \in B_n \}.$$

This finishes the proof.

From now on, whenever we talk about the unitary group $\mathcal{U}(H)$ of a Hilbert space H, we consider it as a Polish group, equipped with the strong (equivalently weak) topology.

For a countable group Γ and a Hilbert space H, we denote by $\operatorname{Rep}(\Gamma, H)$ the set of all group homomorphisms from Γ to the unitary group $\mathcal{U}(H)$. Every element of $\operatorname{Rep}(\Gamma, H)$ is called a unitary representation of Γ on H. We consider the set $\operatorname{Rep}(\Gamma, H)$ as a subspace of the Polish space $\mathcal{U}(H)^{\Gamma}$ under the identification of $\pi \in \operatorname{Rep}(\Gamma, H)$ with $(\pi(\gamma))_{\gamma \in \Gamma} \in \mathcal{U}(H)^{\Gamma}$. If we denote by e the identity element of Γ and by I the identity operator on H, then $\operatorname{Rep}(\Gamma, H)$ corresponds to the set

$$\{(\pi(\gamma))_{\gamma\in\Gamma}\in\mathcal{U}(H)^{\Gamma}\colon\pi(e)=I,\ \forall\gamma_{1},\gamma_{2}\in\Gamma\ \pi(\gamma_{1}\gamma_{2})=\pi(\gamma_{1})\pi(\gamma_{2})\},\$$

which is the intersection of the closed sets

$$\{(\pi(\gamma))_{\gamma\in\Gamma}\in\mathcal{U}(H)^{\Gamma}\colon\pi(e)=I\}$$

and

$$\bigcap_{\gamma_1,\gamma_2\in\Gamma} \{(\pi(\gamma))_{\gamma\in\Gamma}\in\mathcal{U}(H)^{\Gamma}\colon \pi(\gamma_1\gamma_2)=\pi(\gamma_1)\pi(\gamma_2)\}.$$

It follows that $\operatorname{Rep}(\Gamma, H)$ is closed in $\mathcal{U}(H)^{\Gamma}$, and so it is a Polish space when equipped with the induced topology.

A measurable space (X, \mathcal{S}) is called a *standard Borel space* if there is a Polish space Y such that (X, \mathcal{S}) is isomorphic to $(Y, \mathcal{B}(Y))$, the space Y endowed with the σ -algebra $\mathcal{B}(Y)$ of its Borel subsets. If (X, \mathcal{S}) is a standard Borel space then we usually call the sets from \mathcal{S} Borel subsets of X. The following known theorem says that there is only one standard Borel space (up to a Borel isomorphism) of a given cardinality (its proof can be found in [9, Theorem 15.6]).

Theorem 2.1.2. Let X, Y be standard Borel spaces. Then X, Y are Borel isomorphic if and only if X and Y have the same cardinality.

A standard probability space is a standard Borel space (X, \mathcal{S}) together with a nonatomic probability measure μ defined on the σ -algebra \mathcal{S} . We denote it shortly by (X, μ) instead of (X, \mathcal{S}, μ) . It is notable that there exists only one standard probability space (up to a Borel isomorphism which preserves measure), namely [0, 1] endowed with m, the restriction of the Lebesgue measure to Borel subsets of [0, 1]. This immediately follows from the next known theorem (its proof can be found in [9, Theorem 17.41]). **Theorem 2.1.3.** Let X be a standard Borel space and μ be a continuous Borel probability measure on X. Then there is a Borel isomorphism $f: X \to [0, 1]$ such that $f\mu = m$.

If (X, μ) is a standard probability space then a Borel automorphism $T: X \to X$ is called a *measure-preserving automorphism* of (X, μ) if for every $B \in S$ we have $\mu(T^{-1}(B)) = \mu(B)$. As usual, we identify two measure-preserving automorphisms T, S of (X, μ) if they agree almost everywhere, i.e. if

$$\mu(\{x \in X : T(x) \neq S(x)\}) = 0.$$

Under this identification, we denote the set of all measure-preserving automorphisms of (X, μ) by $\operatorname{Aut}(X, \mu)$. Every $T \in \operatorname{Aut}(X, \mu)$ can be identified with a unitary operator U_T on the Hilbert space $L^2(X, \mu)$ defined by

$$U_T(f)(x) = f(T^{-1}(x)), \quad f \in L^2(X,\mu), \ x \in X.$$

The following lemma is a well known fact but I did not find any complete proof in the literature.

Lemma 2.1.4. Let (X, μ) be a standard probability space. Then the set

$$\{U_T \in \mathcal{U}(L^2(X,\mu)) \colon T \in \operatorname{Aut}(X,\mu)\}$$

is closed in $\mathcal{U}(L^2(X,\mu))$.

Proof. We will show that $U \in \mathcal{U}(L^2(X,\mu))$ is of the form U_T for some $T \in \operatorname{Aut}(X,\mu)$ if and only if it is a positivity preserving operator fixing 1 (= the constant function on X whose value is 1), i.e. if we have

$$U \in \{ V \in \mathcal{U}(L^2(X,\mu)) \colon \forall f \in L^2(X,\mu) \ (f \ge 0 \Rightarrow V(f) \ge 0), \ V(1) = 1 \}.$$

