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KOMBINATORIKA FILTRŮ NA PŘIROZENÝCH
ČÍSLECH

COMBINATORICS OF FILTERS ON THE
NATURAL NUMBERS

Bakalářská práce

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Prohlašuji, že jsem bakalářskou práci vypracoval samostatně a že jsem uvedl všechny použité prameny a literaturu. Děkuji svému vedoucímu práce panu Jonathanu Vernerovi za vedení bakalářské práce.

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Abstract

Práce se věnuje kombinatorickým vlastnostem filtrů na přirozených číslech. Obsahuje úvod a motivaci do problematiky definovatelnosti filtrů a jejich kombinatorikou, definice základních typů filtrů: P-filtr, Q-filtr, Rapid filtr; upořádání: Rudin-Kiesler, Rudin-Blass, Katětov a Tukey; konstrukce filtrů; základní definice z kombinatoriky na ω ; úvod do deskriptivní teorie množin, topologie a základní výsledky.

Abstract

The work is devoted to combinatorial properties of filters on natural numbers as an introduction and motivation to the definability of the filters and its combinatorics. Basic filter types: P-filter, Q-filter, Rapid filter; orders: Rudin-Kiesler, Rudin-Blass, Katětov and Tukey; filter constructions; basic definitions related to combinatorics on ω ; introduction to basic descriptive set theory and topology and some specific results.

Contents

1	Introduction	4
2	Chapter I	6
2.1	Filters	6
3	Chapter II	11
3.1	Orders on filters on ω	12
3.2	Ultrafilter constructions	14
3.3	Standard combinatorial properties	15
4	Chapter III	18
4.1	Topology	18
4.2	Definable sets	23
4.3	Meager sets	24
4.4	Filters and convergence	25
5	Chapter IV	26
5.1	Ideals and filters	26
5.2	Submeasure	27
	References	30

Introduction

The goal of this work is to show the Mazur theorem from [Mazu91] as a bridge between the topology and combinatorics. In the Chapters I and II there are basic definitions related to combinatorics on ω . Chapter III contains an introduction to topology and basic descriptive set theory. Chapter IV focuses to Mazur's specific result.

Set theory is a domain of mathematical logic that studies sets. Georg Cantor as the inventor of the Set theory, a theory of actual infinity, commonly based on *ZFC* (the Zermelo-Fraenkel axioms with the axiom of choice). The first infinite ordinal number in ZFC (the first after all natural numbers) is denoted ω and relates to common natural numbers \mathbb{N} .

Definition 1.1. The set X is *strictly larger* than Y , denoted $X \succ Y$, if there exists one-to-one function from Y to X and there is not bijection from Y to X .

The *Power set* axiom says there exists the set of all subsets of any set X denoted $\mathcal{P}(X)$.

Theorem 1.2. $\mathcal{P}(X) \succ X$ The power set of any set is strictly larger than given set.

Proof. Obviously there is a one-to-one function from X to $\mathcal{P}(X)$. Using a contradiction, let there is a bijection $f : X \rightarrow \mathcal{P}(X)$, so there is a pairing between these sets.

Let imagine the set $A = \{Y \in X \mid Y \notin f(Y)\}$. A is a subset of $\mathcal{P}(X)$, so there must be some element $z \in X$ such that $f(z) = A$.

If $z \in A$, then $z \notin f(z) = A$.
If $z \notin A$, then $z \in A$ by definition of A .

□

In this sense the set $\mathcal{P}(\omega)$ is strictly larger than ω . So there are many infinities. Cantor's development of the theory of infinite sets with various sizes of infinity denoted $\aleph_0, \aleph_1, \aleph_2, \dots$ stimulates a discussion of the philosophy of mathematics. This theorem hasn't probably anything with physics otherwise there are various philosophical views, such as *Finitism*, *Constructivism*, *Platonism*, *Formalism*, regarding what this theorem really means.[Kun11] Finitism says that it's meaningless. The sense of an infinite set is a fiction. Finitism is an extreme form of constructivism, according to which a mathematical object does not exist unless it can be constructed from natural numbers in a finite number of steps. Platonism has come to mean that the

infinitistic concepts really do have some meaning in the abstract mathematical universe.

The formalist philosophy of mathematics maintains the interest in studying the consequences of some collections of axioms. In this case it is ZFC. The focus to the relation between truth and proof appears after Gottlob Frege's invention of the concept of formal logic, and Kurt Gödel's proofs of arithmetics incompleteness in 1931.

In ZFC is not provable which cardinality equals the cardinality of $\mathcal{P}(\omega)$. This can be assumed as the additional axiom $2^{\aleph_0} = \aleph_1$ which is called the *Continuum hypothesis*¹. For this, the size of continuum 2^{\aleph_0} is abbreviated \mathfrak{c} , and the first uncountable cardinal \aleph_1 (the first uncountable ordinal ω_1). There are many cardinal characteristic of the continuum. The continuum could mean \mathbb{R} , *Cantor space* 2^ω , $[\omega]^\omega$ or Baire space ω^ω . These spaces are essentially the same after removal of at most a countable set from each space, there exists a homeomorphism between the modified spaces.

The mathematical aspect of ZFC is called *infinitary combinatorics*.[\[Kun11\]](#) It means proving the theorems using ordinary reasoning. The concept of *Filter* realizes the notion of big sets. It could be imagined as a decision procedure of the majority. For example if the set of people, who voted for some alternative, is in the filter, then they form a majority.

An *ultrafilter* contains every subset or its complement so it is a truth-value assignment. It has a connection to two-valued logic. In this text all ultrafilters are on the set ω . This set of ultrafilters has size same as $\mathcal{P}(\mathcal{P}(\omega))$. It is a question: How can be distinguished the ultrafilters?

In the lecture at the fourth international congress held in Rome in 1908, Frigyes Riesz (1880 - 1956) introduced the concept of ultrafilter. Henri Cartan (1904 - 2008) pointed out the usefulness of this concept nearly thirty years after in the articles of *Théorie des filters* and *Filters et ultrafilters* published in *Compt. rend. Acad. Sci. Paris* (1937).

The theory of definability plays important role here, it develops the topological hierarchy which classifies the sets over real numbers \mathbb{R} . As the real number it is possible to take the points from Cantor space and an ultrafilter could be regarded as a subspace of Cantor space.

¹Paul Cohen in 1963 showed that *Continuum hypothesis* is independent to ZFC axioms.

Chapter I

In this chapter there is an introduction of basic definitions and facts related to the concept of an ultrafilter.

2.1 Filters

Definition 2.1 (Filter on a set). A *filter* on a set X is a collection \mathcal{F} of subsets of X such that:

1. $X \in \mathcal{F}$;
2. if $A \in \mathcal{F}$ and $B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;
3. if $A, B \subseteq X$, $A \in \mathcal{F}$, and $A \subseteq B$, then $B \in \mathcal{F}$.
4. $A \in \mathcal{F}$ or $X \setminus A \in \mathcal{F}$ for all $A \subseteq X$ then \mathcal{F} is called *ultrafilter*.

A filter \mathcal{F} is *proper* if $\emptyset \notin \mathcal{F}$. Only proper filters are considered. A filter \mathcal{F} is *principal*² if there is an $x \in X$ such that $\mathcal{F} = \{A \subseteq X \mid x \in A\}$.

Observation 2.2. Principal filter is ultrafilter.

Proposition 2.3. An ultrafilter is principal if and only if it contains a finite set.

Proof. The right direction is obvious. If there is an ultrafilter which contains finite set A . By dividing the part of A and checking if the part is in the ultrafilter then there is a singleton of some element x . So ultrafilter is principal. \square

Corollary 2.4. An ultrafilter is non-principal (free) if and only if it contains all cofinite sets.

Definition 2.5. A filter \mathcal{F} is *Fréchet filter* on a infinite set X if $\mathcal{F} = \{A \subseteq X \mid |X \setminus A| < \omega\}$.

A filter containing Fréchet filter is called *Free filter*.

Proposition 2.6. A filter is a free filter if the intersection of all its members is empty.

Proof. If the intersection is not empty so there is some set A and the filter contains all cofinite sets. The complements of finite parts of A give a contradiction. \square

²The terminology "principal" is imported from ring theory.

Observation 2.7. If \mathcal{A} is a nonempty family of filters over X , then $\bigcap \mathcal{A}$ is a filter over X .

