FACULTY OF MATHEMATICS AND PHYSICS
Charles University

MASTER THESIS

Ondřej Skácel

Modern amplitude methods

Institute of Theoretical Physics

Supervisor of the master thesis:	RNDr. Jiří Novotný, CSc.
Study programme:	Physics
Study branch:	Theoretical Physics

Prague 2019

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I would like to thank my advisor for many helpful discussions.

Title: Modern amplitude methods

Author: Ondřej Skácel

Institute: Institute of Theoretical Physics

Supervisor: RNDr. Jiří Novotný, CSc., Institute of Particle and Nuclear Physics

Abstract: The work is centered on the study of dimensionally reduced vector effective field theories from the point of view of soft scalar limits using the spinor-helicity formalism. The Dirac-Born-Infeld model is singled out by its enhanced soft limit at the level of four and six-point amplitudes. In the process, the spinor-helicity formalism is outlined and its use illustrated on explicit examples. The remainder of the work is focused on the corresponding questions in six dimensions. The relevant version of spinor-helicity formalism is presented, followed by a discussion of little group invariants and of the (im)possibility of their use on characterization of theories. Lastly, attempts at formalizing the process of taking the soft limit are made, with inspiration from the four-dimensional case.

Keywords: spinor-helicity formalism, soft limits, DBI

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Introduction

The standard Feynman-diagrammatic approach to Quantum Field Theory, for all its successes, has struggled in certain situations with a combinatorial explosion of diagrams. Furthermore, its highly redundant formalism for describing spin-1 fields makes it unfeasible to go beyond the most elementary processes even in a theory as symmetric as QCD (and the situation is even worse in the effective field theory for gravity). The modern approaches aspire to alleviate these difficulties, expressing the amplitudes in the right variables and looking for deeper relations both within theories and between (sometimes only seemingly) different theories.

The spinor-helicity formalism identifies the helicities of massless spin-1 particles as the right variables to employ and brings to light the usefulness of expressing momenta in terms of spinors. It has immense consequences for Yang-Mills theories, simplifying computations significantly, providing a recursion relation for tree-level amplitudes and uncovering previously hidden relations.

The implications of the formalism extend beyond the original context of Yang-Mills theories. Its advantages have been shown for a wide range of effective field theories (EFTs), from the non-linear σ-model to the theories of a single vector field. In contrast with renormalizable QFTs, the space of possible EFTs is vast, even in the simplest conceivable setting of a single scalar. A natural question arises of whether the theories can be classified based on the properties of their scattering amplitudes. The study of soft limits of the amplitudes has frequently brought an affirmative answer. Pleasingly, the theories with the best-behaved amplitudes tend to have a geometrical interpretation. This work is focused on attempting to characterize dimensionally reduced vector field theories by their soft limits, the Dirac-Born-Infeld (DBI) model, another theory with a geometrical origin, being at the center of our attention.

The first chapter outlines the spinor-helicity formalism and its use in the context of Yang-Mills theories. Its main purpose is to lay the groundwork for its practical use in further chapters. The second chapter discusses the process of dimensional reduction as well as the theories this work is concerned with, notably the DBI model. The third chapter proves the unique characterization of DBI as a dimensionally reduced theory with an enhanced soft scalar limit at the level of four and six-point amplitudes. In the process, the usefulness of the spinor-helicity formalism is highlighted and the procedure of taking the soft limits formalized. The last chapter deals with the spinor-helicity formalism in six dimensions, outlining it, discussing the little group invariants and attempting to construct a prescription for taking the soft limits. The possibilities of a characterization of DBI similar to the four-dimensional case are examined.

1. The Spinor-Helicity formalism

The Spinor-Helicity formalism was born as an attempt to alleviate some of the computational difficulties inherent to the standard Feynman-diagrammatic way of obtaining scattering amplitudes. In short, for Yang-Mills theories, arguably the most symmetrical nonSUSY theories, even tree-level computations quickly become unfeasible due to an overwhelming number of combinations to consider. However, once summed over polarizations and colors, the physical results turn out to be rather simple in form. This is strongly suggestive of a possible hidden simplicity. As it turns out, the right approach lies in expressing the results in terms of helicities instead of polarizations and in using momentum spinors rather than four-momenta. The degree of simplification not only allows the computation of higher-point amplitudes, but goes over to the loop-level. More importantly, the formalism uncovers deep relations between different theories and is generally useful even beyond the original context of Yang-Mills.

This chapter serves as a brief introduction to the formalism, starting with the section 1.1, wherein the original motivation is explained in more detail. The following section 1.2 suggests a possible direction and the section 1.3 outlines the formalism. In the concluding section 1.4, the resulting simplifications and further implications are discussed.

Most of the content of this chapter is standard and an interested reader is referred to either the Part N of Zee [1] for a more detailed outline, the chapter 27 of Schwartz [2] for a textbook approach, a comprehensive review by Dixon [3] or to an almost encyclopedic treatment by Weinzierl [4]. However, beware that there are as many conventions for the spinor-helicity formalism as there are authors (and this work makes no exception).

1.1 The original raison d'être

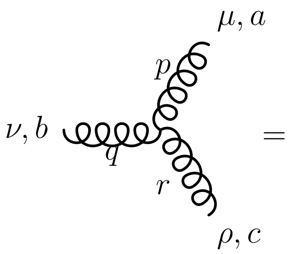
To outline the problems the formalism was designed to solve, we'll consider a specific process in a specific theory, namely the $gg \to gg$ scattering in pure QCD. When written out explicitly, the field strength is given as

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu,$$

while the pure QCD Lagrangian is

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}.$$

Extracting the vertices is then a straightforward task, for example the 3-point vertex takes the form



$$= g f^{abc} \left(\eta^{\mu\nu} (k-p)^{\rho} + \eta^{\nu\rho} (p-q)^{\mu} + \eta^{\rho\mu} (q-k)^{\nu} \right). \tag{1.1}$$

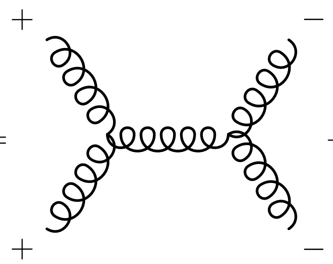
For the $gg \to gg$ scattering at the tree-level, there are the s, t and u-channel diagrams to consider as well as the 4-point vertex on its own. Each 3-point vertex corresponds to three terms, so the s, t and u diagrams each consist of 9 terms. The 4-point vertex contains 6 terms, leading to $3 \times 9 + 6 = 33$ terms in total (although some might cancel). Computing the physically meaningful squared amplitude therefore means dealing with an order of $33^2 \approx 10^3$ terms.

This might be unpleasant, but it's not by any means a reason for concern. There is no physical law prohibiting a computation of a simple process to be elaborate. However, if we sum over colors and polarizations, the following notably uncomplicated result is obtained:

$$\frac{1}{256} \sum_{\epsilon, a} |\mathcal{M}|^2 = \frac{9}{2} g^4 \left(3 - \frac{tu}{s^2} - \frac{us}{t^2} - \frac{st}{u^2} \right).$$

To attain similarly benign expressions even for the intermediary results, it's necessary to consider helicities instead of polarization vectors as the relevant variables. As an example, the following holds (where the ellipsis signifies the various crossings as well as the contribution of the 4-point diagram):

$$\mathcal{M}_{g^+g^+ \to g^-g^-} =$$



$$+ \ldots = 0.$$

The development proceeds first by separating the group factors from the kinematic factors (see [3]) for the tree-level Yang-Mills amplitudes. The resulting formula is as follows:

$$\mathcal{M}_n \left(\{k_i, \lambda_i, a_i\} \right) = \sum_{\sigma \in S_n / Z_n} \mathrm{Tr} \left(T^{a_{\sigma(1)}} \ldots T^{a_{\sigma(n)}} \right) A_n^{tree} \left(\sigma \left(1^{\lambda_1} \right), \ldots, \sigma \left(1^{\lambda_n} \right) \right),$$

where the k, λ and a stand for the momenta, helicities and colors respectively.

The partial amplitudes A_n^{tree} turn out to be rather simple in form. In a notation which is to be explained in this chapter, the partial amplitude for a $--++\to 0$ scattering (where from now on, all the particles are treated as incoming, so there is no contradiction with the vanishing process above) can be expressed as

$$A_n^{tree}\left(1^-,2^-,3^+,4^+\right)=\frac{\langle 1|2\rangle^4}{\langle 1|2\rangle\langle 2|3\rangle\langle 3|4\rangle\langle 4|1\rangle}. \tag{1.2}$$

On the other hand, scattering processes with at most one particle with a different helicity from the rest (e.g. the $-++++\to 0$ scattering) vanish identically. The above contribution is therefore the only nontrivial one up to crossing.

1.2 An inspiration

Thinking in terms of helicities is by itself an improvement, but the real power of the spinor-helicity formalism comes from the "spinor" part. Four-momenta come with an implicit mass-shell condition. In contrast, momentum spinors solve it automatically. To motivate their introduction, we review a somewhat similar example of a reduction of an object to simpler components, which might be more familiar to the reader uninitiated with the spinor-helicity formalism.

The starting point is the Dirac equation

$$\left(i\gamma^\mu\partial_\mu-m\right)\Psi=0.$$

The gamma matrices take the following form in the Weyl representation

$$\gamma^\mu=\begin{pmatrix}0 & \bar\sigma^\mu\\ \sigma^\mu & 0\end{pmatrix}, \tag{1.3}$$

where $\sigma^\mu=(1_{2\times2},\sigma^i)$, $\bar\sigma^\mu=(1_{2\times2},-\sigma^i)$.

In the standard treatment, Ψ is a four component object. For massless particles, it's however useful to treat it instead as two two-dimensional spinors. Introducting a bit of notation in the process, we have

$$\Psi=\begin{pmatrix}\chi_A\\ \theta^{\dot A}\end{pmatrix}.$$

Having presented the dotted and undotted indices (and the so-called left and right-handed spinors respectively), it's straightforward to infer their placement for the σ-matrices. Rewriting the original Dirac equation in terms of the two-spinors, two coupled equations are obtained:

$$i\bar\sigma^\mu_{B\dot A}\partial_\mu\theta^{\dot A}=m\chi_B$$
$$i\sigma^{\mu\dot BA}\partial_\mu\chi_A=m\theta^{\dot B}.$$

Taking $m = 0$, the dynamics of the left and right-handed spinors decouples. The analysis can be taken further in order to establish the transformation properties of the spinors with respect to the Lorentz group. It turns out that they correspond to the $(1/2, 0)$ and $(0, 1/2)$ representations of the underlying $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ respectively. This means that the transformations are

$$\begin{aligned} \theta'^{\dot{A}} &= U^{\dot{A}}{}_{\dot{B}} \theta^{\dot{B}} \\ \chi'^{A} &= V^{A}{}_{B} \chi^{B}, \end{aligned} \tag{1.4}$$

where U and V are two independent $SL(2, \mathbb{C})$ matrices and the raising and lowering of indices is facilitated by the totally antisymmetric tensor ε, which will be discussed in more detail in the next section. Note that in the usual treatment of QFT, the development proceeds the other way around - starting from the representations of the Lorentz group, increasingly complex objects are built up and their dynamics studied.