This is all we need since this set is the intersection of the closed sets

$$\bigcap_{\substack{f \in L^2(X,\mu) \\ f > 0}} \{ V \in \mathcal{U}(L^2(X,\mu)) \colon V(f) \ge 0 \}$$

and

$$\{V \in \mathcal{U}(L^2(X,\mu)): V(1) = 1\}.$$

The fact that for every $T \in \operatorname{Aut}(X,\mu)$, the operator U_T preserves positivity and fixes 1 is easy. So let us choose a positivity preserving operator $U \in \mathcal{U}(L^2(X,\mu))$ fixing 1. Let \mathcal{S} denote the σ -algebra of all Borel subsets of X and let $A \in \mathcal{S}$. Then we have

$$\langle U(\chi_A), U(\chi_{X\setminus A}) \rangle = \langle \chi_A, \chi_{X\setminus A} \rangle = 0,$$

$$0 = U(0) \le U(\chi_A) \le U(1) = 1$$

and

$$0 = U(0) \le U(\chi_{X \setminus A}) \le U(1) = 1$$

These facts easily imply that only on a μ -null set, both $U(\chi_A)$ and $U(\chi_{X\setminus A})$ can have nonzero values. But we also have

$$U(\chi_A) + U(\chi_{X \setminus A}) = U(1) = 1,$$

and so $U(\chi_A)$ can be represented by a characteristic function of some $B \in \mathcal{S}$ (and $U(\chi_{X\setminus A})$) can be represented by the characteristic function of its complement). For every $A \in \mathcal{S}$, choose $B(A) \in \mathcal{S}$ such that $U(\chi_A)$ can be represented by the characteristic function of B(A) (so B(A) is uniquely determined up to a μ -null set).

Claim 2.1.5. The mapping $A \in \mathcal{S} \mapsto B(A) \in \mathcal{S}$ has the following properties:

- (i) For every $A \in \mathcal{S}$, we have $\mu(B(A)) = \mu(A)$.
- (ii) For every $A \in \mathcal{S}$, the sets $B(X \setminus A)$ and $X \setminus B(A)$ differ only by a μ -null set.
- (iii) For every sequence $\{A_n\}_{n\in\mathbb{N}}$ of sets from \mathcal{S} , the sets $B(\bigcup_{n\in\mathbb{N}}A_n)$ and $\bigcup_{n\in\mathbb{N}}B(A_n)$ differ only by a μ -null set.

Proof. (i) For every $A \in \mathcal{S}$, we have

$$\mu(B(A)) = ||\chi_{B(A)}||^2 = ||U(\chi_A)||^2 = ||\chi_A||^2 = \mu(A).$$

(ii) Let $A \in S$. The set B(A) was defined such that $U(\chi_A)$ can be represented by the characteristic function of B(A) and $U(\chi_{X\setminus A})$ can be represented by the characteristic function of $X\setminus B(A)$. So the sets $B(X\setminus A)$ and $X\setminus B(A)$ differ only by a μ -null set.

(iii) Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of sets from \mathcal{S} . For $n\in\mathbb{N}$, let us define

$$A'_n = A_n \setminus \bigcup_{m < n} A_m,$$

so that the sets A'_n , $n \in \mathbb{N}$, are pairwise disjoint. Since disjointness (up to a μ -null set) of two sets corresponds to the orthogonality of their characteristic functions, the sets $B(A'_n)$, $n \in \mathbb{N}$, are also pairwise disjoint (up to μ -null sets). So we have

$$\chi_{B\left(\bigcup_{n\in\mathbb{N}}A_{n}\right)} = U\left(\chi_{\bigcup_{n\in\mathbb{N}}A_{n}}\right) = U\left(\chi_{\bigcup_{n\in\mathbb{N}}A'_{n}}\right) = U\left(\sum_{n=1}^{\infty}\chi_{A'_{n}}\right)$$
$$= \sum_{n=1}^{\infty}U\left(\chi_{A'_{n}}\right) = \sum_{n=1}^{\infty}\chi_{B(A'_{n})} = \chi_{\bigcup_{n\in\mathbb{N}}B(A'_{n})},$$

and so the sets $B\left(\bigcup_{n\in\mathbb{N}}A_n\right)$ and $\bigcup_{n\in\mathbb{N}}B\left(A'_n\right)$ differ only by a μ -null set. Further, for every $n\in\mathbb{N}$ we have

$$\chi_{B(A'_n)} = U\left(\chi_{A'_n}\right) \le U\left(\chi_{A_n}\right) = \chi_{B(A_n)},$$

and so

$$\mu\left(\bigcup_{n\in\mathbb{N}}B(A'_n)\setminus\bigcup_{n\in\mathbb{N}}B(A_n)\right)=0.$$

On the other hand, for every $m \in \mathbb{N}$ we have

$$\chi_{B(A_m)} = U\left(\chi_{A_m}\right) \le U\left(\chi_{\bigcup_{n\in\mathbb{N}}A_n}\right) = \chi_{B\left(\bigcup_{n\in\mathbb{N}}A_n\right)},$$

and so

$$\mu\left(\bigcup_{m\in\mathbb{N}}B(A_m)\backslash B\left(\bigcup_{n\in\mathbb{N}}A_n\right)\right)=0.$$

It follows that $\bigcup_{n \in \mathbb{N}} B(A_n)$ also differs from the sets $B\left(\bigcup_{n \in \mathbb{N}} A_n\right)$ and $\bigcup_{n \in \mathbb{N}} B(A'_n)$ only by μ -null sets, as we wanted.