Proof. If $\bigcap \mathcal{A}$ is not a filter, then there is some member of \mathcal{A} which doesn't satisfy filter conditions. \square

Observation 2.8. If \mathcal{A} is a \subseteq -chain of filters over X , then $\bigcup \mathcal{A}$ is a filter over X .

Proof. If $\bigcup \mathcal{A}$ is not a filter, then there is some member of \mathcal{A} which doesn't satisfy filter conditions. \square

Observation 2.9. If \mathcal{F} is a filter and $X \in \mathcal{F}$, then $\mathcal{P}(X) \cap \mathcal{F}$ is a filter over X .

Proof. If $\mathcal{P}(X) \cap \mathcal{F}$ is not a filter, obviously then \mathcal{F} can't be a filter. \square

Observation 2.10. Let κ be an infinite cardinal, $|S| \leq \kappa$. The set $\{X \subset S \mid |X| > \kappa\}$ is a nonprincipal filter over S .

Definition 2.11 (Finite intersection property FIP). A nonempty system E of sets has the *Finite intersection property*, FIP; if for every $n \in \omega$ and every family $e_0, \dots, e_n \in E$ is true:

$$e_0 \cap \dots \cap e_n \neq \emptyset.$$

Observation 2.12. Every $E \subseteq \mathcal{P}(X)$ with the FIP can be extended to a proper filter.

Proof. \mathcal{F} is defined: $\mathcal{F} = \{A \subseteq X \mid \exists n \in \omega \exists e_0, \dots, \exists e_n \in E (e_0 \cap \dots \cap e_n \subseteq A)\}$. \mathcal{F} is closed under intersection, i.e. that for $A, B \in \mathcal{F}$ there is $X \cap Y \in \mathcal{F}$ because if

$$e_0 \cap \dots \cap e_n \subseteq A \text{ and } f_0 \cap \dots \cap f_m \subseteq B$$

then

$$e_0 \cap \dots \cap e_n \cap f_0 \cap \dots \cap f_m \subseteq A \cap B$$

\square

Lemma 2.12.1. A filter \mathcal{F} over X is an ultrafilter if and only if it is maximal in the order \subseteq .

Proof. Let \mathcal{U} is ultrafilter. For contradiction, there is a $\mathcal{F} \supset \mathcal{U}$ and is some $A \in \mathcal{F} \setminus \mathcal{U}$ and $X \setminus A \in \mathcal{U}$. Then $X \setminus A \in \mathcal{F}$ and $A \in \mathcal{F}$ is contradiction.

For other side assume \mathcal{F} is a filter that is not an ultrafilter. To find $\mathcal{F}' \supset \mathcal{F}$: Let $B \subseteq X$ be such that neither B nor $X \setminus B$ is in \mathcal{F} . Consider the family $\mathcal{G} = \mathcal{F} \cup \{B\}$, \mathcal{G} has the finite intersection property because if $A \in \mathcal{F}$, then $A \cap B \neq \emptyset$, otherwise there is $A \subseteq X \setminus B$ and $X \setminus B \in \mathcal{F}$. If $A_1, \dots, A_n \in \mathcal{F}$, we have $A_1 \cap \dots \cap A_n \in \mathcal{F}$ and so

$$B \cap A_1 \cap \dots \cap A_n \neq \emptyset$$

\mathcal{G} has finite intersection property, so there is a filter $\mathcal{F}' \subseteq \mathcal{G}$.

Since $B \in \mathcal{F}' \setminus \mathcal{F}$, \mathcal{F} is not maximal. □

The *Axiom of choice* implies following useful theorem.

Theorem 2.13 (Zorn's lemma). If X is a partially ordered set such that every chain in X has an upper bound, then X contains a maximal element.

Theorem 2.14 (Tarski's Ultrafilter Theorem). Every filter can be extended to an ultrafilter

Proof. Let \mathcal{F}_0 be a filter. $P = \{\mathcal{F} \mid \mathcal{F}_0 \subseteq \mathcal{F} \text{ and } \mathcal{F} \text{ is filter}\}$. $\langle P, \subset \rangle$ is partially ordered set. Let C is a chain in P . Let C is a chain in P , then $\bigcup C$ is a filter and an upper bound of C in P . By the Zorn's lemma there exists a maximal element \mathcal{U} in P . This \mathcal{U} is an ultrafilter. □

A filter \mathcal{F} over S is countably complete (σ -complete) if it is closed under countable intersections. Every principal filter is closed under arbitrary intersections.

Definition 2.15 (Filter Base). A filter *Base* over a set X is a collection \mathcal{B} of subsets of X such that:

1. if $A \in \mathcal{B}$ and $A' \in \mathcal{B}$, then $A \cap A' \in \mathcal{B}$;
2. $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$.

Given a filter base \mathcal{B} , the filter generated by \mathcal{B} is defined as the smallest filter containing \mathcal{B} . Every filter is also a filter base, so the process of passing from base to the filter generated by it.

Let X be a non-empty set and C be a non-empty subset of X . Then $\{C\}$ is a filter base. The filter generated by C (i.e., the collection of all subsets containing C) is called the filter generated by C .

Definition 2.16. An ultrafilter \mathcal{F} is a *uniform* ultrafilter in X if $|A| = |X|$ for every $A \in \mathcal{F}$.

Definition 2.17 (Filter Generators). The set S is said to *generate* a filter \mathcal{F} (or it is called a set of *filter generators* of \mathcal{F}) if the family all finite intersections of elements of S forms a filter base of \mathcal{F} .

For the answer how many ultrafilters are possible on ω let define following concept.

Definition 2.18. A set $\mathcal{C} \subseteq \mathcal{P}(\omega)$ is *uniformly independent* if for any distinct sets $X_1, \dots, X_n, Y_1, \dots, Y_m \in \mathcal{C}$

$$|X_1 \cap \dots \cap X_n \cap (\omega \setminus Y_1) \cap \dots \cap (\omega \setminus Y_m)| = \omega.$$

It means that for all finite boolean combinations of distinct sets the intersection has cardinality ω .

Firstly let be proven the following lemma.

Lemma 2.18.1. There exist 2^ω uniformly independent subsets of ω .

Proof. Let Fin is the set of all finite subsets of ω and let

$$A = \{\langle F, F' \rangle \mid F \in Fin \text{ and } F' \subseteq Fin \text{ and } |F'| \in Fin\},$$

the size of $Fin \times Fin^{<\omega}$ is ω , so $|A| = \omega$.

For each $X \subseteq \omega$, let

$$A_X = \{\langle F, F' \rangle \in A \mid F \cap X \in F'\}$$

and let

$$\mathcal{C} = \{A_X \mid X \subseteq \omega\}$$

If X and Y are distinct subsets of ω , then $A_X \neq A_Y$. For example, if $n \in X$ but $n \notin Y$, then let $F = \{n\}$, $F' = \{F\}$, and $\langle F, F' \rangle \in A_X$ and $\langle F, F' \rangle \notin A_Y$, so $|\mathcal{C}| = 2^\omega$.

To show that \mathcal{C} is uniformly independent, let $X_1, \dots, X_n, Y_1, \dots, Y_m$ are distinct subsets of ω . For each $i \leq n$ and each $j \leq m$, let $a_{ij} \in \omega$ such that either $a_{ij} \in X_i \setminus Y_j$ or $a_{ij} \in Y_j \setminus X_i$.

Now let $F \in \text{Fin}$ such that $\{a_{ij} \mid i \leq n \text{ and } j \leq m\} \subseteq F$.
 $\forall i \leq n, j \leq m (F \cap X_i \neq F \cap Y_j)$. So if $F' = \{F \cap X_i \mid i \leq n\}$, then

$$\begin{aligned} \forall i \leq n \langle F, F' \rangle &\in A_{X_i}, \\ \forall j \leq m \langle F, F' \rangle &\notin A_{Y_j}, \end{aligned}$$

then,

$$|A_{X_1} \cap \dots \cap A_{X_n} \cap (\omega \setminus A_{Y_1}) \cap \dots \cap (\omega \setminus A_{Y_m})| = \omega.$$

□

Theorem 2.19 (Pospíšil³). The number of uniform ultrafilters on ω is 2^{2^ω} .