In this section, a four-spinor Ψ was successfully decomposed into simpler objects. The spinor-helicity formalism goes further and decomposes the four-momenta into momentum two-spinors. The idea is similar - suitable objects are identified and their transformation properties established.

1.3 The formalism

As suggested in the preceding section, the right approach is to decompose the four-momenta into objects transforming as spinors. The only objects with spinorial indices available are the σ-matrices. Contracting with the four-momentum, it's possible to define

$$p^{\dot{A}A} \equiv \sigma^{\mu \dot{A}A} p_{\mu}. \tag{1.5}$$

By itself, this brings no improvement, however, a key observation is the following:

$$\det\left(p^{\dot{A}A}\right) = \det\begin{pmatrix} p_0 + p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 - p_3 \end{pmatrix} = p^2 = m^2.$$

For massless particles, the $p^{\dot{A}A}$ matrix is rank-1 and therefore can be decomposed into an outer product of two vectors (meant in the algebraic sense as "columns of numbers". In terms of their transformation properties, they will be shown to be spinors). In the massive case, $p^{\dot{A}A}$ can be written as a sum of two rank-1 matrices, an approach which is developed in [5], but lies beyond the scope of this text. For our purposes,

$$p^{\dot{A}A} = \tilde{\lambda}^{\dot{A}} \lambda^{A} \equiv |\lambda]\langle\lambda|. \tag{1.6}$$

The relation between $\tilde{\lambda}^{\dot{A}}$ and λ^A warrants an explanation. Further development of the spinor-helicity formalism brought to light the surprising usefulness of complex momenta, which will be briefly discussed later in this chapter. In that case, $\tilde{\lambda}$ and λ are unrelated. For real momenta however, the hermiticity of the σ-matrices leads to the requirement $\tilde{\lambda}^{\dot{A}} = \left(\lambda^A\right)^\star$.

The assignment of $\tilde{\lambda}^{\dot{A}}$ and λ^A is not unique - it's possible to scale one (conventionally λ) by z while scaling the other by $1/z$ without violating (1.6). For complex momenta, z can be arbitrary, while the $\lambda = \tilde{\lambda}^*$ requirement for real momenta restricts z to be a phase. These are the little group transformations associated with the momentum p.

The spinorial indices on $\tilde{\lambda}$ and λ suggest their transformation properties, but they're not by any means automatic. It's possible to establish them by explicitly calculating the λs in terms of the original four-momenta, but that's a rather laborious way. Instead, we'll use the characterization of spinors as objects which satisfy the Weyl equation. First, an inverse relation to (1.6) is derived:

$$\frac{1}{2}p^{\dot{A}A}\bar{\sigma}^{\mu}_{A\dot{A}} = \frac{1}{2}p_\nu \bar{\sigma}^{\mu}_{A\dot{A}}\sigma^{\nu\dot{A}A} = \frac{1}{2}p_\nu \left(2\eta^{\mu\nu}\right) = p^\mu.$$

Using the above and a Fiertz identity $\bar{\sigma}_{\mu B\dot{B}}\bar{\sigma}^{\mu}_{A\dot{A}} = 2\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}$, where ϵ is the totally antisymmetric 2×2 tensor (with a convention $\epsilon_{12} = 1$), we show that the Weyl equation (in the momentum space) is indeed satisfied:

$$i\bar{\sigma}^{\mu}_{A\dot{A}}\partial_\mu \tilde{\lambda}^{\dot{A}} \propto \bar{\sigma}^{\mu}_{A\dot{A}} p_\mu \tilde{\lambda}^{\dot{A}} = \frac{1}{2}\tilde{\lambda}^{\dot{B}}\lambda^B \bar{\sigma}_{\mu B\dot{B}}\bar{\sigma}^{\mu}_{A\dot{A}}\tilde{\lambda}^{\dot{A}} = \epsilon_{AB}\epsilon_{\dot{A}\dot{B}}\tilde{\lambda}^{\dot{A}}\tilde{\lambda}^{\dot{B}}\lambda^B = 0.$$

For λ_A (note that the index has been lowered, which is the next topic of discussion) a similar result can be derived.

Constructing Lorentz invariants requires raising and lowering indices. The only suitable object around is the antisymmetric tensor ϵ. It can be checked that the following definitions induce the correct transformation properties.

$$\lambda_A \equiv \epsilon_{BA}\lambda^B$$
$$\tilde{\lambda}_{\dot{A}} \equiv \epsilon_{\dot{B}\dot{A}}\tilde{\lambda}^{\dot{B}}$$

This in turn allows the introduction of scalar products of spinors (dropping the tildas from now on):

$$\langle\lambda|\chi\rangle \equiv \lambda^A \chi_A = \epsilon_{BA}\lambda^A\chi^B$$
$$[\lambda|\chi] \equiv \lambda_{\dot{A}}\chi^{\dot{A}} = \epsilon^{\dot{A}\dot{B}}\lambda_{\dot{A}}\chi_{\dot{B}}.$$

The invariance can be easily checked (although it follows directly from the previous claims):

$$[\lambda'|\chi'] = \lambda'_{\dot{1}}\chi'_{\dot{2}} - \lambda'_{\dot{2}}\chi'_{\dot{1}} = \det\begin{pmatrix}\lambda'_{\dot{1}} & \chi'_{\dot{1}}\\ \lambda'_{\dot{2}} & \chi'_{\dot{2}}\end{pmatrix} = \det\begin{pmatrix}\lambda_{\dot{1}} & \chi_{\dot{1}}\\ \lambda_{\dot{2}} & \chi_{\dot{2}}\end{pmatrix} = [\lambda|\chi]\,,$$

where the equality of determinants is a consequence of the left-handed spinors transforming via a multiplication by a $SL(2,\mathbb{C})$ matrix, as in (1.4). On the other hand, the scalar product of four-momenta can be expressed as

$$p_\mu q^\mu = \frac{1}{4}p^{\dot{A}A}q^{\dot{B}B}\bar{\sigma}_{\mu A\dot{A}}\bar{\sigma}^{\mu}_{B\dot{B}} = \frac{1}{2}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}p^{\dot{A}A}q^{\dot{B}B} = \frac{1}{2}\langle p|q\rangle\,[q|p]\,, \tag{1.7}$$

which shows that it's possible to regard the spinor products as "square roots" of the vectorial scalar product.

As a last step towards practical application, we'll discuss polarization vectors. For massless vectors in four dimensions, there are two polarization vectors, which satisfy $\epsilon_1 \cdot (\epsilon_2)^\star = 0$, both required to be orthogonal to the momentum of the particle and a fixed lightlike direction (the so-called reference vector). The correct expressions are as follows:

$$\begin{aligned}\epsilon_+^{\dot{B}B}(p|q) &\equiv \sqrt{2}\frac{p^B q^{\dot{B}}}{[p|q]}\\ \epsilon_{-B\dot{B}}(p|q) &\equiv -\sqrt{2}\frac{p_{\dot{B}}q_B}{\langle p|q\rangle},\end{aligned} \tag{1.8}$$

where q^μ is the reference vector. As a check of the validity of ϵ_+ and ϵ_- as polarization vectors (or bispinors, when written as above), note that their orthogonality to both p and q comes trivially from the way the scalar product is formed (see (1.7)). Furthermore, their scaling $\epsilon_\pm \to z^{\pm 2}\epsilon_\pm$ under a little group transformation of p implies that they correspond to helicity eigenstates (as the choice of indices suggests).

The choice of the reference vector is arbitrary (aside from the condition $p \cdot q \neq 0$, which makes the above expressions well-defined) and corresponds to a selection of gauge. It can be verified that upon changing q, the polarization vectors are modified by an amount proportional to p, as expected. For example,

$$\begin{aligned}\sqrt{2}\frac{p^B q^{\dot{B}}}{[p|q]} - \sqrt{2}\frac{p^B r^{\dot{B}}}{[p|r]} &= \sqrt{2}p^B\frac{[p|r]\,q^{\dot{B}} - [p|q]\,r^{\dot{B}}}{[p|q]\,[p|r]}\\ &= \sqrt{2}p^B p^{\dot{B}}\frac{[q|r]}{[p|q]\,[p|r]}.\end{aligned}$$

On the second line, the Shouten identity

$$|1] = \frac{[3|1]}{[3|2]}|2] + \frac{[2|1]}{[2|3]}|3] \tag{1.9}$$

was used. It is easily verified by noting the two-dimensionality of spinors and contracting by $[1|$, $[2|$ or $[3|$.

The advantage of using spinor-helicity formalism is apparent. The dependence on the reference vector is explicit, which can be used to great effect. As an example, note that

$$\begin{aligned} \epsilon_+ (p|q) \cdot \epsilon_+ (r|q) &= 0 \\ \epsilon_- (q|r) \cdot \epsilon_+ (p|q) &= 0. \end{aligned} \tag{1.10}$$

Thus a clever choice of reference vectors can often lead to significant simplifications. In the case of tree-level Yang-Mills scattering, a simple observation shows that any diagram has at least one contraction of polarizations (trying to construct a counterexample using (1.1) is probably the easiest way to see this). Suppose now that we're concerned with an all-but-one plus-helicity scattering. Choosing the momentum of the negative-helicity gluon as the reference vector for all the positive-helicity gluons and using the equations (1.10), we see that all the contractions vanish and so does the scattering amplitude.