Let us choose $f \in L^2(X,\mu)$ such that $f \geq 0$. Since U preserves positivity, it also preserves real, resp. purely imaginary functions. In particular, $U^{-1}(f)$ is real (since U maps the imaginary part of $U^{-1}(f)$ to 0). Let us denote by g^+ , resp. g^- the positive, resp. the negative part of $U^{-1}(f)$. Then g^+ and g^- are orthogonal to each other, and so the same holds for $U(g^+)$ and $U(g^-)$. Since both $U(g^+)$ and $U(g^-)$ are non-negative, this can happen only if

$$\mu\{x \in X : U(g^+)(x) \neq 0 \text{ and } U(g^-)(x) \neq 0\} = 0.$$

But we have $U(g^+) - U(g^-) = U(g^+ - g^-) = f \ge 0$, and so $U(g^-) = 0$ μ -almost everywhere. Then we also have $g^- = 0$ μ -almost everywhere, and so $U^{-1}(f) = g^+ \ge 0$. This shows that U^{-1} is a positivity preserving operator, as well as U.

Now, let $B \in \mathcal{S}$. Since U^{-1} is a positivity preserving operator fixing 1, we know that $U^{-1}(\chi_B)$ can be represented by a characteristic function of some $A(B) \in \mathcal{S}$. Moreover, the mapping $B \in \mathcal{S} \mapsto A(B) \in \mathcal{S}$ has analogous properties to those of the mapping $A \in \mathcal{S} \mapsto B(A) \in \mathcal{S}$ proved in Claim 2.1.5. For every $A \in \mathcal{S}$, we also have

$$\chi_A = U^{-1}(\chi_{B(A)}) = \chi_{A(B(A))},$$

and so the sets A and A(B(A)) differ only by a μ -null set. On the other hand, for every $B \in \mathcal{S}$, we have

$$\chi_B = (U^{-1})^{-1}(\chi_{A(B)}) = U(\chi_{A(B)}) = \chi_{B(A(B))},$$

and so the sets B and B(A(B)) differ only by a μ -null set.

Let us denote by \mathcal{N} the σ -ideal of all μ -null subsets of X. Let us consider the equivalence relation on \mathcal{S} defined such that two sets from \mathcal{S} are equivalent if they differ only by a μ null set. Let \mathcal{S}/\mathcal{N} be the quotient set of \mathcal{S} induced by this equivalence relation. Let us define a mapping from \mathcal{S}/\mathcal{N} to \mathcal{S}/\mathcal{N} such that for every $A \in \mathcal{S}$, the equivalence class \tilde{A} of A is mapped to the equivalence class $\tilde{B}(A)$ of B(A). By the Claim 2.1.5 and the consequent considerations, we have verified all assumptions of [17, p. 413, Theorem 21] on the mapping $\tilde{A} \mapsto \tilde{B}(A)$ (i.e., we have verified that this mapping is a σ -isomorphism of the Boolean σ -algebra \mathcal{S}/\mathcal{N} onto itself in the terminology of [17], the mapping $\tilde{B} \mapsto \tilde{A}(B)$ being its inverse). By this theorem, there exists a Borel isomorphism T of X onto itself such that the sets B(A) and T(A) differ only by a μ -null set for every $A \in \mathcal{S}$ (and so Tpreserves the measure μ). Now, for every $A \in \mathcal{S}$ we have

$$U_T(\chi_A)(x) = 1 \Leftrightarrow \chi_A(T^{-1}(x)) = 1 \Leftrightarrow T^{-1}(x) \in A \Leftrightarrow x \in T(A),$$

and so the unitary operators U and U_T coincide on the characteristic functions of the sets from S. It follows that the operators U and U_T are the same.

By the previous lemma, we can view $\operatorname{Aut}(X, \mu)$ as a closed subgroup of the Polish group $\mathcal{U}(L^2(X, \mu))$. Then it becomes a Polish group, too.

Let Γ be a group and X be a set. Denote by e the identity element of Γ . Let a be a mapping from $\Gamma \times X$ to X. Then a is called a (group) action of Γ on X if the following two conditions hold:

(i)
$$a(gh, x) = a(g, a(h, x))$$
 for every $g, h \in \Gamma$ and $x \in X$,

(ii)
$$a(e, x) = x$$
 for every $x \in X$.

If this is the case, we also say that Γ acts on X (by the action a).

An action a of a countable group Γ on a standard probability space (X, μ) is called a measure preserving action of Γ on (X, μ) if all the maps $a(\gamma, \cdot) \colon X \to X, \gamma \in \Gamma$, defined by $a(\gamma, \cdot) \colon x \mapsto a(\gamma, x)$ are in Aut (X, μ) . Again, we identify two measure preserving actions a, b of Γ on (X, μ) if they agree almost everywhere, i.e. if for every $\gamma \in \Gamma$ we have

$$\mu(\{x \in X \colon a(\gamma, x) \neq b(\gamma, x)\}) = 0.$$

Under this identification, we denote the set of all measure preserving actions of Γ on (X, μ) by $A(\Gamma, X, \mu)$. Then $A(\Gamma, X, \mu)$ is a closed subset of the Polish space $\operatorname{Aut}(X, \mu)^{\Gamma}$, since it is the intersection of the closed sets

$$\{(T(\gamma))_{\gamma\in\Gamma}\in \operatorname{Aut}(X,\mu)^{\Gamma}\colon T(e)=I\}$$

and

$$\{(T(\gamma))_{\gamma\in\Gamma}\in \operatorname{Aut}(X,\mu)^{\Gamma}\colon \forall\gamma_1,\gamma_2\in\Gamma\ T(\gamma_1\gamma_2)=T(\gamma_1)T(\gamma_2)\}.$$

So $A(\Gamma, X, \mu)$ equipped with the induced topology is also a Polish space.