Proof. Let \mathcal{C} be an uniformly independent family of subsets of ω . For every function $f : \mathcal{C} \rightarrow \{0, 1\}$, consider this family of subsets of ω :

$$G_f = \{X \mid \omega \setminus X \mid \leq \omega\} \cup \{X \mid f(X) = 1\} \cup \{\omega \setminus X \mid f(X) = 0\}$$

The family G_f has the finite intersection property, and so there exists an ultrafilter D_f such that $D_f \supseteq G_f$. D_f is uniform. If $f \neq g$, then for some $X \in \mathcal{C}$, $f(X) \neq g(X)$; e.g. $f(X) = 1$ and $g(X) = 0$ and then $X \in D_f$ while $\omega \setminus X \in D_g$. So there are 2^{2^ω} distinct uniform ultrafilters over ω .[\[Jech78\]](#) □

³ *Bedřich Pospíšil (1912-1944) proved the theorem in a work On bicomact spaces published in 1939 at Masaryk University periodicals in Brno. On the request of the most significant set theory magazine published at the time, Fundamenta Mathematicae. Pospíšil published a revised version of his paper in this magazine. In 1941 he was arrested by the Gestapo and sentenced to three years in a concentration camp, from where he returned on May 17, 1944 but he soon succumbed to the consequences of long imprisonment.*

Chapter II

At first sight it is not clear that all non-principal ultrafilters are not the same (up to permutation of ω). However a simple cardinality argument shows that this can't be the case. There are too many ultrafilters and not enough permutations so that there are non-isomorphic non-principal ultrafilters on ω . It is an important problem to find the properties that distinguish them.

The analysis of different orders on the set of all ultrafilters on ω gives some view on complicated structure of this set. There is a ordering of the ultrafilters which says that \mathcal{U} is less than \mathcal{V} if it is a quotient of \mathcal{V} under some mapping of the natural numbers.

Let define following useful order concepts.

Definition 3.1. A *quasiorder* is a set with a transitive reflexive relation \leq .

Definition 3.2. A *partial order* is antisymmetric quasiorder.

Definition 3.3. A partial order is *directed* if for any two members there is another member such is above both.

Definition 3.4. A subset $A \subseteq X$ of partially ordered set $\langle X, \leq \rangle$ is *cofinal* if $\forall x \in X \exists a \in A (x \leq a)$.

Definition 3.5. A subset $A \subseteq X$ of partially ordered set $\langle X, \leq \rangle$ is *bounded* if $\exists x \in X \forall a \in A (a \leq x)$.

The cofinal and unbounded set is same in linear order.

Observation 3.6. If $\langle X, \leq \rangle$ is directed order, and $A \subseteq X$ is cofinal, then A is directed.

Proof. For any two $a, b \in A$ there is another $c \in X$ above. The cofinality gives some $d \in A$ above c . From transitivity $a, b \leq d$. \square

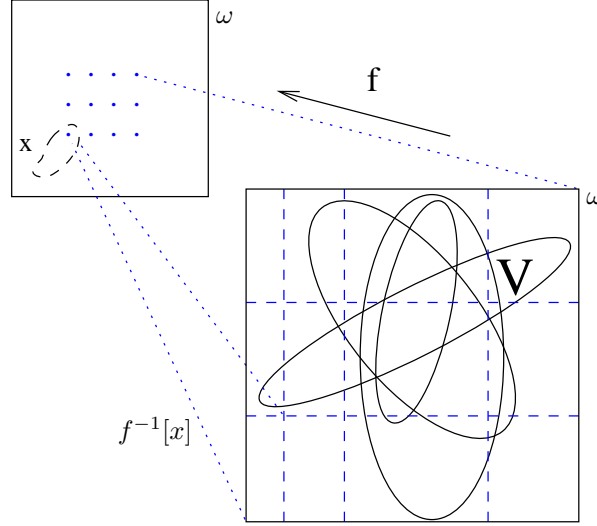
Definition 3.7. A function $f : X \rightarrow Y$ is cofinal if the image of each cofinal subset of X is cofinal in Y .

Definition 3.8. A partial ordering $\langle Y, \leq_Y \rangle$ is *Tukey reducible* to a partial ordering $\langle X, \leq_X \rangle$, $X \leq_T Y$, if there is a cofinal function $f : Y \rightarrow X$. [Dobrinen]

3.1 Orders on filters on ω

Definition 3.9 (image of a filter under a function $f : \omega \rightarrow \omega$). For $f \in {}^\omega\omega$ and a filter $\mathcal{V} \subseteq \mathcal{P}(\omega)$ let

$$f(\mathcal{V}) = \{x \subseteq \omega \mid \exists y \in \mathcal{V} f[y] \subseteq x\}$$



Picture illustrates a factoring on ω induced by the function f and pre-image of some set x .

Observation 3.10. $f(\mathcal{V}) = \{x \subseteq \omega \mid f^{-1}[x] \in \mathcal{V}\}$

Observation 3.11. If $\mathcal{V} \subseteq \mathcal{P}(\omega)$ is an ultrafilter over ω , then $\mathcal{U} = f(\mathcal{V})$ is also an ultrafilter over ω .

Proof. Since $f^{-1}[\omega] = \omega$, so $\omega \in \mathcal{U}$, and since $f^{-1}[\emptyset] = \emptyset$, so $\emptyset \notin \mathcal{U}$.

If $x \subseteq x'$ and $x \in f(\mathcal{V})$, then $f[y] \subseteq x$ for some $y \in \mathcal{V}$, and therefore $f[y] \subseteq x'$, which shows that $x' \in f(\mathcal{V})$.

If $x, x' \in f(\mathcal{V})$, then $f^{-1}[x], f^{-1}[x'] \in \mathcal{V}$, and since \mathcal{V} is a ultrafilter,

$f^{-1}[x] \cap f^{-1}[x'] \in \mathcal{V}$. Since $f^{-1}[x \cap x'] \in \mathcal{V}$ we get $x \cap x' \in f(\mathcal{V})$.

if $x \notin f(\mathcal{V})$, then $f^{-1}[x] \notin \mathcal{V}$, and $\omega \setminus f^{-1}[x] \in \mathcal{V}$, then $f^{-1}[\omega] \setminus f^{-1}[x] \in \mathcal{V}$, and $f^{-1}[\omega \setminus x] \in \mathcal{V}$, so $\omega \setminus x \in \mathcal{V}$. \mathcal{U} is ultrafilter. \square

Lemma 3.11.1. if \mathcal{U} is ultrafilter and $f(\mathcal{U}) = \mathcal{U}$, then $\{n \mid f(n) = n\} \in \mathcal{U}$, f is identity.

Proof. Let $A = \{n \mid f(n) = n\}$, $B = \{n \mid f(n) < n\}$, and $C = \{n \mid f(n) > n\}$. $f^{(n)}$ abbreviates n -th iteration of f .

If $B \in \mathcal{U}$, let $B_n = \{m \mid \forall n' < n (f^{(n')}(m) \in B) \text{ and } f^{(n)}(m) \notin B\}$.

$$\bigcup_{1 \leq n} B_n = B.$$

One of $B_E = \bigcup_{1 \leq n} B_{2n}$ and $B_O = \bigcup_{1 \leq n} B_{2n+1}$ is in \mathcal{U} because \mathcal{U} is ultrafilter.

If $B_E \in \mathcal{U}$, then $f[B_E] \in \mathcal{U}$, and if $B_O \in \mathcal{U}$, then $f[B_O] \in \mathcal{U}$, so both cases are impossible, $B \notin \mathcal{U}$.

If $C \in \mathcal{U}$, let $C_n = \{m \mid \forall n' < n (f^{(n')}(m) \in C) \text{ and } f^{(n)}(m) \notin C\}$.

$\bigcup_{1 \leq n} C_n = C$. Same as for B , $C \notin \mathcal{U}$.