1.4 The advantages and further directions

The spinor-helicity formalism makes much more structure apparent in the scattering amplitudes. It was shown that the tree-level Yang-Mills amplitudes vanish in certain cases, but even non-trivial processes lead to expressions which are relatively simple in form. For example, in the first nontrivial case - the processes with all but two gluons of the same helicity (the so-called Parke-Taylor amplitudes [6]), the following generalization of (1.2) holds:

$$A_n^{tree}\left(1^-, 2^-, 3^+, 4^+, \ldots, n^+\right) = \frac{\langle 1|2\rangle^4}{\langle 1|2\rangle\langle 2|3\rangle \ldots \langle n|1\rangle}.$$

To emphasize: a n-gluon amplitude, an object beyond any possibility of calculation using the standard methods for even a moderately large n, as a closed expression which fits on one line. However, the simplifications reach even further.

Previously, we mentioned the complexification of momenta. This approach has proved extremely useful for a large number of theories, allowing to derive recursion relations for higher-point amplitudes in terms of simpler ones. The usual way can be summarized as follows.

Two of the momenta are complexified ($p' = p + zq$ for z complex), while respecting the conservation laws and the masslessness of particles. When a diagram can be separated in two by cutting a single propagator in such a way as to have each of the particles in a different part, the propagator has a simple pole. The amplitude as a function of z has poles only in such cases. If, in addition, it vanishes as $|z| \to \infty$, it can be expressed as a sum of the form

$$A(z) = \sum_{poles} \frac{Res_{z_j} A}{z - z_j}.$$

It can be shown that each residue is a product of the on-shell scattering amplitudes of both parts of the original diagram (which was separated by cutting a propagator). Thus, for every theory (or a situation within a theory), for which we can ensure that $A(z) \to 0$, there is a recursion relation for the tree-level scattering amplitudes. For a detailed (but very readable) treatment, the reader is referred to [7].

In general, the whole machinery of modern amplitude methods is constantly growing more powerful. Twistors have been identified as the right variables for gauge theories [8], deep relations between gravity and Yang-Mills uncovered [9] and amplitudes in $\mathcal{N} = 4$ supersymmetric Yang-Mills represented as geometrical objects in the projective space [10].

This work is mostly concerned with effective field theories and their classification using soft limits. Impressive progress has been made in this direction, especially in the case of theories of a single scalar field [11; 12]. The subject of EFT's will be discussed in more detail in further chapters.

2. The theories of interest

The main purpose of this work is the study and characterization of dimensionally reduced vector field theories from the point of view of soft limits using the spinor-helicity formalism. This chapter deals with the process of dimensional reduction, introduces the notation and the parametrization used and discusses the Dirac-Born-Infeld model (DBI), which is at the center of our interest. The chapter is, in the context of the whole work, an interlude and is meant to serve mostly as a reference.

2.1 Dimensional reduction

The main idea can be summarized as follows. Suppose some of the dimensions of the spacetime manifold our theory lives on were compact. From dimensional analysis, loosely speaking, $\partial_D \sim \frac{1}{R} \sim p_D$, where R is a characteristic length of the compact dimension D. For R small the dynamics in the compact dimensions decouples, however, the components of the tensorial fields on the original manifold in the "small directions" do not. This was first illustrated by Kaluza and Klein who dimensionally reduced gravity from 5 to 4 dimensions, obtaining both the 4-dimensional gravity and electrodynamics, the vector potential corresponding to the fifth component of the original metric tensor ($A_\mu \equiv h_{\mu 5} = g_{\mu 5} - \eta_{\mu 5}$).

Interestingly, reducing gravity by more than one dimension leads to the emergence of Yang-Mills fields in the same way. However, as Witten showed in [13], it's unlikely that the standard model is simply a reduction of a single supergravity theory.

Generically, compactifying a dimension leads to an emergence of a tower of massive particles [14]. In the specific, tractable, case of $M \times S^1$, the particles correspond to the Fourier modes in the small direction, the masses being proportional to $\frac{1}{R}$. The dimensional reduction is the process of neglecting the massive fields, i.e. leaving only the zero mode in. As long as R is small, this is evidently reasonable.

In this work, the theories that result as a dimensional reduction of 5- and 7-dimensional vectorial effective field theories are studied (in chapters 3 and 4 respectively). Before compactification, the general theory is given by $\mathcal{L} = \mathcal{L}\left(F_{\mu\nu}\right)$, which is an infinite sum of products of traces of powers of $F^{\mu}{}_{\nu}$. A slight simplification results from the Cayley-Hamilton theorem, which allows to equate the d-th power of a $d \times d$ matrix to a weighted sum of its first $d-1$ powers. Additionally, the antisymmetry of $F_{\mu\nu}$ makes the traces of odd powers vanish, leading to $\mathcal{L}_5\left(F_{\mu\nu}\right) = \mathcal{L}_5\left(\langle F^2\rangle, \langle F^4\rangle\right)$ and $\mathcal{L}_7\left(F_{\mu\nu}\right) = \mathcal{L}_7\left(\langle F^2\rangle, \langle F^4\rangle, \langle\langle F^6\rangle\right)$ respectively, where the angle brackets denote the trace.

In our case, the dimensional reduction (from 5D) proceeds as follows. First, note that

$$F_{5A5} = \partial_A A_5 - \partial_5 A_A.$$

The $\partial_5 A_A$ part corresponds to the neglected higher Fourier modes. Renaming A_5 to φ, the dimensional reduction ansatz is therefore

$$(F_5)_{AB} = \begin{pmatrix} F_{4\mu\nu} & \partial_\mu\varphi \\ -\partial_\nu\varphi & 0 \end{pmatrix}.$$

The traces of the powers of $F^\mu{}_\nu$ can be expressed in terms of the dimensionally reduced fields:

$$\begin{aligned} \langle {F_5}^2 \rangle &= F_{4\mu\nu} F_4^{\nu\mu} + 2\partial_\mu\varphi\partial^\mu\varphi \\ \langle {F_5}^4 \rangle &= \langle {F_4}^4 \rangle + 4\partial\varphi \cdot F_4 \cdot F_4 \cdot \partial\varphi + 2\left(\partial\varphi \cdot \partial\varphi\right)^2. \end{aligned} \tag{2.1}$$

A general parametrization of a dimensionally reduced 5-dimensional vectorial theory, which is to be used in the rest of this work, is then

$$\mathcal{L} = A\langle {F_5}^2 \rangle + B\langle {F_5}^4 \rangle + C\langle {F_5}^2 \rangle^2 + D\langle {F_5}^2 \rangle^3 + E\langle {F_5}^2 \rangle\langle {F_5}^4 \rangle + \dots \; . \tag{2.2}$$

Note that $A = \frac{1}{4}$ gives the correct kinetic terms. In the $(7 \to 6)$-dimensional case, the same parametrization (with $F_5 \to F_7$) is used for the four and six particle vertices, but there is an additional term $F\langle {F_7}^6 \rangle$.

2.2 The Dirac-Born-Infeld model

The Dirac-Born-Infeld model is a dimensional reduction of the Born-Infeld electrodynamics

$$\mathcal{L}^{BI} = -\sqrt{(-1)^{d-1} \det\left(\eta_{AB} + F_{AB}\right)}. \tag{2.3}$$

The original motivation was to ensure a finite self-energy of a point-charge [15]. This was achieved by postulating an upper limit for the strength of the electric field, a key property of the above Lagrangian. In modern physics, the model arises naturally in the context of string theory as an action for the gauge fields on D-branes. In the context of this thesis, the DBI will be singled out by the soft limits of amplitudes with 2 scalars and an arbitrary number of photons (at least in 4D). As in the previous section, a compactification of one of the dimensions will give rise to the scalar fields.

The dimensional reduction of BI can be expressed using

$$\begin{aligned} \det\begin{pmatrix} \eta_{\mu\nu} + F_{\mu\nu} & \partial_\mu\varphi \\ -\partial_\nu\varphi & -1 \end{pmatrix} &= \det\begin{pmatrix} \eta_{\mu\nu} + F_{\mu\nu} - \partial_\mu\varphi\partial_\nu\varphi & 0 \\ -\partial_\nu\varphi & -1 \end{pmatrix} \\ &= -\det\left(\eta_{\mu\nu} + F_{\mu\nu} - \partial_\mu\varphi\partial_\nu\varphi\right). \end{aligned} \tag{2.4}$$

In what follows, the notation $g_{\mu\nu} \equiv \eta_{\mu\nu} - \partial_\mu\varphi\partial_\nu\varphi$ will be used. Note that $g_{\mu\nu}$ can be identified as the induced metric of a $(d-1)$-brane in a d-dimensional flat spacetime with $\varphi(x)$ its d-th coordinate. The inverse metric takes the form

$$g^{\mu\nu} = \eta^{\mu\nu} + \frac{\partial^\mu\varphi\partial^\nu\varphi}{1-\partial\varphi\cdot\partial\varphi}.$$

The expression (2.4) can be simplified as

$$-\det\left(\eta_{\mu\nu} + F_{\mu\nu} - \partial_\mu\varphi\partial_\nu\varphi\right) = -\det\left(g_{\alpha\mu}\right)\det\left(\delta^\mu_\nu + g^{\mu\beta}F_{\beta\nu}\right),$$

with

$$\begin{aligned}\det\left(\delta^\mu_\nu + g^{\mu\beta}F_{\beta\nu}\right) &= \exp\left(\mathrm{Tr}\left(\log\left(\delta^\mu_\nu + g^{\mu\beta}F_{\beta\nu}\right)\right)\right)\\ &= \exp\left(\sum_1^\infty (-1)^{n-1}\frac{\langle\left(g^{\nu\beta}F_{\beta\alpha}\right)^n\rangle}{n}\right).\end{aligned}$$

Similarly, the other term can be manipulated as follows:

$$\begin{aligned}\det\left(g_{\mu\nu}\right) &= \det\left(\eta_{\alpha\mu}\right)\det\left(\delta^\alpha_\nu - \partial^\alpha\varphi\partial_\nu\varphi\right)\\ &= (-1)^{D-1}\exp\left(\mathrm{Tr}\left(\log\left(\delta^\alpha_\nu - \partial^\alpha\varphi\partial_\nu\varphi\right)\right)\right)\\ &= (-1)^{D-1}\exp\left(-\mathrm{Tr}\left(\sum_1^\infty\frac{\left(\partial^\alpha\varphi\partial_\nu\varphi\right)^n}{n}\right)\right)\\ &= (-1)^{D-1}\exp\left(-\sum_1^\infty\frac{\left(\partial\varphi\cdot\partial\varphi\right)^n}{n}\right)\\ &= (-1)^{D-1}\left(1-\partial\varphi\cdot\partial\varphi\right).\end{aligned}$$