If (X, μ) is a standard probability space, we denote by $L_0^2(X, \mu)$ the orthogonal complement of the constant functions in the Hilbert space $L^2(X, \mu)$, i.e.

$$L_0^2(X,\mu) = \{ f \in L^2(X,\mu) \colon \int_X f d\mu = 0 \}.$$

If a is a measure-preserving action of a countable group Γ on a standard probability space (X, μ) then the formula

$$\kappa_0^a(\gamma)(f)(x) = f(a(\gamma^{-1}, x)), \quad \gamma \in \Gamma, \ f \in L_0^2(X, \mu), \ x \in X$$

defines a unitary representation of Γ on $L_0^2(X,\mu)$. Indeed, for every $\gamma \in \Gamma$, the mapping $a(\gamma, \cdot)$ is in $\operatorname{Aut}(X,\mu)$ and the unitary operator $U_{a(\gamma,\cdot)}$ on $L^2(X,\mu)$ clearly maps the subspace $L_0^2(X,\mu)$ onto itself. It follows that the restriction of $U_{a(\gamma,\cdot)}$ on $L_0^2(X,\mu)$, which is the mapping $f \in L_0^2(X,\mu) \mapsto \kappa_0^a(\gamma)(f) \in L_0^2(X,\mu)$, is a unitary operator on $L_0^2(X,\mu)$. Finally, the fact that the mapping $\gamma \in \Gamma \mapsto \kappa_0^a(\gamma) \in \mathcal{U}(L_0^2(X,\mu))$ is a group homomorphism is obvious. This unitary representation is called *Koopman unitary representation* κ_0^a of Γ on $L_0^2(X,\mu)$ associated with the action a.

If Γ is a countable group and H is a Hilbert space then the unitary group $\mathcal{U}(H)$ acts on $\operatorname{Rep}(\Gamma, H)$ by conjugation, which is a mapping from $\mathcal{U}(H) \times \operatorname{Rep}(\Gamma, H)$ to $\operatorname{Rep}(\Gamma, H)$ defined by

$$T \cdot \pi = T\pi T^{-1}, \quad T \in \mathcal{U}(H), \ \pi \in \operatorname{Rep}(\Gamma, H),$$

where

$$T\pi T^{-1}(\gamma) = T\pi(\gamma)T^{-1}, \quad \gamma \in \Gamma.$$

Two unitary representations π , ρ of a countable group Γ on Hilbert spaces H, K respectively are called *unitarily equivalent* if there exists a unitary operator U from H onto K such that we have

$$U\pi(\gamma) = \rho(\gamma)U, \quad \gamma \in \Gamma.$$

We say that a unitary representation π of a countable group Γ on a Hilbert space H is *realizable by an action* if there is a standard probability space (X, μ) and $a \in A(\Gamma, X, \mu)$ such that π is unitarily equivalent to the Koopman representation κ_0^a of Γ on $L_0^2(X, \mu)$ associated with a.

Some interesting facts concerning the notion of realizability by an action can be found in [10, Appendix H, (F)]. It is shown there that if Γ is a countable group and H is a Hilbert space then the set of realizable by an action unitary representations of Γ on H is dense in $\operatorname{Rep}(\Gamma, H)$ (see [10, Proposition H.14]). It is also stated there (without proof) that this set is either meager or comeager in $\operatorname{Rep}(\Gamma, H)$ since it is invariant under conjugacy by elements of the unitary group $\mathcal{U}(H)$. Here, we prove this fact.

Lemma 2.1.6. Let H be a Hilbert space and Γ be a countable group. Then the set of realizable by an action $\pi \in \text{Rep}(\Gamma, H)$ is either meager or comeager in $\text{Rep}(\Gamma, H)$.

Proof. Denote the set of all realizable by an action unitary representations of Γ on H by A. By Theorem 2.1.3, a representation $\pi \in \operatorname{Rep}(\Gamma, H)$ is in A if and only if there is $a \in A(\Gamma, [0, 1], m)$ such that π is unitarily equivalent to the Koopman representation κ_0^a of Γ on $L_0^2([0, 1], m)$ associated with a. If we fix an arbitrary unitary operator U from H onto $L_0^2([0, 1], m)$, this is equivalent to the proposition

$$\exists a \in A(\Gamma, [0, 1], m) \; \exists V \in \mathcal{U}(H) \; \forall \gamma \in \Gamma \; UV\pi(\gamma) = \kappa_0^a(\gamma)UV.$$

From this, it easily follows that A is an analytic subset of $\operatorname{Rep}(\Gamma, H)$. But every analytic set has the Baire property (see [9, Corollary 29.14]), and so either A is meager in $\operatorname{Rep}(\Gamma, H)$ or there is a nonempty open set $G \subseteq \operatorname{Rep}(\Gamma, H)$ such that $A \cap G$ is comeager in G (see [9, Proposition 8.26]). There is nothing to prove in the former case, so suppose the latter. By [10, Proposition H.1], there is $\pi \in \operatorname{Rep}(\Gamma, H)$ such that its conjugacy class $\{U \cdot \pi : U \in \mathcal{U}(H)\}$ is dense in $\operatorname{Rep}(\Gamma, H)$. Without loss of generality, we may assume that $\pi \in G$. Now, whenever G_1 is a nonempty open subset of $\operatorname{Rep}(\Gamma, H)$, there is $U \in \mathcal{U}(H)$ such that $U \cdot \pi \in G_1$. The set A is comeager in G and the mapping $\rho \in \operatorname{Rep}(\Gamma, H) \mapsto U \cdot \rho \in \operatorname{Rep}(\Gamma, H)$ is a self-homeomorphism of $\operatorname{Rep}(\Gamma, H)$ which preserves realizability by an action, and so A is also comeager in $\{U \cdot \rho : \rho \in G\}$. So the intersection $\{U \cdot \rho : \rho \in G\} \cap G_1$ is a nonempty (it contains $U \cdot \pi$) open subset of G_1 in which A is comeager. We have shown that for every nonempty open subset G_1 of $\operatorname{Rep}(\Gamma, H)$, there is a nonempty open subset G_2 of G_1 such that A is comeager in G_2 . Let us define

$$\tilde{G} = \bigcup \{ G \subseteq \operatorname{Rep}(\Gamma, H) \colon G \text{ is open in } \operatorname{Rep}(\Gamma, H) \text{ and } A \text{ is comeager in } G \}.$$