Let $C^c = \omega \setminus C$ and $C_0^c = \{n \in C^c \mid n \notin f[C^c]\}$,

$C_n^c = \{m \in C^c \mid \forall n' < n (m \in f^{(n')}[C_0^c]) \text{ and } m \notin f^{(n)}[C_0^c]\}$, so $C^c \notin \mathcal{U}$, then $A \in \mathcal{U}$. \square

Definition 3.12 (Rudin-Keisler order). Let \mathcal{F}, \mathcal{G} are filters. If there is a function $f : \omega \rightarrow \omega$ such that $A \in \mathcal{F}$ if and only if $f^{-1}[A] \in \mathcal{G}$, then $\mathcal{F} \leq_{RK} \mathcal{G}$. [Hrus11]

Definition 3.13. $\mathcal{F} \equiv_{RK} \mathcal{G}$ if and only if $\mathcal{F} \leq_{RK} \mathcal{G}$ and $\mathcal{G} \leq_{RK} \mathcal{F}$

Observation 3.14. $\mathcal{F} \equiv_{RK} \mathcal{G}$ if and only if there is a permutation $f : \omega \rightarrow \omega$ such that $\mathcal{F} = \{A \subset \omega \mid f^{-1}(A) \in \mathcal{G}\}$.

Proof. By definition of RK order, $A \in \mathcal{F}$ if and only if $g^{-1}[f^{-1}[A]] \in \mathcal{F}$, then $\omega = g^{-1}[f^{-1}[\omega]]$, so $f \circ g$ is a permutation. \square

Ultrafilters that are RK equivalent are said to be isomorphic. There are several partial orders on *isomorphism types* of ultrafilters in the following definitions. The given isomorphism type means the set of all isomorphic ultrafilters.

Observation 3.15. If \mathcal{U} and \mathcal{V} are ultrafilters on ω and $\forall A \in \mathcal{V} (f[A] \in \mathcal{U})$, then f witnesses that $\mathcal{U} \leq_{RK} \mathcal{V}$.

Proof. Let $B \in \mathcal{U}$, for contradictory let $f^{-1}[B] \notin \mathcal{V}$, then $\omega \setminus f^{-1}[B] \in \mathcal{V}$, so $f^{-1}[\omega \setminus B] \in \mathcal{V}$, then $f[f^{-1}[\omega \setminus B]] \subseteq \omega \setminus B \in \mathcal{U}$, then $B \notin \mathcal{U}$.

The other side, let $f^{-1}[A] \notin \mathcal{V}$, then $\omega \setminus f^{-1}[A] \in \mathcal{V}$, and $f^{-1}[\omega \setminus A] \in \mathcal{V}$, so $f[f^{-1}[\omega \setminus A]] \subseteq \omega \setminus A \in \mathcal{U}$, and then $A \notin \mathcal{U}$. \square

The relation \leq_{RK} is a quasiorder since the relation is not antisymmetric.

Definition 3.16 (Katětov order). Let \mathcal{F}, \mathcal{G} are filters. If there is a function $f : \omega \rightarrow \omega$ such that $f^{-1}[A] \in \mathcal{G}$, for all $A \in \mathcal{F}$ then $\mathcal{F} \leq_K \mathcal{G}$. [Hrus11]

The Katětov order was introduced by Miroslav Katětov⁴ together with the Rudin-Keisler order. [Hrus11]

⁴Since 1953 to 1957 he was rector of Charles University in Prague.

On ultrafilters the orders became the same. The filters are Katětov equivalent, $\mathcal{F} \equiv_K \mathcal{G}$, analogously as Rudin-Keisler equivalence.

The same for Katětov-Blass and Tukey orders.

Observation 3.17. If $\mathcal{F} \subseteq \mathcal{G}$, then $\mathcal{F} \leq_K \mathcal{G}$, f is identity.

Let consider a variant of Katětov order.

Definition 3.18 (Katětov-Blass order). Let \mathcal{F}, \mathcal{G} are filters. If there is a finite-to-one function $f : \omega \rightarrow \omega$ such that $f^{-1}[A] \in \mathcal{G}$, for all $A \in \mathcal{F}$ then $\mathcal{F} \leq_{KB} \mathcal{G}$. [Hrus11]

Let consider a variant of Rudin-Keisler order.

Definition 3.19 (Rudin-Blass order). Let \mathcal{F}, \mathcal{G} are filters. If there is a finite-to-one function $f : \omega \rightarrow \omega$ such that $A \in \mathcal{F}$ if and only if $f^{-1}[A] \in \mathcal{G}$, then $\mathcal{F} \leq_{RB} \mathcal{G}$. [Hrus11]

An ultrafilter can be considered as a partial ordering by reverse inclusion. So $\langle \mathcal{U}, \supseteq \rangle$ is a directed partial ordering.

Definition 3.20 (Tukey order). Let \mathcal{U}, \mathcal{V} are ultrafilters. If there is a cofinal function $f : \mathcal{V} \rightarrow \mathcal{U}$, then $\mathcal{U} \leq_T \mathcal{V}$.

Observation 3.21. Let \mathcal{U}, \mathcal{V} are ultrafilters. If $\mathcal{U} \leq_{RK} \mathcal{V}$, then $\mathcal{U} \leq_T \mathcal{V}$

Proof. Let $\forall A \in \mathcal{V} (f[A] \in \mathcal{U})$. The Tukey function $f'(A) = f[A]$ for all $A \in \mathcal{V}$. f' is obviously cofinal, so $\mathcal{U} \leq_T \mathcal{V}$. \square

Tukey ordering on ultrafilters is a weakening of Rudin-Keisler ordering. The Tukey equivalence class of an ultrafilter is called its Tukey type.

3.2 Ultrafilter constructions

Above there are constructions of filter via function f . Following constructions operate with the set of filters.

Definition 3.22 (Fubini product). Let \mathcal{F}, \mathcal{G} are filters on ω .

The $\mathcal{F} \times \mathcal{G} = \{A \subseteq \omega \times \omega \mid \{n \mid A(n) \in \mathcal{G}\} \in \mathcal{F}\}$ such $A \subseteq \omega \times \omega$ where $A(n)$ is vertical section at n ; $A^x(n) = \{m \mid (n, m) \in A\}$

The product $\mathcal{F} \times \mathcal{G}$ is induced by the base $\{a \times b \mid a \in \mathcal{F} \text{ and } b \in \mathcal{G}\}$.

This filter can be viewed as a filter on ω by fixing a bijection $b : \omega \times \omega \rightarrow \omega$.

Definition 3.23 (F-sum). If $\{\mathcal{F}_s \mid s \in S\}$ is a set of filters and \mathcal{F} is a filter on ω . Then the F-sum of the filters is:

$$\mathcal{F} \setminus \sum_{s \in \omega} F_s = \{A \subseteq \bigcup_{s \in \omega} \{s\} \times S_s \mid \{s \mid A_x(s) \in \mathcal{F}_s\} \in \mathcal{F}\}$$

Definition 3.24 (Free-product filter). Let \mathcal{F}, \mathcal{G} are filters on ω . $\mathcal{F} \otimes \mathcal{G} = \{(A, B) \mid A \in \mathcal{F} \text{ and } B \in \mathcal{G}\}$

Note. If \mathcal{U}, \mathcal{V} are ultrafilters then so is $\mathcal{U} \times \mathcal{V}$. However $\mathcal{U} \otimes \mathcal{V}$ is never an ultrafilter (e.g. because of the set $\sum_{n < \omega} \{n\} \otimes \omega \setminus n$)

3.3 Standard combinatorial properties

Let define some special sorts of ultrafilters. The first combinational property of filters, a generalization of the standard P-point property of ultrafilters.

Definition 3.25 (P-filter). A filter \mathcal{F} is *P-filter* if for every (descending: $A_0 \supseteq A_1 \supseteq A_2 \dots$) countable sequence $\langle A_n \in \mathcal{F} \mid n < \omega \rangle$ of elements of \mathcal{F} there exists $X \in \mathcal{F}$ such that $X \subseteq^* A_n$ for all $n < \omega$. $X \setminus \mathcal{F}_n$ is finite.

P-ultrafilters are called P-points (weakly selective). A P-point is a non-principal ultrafilter (A point of topological space is a P-point if its filter of neighbourhoods is closed under countable intersections.)