Altogether, the DBI Lagrangian takes the form

$$\mathcal{L}^{DBI} = -\sqrt{1-\partial\varphi\cdot\partial\varphi}\;\exp\left(\frac{1}{2}\sum_1^\infty(-1)^{n-1}\frac{\langle\left(g^{\nu\beta}F_{\beta\alpha}\right)^n\rangle}{n}\right).$$

The coefficients of DBI in the canonical parametrization (2.2) are most easily obtained by expressing the Born-Infeld Lagrangian (2.3) in a form similar to the above for DBI and expanding the exponential.

$$\mathcal{L}^{BI} = -\exp\left(-\frac{1}{2}\sum_1^\infty\frac{\langle F^n\rangle}{n}\right) \tag{2.5}$$

In the 4-dimensional case, the Cayley-Hamilton theorem is used to decompose $\langle F^6\rangle$. For 5×5 antisymmetric matrices the theorem results in

$$F^5 = \frac{1}{2}\langle F^2\rangle F^3 - \frac{1}{8}\left(\langle F^2\rangle^2 - 2\langle F^4\rangle\right)F.$$

The algebraic manipulations described above lead to the following coefficients in the 5-dimensional case.

$$\begin{array}{llllll} A_5 & = \frac{1}{4} & B_5 & = \frac{1}{8} & C_5 & = -\frac{1}{32} \\ D_5 & = -\frac{1}{128} & E_5 & = \frac{1}{32} & & \end{array} \tag{2.6}$$

In the 7-dimensional case, the situation is even easier as F^6 cannot be decomposed.

$$\begin{array}{llllll} A_7 & = \frac{1}{4} & B_7 & = \frac{1}{8} & C_7 & = -\frac{1}{32} \\ D_7 & = \frac{1}{384} & E_7 & = -\frac{1}{32} & F_7 & = \frac{1}{12} \end{array} \tag{2.7}$$

3. The four dimensional case

In the previous chapter, we discussed the dimensional reduction of vector effective field theories. The class of such theories is vast and a question arises of whether some specific theories can be singled out on account of their special properties. The answer is affirmative, the special properties are the soft limits of two-scalar amplitudes and the theory singled out is the DBI model, which was described in section 2.2.

A direct proof of the $O(t^2)$ soft limit for scalar particles in DBI is possible [16], as it is invariant with respect to the Lorentz transformation in the original five dimensions, which are realized non-linearly in the dimensionally reduced situation:

$$\begin{aligned}\delta\varphi &= \omega^\mu\left(-x_\mu + \varphi\partial_\mu\varphi\right)\\ \delta F_{\mu\nu} &= -\omega^\alpha\left(\varphi\partial_\alpha F_{\mu\nu} + \left(\partial_{[\mu}\varphi\right)F_{\alpha\nu]}\right).\end{aligned}$$

Invariance under these transformations is sufficient for the $O(t^2)$ soft limit, but not all theories which possess it come from a dimensional reduction. In fact, it turns out that DBI is the only such theory.

In this chapter, the unique characterization of DBI among the dimensionally reduced theories for 4- and 6- point amplitudes will be proved. This is not interesting for its own sake, because, as outlined above, a complete proof is possible by other means. However, the computations involved will show the extreme efficiency of the spinor-helicity formalism. Furthermore, for colorless vector fields, there are additional simplifications compared to the situation described in section 1.3. These will be discussed in the first section 3.1. The computation of four-point amplitudes, as well as an examination of the process of soft limits, will follow in 3.2 and the chapter will conclude with the computation of six-point amplitudes in 3.3.

3.1 Decomposing the field-strength tensor

It will prove advantageous to decompose $F_{\mu\nu}$ into positive and negative-helicity components. We will outline the procedure, neglecting the various normalizations in the process. They will be present only in the final results and a reader is invited to verify them.

The obvious first step is contracting the field-strength tensor (in the momentum representation) with the σ-matrices, as these are the only objects available with the right indices.

$$F_{\mu\nu}\bar{\sigma}^{\mu}_{A\dot{A}}\sigma^{\nu\dot{B}B} \sim \lambda_A\bar{\lambda}_{\dot{A}}\epsilon^{\dot{B}B} - \epsilon_{A\dot{A}}\bar{\lambda}^{\dot{B}}\lambda^B$$

Recalling that $\epsilon_+^{\dot{B}B} \sim \bar{\chi}^{\dot{B}}\lambda^B$ and $\epsilon_{-A\dot{A}} \sim \bar{\lambda}_{\dot{B}}\chi_B$, where λ corresponds to the momentum of the particle and χ to its reference vector, it's clear that by contracting over the dotted indices in the above expression the negative-helicity case vanishes. Vice-versa, contracting over the undotted indices makes the expression vanish for positive helicity. This leads us to define (now, in general, even off-shell)

$$\begin{aligned} \phi_{AB} &\equiv \frac{1}{2}\epsilon^{\dot{A}\dot{B}}F_{A\dot{A}B\dot{B}} = \frac{1}{2}\epsilon^{\dot{A}\dot{B}}F_{\mu\nu}\bar{\sigma}^{\mu}_{A\dot{A}}\bar{\sigma}^{\nu}_{B\dot{B}} \\ \bar{\phi}_{\dot{A}\dot{B}} &\equiv \frac{1}{2}\epsilon^{AB}F_{A\dot{A}B\dot{B}}. \end{aligned} \tag{3.1}$$

What do the ϕ and $\bar{\phi}$ fields correspond to? Noting that $\epsilon_-(p|q)^\star = \epsilon_+(p|q)$ for real momenta, the above reasoning shows that ϕ annihilates positive-helicity photons and creates negative-helicity ones (the converse is true for $\bar{\phi}$). In order to use the fields in practice, it's necessary to establish their translation into Feynman rules. Their index structure suggests $\phi_{AB} \leftrightarrow c\lambda_A\lambda_B$ and a detailed computation using their definition (3.1) and the expressions for polarization vectors (1.8) confirms it. The results are

$$\begin{aligned} \phi_{AB} &\leftrightarrow \pm i\sqrt{2}\lambda_A\lambda_B \\ \bar{\phi}_{\dot{A}\dot{B}} &\leftrightarrow \pm i\sqrt{2}\bar{\lambda}_{\dot{A}}\bar{\lambda}_{\dot{B}}, \end{aligned} \tag{3.2}$$

where for outgoing particles, the plus sign is taken and vice-versa (e.g., an incoming negative-helicity photon gets $-i\sqrt{2}\bar{\lambda}_{\dot{A}}\bar{\lambda}_{\dot{B}}$).

As defined, both ϕ and $\bar{\phi}$ are symmetric. Making use of the identity $\epsilon_{AB}\chi_C^C = \chi_{AB} - \chi_{BA}$, which can be easily verified component-wise, the following is obtained:

$$\begin{aligned} \phi_{AB}\epsilon_{\dot{A}\dot{B}} &= \frac{1}{2}\left(F_{A\dot{A}B\dot{B}} - F_{A\dot{B}B\dot{A}}\right) \\ \bar{\phi}_{\dot{A}\dot{B}}\epsilon_{AB} &= \frac{1}{2}\left(F_{A\dot{A}B\dot{B}} - F_{B\dot{A}A\dot{B}}\right). \end{aligned}$$

This, in combination with the antisymmetry of $F_{\mu\nu}$ leads to the final decomposition of the field strength tensor

$$F_{A\dot{A}B\dot{B}} = \phi_{AB}\epsilon_{\dot{A}\dot{B}} + \bar{\phi}_{\dot{A}\dot{B}}\epsilon_{AB}.$$

If F is on-shell, the above is its decomposition into positive and negative helicity parts (recall that all particles are treated as incoming in this work). Analogical computation shows that the dual tensor $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ is expressed in terms of ϕ and $\bar{\phi}$ as

$$\tilde{F}_{A\dot{A}B\dot{B}} = i\phi_{AB}\epsilon_{\dot{A}\dot{B}} - i\bar{\phi}_{\dot{A}\dot{B}}\epsilon_{AB}.$$

The next order of business is translating the building blocks of our theories - the traces of $(F_5)^n$ (see (2.1)) - into the new variables. For the second power, it's straightforward to establish

$$F_{\mu\nu}F^{\nu\mu} = -\frac{1}{4}F_{A\dot{A}B\dot{B}}F^{A\dot{A}B\dot{B}} = \frac{1}{2}\left(\phi^A{}_B\phi^B{}_A + \bar{\phi}^{\dot{A}}_{\dot{B}}\bar{\phi}^{\dot{B}}_{\dot{A}}\right) \equiv \frac{1}{2}\left(\langle\phi^2\rangle + \langle\bar{\phi}^2\rangle\right). \tag{3.3}$$

The evaluation of $\langle F^4 \rangle$ is rather involved (including a use of the Cayley-Hamilton theorem), so we'll only state the final result:

$$\langle F^4 \rangle = \frac{1}{16}\left(\langle \phi^2 \rangle^2 + \langle \bar{\phi}^2 \rangle^2\right) + \frac{3}{8}\langle \phi^2 \rangle \langle \bar{\phi}^2 \rangle. \tag{3.4}$$

For the 2-photon, 2-scalar term $\partial\varphi \cdot F \cdot F \cdot \partial\varphi$, the starting point is

$$F_{\mu\nu} = \frac{1}{4}\sigma_\mu^{\dot{A}B}\sigma_\nu^{\dot{C}D}\left(\phi_{BD}\epsilon_{\dot{A}\dot{C}} + \bar{\phi}_{\dot{A}\dot{C}}\epsilon_{BD}\right).$$

Contracting two field strengths gives

$$\begin{aligned} F_{\nu\alpha}F^{\alpha\beta} &= \frac{1}{16}\sigma_\nu^{\dot{A}B}\left(2\epsilon^{\dot{C}\dot{K}}\epsilon^{DL}\right)\sigma^{\mu\dot{M}N}\left(\phi_{BD}\epsilon_{\dot{A}\dot{C}} + \bar{\phi}_{\dot{A}\dot{C}}\epsilon_{BD}\right)\left(\phi_{LN}\epsilon_{\dot{K}\dot{M}} + \bar{\phi}_{\dot{K}\dot{M}}\epsilon_{LN}\right) \\ &= -\frac{1}{8}\sigma_\nu^{\dot{A}B}\sigma^{\mu\dot{M}N}\left(\epsilon_{\dot{A}\dot{M}}\phi_{BD}\phi^D{}_N - 2\bar{\phi}_{\dot{A}\dot{M}}\phi_{BN} + \epsilon_{BN}\bar{\phi}_{\dot{A}\dot{C}}\bar{\phi}^{\dot{C}}_{\dot{M}}\right). \end{aligned}$$