Then the complement of \tilde{G} in $\operatorname{Rep}(\Gamma, H)$ is a closed subset of $\operatorname{Rep}(\Gamma, H)$ with empty interior by the previous considerations, so it is nowhere dense. Further, $\tilde{G} \setminus A$ is meager (see [9, Theorem 8.29]). Finally, the complement of A is a subset of $(\operatorname{Rep}(\Gamma, H) \setminus \tilde{G}) \cup (\tilde{G} \setminus A)$, and so it is meager, too.

It is also shown in [10] that if Γ is torsion-free abelian then the set of realizable by an action $\pi \in \operatorname{Rep}(\Gamma, H)$ is meager in $\operatorname{Rep}(\Gamma, H)$. Here, we consider the following question.

Question 2.1.7. Let Γ be a countable group and H be an infinite-dimensional separable complex Hilbert space. Is the set of all realizable by an action $\pi \in \operatorname{Rep}(\Gamma, H)$ meager in $\operatorname{Rep}(\Gamma, H)$?

This question was posed in [10, Problem H.16] for an infinite group Γ . Here, we show that if Γ is finite abelian then the answer is negative. This will be done by proving the following theorem which is the main result of this chapter.

Theorem 2.1.8. Let Γ be a finite abelian group and let H be an infinite dimensional separable complex Hilbert space. Then the set

 $\{\pi \in \operatorname{Rep}(\Gamma, H) \colon \pi \text{ is realizable by an action}\}\$

is comeager in $\operatorname{Rep}(\Gamma, H)$.

2.2 Proof of the main theorem

In this section, we provide a negative answer to Question 2.1.7 in case of a finite abelian group Γ by proving Theorem 2.1.8.

Proof of Theorem 2.1.8. Suppose that Γ has n elements g_1, \ldots, g_n . Let us define

$$\phi \colon [0,1) \to [0,\frac{1}{n})$$

by

$$\phi(x) = x - \frac{\lfloor nx \rfloor}{n}$$

(where $\lfloor y \rfloor$ denotes the integer part of y). This means that for every $x \in [0,1)$ there is

some $j \in \{1, \dots, n\}$ such that $x = \frac{j-1}{n} + \phi(x)$. We define $a: \Gamma \times [0, 1) \to [0, 1)$ such that for $j, k \in \{1, \dots, n\}$ and $x \in [\frac{k-1}{n}, \frac{k}{n})$ we have $a(g_j, x) = \frac{l-1}{n} + \phi(x)$ where $l \in \{1, \dots, n\}$ is such that $g_l = g_k g_j^{-1}$. We verify that a is an action of Γ on [0,1). Let $x \in [0,1)$ and $j, k, l \in \{1,\ldots,n\}$ be such that $x \in [\frac{k-1}{n}, \frac{k}{n})$ and $g_l = g_k g_j^{-1}$. Let $m \in \{1,\ldots,n\}$. We want to show that $a(g_m, a(g_j, x)) = a(g_m g_j, x)$. But $a(g_j, x) = \frac{l-1}{n} + \phi(x) \in [\frac{l-1}{n}, \frac{l}{n})$ and so we have $a(g_m, a(g_j, x)) = \frac{p-1}{n} + \phi(a(g_j, x)) = \frac{p-1}{n} + \phi(a(g_j, x)) = \frac{p-1}{n} + \phi(x)$ where $p \in \{1,\ldots,n\}$ is such that $g_p = g_l g_m^{-1} = g_k (g_m g_j)^{-1}$, and so $a(g_m, a(g_j, x)) = a(g_m g_j, x) = a(g_m g_j, x)$. Finally, if g_j is the identity element of Γ then $g_l = g_k g_j^{-1}$ holds if and only if $a(g_m g_j, x) = a(g_m g_j, x)$. k = l and so easily $a(g_i, x) = x$ for every $x \in [0, 1)$.

Moreover, the action a preserves the Lebesgue measure of [0,1) since for every $j \in$ $\{1,\ldots,n\}$, the mapping $a(g_i,\cdot)\colon [0,1)\to [0,1)$ acts as a permutation of the intervals $[0,\frac{1}{n}),\ldots,[\frac{n-1}{n},1)$. Let κ_0^a denote the Koopman representation of Γ on $L_0^2[0,1) = \{f \in I\}$ $L^2[0,1): \int_0^1 f = 0$ associated with a. For every character γ from the dual group $\hat{\Gamma}$ of Γ , let us define a subspace K_{γ} of $L_0^2[0,1)$ by

$$K_{\gamma} = \{ f \in L^2_0[0,1) \colon \forall g \in \Gamma \; \kappa^a_0(g)(f) = \gamma(g)f \}.$$

Claim 2.2.1. For every $\gamma \in \hat{\Gamma}$, we have $\dim K_{\gamma} = \infty$.