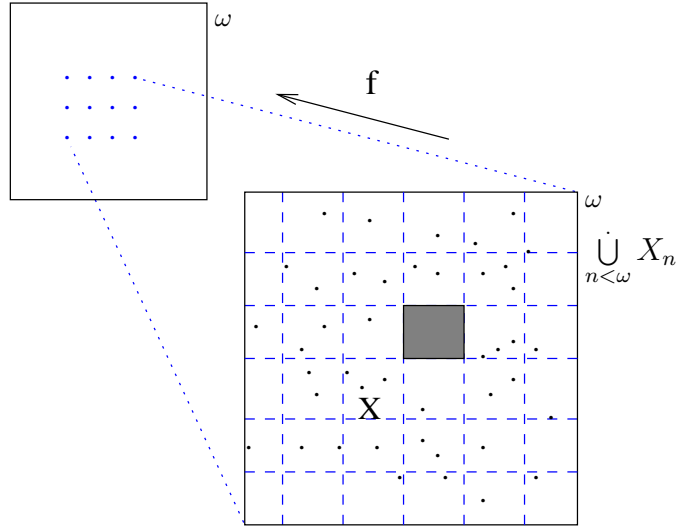
Definition 3.26 (P-ultrafilter 1). An ultrafilter \mathcal{U} is *P-ultrafilter* (weakly selective): Let there is a ω factoring: $\omega = \bigcup_{n < \omega} X_n$ and for \mathcal{U} there are satisfied one of following items:

1. $\exists n < \omega (X_n \in \mathcal{U})$
2. $\exists (X \in \mathcal{U})(\forall n)(|X \cap X_n| < \omega)$

Definition 3.27 (P-ultrafilter 2). An ultrafilter \mathcal{U} is *P-ultrafilter* (weakly selective)

if $(\forall f : \omega \rightarrow \omega)(\exists X \in \mathcal{U})(\forall n \in \omega)(|(f \upharpoonright X)^{-1}(n)| < \omega)$,
it means $f \upharpoonright X$ is constant or finite-to-one.

Every function on ω becomes finite-to-one or constant when restricted to some set in \mathcal{U} .



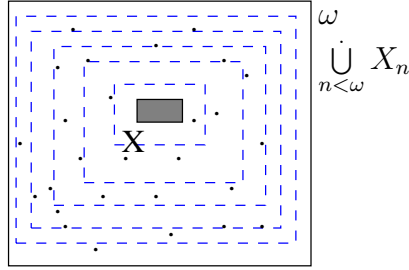
The both previous P-ultrafilter definitions are equivalent using $f(x) = n \Leftrightarrow x \in X_n$. $\{X_n \mid n < \omega\}$ is ω factoring. If some $X_n \in \mathcal{U}$, it is done. If not, there is a $X \in \mathcal{U}$ such that $|f \upharpoonright X| < \omega$. This means $X \cap X_n < \omega$.

Observation 3.28. The definitions of P-filter and P-ultrafilter are equivalent.

Proof. Let there is a factoring $\omega = \bigcup_{n < \omega} X_n$. If some set $X_n \in \mathcal{U}$ it is finished.

If no partition is in the ultrafilter, let there is an enumeration of their complements: $\langle X'_n \mid X'_n = \omega \setminus X_n \text{ for } n \in \omega \rangle$. For this set exists $X \in \mathcal{U}$, and for every $n \in \omega$, $|X \cap X'_n| < \omega$.

The other direction, let $\langle A_n \in \mathcal{U} \mid n < \omega \rangle$ is a sequence in \mathcal{U} . Without loss of generality the sequence is strictly decreasing, and $A_0 = \omega$. If \mathcal{U} contains the intersection it is finished. If not, let consider the factoring defined $X_n = A_n \setminus A_{n+1}$ illustrated on the following picture.



No part this factoring of ω is in \mathcal{U} since if $X_n \in \mathcal{U}$ then $X_n \cap A_{n+1} = \emptyset \in \mathcal{U}$. There is some $X \in \mathcal{U}$ where $|X \cap A_n| < \omega$. By induction, $X \subseteq A_0$. Suppose $X \subseteq^* A_n$. $X \cap A_{n+1} = (X \cap A_n) \setminus X_n$, since $X_n \cap X$ is finite, then $X \cap A_n =^* X \cap A_{n+1}$, so $X \subseteq^* A_{n+1}$.

□

Definition 3.29 (Q-filter). A filter \mathcal{F} is *Q-filter* if for every partition P of ω into finite sets there is a selector $A \in \mathcal{F}$, set $\forall p \in P (A \cap p \neq \emptyset)$.

Definition 3.30 (Rapid-filter). A filter \mathcal{F} is *Rapid-filter* if for each function $h : \omega \rightarrow \omega$, there is $A \in \mathcal{F}$ with $|A \cap h(n)| \leq n$ for every $n < \omega$.

Chapter III

Descriptive set theory deals with sets of real numbers that are described in some simple way: sets that have simple topological structure (e.g., continuous images of closed sets) or are definable in a simple way. The goal of this chapter presents filters on ω in the context of their topological properties. It means to identify filters on ω with subsets of 2^ω via characteristic functions of their elements.

4.1 Topology

Definition 4.1 (Topological space). A *Topological space* is an ordered pair $\langle X, \tau \rangle$, where X is a set and $\tau \subseteq \mathcal{P}(X)$ such that:

1. $\emptyset, X \in \tau$;
2. if $A \subseteq \tau$, then $\bigcup A \in \tau$;
3. if $A, B \in \tau$, then $A \cap B \in \tau$.

The collection τ is called *topology*. Members of the topology are called *open* sets. The set is called *closed* if its complement is open. The idea behind this definition, at least for the standard spaces, is that an open set is one which contains no point of its boundary. For instance, in 2-dimensional euclidean space, an open disc, meaning the set of points having distance strictly less than some fixed number from a fixed point, forms an open set. Another way of explain this is that wherever in the set is possible to move a little in any direction, and stay in the set. For the closed disc moving any distance may possible leave the set. [\[Truss97\]](#)

Though the definition of closed as the complement of open, it is possible for a set to be both closed and open. In this case the set is called *clopen*. Obvious examples of clopen sets in all spaces are \emptyset and X , but there may be many more clopen sets than that. The more clopen sets are in the more disconnected spaces.

Definition 4.2 (Neighbourhood). \mathcal{N}_x is *neighbourhood* of $x \in X$ if there is an open set O containing x such that $O \subseteq \mathcal{N}_x$. If \mathcal{N}_x is open, we call it open neighbourhood O_x .

Observation 4.3. Directly from definition, the system of closed sets contains X and \emptyset and is closed under arbitrary intersections and finite unions (De Morgan's laws).

Definition 4.4 (Interior). If Y is a subset of X , let $\text{int}(Y)$ be the union of open sets contained in Y .

$$\text{int}(Y) = \bigcup \{O \in \tau \mid O \subseteq Y\}$$

Definition 4.5 (Closure). Let \bar{Y} be the intersection of all closed sets containing Y .

$$\bar{Y} = \bigcap \{C \mid C \text{ is closed and } Y \subseteq C\}$$

Observation 4.6. $\text{int}(Y)$ is the greatest open set contained in Y and \bar{Y} is the smallest closed set containing Y in the ordering under inclusion.

Definition 4.7. Set $D \subseteq X$ is *dense* in (X, τ) if $\bar{D} = X$.

Definition 4.8. Set $\mathcal{B} \subseteq \mathcal{P}(X)$ is *topology base* if

1. for $U, V \in \mathcal{B}$ and $x \in U \cap V$ then $\exists W \in \mathcal{B} (x \in W \subseteq U \cap V)$.
2. $\forall x \in X \exists U \in \mathcal{B} (x \in U)$

Definition 4.9 (Compactness). $\langle X, \tau \rangle$ is *compact* if every open cover of X has a finite subcover, where C is an open cover if $C \subseteq \tau$ and $\bigcup C = X$. Conversely if F is a system of closed sets and has FIP then $\bigcap F$ is non-empty.

Definition 4.10. $\langle X, \tau \rangle$ is *locally compact* if every point x has a compact neighbourhood.

Definition 4.11 (Filter converges to x). Let \mathcal{F} be a filter and $x \in X$. Filter converges to x , or that x is a limit of \mathcal{F} if all $N_x \subseteq \mathcal{F}$.

Example 4.12. Frechet filter \mathcal{F} in discrete topology on ω is non-convergent filter. Singleton set $\{n\}$ cannot belong to \mathcal{F} .

Definition 4.13 (Hausdorff space). A *Hausdorff space*⁵ is a topological space with a separation property: any two distinct points can be separated by disjoint open sets.

Lemma 4.13.1. X is Hausdorff space if every filter has at most one limit.