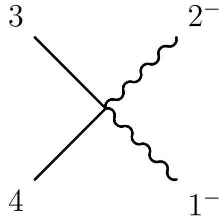
This results in the following vertex:

$$\partial\varphi_1 \cdot F \cdot F \cdot \partial\varphi_2 \leftrightarrow \frac{i}{8}p_1{}^{\dot{A}B}p_2^{\dot{M}N}\left(\epsilon_{\dot{A}\dot{M}}\phi_{BD}\phi^D{}_N - 2\phi_{BN}\bar{\phi}_{\dot{A}\dot{M}} + \epsilon_{BN}\bar{\phi}_{\dot{A}\dot{C}}\bar{\phi}^{\dot{C}}_{\dot{M}}\right). \tag{3.5}$$

In order to check that the decomposition is derived correctly and to explain its use, a simple process will be computed in both the standard way (i.e. as in section 1.3) and using the machinery described above. The process is the two-scalar, two-photon scattering (with a helicity configuration 1^-2^-34) using the interaction Lagrangian

$$\mathcal{L}_{INT} = \langle F^2 \rangle \partial\varphi \cdot \partial\varphi.$$

In the classical approach, we'll first set the reference vector of gluon 1 to p_2 and vice-versa. This leads to $\langle F^2 \rangle \to -2i^2 p_{1\mu}\epsilon_{1\nu}p^{2\mu}\epsilon_2^\nu$ (all particles are considered to be incoming). Taking all possible field-particle pairings into account, the following result is obtained:



$$\begin{aligned} i\mathcal{M} &= -8i\ p_1 \cdot p_2\ p_3 \cdot p_4\ \epsilon_1 \cdot \epsilon_2 \\ &= 8i\,(p_1 \cdot p_2)^2\,\frac{[1|2]}{\langle 1|2 \rangle} = -4i\ p_1 \cdot p_2\ [1|2]^2. \end{aligned}$$

On the other hand, using the decomposition of F, the contribution is

$$i\mathcal{M} = i\frac{1}{2}\,(-1)\,2\left(-\sqrt{2}i\right)^2[1|2]^2 2\,(-i)^2\,\frac{1}{2}\langle 3|4\rangle\,[4|3] = -4ip_1\cdot p_2[1|2]^2,$$

where, in order from left to right, the i is for the first-order Dyson expansion, the $\frac{1}{2}$ for (3.3), the (-1) from $\bar{\phi}_{\dot{A}\dot{B}}\bar{\phi}^{\dot{A}\dot{B}} = -\bar{\phi}^{\dot{A}}_{\dot{B}}\bar{\phi}^{\dot{B}}_{\dot{A}}$, the 2 for the two possible ways to pair the particles with the $\bar{\phi}$ fields, the $\left(-\sqrt{2}i\right)^2$ from (3.2) and the rest is standard.

The last remaining practical concerns are the propagators of the ϕ and $\bar{\phi}$ fields. The computation is long but straightforward, starting at (3.1) and using the propagator of photons in the process. The results are (for a detailed treatment, see [17])

$$\begin{aligned}\langle\phi_{AB}\bar{\phi}_{\dot{C}\dot{D}}\rangle &= \frac{i}{p^2+i\epsilon}\left(p_{A\dot{C}}p_{B\dot{D}}+p_{A\dot{D}}p_{B\dot{C}}\right)\\ \langle\phi_{AB}\phi_{CD}\rangle &= -i\left(\epsilon_{AC}\epsilon_{BD}+\epsilon_{AD}\epsilon_{BC}\right)\\ \langle\bar{\phi}_{\dot{A}\dot{B}}\bar{\phi}_{\dot{C}\dot{D}}\rangle &= -i\left(\epsilon_{\dot{A}\dot{C}}\epsilon_{\dot{B}\dot{D}}+\epsilon_{\dot{A}\dot{D}}\epsilon_{\dot{B}\dot{C}}\right).\end{aligned} \tag{3.6}$$

One unexpected (and uncommon) feature are the contact terms. They'll prove to be an inconvenience when we get to 6-point amplitudes.

3.2 4-point amplitudes

This section is concerned with the study of 4-point amplitudes in theories of the form (2.2) from the point of view of soft limits. A proper explanation of the term "soft limit" is however left until several amplitudes have been computed, as examples are helpful in explaining the concept. For now, the reader can regard the soft limit vaguely as a process of scaling the momentum of a particle to 0.

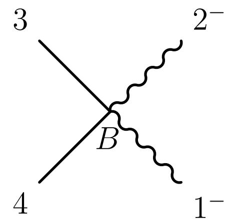
There are two terms which can contribute to the 4-point 2-photon, 2-scalar scattering at the tree-level. These are (see (2.1)):

$$\mathcal{L}_{INT4} = 4B\partial\varphi\cdot F\cdot F\cdot\partial\varphi + 4C\langle F^2\rangle\partial\varphi\cdot\partial\varphi.$$

Considering first the helicity configuration 1^-2^-34, the C-term has already been dealt with in the last section. The contribution of the B-term will be evaluated in two steps, the first being the computation in a specific field operator-particle assignment, namely (loosely speaking) $\partial\varphi\cdot F\cdot F\cdot\partial\varphi \leftrightarrow 3\cdot 1\cdot 2\cdot 4$. The result is

$$i\mathcal{M}_{B3124} = 4B\frac{i}{8}3^{\dot{A}}3^{B}4^{\dot{M}}4^{N}\left(-\sqrt{2}i\right)^2\epsilon_{BN}1_{\dot{A}}1_{\dot{C}}2^{\dot{C}}2_{\dot{M}} = iB\,[1|3]\,[2|4]\,[1|2]\,\langle 3|4\rangle.$$

Taking into account all such assignments, the total contribution is

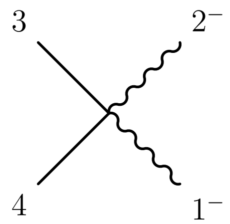


$$i\mathcal{M}_{B--} = 2iB\langle 3|4\rangle\,[1|2]\,([1|3]\,[2|4] + [1|4]\,[3|2])\,.$$

This can be further simplified using the Shouten identity (1.9), which leads to

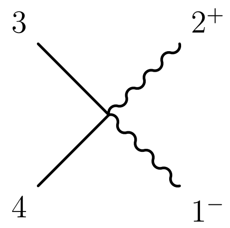
$$[1|2]\,[3|4] + [1|3]\,[4|2] + [1|4]\,[2|3] = 0.$$

The final result for a combination of both the B- and C- terms is



$$i\mathcal{M}_{--} = -2i\,(4C + B)\,[1|2]^2\langle 3|4\rangle\,[4|3]\,.$$

Analogical computation for the 1^-2^+34 helicity configuration gives



$$i\mathcal{M}_{-+} = 4iB\langle 3|2\rangle\langle 4|2\rangle\,[1|3]\,[1|4] = -4iB[1|4]^2\langle 4|2\rangle^2,$$

where the last equality follows from the momentum conservation

$$|1]\langle 1| + |2]\langle 2| + |3]\langle 3| + |4]\langle 4| = 0.$$

What is the behavior of these amplitudes in the soft limit (for $p_4 \to 0$)? The forms the expressions take suggest $i\mathcal{M}_{--} = O\,(t)$ and $i\mathcal{M}_{-+} = O\,(t^2)$, however, the situation is not as simple (even though in the end, these initial impressions will turn out to be correct).

There is, in principle, an infinite number of ways to take the limit - if we have a prescription for taking the limit, it's always possible to add continuous little group transformations to selected momenta and change the resulting $O\,(t^m)$ behavior in any desired way. One of the possible ways to take the limit is to compensate the scaling of one particle's momentum by changing the momenta of two other particles in the following way:

$$\begin{aligned} |\hat{s}\rangle &= (1-\epsilon)\,|s\rangle = t|s\rangle \\ |\hat{\imath}] &= |i] + \epsilon\frac{\langle j|s\rangle}{\langle j|i\rangle}|s] \\ |\hat{\jmath}] &= |j] + \epsilon\frac{\langle i|s\rangle}{\langle i|j\rangle}|s]. \end{aligned} \tag{3.7}$$

Another option would be the result of exchanging the square and angle brackets in the above equations.

The conservation of momentum in (3.7) can be verified using the Shouten identity (1.9):

$$\begin{aligned}|\hat{s}\rangle[\hat{s}| + |\hat{i}\rangle[\hat{i}| + |\hat{j}\rangle[\hat{j}| &= |s\rangle[s| + |i\rangle[i| + |j\rangle[j| + \epsilon[s|\left(\frac{\langle j|s\rangle}{\langle j|i\rangle}|i\rangle + \frac{\langle i|s\rangle}{\langle i|j\rangle}|j\rangle - |s\rangle\right)\\ &= |s\rangle[s| + |i\rangle[i| + |j\rangle[j|.\end{aligned}$$

In applying the limit to our amplitudes, the following computation will prove useful:

$$[\hat{i}|\hat{j}] = [i|j] + \frac{\epsilon}{\langle j|i\rangle}\left(\langle j|s\rangle\,[s|j] + \langle i|s\rangle\,[s|i]\right) = [i|j] + \frac{2\epsilon}{\langle j|i\rangle}\left(p_i + p_j\right)\cdot p_s.$$

Using the prescription (3.7) on $i\mathcal{M}_{--}$ with $(i,j,s) = (1,2,4)$ and noting that

$$[1|2] + \frac{2\epsilon}{\langle 2|1\rangle}\left(p_1 + p_2\right)\cdot p_4 = [1|2] - \frac{2\epsilon}{\langle 2|1\rangle}\,p_1\cdot p_2 = (1-\epsilon)\,[1|2]\,,$$

the soft limit defined this way turns out to be $O\left(t^3\right)$. This is rather surprising, especially taking into account that using the same prescription on $i\mathcal{M}_{-+}$ gives only $O\left(t^2\right)$. The same holds even if we chose 1 (or 2) and 3 as the compensating spinors. On the other hand, taking the limit using "the conjugate" of (3.7), i.e.,

$$\begin{aligned}|\hat{s}] &= (1-\epsilon)\,|s] = t|s]\\ |\hat{i}\rangle &= |i\rangle + \epsilon\frac{[j|s]}{[j|i]}|s\rangle\\ |\hat{j}\rangle &= |j\rangle + \epsilon\frac{[i|s]}{[i|j]}|s\rangle,\end{aligned}\tag{3.8}$$

leads to an $O(t)$ behavior for $i\mathcal{M}_{--}$. Here lies the resolution of the problems in defining the limit. The $O\left(t^3\right)$ limit came from modifying the $|1]$ and $|2]$ spinors, while modifying the $|1\rangle$ and $|2\rangle$ spinors led to the expected result. The soft limit is therefore taken in such a way as not to modify the left-handed spinors associated with negative-helicity photons and vice-versa. It turns out that this requirement specifies the limit without any ambiguity.