Proof. Let us choose $\gamma \in \hat{\Gamma}$. Define $T: L_0^2[0, \frac{1}{n}) \to L_0^2[0, 1)$ by

$$T(f)(x) = \gamma(g_j)f(\phi(x))$$

where $f \in L^2_0[0, \frac{1}{n}), j \in \{1, \ldots, n\}$ and $x \in [\frac{j-1}{n}, \frac{j}{n}]$. The definition is correct since for every $f \in L_0^2[0, \frac{1}{n})$ and $j \in \{1, \ldots, n\}$, we have

$$\int_{\frac{j-1}{n}}^{\frac{j}{n}} |T(f)|^2 = \int_{0}^{\frac{1}{n}} |f|^2 < \infty$$

and

$$\int_{\frac{j-1}{n}}^{\frac{j}{n}} T(f) = \gamma(g_j) \int_{0}^{\frac{1}{n}} f = 0,$$

and so $T(f) \in L_0^2[0,1)$. Moreover, T is injective since $T(f)|_{[0,\frac{1}{n})} = \gamma(g_1)f$ for every $f \in L_0^2[0,\frac{1}{n})$. The operator T is also obviously linear and so it is enough to show that $T(L_0^2[0,\frac{1}{n})) \subseteq K_{\gamma}$. To do this, let us choose $f \in L_0^2[0,\frac{1}{n}), j, k \in \{1,\ldots,n\}$ and $x \in [\frac{k-1}{n}, \frac{k}{n})$. Let $l \in \{1,\ldots,n\}$ be such that $g_l = g_k g_j = g_k (g_j^{-1})^{-1}$. Then

$$\kappa_0^a(g_j)(T(f))(x) = T(f)(a(g_j^{-1}, x)) = T(f)\left(\frac{l-1}{n} + \phi(x)\right)$$
$$= \gamma(g_l)f\left(\phi\left(\frac{l-1}{n} + \phi(x)\right)\right) = \gamma(g_l)f\left(\phi(x)\right)$$
$$= \gamma(g_k)\gamma(g_j)f\left(\phi(x)\right) = \gamma(g_j)T(f)(x),$$

as we wanted.

For every $\pi \in \operatorname{Rep}(\Gamma, H)$ and $\gamma \in \widehat{\Gamma}$, define (similarly as above) a subspace H^{π}_{γ} of H by

$$H^{\pi}_{\gamma} = \{ h \in H \colon \forall g \in \Gamma \ \pi(g)(h) = \gamma(g)h \}.$$

Claim 2.2.2. Let $\pi \in \operatorname{Rep}(\Gamma, H)$ be such that for every $\gamma \in \hat{\Gamma}$, we have $\dim H^{\pi}_{\gamma} = \infty$. Then π is unitarily equivalent to κ_0^a .

Proof. For every $g \in \Gamma$, let us denote by o(g) the order of g in Γ . Now, for every $j \in \{1, \ldots, n\}$ and $r \in \{1, \ldots, o(g_j)\}$, let us define

$$\lambda_{j,r} = \exp \frac{2r\pi i}{o(g_j)}$$

This means that $\lambda_{j,1}, \ldots, \lambda_{j,o(g_i)}$ are all $o(g_j)$ th roots of unity. It follows that

 $\sigma(\pi(g_j)) \subseteq \{\lambda_{j,1}, \dots, \lambda_{j,o(g_j)}\}, \quad j \in \{1, \dots, n\}$

since for every $j \in \{1, \ldots, n\}$, we have

$$\lambda \in \sigma\left(\pi(g_j)\right) \Rightarrow \lambda^{o(g_j)} \in \sigma\left(\pi(g_j)^{o(g_j)}\right) = \sigma\left(I\right) = \{1\}.$$

Let $j \in \{1, \ldots, n\}$, $r \in \{1, \ldots, o(g_j)\}$. If $\lambda_{j,r}$ is an eigenvalue of $\pi(g_j)$, let H_r^j be its eigenspace. And if $\lambda_{j,r}$ is not an eigenvalue of $\pi(g_j)$, let $H_r^j = \{0\}$. We show by induction on j that for every $j \in \{1, \ldots, n\}$, we have

$$H = \bigoplus_{\substack{r_1 \in \{1, \dots, o(g_1)\}\\ \vdots\\ r_j \in \{1, \dots, o(g_j)\}}} H^1_{r_1} \cap \dots \cap H^j_{r_j},$$

where the symbol \bigoplus refers to the direct sum of pairwise orthogonal subspaces. By the spectral theorem (see [18, Theorem 12.23]) applied to the unitary operator $\pi(g_1)$, there

exists a unique resolution E of the identity on the subsets of $\sigma(\pi(g_1)) \subseteq \{\lambda_{1,1}, \ldots, \lambda_{1,o(g_1)}\}$ such that

$$\pi(g_1) = \int_{\sigma(\pi(g_1))} \lambda dE(\lambda) = \sum_{\lambda \in \sigma(\pi(g_1))} \lambda E(\{\lambda\}).$$

By properties of a resolution of the identity, the operators $E(\{\lambda\})$ from the previous sum are orthogonal projections with pairwise orthogonal ranges. It follows that if $\lambda_{1,r} \in \sigma(\pi(g_1))$ then the range of $E(\{\lambda_{1,r}\})$ is nontrivial and equals H_r^1 . So we have

$$H = \bigoplus_{r_1 \in \{1, \dots, o(g_1)\}} H^1_{r_1}.$$

Suppose that for some $j \in \{2, ..., n\}$, we already know that

$$H = \bigoplus_{\substack{r_1 \in \{1, \dots, o(g_1)\}\\ \vdots\\ r_{j-1} \in \{1, \dots, o(g_{j-1})\}}} H^1_{r_1} \cap \dots \cap H^{j-1}_{r_{j-1}}.$$