⁵ Hausdorff included the separation property in his axiomatic description of general spaces in *Grundzüge der Mengenlehre* (1914; “Elements of Set Theory”). Although later it was not accepted as a basic axiom for topological spaces, the Hausdorff property is often assumed in certain areas of topological research. It is one of a long list of properties that have become known as “separation axioms” for topological spaces.

Proof. Suppose X is Hausdorff and let $x \neq y$. Then there are neighbourhoods U and V of x and y respectively with $U \cap V = \emptyset$. No filter contains both U and V , and so no filter can converge to both x and y . Hence all filters have at most one limit.

Conversely, suppose that x and y do not have disjoint neighbourhoods. Then $N_x \cup N_y$ forms a subbase for a filter which converges to both x and y . So if every filter has at most one limit then X is Hausdorff. \square

So requiring X to be Hausdorff is equivalent to requiring unique limits. In Hausdorff space $\lim_{\mathcal{F}} = x$ means x is unique limit of \mathcal{F} . Note that not all filters have a limit.

Definition 4.14 (Regular space). A *regular space* is a topological space with a separation property: Any point and closed set can be separated by disjoint open sets.

Lemma 4.14.1. Locally compact Hausdorff space $\langle X, \tau \rangle$ is regular.

Proof. Let there is closed set $F \subseteq X$ and point $x \in X \setminus F$. Let $\langle N_x, \tau_{N_x} \rangle$ is compact neighbourhood subspace with the following disjoint open sets $O_x = \bigcup \{O \in \tau_{N_x} \mid \overline{O} \cap F = \emptyset\}$ and $O_F = X \setminus \overline{O_x}$ \square

Definition 4.15 (Normal space). A *normal space* is a topological space with a separation property: Any two distinct closed sets can be separated by disjoint open sets.

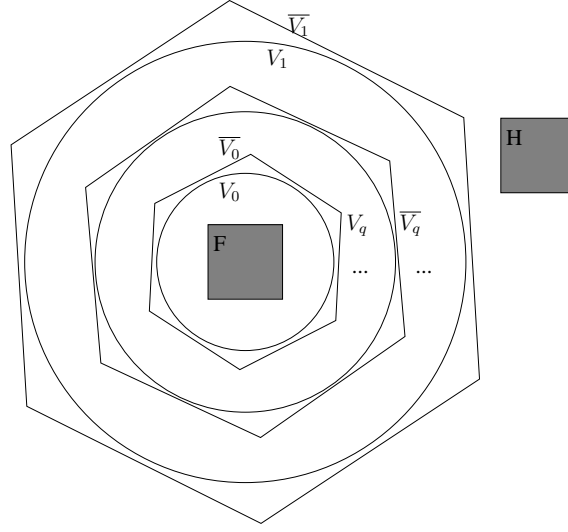
Definition 4.16 (Continuous function). Let $\langle X, \tau \rangle, \langle Y, \sigma \rangle$ are topological spaces and $f : X \rightarrow Y$ is function. f is *continuous* if for every open set U in Y , $f^{-1}[U]$ is open in X .

Observation 4.17. A topological space is normal if and only if for every open set U and every closed $C \subseteq U$, there is an open set V such $C \subseteq V \subseteq \overline{V} \subseteq U$.

Theorem 4.18 (Urysohn's lemma⁶). Let $\langle X, \tau \rangle$ is normal space and F, H are closed sets such that $F \cap H = \emptyset$, then exists a continuous function which separates F and H .

Proof. Firstly let construct a system of open sets $\{V_q \mid q \in \mathbb{Q} \cap [0, 1]\}$, where is satisfied $V_q \subseteq \overline{V_q} \subseteq V_p \iff q < p \in \mathbb{Q} \cap [0, 1]$ and $\forall q \in \mathbb{Q} \cap [0, 1] (F \subseteq V_q \text{ and } H \cap V_q = \emptyset)$.

⁶Urysohn's lemma has the usefull applications. For example Urysohn Metrization Theorem. If X is a normal space with a countable basis, then there is the continuous function from X to $[0, 1]$ to assign numerical coordinates to the points of X and obtain an embedding of X into \mathbb{R}^ω . From this, every countable normal space is a metric space.



The construction uses the induction. Firstly there is enumeration $\mathbb{Q} \cap [0, 1]$ as $\langle q_n \mid n < \omega \rangle$ where $q_0 = 0$ and $q_1 = 1$. Using the previous observation we setup V_0 and V_1 as the first step.

The induction step:

For V_{q_i} there is $k > i$ maximal where $q_i < q_{k+1}$. For $\overline{V_{q_i}} \subseteq V_{q_{i+}}$ there is $V_{q_{k+1}}$ where

$$V_{q_i} \subseteq \overline{V_{q_i}} \subseteq V_{q_{k+1}} \subseteq \overline{V_{q_{k+1}}} \subseteq V_{q_{i+1}}$$

The function f is defined:

$$f(x) = \begin{cases} \inf(\{q \mid x \in \overline{V_q}\}) & \text{if } x \in \bigcup_{q \in \mathbb{Q} \cap [0, 1]} \overline{V_q} \\ 1 & \text{otherwise} \end{cases}$$

For showing the function f is continuous. Let (q_1, q_2) is open interval in $\mathbb{Q} \cap [0, 1]$. Firstly let $q_2 < 1$ then

$$f^{-1}[(q_1, q_2)] = U_{(q_1, q_2)} = \bigcup_{q \in (q_1, q_2)} V_q \setminus \overline{V_{q_1}}$$

is open set. Now let $q_2 = 1$ then

$$f^{-1}[(q_1, q_2)] = U_{(q_1, q_2)} \cup X \setminus \overline{V_1}$$

is open set. □

Theorem 4.19. A topological space X is compact if and only if every ultrafilter on X converges to at least one point.

Proof. Suppose that X is compact, and let \mathcal{U} be an ultrafilter on X . Then \mathcal{U} has FIP, since it is closed under finite intersections, and $\emptyset \notin \mathcal{U}$. Compactness causes that there is some point $x \in \bigcup_{B \in \mathcal{U}} \overline{B}$. This means that every open neighbourhood of x meets every $B \in \mathcal{U}$. Let N_x be an open neighbourhood of x . Since no member of \mathcal{U} is disjoint from N_x , in particular $\omega \setminus N_x \notin \mathcal{U}$. Since \mathcal{U} is an ultrafilter, it must be that $N_x \in \mathcal{U}$. This proves that \mathcal{U} converges to x .

For the converse, suppose that every ultrafilter converges and let F be a family of subsets of X that has FIP. Then F generates a filter, which can be extended to an ultrafilter \mathcal{U} . By assumption, \mathcal{U} converges to some point x . Consider $B \in F$. Since \mathcal{U} converges to x , every neighbourhood of x meets B . This says exactly that $x \in \overline{B}$, so, since this is true of every $B \in F$, so $x \in \bigcup_{B \in F} \overline{B}$. This proves that X is compact. \square

Definition 4.20 (P-point). A point x in topological space X is called a *P-point* if the intersection of countably many neighbourhoods of x is again a neighbourhood of x .

Definition 4.21 (Weak P-point). A point x in a topological space that is not an accumulation point of any countable subset of the space is called a *weak P-point*. Every P-point is a weak P-point.

Let $2^{<\omega}$ denotes the set of all finite sequences of 0,1. The ordering by inclusion of these sequences turns $\langle 2^{<\omega}, \subseteq \rangle$ into a tree. $\langle 2^{<\omega}, \subseteq \rangle$ is the full binary tree of height ω .

Definition 4.22 (Cantor space). The pair $\langle 2^\omega, \tau \rangle$ is called *Cantor space* with topology generated by base set $\mathcal{B} = \{B \mid B \supset A \in 2^{<\omega}\}$. (the set of all cofinal branches).

Observation 4.23. Cantor space has a countable base.
(A set of all finite sequences is countable.)

Observation 4.24. Cantor space has a base composed of clopen sets.⁷
(A complement of any base set is union of base sets which have different initial sequence.)

⁷The space is totally disconnected.

Definition 4.25 (Product topology of two Cantor spaces). $\langle 2^\omega \times 2^\omega, \sigma \rangle$

Let \mathcal{B} is a base of Cantor space.

The base of the product is $\mathcal{B} \times \mathcal{B}$, and $\sigma = \{\bigcup \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B} \times \mathcal{B}\}$.