Employing the procedure described above, $i\mathcal{M}_{--} = O(t)$, $i\mathcal{M}_{-+} = O\left(t^2\right)$. If the $O\left(t^2\right)$ behavior is assumed, the following requirement is obtained:

$$4C + B = 0.$$

This is satisfied in DBI (see (2.6)). The characterization of DBI as the unique dimensionally reduced theory with a $O\left(t^2\right)$ soft scalar limit has therefore been proven at the level of four-point amplitudes.

In addition, the 4-photon vertex can now be rewritten using (3.3) and (3.4) as

$$B\langle F^4\rangle + C\langle F^2\rangle^2 = \frac{B}{4}\langle\phi^2\rangle\langle\bar{\phi}^2\rangle,$$

which manifestly conserves helicity.

3.3 6-point amplitudes

Having established that $4C + B = 0$ and the helicity conservation at the 4-point level, it's important to examine whether the helicity conservation holds even for the 4-point vertex embedded in a larger diagram, i.e. with off-shell particles. For our purposes, it's sufficient to consider only one off-shell leg.

The relevant terms for the $++$ choice of helicities are

$$\begin{aligned}&\frac{4iB}{8}p_L^{\dot{A}B}p_R^{\dot{M}N}\phi_{BD}\phi_N^D\epsilon_{\dot{A}\dot{M}} + iCp_{LN\dot{M}}p_R^{\dot{M}N}\phi_{AB}\phi^{AB}\\ =&\frac{iB}{2}\left(p_L^{\dot{A}B}p_R^{\dot{M}N}\phi_{BD}\phi_N^D\epsilon_{\dot{A}\dot{M}} - \frac{1}{2}p_{LN\dot{M}}p_R^{\dot{M}N}\phi_{AB}\phi^{AB}\right).\end{aligned}$$

First, let's assume a single photon is off-shell. The momenta of the remaining particles are denoted as L, R for the scalars and P for the second photon. Neglecting the $\left(-\sqrt{2}i\right)$ factor per each photon, the contribution of the $(p_L, p_R, \varphi, \varphi) \to (L, R, \varphi, P)$ assignment is proportional to

$$[L|R]\,\langle R|P\rangle L^B P^D\phi_{BD} - \frac{1}{2}\,[L|R]\,\langle R|L\rangle P^B P^D\phi_{BD}.$$

Taking into account all possible assignments and using the Shouten identity leads to

$$2\phi_{BD}\,[L|R]\,P^D\left(\langle R|P\rangle L^B - \langle L|P\rangle R^B - \langle L|R\rangle P^B\right) = 0.$$

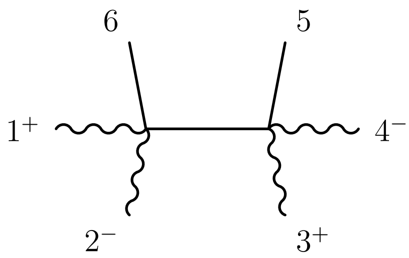
Analogically, taking one of the scalars off-shell, denoting the other scalar as L and the photons as 1 and 2 respectively and considering the assignment $(p_L, p_R, \varphi, \varphi) \to (L, p, 1, 2)$ leads to a contribution proportional to

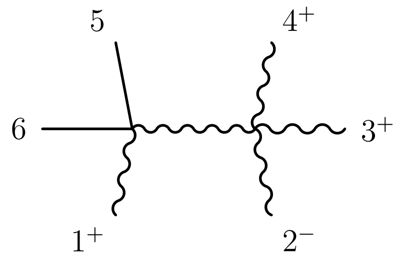
$$L_{\dot{M}}\langle 2|1\rangle p_R^{\dot{M}N}\left(\langle L|1\rangle 2_N + \frac{1}{2}\langle 1|2\rangle L_N\right).$$

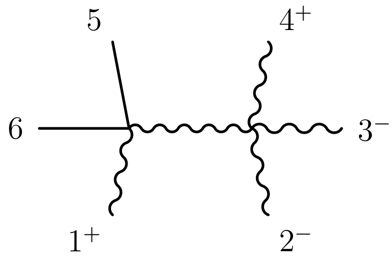
As before, summing over all possibilities, the vertex vanishes:

$$2L_{\dot{M}}\langle 2|1\rangle p_R^{\dot{M}N}\left(\langle L|1\rangle 2_N - \langle L|2\rangle 1_N + \langle 1|2\rangle L_N\right) = 0.$$

The fact that the helicity is conserved for 4-point vertices leads to a vast reduction in the number of diagrams relevant to the six-point amplitudes. Roughly speaking, the only interesting possibilities aside from the contact 6-point vertices, up to various crossings and helicity flips, are the following.







We'll now compute the $1^+2^+3^+4^+56$, $1^-2^+3^+4^+56$ and $1^-2^-3^+4^+56$ amplitudes explicitly, with the expectation that the requirement of a $O\left(t^2\right)$ soft scalar limit could specify a particular theory or a subset of theories.

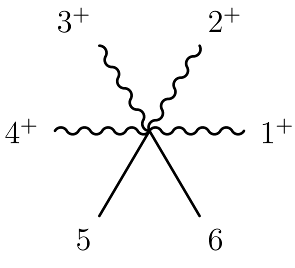
The relevant terms from (2.2), taking into account the results derived at the level of 4-point amplitudes and utilizing (3.3), (3.4) and (3.5), are

$$\begin{aligned}\mathcal{L}_{INT6} = {}& \frac{B}{4}\langle\phi^2\rangle\langle\bar{\phi}^2\rangle + B\partial^{\dot{A}B}\varphi\;\partial^{\dot{M}N}\varphi\;\phi_{BN}\bar{\phi}_{\dot{A}\dot{M}} \\ &+ \partial\varphi\cdot\partial\varphi\left(6D\langle F^2\rangle^2 + 2E\langle F^4\rangle\right) + 4E\langle F^2\rangle\partial\varphi\cdot F\cdot F\cdot\partial\varphi.\end{aligned}$$

In the $1^+2^+3^+4^+56$ case, only contact 6-point vertices contribute. The relevant terms are

$$\frac{12D+E}{8}\langle\phi^2\rangle^2\partial\varphi\cdot\partial\varphi - \frac{E}{4}\langle\phi^2\rangle\partial_{\dot{M}}{}^{B}\varphi\;\partial^{\dot{M}N}\varphi\phi_{BD}\phi^{D}{}_{N}.$$

By now, the procedure for evaluating amplitudes for contact vertices has been illustrated several times, so it's sufficient just to state the final result, skipping the straightforward computation:



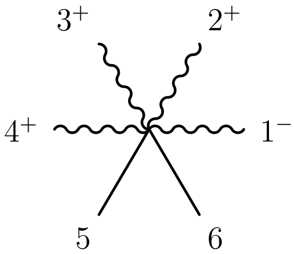
$$= -12i\left(4D+E\right)\langle 5|6\rangle\left[6|5\right]\langle 1|2\rangle^2\langle 3|4\rangle^2 + (1\leftrightarrow 3) + (1\leftrightarrow 4)\,.$$

Taking the soft limit $|6\rangle \to 0$, the expression above behaves as $O(t)$. This leads to the first requirement imposed by the $O(t^2)$ limit, namely

$$4D + E = 0.$$

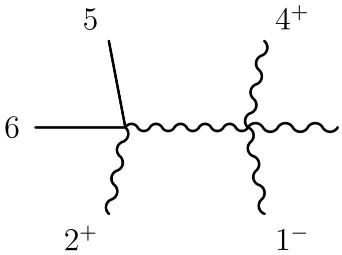
This relation is satisfied in DBI (see (2.6)).

The $1^-2^+3^+4^+56$, amplitude has only a single contribution from contact 6-point vertices:



$$= 8iE\,[1|5]\,[1|6]\,\langle 5|2\rangle\langle 6|2\rangle\langle 3|4\rangle^2 + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)\,.$$

There is however even a nontrivial term from the third order of the Dyson expansion, involving one of the degenerate contact propagators of (3.6):



$$\begin{aligned} &= 8B^2\langle 5|2\rangle\langle 6|2\rangle\langle 3|4\rangle^2 5^{\dot{A}}6^{\dot{M}}\langle \bar{\phi}_{\dot{A}\dot{M}}\bar{\phi}_{\dot{C}\dot{D}}\rangle 1^{\dot{C}}1^{\dot{C}} \\ &= -16iB^2\,[1|5]\,[1|6]\,\langle 5|2\rangle\langle 6|2\rangle\langle 3|4\rangle^2. \end{aligned}$$

Taking into account the other two possibilities for the positive helicity spinor on the same vertex as the scalars, the total result is

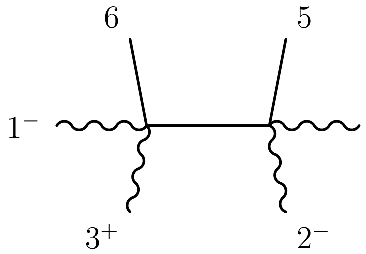
$$i\mathcal{M}_{-+++} = 8i\left(E - 2B^2\right)[1|6]\,\langle 5|2\rangle\langle 6|2\rangle\langle 3|4\rangle^2 + (2 \leftrightarrow 3) + (2 \leftrightarrow 4)\,.$$

Taking $|6\rangle \to 0$ and compensating using $|3]$ and $|4]$, it's readily seen that the limiting behavior is $O(t)$. Therefore, if our assumption of the enhanced soft limit is to be satisfied, it's required that

$$E - 2B^2 = 0.$$

This holds in DBI (see (2.6)), singling it out as the only possible candidate for a dimensionally reduced theory (of the form discussed in chapter 2) with an $O(t^2)$ soft scalar limit. It's however necessary to check that the $1^-2^-3^+4^+56$ amplitudes have the desired behavior.