Let us choose one of the subspaces $H_{r_1}^1 \cap \ldots \cap H_{r_{j-1}}^{j-1}$ from the previous direct sum and pick one of its elements h and $k \in \{1, \ldots, j-1\}$. Then we have

$$\pi(g_k) (\pi(g_j)(h)) = \pi(g_j) (\pi(g_k)(h)) = \pi(g_j) (\lambda_{k,r_k} h) = \lambda_{k,r_k} \pi(g_j)(h)$$

which shows that $H_{r_1}^1 \cap \ldots \cap H_{r_{j-1}}^{j-1}$ is invariant for the unitary operator $\pi(g_j)$. So we can apply the spectral theorem to the restriction of $\pi(g_j)$ to $H_{r_1}^1 \cap \ldots \cap H_{r_{j-1}}^{j-1}$ similarly as above to get

$$H_{r_1}^1 \cap \ldots \cap H_{r_{j-1}}^{j-1} = \bigoplus_{r_j \in \{1, \dots, o(g_j)\}} H_{r_1}^1 \cap \ldots \cap H_{r_{j-1}}^{j-1} \cap H_{r_j}^j$$

which easily finishes the induction step.

Choose a subspace of H of the form $H_{r_1}^1 \cap \ldots \cap H_{r_n}^n$ from the decomposition we have just proven (for j = n) and suppose that it contains a nonzero vector h. Pick any $j, k, l \in \{1, \ldots, n\}$ such that $g_j g_k = g_l$. Then we have

$$\lambda_{l,r_l}h = \pi(g_l)h = \pi(g_j)\pi(g_k)h = \lambda_{j,r_j}\lambda_{k,r_k}h$$

and so $\lambda_{l,r_l} = \lambda_{j,r_j}\lambda_{k,r_k}$. Also, if g_j is the identity element of Γ then $o(g_j) = 1$ and so $\lambda_{j,r_j} = 1$. These two facts imply that the mapping $\gamma \colon \Gamma \to \mathbb{C}$ defined by

$$\gamma(g_j) = \lambda_{j,r_j}, \quad j \in \{1, \dots, n\}$$

is in the dual group $\hat{\Gamma}$ of Γ . So we have $H^1_{r_1} \cap \ldots \cap H^n_{r_n} \subseteq H^{\pi}_{\gamma}$.

For every $j \in \{1, \ldots, n\}$, the subspaces H_r^j , $r = 1, \ldots, o(g_j)$ are pairwise orthogonal to each other and so it easily follows that the subspaces H_{γ}^{π} , $\gamma \in \hat{\Gamma}$, are pairwise orthogonal to each other, too. So we have

$$\bigoplus_{\gamma \in \hat{\Gamma}} H_{\gamma}^{\pi} \subseteq H = \bigoplus_{\substack{r_1 \in \{1, \dots, o(g_1)\}\\ \vdots\\ r_n \in \{1, \dots, o(g_n)\}}} H_{r_1}^1 \cap \dots \cap H_{r_n}^n \subseteq \bigoplus_{\gamma \in \hat{\Gamma}} H_{\gamma}^{\pi}$$

and consequently

$$H = \bigoplus_{\gamma \in \hat{\Gamma}} H_{\gamma}^{\pi}$$

By repeating the previous considerations for κ_0^a and $L_0^2[0,1)$ instead of π and H, we obtain that

$$L_0^2[0,1) = \bigoplus_{\gamma \in \hat{\Gamma}} K_{\gamma}.$$

Since all the subspaces H^{π}_{γ} of H and K_{γ} of $L^{2}_{0}[0,1)$, $\gamma \in \hat{\Gamma}$, are infinite dimensional there exists a unitary operator from H onto $L^{2}_{0}[0,1)$ which maps H^{π}_{γ} onto K_{γ} for every $\gamma \in \hat{\Gamma}$. Such an operator easily witnesses that π and κ^{a}_{0} are unitarily equivalent.

Claim 2.2.3. The set $A = \{\pi \in \operatorname{Rep}(\Gamma, H) : \forall \gamma \in \widehat{\Gamma} \dim H^{\pi}_{\gamma} = \infty\}$ is dense G_{δ} in $\operatorname{Rep}(\Gamma, H)$.

Proof. First, we prove that A is dense. Let us pick an open set U in $\operatorname{Rep}(\Gamma, H)$ of the form

 $U = \left\{ \pi \in \operatorname{Rep}(\Gamma, H) \colon \forall g \in \Gamma \; \forall j \in \{1, \dots, p\} \; \|\pi(g)(h_j) - \rho(g)(h_j)\| < \varepsilon \right\}$

where $\rho \in \operatorname{Rep}(\Gamma, H)$, $p \in \mathbb{N}$, $h_1, \ldots, h_p \in H$ and $\varepsilon > 0$. We want to find some $\pi \in A \cap U$. Let us define

$$H' = \operatorname{span} \left\{ \rho(g)(h_j) \colon g \in \Gamma, j \in \{1, \dots, p\} \right\}.$$

Then H' is a finite dimensional subspace of H which is invariant for the representation ρ . Let T be a unitary operator of $L_0^2[0,1)$ onto the orthogonal complement $(H')^{\perp}$ of H' in H. Define $\pi \in \operatorname{Rep}(\Gamma, H)$ such that it coincides with ρ on H' and such that the restriction of π to $(H')^{\perp}$ is defined as the conjugation of κ_0^a by the unitary operator T. Then $\pi \in U$ and $\dim H_{\gamma}^{\pi} \geq \dim K_{\gamma} = \infty$ for every $\gamma \in \Gamma$.