Observation 4.26. The intersection and union, as the functions $f : \mathcal{P}(\omega) \times \mathcal{P}(\omega) \rightarrow \mathcal{P}(\omega)$, are continuous as the function from $\langle 2^\omega \times 2^\omega, \sigma \rangle$ to $\langle 2^\omega, \tau \rangle$, where $2^\omega \approx \mathcal{P}(\omega)$.

Proof. Let $O = \{\bigcup \mathcal{A} \mid \mathcal{A} \subseteq \mathcal{B}\}$ is an open set.

Using base sets contained in O , so let $X \in \mathcal{B}$ and $X \subseteq O$,

Pre-image $\bigcap^{-1}[X]$ is obviously a subset of $\mathcal{B} \times \mathcal{B}$ and $\bigcup^{-1}[X]$ is a subset of $\mathcal{B} \times \mathcal{B}$,

(all possible boolean combinations of initial segments of the pairs from the base set according to \cap or \cup).

Then for any base subset $\mathcal{A} \subseteq \mathcal{B}$,

$\exists \mathcal{A}' \subseteq \mathcal{B} \times \mathcal{B} \forall A, B (A \cap B \in \bigcup \mathcal{A} \text{ and } \langle A, B \rangle \in \bigcup \mathcal{A}')$,

$\exists \mathcal{A}' \subseteq \mathcal{B} \times \mathcal{B} \forall A, B (A \cup B \in \bigcup \mathcal{A} \text{ and } \langle A, B \rangle \in \bigcup \mathcal{A}')$,

then $\bigcap^{-1}[O] = \{\langle A, B \rangle \mid A \cap B \in \bigcup \mathcal{A} \text{ and } \mathcal{A} \subseteq \mathcal{B}\}$ is open,

$\bigcup^{-1}[O] = \{\langle A, B \rangle \mid A \cup B \in \bigcup \mathcal{A} \text{ and } \mathcal{A} \subseteq \mathcal{B}\}$ is open.

□

4.2 Definable sets

Descriptive set theory clasifies the sets according to the complexity of their definitions. *Borel hierarchy* is used to describe a collection of subsets of \mathbb{R} , Baire space or Cantor space, etc. Level one consists of all open (Σ_1^0) and closed (Π_1^0) sets, and levels 2, 3, 4, ... are obtained by taking countable unions and intersections of previous level. More complex definable sets are *projective sets*, those obtained from Borel sets by the operation of continuous image and complementation.

Definition 4.27 (F_σ). A set $A \subseteq \mathbb{R}$ is F_σ ⁸ if it is countable union of closed sets F . The class is denoted Σ_2^0 in logical notation.

Definition 4.28 (G_δ). A set $A \subseteq \mathbb{R}$ is G_δ ⁹ if it is countable intersection of open sets G . The class is denoted Π_2^0 in logical notation.

The next levels are $F_{\sigma\delta}$, it is a countable intersections of F_σ . And $G_{\delta\sigma}$, it is a countable unions of G_δ .

⁸ F_σ comes from French: The F stands for *fermé*, meaning "closed," while the sigma stands for *somme*, meaning "sum."

⁹ G_δ comes from German: The G stands for *Gebiet*, meaning "area," while the delta stands for *Durchschnitt*, meaning "intersection."

4.3 Meager sets

Meager set of first category is a set that, considered as a subset of a topological space, is in a precise sense small or negligible.

Definition 4.29 (Nowhere dense set). Given a topological space X , a subset A of X is *nowhere dense* if for every non-empty open set O there is a non-empty open set $O' \subseteq O$ such that $O' \cap A = \emptyset$.

A subset B of X is nowhere dense if there is no neighbourhood on which B is dense: for any nonempty open set U in X , there is a nonempty open set V contained in U such that V and B are disjoint.

Definition 4.30 (Meager set). Given a topological space X , a subset A of X is *meager* (the first category) if it can be expressed as the union of countably many nowhere dense subsets of X .

The rational numbers are meager as a subset of \mathbb{R} . The Cantor set is meager as a subset of \mathbb{R} , but not as a space, since it is complete metric space.

Definition 4.31 (Baire space). A topological space is called a *Baire space* if the complements of meager sets in X are dense.

Observation 4.32. A topological space is Baire if and only if the intersection of countable many open dense sets in X is dense in X .

Theorem 4.33 (Baire category theorem). Every locally compact Hausdorff space $\langle X, \tau \rangle$ is Baire.

Proof. Let there are countable many open dense sets

$$\mathcal{D} = \{D_{n \in \omega} \in \tau \mid D_n \text{ is dense}\},$$

and open set O , so $O \cap D_0$ is not empty, then there exists open set O_0 ,

$$\overline{O_0} \subseteq O \cap D_0,$$

by the regularity of locally compact Hausdorff space. Inductively there exists

$$\overline{O_{n+1}} \subseteq \overline{O_n} \cap D_n$$

$\bigcap_{n \in \omega} \overline{O_n}$ has FIP and by the local compactness is not empty.

$\bigcap_{n \in \omega} \overline{O_n} = \bigcap_{n \in \omega} O_n \subseteq \bigcap \mathcal{D} \cap O$, so $\bigcap \mathcal{D}$ is dense. □

4.4 Filters and convergence

Standard limit (convergence) of a sequence $\langle x_{n \in \omega} \mid x_n \in \mathbb{R} \rangle$ is defined:

$$\lim_{n \rightarrow \infty} x_n = a \text{ if } \forall \varepsilon \exists n_0 \forall n > n_0 (|a_n - a| < \varepsilon)$$

The notion of the filter convergence is a generalization of the classical notion of the convergence of a sequence. The use of filter is way how to talk about convergence in arbitrary topological space. Let \mathcal{N}_a be a set of all open neighbourhoods of a . \mathcal{N}_a has following properties:

1. $X \in \mathcal{N}_a$;
2. if $A \in \mathcal{N}_a$ and $B \in \mathcal{N}_a$, then $A \cap B \in \mathcal{N}_a$;
3. if $A, B \subseteq \mathcal{N}_a$, $A \in \mathcal{N}_a$, and $A \subseteq B$, then $B \in \mathcal{N}_a$.
4. $\emptyset \notin \mathcal{N}_a$.

The neighbourhood satisfies the filter properties and is called a *neighbourhood filter*.

Definition 4.34. F-lim $x_n = a$ if $\forall A \in \mathcal{N}_a (\{n \mid x_n \in A\} \in \mathcal{F})$, for $\langle x_n \mid n \in \omega \rangle$.¹⁰

In other words for all neighbourhoods A of the point a almost all sequence members are in \mathcal{N}_a . Standard limit definition is equivalent to F-lim where \mathcal{F} is Fréchet filter.

Observation 4.35. Let S is a sequence $\langle x_{n \in \omega} \mid x_n \in \mathbb{R} \rangle$ and a is a limit point. $a \in \overline{\{x_n \mid n < \omega\} \setminus \{a\}}$ and $A = \{X \subseteq \omega \mid \lim_{n \in X} x_n = a\}$

If A is non-empty, A is closed under union and subsets. It leads to the following chapter.

¹⁰Filter convergence was formulated by Henri Cartan around 1937 and explored by Bourbaki in the 1940s.

Chapter IV

In this chapter there is a basic result of the relation between submeasures and ideals on ω .

5.1 Ideals and filters

Definition 5.1 (Ideal over a set). An *ideal* over a set X is a collection \mathcal{I} of subsets of X such that:

1. $\emptyset \in \mathcal{I}$;
2. if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$;
3. if $A, B \subseteq X$, $A \in \mathcal{I}$, and $A \subseteq B$, then $B \in \mathcal{I}$.

Given an ideal \mathcal{I} , \mathcal{I}^* is the dual filter, consisting of complements of the sets in \mathcal{I} . Similarly, if \mathcal{F} is a filter on X , \mathcal{F}^* denotes the dual ideal.

$$\mathcal{I}^* = \{A \subseteq X \mid X \setminus A \in \mathcal{I}\}$$

Duality between ideals and filters allows to examine only one of this concepts which is in some particular situation better. The sentences could be transformed using De Morgan's laws.