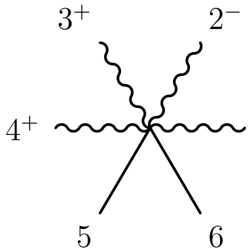
The term with a scalar propagator has the desired behavior on its own:



$$= \frac{-4iB^2}{\left(p_1+p_3+p_6\right)^2+i\epsilon} \langle 3|6\rangle^2 [6|1]^2 \langle 4|5\rangle^2 [5|2]^2 .$$

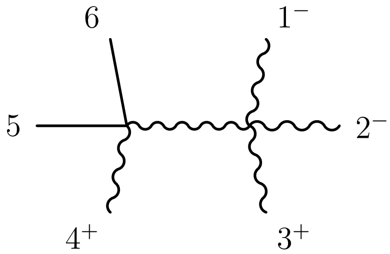
Note that this holds even for all the crossings of the above diagram.

The sole contribution of six-particle contact vertices originates from the $4E\langle F^2\rangle \partial\varphi \cdot F \cdot F \cdot \partial\varphi$ term (the $\partial\varphi \cdot \partial\varphi \langle\phi^2\rangle\langle\bar{\phi}^2\rangle$ term cancels out in DBI):



$$= -8iE\,[1|2]^2 \langle 3|4\rangle^2 [5|6] \langle 6|5\rangle. \tag{3.9}$$

The following diagram (and the last one to consider) involves the non-degenerate propagator from (3.6):



$$\begin{aligned} =&\frac{16iB^2}{\left(p_4+p_5+p_6\right)^2+i\epsilon}\langle 4|5\rangle\langle 4|6\rangle [1|2]^2 \\ &\cdot [5|\left(4+6\right)|3\rangle [6|\left(4+5\right)|3\rangle, \end{aligned} \tag{3.10}$$

where the momenta in brackets, as the structure suggests, represent $i \leftrightarrow |i]\langle i|$. Note that there are four such terms, one for each possible particle in the vertex with the two scalars.

The latter two contributions don't have the required limit on their own. The $O\left(t\right)$-part of (3.10) will be extracted to see that it cancels (3.9). This was also verified numerically (see appendix A for details on the way the configurations of momenta were generated). The limit is taken by scaling $|6] \to 0$ and using $|5\rangle$ and $|1\rangle$ to compensate. In contrast with (3.8), the hats are dropped from the modified spinors. First, note that

$$\frac{1}{\left(p_4+p_5+p_6\right)^2} = \frac{1}{\langle 4|5\rangle [5|4] + \langle 4|6\rangle [6|4] + \langle 5|6\rangle [6|5]} = \frac{1}{\langle 4|5\rangle [5|4]} + O\left(t\right).$$

The above allows to express (3.10) as

$$\begin{aligned} &\frac{16iB^2}{[5|4]}\langle 4|6\rangle [1|2]^2 [5|\left(4+6\right)|3\rangle [6|\left(4+5\right)|3\rangle + O\left(t^2\right) \\ &= 16iB^2 \langle 4|6\rangle [1|2]^2 \langle 4|3\rangle [6|\left(4+5\right)|3\rangle + O\left(t^2\right). \end{aligned}$$

Adding the result to its $3 \leftrightarrow 4$ cross and combining the terms not proportional to $\langle 3|4\rangle^2$ using the Shouten identity, the following is obtained:

$$16iB^2 \, [1|2]^2 \, \langle 3|4\rangle^2 [6| \, (3+4+5) \, |6\rangle + O\left(t^2\right).$$

Analogical contributions for 1^- and 2^- in the left-hand side vertex in (3.10) (instead of 4^+) add up to

$$\begin{aligned} &\frac{16iB^2}{(p_1+p_5+p_6)^2+i\epsilon} [5|1] \, [6|1] \, \langle 3|4\rangle^2 \cdot [2| \, (1+6) \, |5\rangle [2| \, (1+5) \, |6\rangle + (1 \leftrightarrow 2) \\ =&16iB^2 \, [1|2]^2 \, \langle 3|4\rangle^2 [6| \, (1+2+5) \, |6\rangle + O\left(t^2\right). \end{aligned}$$

The final result, taking into account all four possibilities, is therefore

$$16iB^2 \, [1|2]^2 \, \langle 3|4\rangle^2 \langle 5|6\rangle \, [6|5] + O\left(t^2\right).$$

The $O\,(t)$-part cancels (3.9) exactly, leaving only a result with the desired $O\,(t^2)$ behavior. This means that uniqueness of DBI as a dimensionally reduced theory with enhanced soft scalar limit has been proven at the level of six-point amplitudes (or, more precisely, that if there is to be such a theory, it has to agree with DBI at least up to six-particle vertices).

4. The six dimensional case

In the first chapter, the spinor-helicity formalism and its use was outlined in four dimensions. A similar construction is possible in six dimensions, which naturally leads to the question of whether the unique characterization of DBI among the dimensionally reduced vector effective field theories could be extended from four dimensions. It will be shown that the main difficulty lies in formalizing the notion of a soft limit properly. In 4 dimensions, roughly speaking, it was sufficient not to make any unnecessary little group transformations in the limiting process. The problem in 6 dimensions is that it's not clear what does "not making unnecessary little group transformations" precisely mean.

The first section introduces the spinor-helicity formalism in six dimensions. It's followed by a section which discusses the construction of little group invariants and the impossibility of their use on characterization of theories. The concluding section deals with the problems outlined above concerning the soft limits.

4.1 The spinor-helicity formalism

The formalism, as outlined here, was introduced in [18]. The reader is referred there for a more detailed treatment, which is centered on establishing the amplitudes in Yang-Mills.

The starting point of our discussion are the 8×8 γ-matrices. Decomposing them into the six-dimensional analogues of the Pauli matrices, in parallel with (1.3), the defining anticommutation relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ leads to

$$\sigma^\mu_{AB}\tilde{\sigma}^{\nu BC} + \sigma^\nu_{AB}\tilde{\sigma}^{\mu BC} = 2\eta^{\mu\nu}\delta^C_A. \tag{4.1}$$

It's possible to find a basis in which σ and $\tilde{\sigma}$ are antisymmetric:

$$\begin{array}{llll} \sigma^0 & = i\sigma_1 \otimes \sigma_2 & \tilde{\sigma}^0 & = -i\sigma_1 \otimes \sigma_2 \\ \sigma^1 & = i\sigma_2 \otimes \sigma_3 & \tilde{\sigma}^1 & = i\sigma_2 \otimes \sigma_3 \\ \sigma^2 & = -\sigma_2 \otimes \sigma_0 & \tilde{\sigma}^2 & = \sigma_2 \otimes \sigma_0 \\ \sigma^3 & = -i\sigma_2 \otimes \sigma_1 & \tilde{\sigma}^3 & = -i\sigma_2 \otimes \sigma_1 \\ \sigma^4 & = -\sigma_3 \otimes \sigma_2 & \tilde{\sigma}^4 & = \sigma_3 \otimes \sigma_2 \\ \sigma^5 & = i\sigma_0 \otimes \sigma_2 & \tilde{\sigma}^5 & = i\sigma_0 \otimes \sigma_2. \end{array} \tag{4.2}$$

In addition, they satisfy the following duality:

$$\tilde{\sigma}^{\mu AB} = \frac{1}{2}\epsilon^{ABCD}\sigma^\mu_{CD}. \tag{4.3}$$

Contracting with a momentum (as in chapter 1) allows to define its bispinorial version:

$$\begin{aligned} p_\mu \tilde{\sigma}^{\mu AB} &\equiv p^{AB} \equiv \lambda^{Aa}\lambda^{Bb}\epsilon_{ab} \\ p_\mu \sigma^\mu_{AB} &\equiv p_{AB} \equiv \tilde{\lambda}_{A\dot{a}}\tilde{\lambda}_{B\dot{b}}\epsilon^{\dot{a}\dot{b}}. \end{aligned} \tag{4.4}$$

The second equalities are to be discussed. Note that the convention for ϵ_{ab} will be different from the preceding chapters, namely $\epsilon_{12} = -1$, $\epsilon^{12} = 1$.

First, remark that

$$p_{AB}p^{BC} = \left(\frac{1}{2}\left(\sigma^\mu \tilde{\sigma}^\nu - \sigma^\nu \tilde{\sigma}^\mu\right) + \eta^{\mu\nu}\right) p_\mu p_\nu = 0. \tag{4.5}$$

Neglecting any rigor, it can be argued that both p_{AB} and p^{CD} are rank-2, on account of the duality $p_{AB} = \frac{1}{2}\epsilon_{ABCD}p^{CD}$ (a result of (4.3)), which suggests that both have the same rank, and the above relation, which rules out rank-3.

An antisymmetric rank-2 4×4 matrix can always be expressed as $A = u \otimes v - v \otimes u$. When decomposing p^{AB}, the u and v are the λ^{A1} and λ^{A2} of equation (4.4) (and analogically for p_{AB}). Furthermore, it follows from (4.5) that they're the solutions of the Weyl equation $p_{AB}\lambda^{Ba} = 0$ (and $p^{AB}\tilde{\lambda}_{B\dot{a}} = 0$). This shows that they're indeed spinors (of $SU(4) \sim SO(1,5)$) with respect to the Lorentz group, as their indices suggest.

The assignment of λ^1 and λ^2 is not unique (and the same goes for $\tilde{\lambda}_{\dot{a}}$). They can be mixed into each other by an arbitrary $SL(2,\mathbb{C})$ matrix without violating their defining properties. These are the little group transformations associated with each particle. Note that the little group transformations of λ and $\tilde{\lambda}$ are independent, which is the main source of the problems with the soft limits, as discussed in section 4.3.