Next, we prove that A is G_{δ} . To do this, it is enough to show that for every $\gamma \in \hat{\Gamma}$ and $q \in \mathbb{N}$, the set

$$U_q^{\gamma} = \left\{ \pi \in \operatorname{Rep}(\Gamma, H) \colon \dim H_{\gamma}^{\pi} \ge q \right\}$$

is open in $\operatorname{Rep}(\Gamma, H)$. So let us fix $\gamma \in \hat{\Gamma}$, $q \in \mathbb{N}$ and $\pi \in U_q^{\gamma}$ to show that π is an interior point of U_q^{γ} . There exist linearly independent vectors $h_1, \ldots, h_q \in H$ such that for every $g \in \Gamma$ and $j \in \{1, \ldots, q\}$, we have

$$\pi(g)(h_j) = \gamma(g)h_j.$$

Pick $\eta > 0$ such that whenever $h'_1, \ldots, h'_q \in H$ are such that for every $j \in \{1, \ldots, q\}$ we have $\|h'_j - h_j\| < \eta$, then h'_1, \ldots, h'_q are also linearly independent. Let us define

$$C = \min \left\{ \left| \gamma(g) - \gamma'(g) \right| \colon g \in \Gamma, \gamma, \gamma' \in \widehat{\Gamma}, \gamma(g) \neq \gamma'(g) \right\},\$$
$$\varepsilon = \frac{C\eta}{n}$$

and

$$V = \left\{ \rho \in \operatorname{Rep}(\Gamma, H) \colon \forall g \in \Gamma \; \forall j \in \{1, \dots, q\} \; \|\rho(g)(h_j) - \pi(g)(h_j)\| < \varepsilon \right\}.$$

It is enough to show that $V \subseteq U_q^{\gamma}$. To do this, pick $\rho \in V$. As in the proof of the previous claim, we have

$$H = \bigoplus_{\gamma' \in \hat{\Gamma}} H^{\rho}_{\gamma'},$$

and so for every $j \in \{1, \ldots, q\}$ there is a decomposition

$$h_j = \sum_{\gamma' \in \hat{\Gamma}} h_j^{\gamma'}$$

such that for every $\gamma' \in \hat{\Gamma}$, we have $h_j^{\gamma'} \in H_{\gamma'}^{\rho}$. It is enough to show that for every $j \in \{1, \ldots, q\}$, we have $\|h_j^{\gamma} - h_j\| < \eta$ since then the linearly independent vectors $h_1^{\gamma}, \ldots, h_q^{\gamma}$ witness that $\rho \in U_q^{\gamma}$. Let us pick $j \in \{1, \ldots, q\}$. For every $k \in \{0, 1, \ldots, n\}$, let us denote

$$(\hat{\Gamma})_k = \{\gamma' \in \hat{\Gamma} : \forall l \in \{1, \dots, k\} \; \gamma'(g_l) = \gamma(g_l)\}.$$

Then we have

$$\left\|h_j - h_j^{\gamma}\right\| \le \sum_{k=1}^n \left\|\sum_{\gamma' \in (\hat{\Gamma})_{k-1}} h_j^{\gamma'} - \sum_{\gamma' \in (\hat{\Gamma})_k} h_j^{\gamma'}\right\|$$

For every $k \in \{1, \ldots, n\}$, we also have

$$\begin{split} \|\sum_{\gamma'\in(\hat{\Gamma})_{k-1}} h_{j}^{\gamma'} - \sum_{\gamma'\in(\hat{\Gamma})_{k}} h_{j}^{\gamma'}\|^{2} &= \|\sum_{\gamma'\in(\hat{\Gamma})_{k-1}\setminus(\hat{\Gamma})_{k}} h_{j}^{\gamma'}\|^{2} \\ &= \sum_{\gamma'\in(\hat{\Gamma})_{k-1}\setminus(\hat{\Gamma})_{k}} \|h_{j}^{\gamma'}\|^{2} \leq \frac{1}{C^{2}} \sum_{\gamma'\in(\hat{\Gamma})_{k-1}\setminus(\hat{\Gamma})_{k}} |\gamma'(g_{k}) - \gamma(g_{k})|^{2} \|h_{j}^{\gamma'}\|^{2} \\ &\leq \frac{1}{C^{2}} \sum_{\gamma'\in\hat{\Gamma}} |\gamma'(g_{k}) - \gamma(g_{k})|^{2} \|h_{j}^{\gamma'}\|^{2} = \frac{1}{C^{2}} \|\sum_{\gamma'\in\hat{\Gamma}} \left(\gamma'(g_{k}) - \gamma(g_{k})\right)h_{j}^{\gamma'}\|^{2} \\ &= \frac{1}{C^{2}} \|\sum_{\gamma'\in\hat{\Gamma}} \gamma'(g_{k})h_{j}^{\gamma'} - \gamma(g_{k})\sum_{\gamma'\in\hat{\Gamma}} h_{j}^{\gamma'}\|^{2} = \frac{1}{C^{2}} \|\sum_{\gamma'\in\hat{\Gamma}} \rho(g_{k})(h_{j}^{\gamma'}) - \gamma(g_{k})h_{j}\|^{2} \\ &= \frac{1}{C^{2}} \|\rho(g_{k})(h_{j}) - \pi(g_{k})(h_{j})\|^{2} < \frac{1}{C^{2}} \varepsilon^{2} = \frac{\eta^{2}}{n^{2}}, \end{split}$$

and so

$$\left\|h_j - h_j^{\gamma}\right\| < \sum_{k=1}^n \frac{\eta}{n} = \eta,$$

as we wanted.

Finally, by Claim 2.2.2, Claim 2.2.3 and the Baire category theorem, comeager many unitary representations of Γ on H are unitarily equivalent to κ_0^a . In particular, they are realizable by an action and so the proof is finished.

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