The ideal convergence is dual to the filter convergence. The sequence $\langle x_n \mid n \in \omega \rangle$ is \mathcal{I} -convergent to a if $\forall \varepsilon > 0$ ($\{n \in \omega \mid \varepsilon \leq |x_n - a|\} \in \mathcal{I}$), so $\text{I-lim } x_n = a$. If $\mathcal{I} = \text{Fin}$, then \mathcal{I} -convergence is equivalent to standard convergence.

Definition 5.2 (P-ideal). A ideal \mathcal{I} is *P-ideal* if for every (descending: $A_0 \supseteq A_1 \supseteq A_2 \dots$) countable sequence $\langle A_i \in \mathcal{I} \mid i \in \omega \rangle$ of elements of \mathcal{I} there exists $B \in \mathcal{I}$ such that $B \supseteq^* A_i$ for all $n < \omega$. $A_i \setminus B$ is finite.

5.2 Submeasure

Definition 5.3. A submeasure on ω is a function $\varphi : \mathcal{P}(\omega) \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$ satisfying:

1. $\varphi(\emptyset) = 0$,
2. if $A \subseteq B$ then $\varphi(A) \leq \varphi(B)$,
3. $\varphi(A \cup B) \leq \varphi(A) + \varphi(B)$,

To avoid trivialities, let $\varphi(X) < \infty$ for all finite sets X .

Definition 5.4. If φ submeasure satisfies $\varphi(A) = \lim_{n \rightarrow \infty} \varphi(A \cap \{1, \dots, n\})$, then φ is called a *lower semicontinuous submeasure* (lscsm).

Definition 5.5. $Fin(\varphi) = \{A \subseteq \omega \mid \varphi(A) < \infty\}$, called a *finite ideal* of φ .

Observation 5.6. If φ is lscsm, then $Fin(\varphi)$ is an F_σ ideal.

Proof. $Fin(\varphi) = \bigcup_{m \in \omega} \{A \subseteq \omega \mid \varphi(A) \leq m\}$. For φ lscsm is equal to

$\bigcup_{m \in \omega} \{A \subseteq \omega \mid \lim_{n \rightarrow \infty} \varphi(A \cap \{1, \dots, n\}) \leq m\}$ and
 $\bigcup_{m \in \omega} \bigcap_{n \in \omega} \{A \subseteq \omega \mid \varphi(A \cap \{1, \dots, n\}) \leq m\}$, so $\varphi(A \cap \{1, \dots, n\}) \leq m$ is a condition for closed set A , then $Fin(\varphi)$ is F_σ . \square

Definition 5.7. $Exh(\varphi) = \{A \subseteq \omega \mid \lim_{n \rightarrow \infty} \varphi(A \setminus \{1, \dots, n\}) = 0\}$, called an *exhaustive ideal* of φ .

Observation 5.8. If φ is lscsm, then $Exh(\varphi) \subseteq Fin(\varphi)$.

Observation 5.9. If φ is lscsm, then $Exh(\varphi)$ is an $F_{\sigma\delta}$ P-ideal.

Proof. Let $F_{m,n} = \{A \subseteq \omega \mid \varphi(A \setminus \{1, \dots, m\}) \leq \frac{1}{n}\}$, $F_{m,n}$ is closed set, then $Exh(\varphi) = \bigcap_{n \in \omega} \bigcup_{m \in \omega} F_{m,n}$.

Let $\langle A_i \in \mathcal{I} \mid i \in \omega \rangle$ is in $Exh(\varphi)$, then let have a sequence $\langle n_i \mid \varphi(A_i \setminus \{1, \dots, n_i\}) \leq \frac{1}{2^{n+1}} \rangle$, and $B = \bigcup_{i \in \omega} (A_i \setminus \{1, \dots, n_i\})$, so $A_i \setminus B$ is finite.

For any n there exists k $\varphi(\bigcup_{i \leq n} A_i \setminus \{1, \dots, k\}) \leq \frac{1}{2^{n+1}}$,

so for any n $\varphi(B \setminus \{1, \dots, k\}) \leq \frac{1}{2^n}$, then $B \in Exh(\varphi)$. \square

Definition 5.10. A set $A \subseteq \mathcal{P}(\omega)$ is *hereditary* if it is closed under subsets.

Lemma 5.10.1. For any hereditary F_σ set H there exists a family $\{F_n \mid n \in \omega\}$ of hereditary closed sets such that $H = \bigcup_{n \in \omega} F_n$ and $F_n \subseteq F_{n+1}$ for $n \in \omega$.

Proof. Let $H = \bigcup_{n \in \omega} D_n$ where D_n is closed for $n \in \omega$.

$$F_n = \{A \cap B \mid \exists A \exists B (A \in \bigcup_{k \leq n} D_k \text{ and } B \in \mathcal{P}(\omega))\}$$

If F_n is not closed, then $\bigcup_{k \leq n} D_k$ is not closed by the continuity of \cap function.

So F_n is hereditary closed set. \square

Theorem 5.11 (Mazur). Let \mathcal{I} be an ideal on ω . Then \mathcal{I} is an F_σ if and only if there is a lscsm φ such that $\mathcal{I} = \text{Fin}(\varphi)$. [Mazu91]

The idea of the proof is to define such sets with the indexes satisfying the submeasure conditions.

Proof. For right direction of equivalence let have a F_σ -ideal \mathcal{I} .

$\mathcal{I} = \bigcup_{n \leq \omega} D_n$; where each D_n are closed sets.

$\mathcal{I} = \bigcup_{n \leq \omega} F'_n$; where each F'_n is hereditary closed and $F'_n \subseteq F'_{n+1}$ for each n .

Now let define inductively:

$$F_0 = F'_0$$

$$F_{n+1} = \{x \cup y \mid x, y \in F_n\} \cup F'_{n+1}$$

$\{x \cup y \mid x, y \in F_n\}$ is closed by the continuity of \cup function. For every $x \in \text{Fin}$ there is $\varphi(x) = \min(\{n+1 \mid x \in F_n\})$ which satisfies:

1. $\forall x, y \in \text{Fin} (x \subseteq y \Rightarrow \varphi(x) \leq \varphi(y))$
Let $\varphi(a) > \varphi(b)$, then $\exists n (a \notin F_n \text{ and } b \in F_n)$ where F_n is hereditary, so $a \not\subseteq b$.
2. $\forall x, y \in \text{Fin} (\varphi(x \cup y) \leq \varphi(x) + \varphi(y))$
Let $\varphi(a \cup b) > \varphi(a) + \varphi(b)$, then
 $\exists m \exists n (a \cup b \notin F_{m+n} \text{ and } a \in F_m \text{ and } b \in F_n)$. Then $a \notin F_{m+n}$ and $b \notin F_{m+n}$ is contradictory.
3. φ is total because the function $\min : \mathcal{P}(\omega) \rightarrow \omega$ is total.
(ω is well-ordered)

For the proof of the left direction of equivalence there is a submeasure $\varphi : \text{Fin} \rightarrow \mathbb{R}_0^+ \cup \{\infty\}$, so for every n let

$$F_n = \{x \subseteq \omega \mid \forall k (\varphi(x \cap k) \leq n)\}.$$

For fixed k the set is a finite sum of basic clopen sets, so F_n is closed and $\mathcal{I} = \bigcup_{n \leq \omega} F_n$. \mathcal{I} is hereditary, closed under finite unions and $\omega \notin \mathcal{I}$. \square

In the following examples of some ideals, if the ideal does not consist of subsets of ω but of subsets of some other countable sets, then this countable set is being identical with ω .

Example 5.12. $\mathcal{I}_{\frac{1}{n}} = \{A \subseteq \omega \mid \sum_{n \in A} \frac{1}{n} < \infty\}$ is F_σ P-ideal where submeasure φ is defined: $\varphi(A) = \sum_{n \in A} \frac{1}{n}$

Example 5.13. $\mathcal{I}_{Fin^\omega} = \{A \in 2^{\omega \times \omega} \mid \forall n \in \omega ((\{n\} \times \omega) \cap A \text{ is finite})\}$

Example 5.14. $\mathcal{I}_{nwd} = \{A \subseteq \mathbb{Q} \mid A \text{ is nowhere dense in } \mathbb{R}\}$ is neither a P-ideal nor F_σ .

Example 5.15. $\mathcal{I}_1 = \{A \in 2^{\omega \times \omega} \mid \exists n \in \omega (A \subseteq n \times \omega)\}$ [Sole97]

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