As a small sidestep, we'll introduce the notation for the most frequent contractions of the above described objects:

$$\begin{aligned} \langle p^a | q_{\dot{b}}] &\equiv \lambda^{Aa}\chi_{A\dot{b}} \\ \langle p^a|\sigma^\mu|q^b\rangle &\equiv \lambda^{Aa}\sigma^\mu_{AB}\chi^{Bb} \\ [p_{\dot{a}}|\tilde{\sigma}^\mu|q_{\dot{b}}] &\equiv \tilde{\lambda}_{A\dot{a}}\tilde{\sigma}^{\mu AB}\tilde{\chi}_{B\dot{b}} \\ \langle p^a q^b r^c s^d\rangle &\equiv \epsilon_{ABCD}p^{Aa}q^{Bb}r^{Cc}s^{Dd}. \end{aligned}$$

The inverse relation to (4.4) can be established using the contraction of (4.1), $Tr\left[\sigma^\mu\tilde{\sigma}^\nu\right] = 4\eta^{\mu\nu}$:

$$\begin{aligned} p_\mu &= -\frac{1}{4}\langle p^a|\sigma_\mu|p^b\rangle\epsilon_{ab} \\ &= -\frac{1}{4}[p_{\dot{a}}|\tilde{\sigma}^\mu|p_{\dot{b}}]\epsilon^{\dot{a}\dot{b}}. \end{aligned}$$

The polarization vectors are again required to be orthogonal to both the momentum and a lightlike reference vector. Since $F_{\mu\nu}$ is not reducible, there is no analogue of the 4-dimensional helicity by which one could label the particles and consequently only a single polarization vector endowed with two little group indices:

$$\begin{aligned} \epsilon^{\mu}_{a\dot{a}} &\equiv -\frac{1}{2\sqrt{2}} \langle p_a|\sigma^{\mu}|q_b\rangle \frac{\langle q^b|q_{\dot{a}}]}{p\cdot q} \\ &\equiv -\frac{1}{2\sqrt{2}} \frac{\langle p_a|q^{\dot{b}}]}{p\cdot q} [q_{\dot{b}}|\tilde{\sigma}^{\mu}|p_{\dot{a}}]. \end{aligned} \tag{4.6}$$

The orthogonality can be verified using the identity

$$\sigma^{\mu}_{AB}\tilde{\sigma}^{CD}_{\mu} = -2\left(\delta^{C}_{A}\delta^{D}_{B} - \delta^{D}_{A}\delta^{C}_{B}\right).$$

Finally, the following two identities are often useful

$$\begin{aligned} \det\left(\langle p|q]\right) &= 2p\cdot q \\ \langle q_a|p_{\dot{a}}]\epsilon^{\dot{a}\dot{b}}\langle q_b|p_{\dot{b}}] &= -2\left(p\cdot q\right)\epsilon_{ab}. \end{aligned}$$

4.2 Little group invariants and characterization of theories

As will be discussed in detail in section 4.3, the little group covariance of amplitudes is an obstacle to defining a reasonable prescription for taking the soft limit. It is therefore meaningful to ask whether some sort of a characterization of a subclass of theories (preferrably of DBI), as in chapter 3, could be possible by utilizing little group invariants. Through explicit calculation, it will be argued that the answer is likely negative.

Having no analogue of photons being labeled by helicity, as is the case in four dimensions, means that there is only a single 4-point 2-scalar, 2-photon amplitude to consider. Note that the result will be a tensor with respect to the little group.

The relevant terms of our canonical Lagrangian (2.2) are the B and C terms, repeated here for convenience:

$$\mathcal{L}_{INT4} = 4B\partial\varphi\cdot F\cdot F\cdot\partial\varphi + 4C\partial\varphi\cdot\partial\varphi\langle F^2\rangle.$$

To make the computation more digestible, a convenient choice of reference vectors is made, namely that the momentum of one of the gluons is taken as the reference vector for the other and vice-versa. Numbering our particles as $(123_{a\dot{a}}4_{b\dot{b}})$, this leads to (for F on-shell, as it is to be used here)

$$F_{3\alpha\beta}F_4^{\beta\gamma} = i^2\left(p_3 \cdot p_4 \epsilon_{3\alpha}\epsilon_4^\gamma + \epsilon_3 \cdot \epsilon_4 p_{3\alpha}p^\gamma\right).$$

Utilizing the above and including all four possible field operator - particle assignments, the amplitude is

$$\begin{aligned} i\mathcal{M}_{a\dot{a}b\dot{b}} = -8i\,(&4C\ p_1 \cdot p_2\ p_3 \cdot p_4\ \epsilon_3 \cdot \epsilon_4 + B\ p_1 \cdot \epsilon_3\ p_3 \cdot p_4\ \epsilon_4 \cdot p_2 \\ &+B\ p_1 \cdot p_3\ \epsilon_3 \cdot \epsilon_4\ p_4 \cdot p_2 + B\ p_2 \cdot \epsilon_3\ p_3 \cdot p_4\ \epsilon_4 \cdot p_1 \\ &+B\ p_2 \cdot p_3\ \epsilon_3 \cdot \epsilon_4\ p_4 \cdot p_1). \end{aligned}$$

It's possible to simplify the above expression using simple kinematics. For example, $p_2 \cdot \epsilon_3 = -p_1 \cdot \epsilon_3$, due to the choice of reference vectors and the momentum conservation $p_1 + p_2 + p_3 + p_4 = 0$. The final result is

$$\begin{aligned} -\frac{1}{8}\mathcal{M}_{a\dot{a}b\dot{b}} =& 2B\ p_3 \cdot p_4\ p_1 \cdot \epsilon_3\ p_2 \cdot \epsilon_4 \\ &-2B\ p_1 \cdot p_3\ p_1 \cdot p_4\ \epsilon_3 \cdot \epsilon_4 \\ &+ (4C + B)\, p_1 \cdot p_2\ p_3 \cdot p_4\ \epsilon_3 \cdot \epsilon_4. \end{aligned} \tag{4.7}$$

Note that the above holds in arbitrary dimension if the $a\dot{a}b\dot{b}$ indices of 6d are swapped for the proper indices in the given dimension (e.g. for $+$ or $-$ in 4d). In four dimensions, the only ambiguity in the resulting expression comes from the possible little group transformations of the photons 3 and 4. For example, in the $\mathcal{M}_{--}$ case,

$$\epsilon_{3-} \cdot \epsilon_{4-} = \frac{[3|4]}{\langle 3|4\rangle},$$

so the amplitude in a theory with $B = 0$ would scale as z^{-2} under $\langle 3| \to z\langle 3|$. However, multiplying it by $\mathcal{M}^\star$ and noting that $\lambda = \tilde{\lambda}^\star$ for real momenta, the ambiguity cancels out (as it should) and only the little group-invariant $|\mathcal{M}|^2$ remains.

The situation in six dimensions is more difficult. Contracting the polarization vectors gives

$$\epsilon_{3a\dot{a}} \cdot \epsilon_{4b\dot{b}} = -\frac{1}{2p_3 \cdot p_4}\langle p_a|q_{\dot{b}}]\langle q_b|p_{\dot{a}}].$$

Each of the four indexed objects has independent little group transformations, so assuming once again for simplicity $B = 0$, the result is ambigious. This time however, multiplying by the complex conjugate doesn't save the situation. In the standard choice of σ-matrices (4.2), complex conjugation leads to $\sigma^{\mu\star} = -\eta^{\mu\nu}\tilde{\sigma}^\nu$. Recalling (4.4), this means that $\lambda^\star$ is the "$\tilde{\lambda}$" of a time-reversed momentum and vice-versa. This doesn't lead to any less ambiguity and as numerical evaluation confirms, $|\mathcal{M}_{a\dot{a}b\dot{b}}|^2$ is not little group invariant (i.e. depends on the arbitrary choices made in defining λ^{A1} and other such objects).

This was to be expected, as $F_{\mu\nu}$ is not reducible. The right way to obtain little group-invariant quantities is by contracting the amplitude $\mathcal{M}_{a\dot{a}b\dot{b}}$ with a tensor with the right transformation properties ("the right indices"). These tensors specify the photon configuration in a sense. Note that in the four-dimensional case, it would be possible to proceed similarly, for example to "contract" (in this case multiply) the amplitude propotional to $\frac{[3|4]}{\langle 3|4\rangle}$ by the "tensor" $\frac{\langle 3|4\rangle}{[3|4]}$. This way, $|\mathcal{M}|^2$ would not be ambigious even for complex momenta.

In the case of the 4-point amplitude, there are essentially two choices for the tensor to be used in the contraction:

$$\begin{aligned} T_1^{a\dot{a}b\dot{b}} &= \epsilon_3^{a\dot{a}} \cdot \epsilon_4^{b\dot{b}} \\ T_2^{a\dot{a}b\dot{b}} &= p_1 \cdot \epsilon_3^{a\dot{a}}\ p_2 \cdot \epsilon_4^{b\dot{b}}\ \frac{p_3 \cdot p_4}{p_1 \cdot p_3\ p_1 \cdot p_4}. \end{aligned}$$

All the other options are scalar multiples of these. Note that they're not gauge-invariant, as we're still working in a particular setting of the reference vectors. Their gauge-invariant counterparts are

$$\begin{aligned} &\frac{F_3^{a\dot{a}} \cdot F_4^{b\dot{b}}}{2\ p_3 \cdot p_4} \\ &\frac{p_1 \cdot F_3 \cdot p_2\ p_1 \cdot F_4 \cdot p_2}{p_1 \cdot p_2\ p_1 \cdot p_3\ p_1 \cdot p_4}. \end{aligned}$$

In contracting with the general amplitude (4.7), the "completeness" of polarization vectors is useful

$$\epsilon^{\mu}_{a\dot{a}} \epsilon^{\nu a\dot{a}} = \eta^{\mu\nu} - \frac{1}{p \cdot q} \left(p^{\mu} q^{\nu} + p^{\nu} q^{\mu} \right).$$

Somewhat tedious algebraic manipulations lead to the following results:

$$\begin{aligned} -\frac{1}{8} \mathcal{M}_{a\dot{a}b\dot{b}} T_1^{a\dot{a}b\dot{b}} &= 4\,(4C + B)\, p_1 \cdot p_2\ p_3 \cdot p_4 - 4B\ p_1 \cdot p_3\ p_1 \cdot p_4 \\ -\frac{1}{8} \mathcal{M}_{a\dot{a}b\dot{b}} T_2^{a\dot{a}b\dot{b}} &= 2\,(4C + B)\, p_1 \cdot p_2\ p_3 \cdot p_4 + 4B\ p_1 \cdot p_3\ p_1 \cdot p_4. \end{aligned}$$

Curiously, summing them would lead to a result proportional to $4C + B$, i.e. which vanishes precisely in DBI (see (2.7)). Is this a coincidence or could this be a way to characterize DBI? Unfortunately, as the following arguments show, there's no deep meaning behind the fact.

Firstly, what would the generalization to 6-point (and higher) amplitudes be? If it was the case that the contraction with $p_1 \cdot \epsilon_3^{a\dot{a}}\ p_2 \cdot \epsilon_4^{b\dot{b}}$ vanished in DBI, a natural hypothesis would be that all contraction of amplitudes in DBI with tensors composed as tensor products vanish. The real tensor which makes the amplitude vanish,

$$T^{a\dot{a}b\dot{b}} = \epsilon_3^{a\dot{a}} \cdot \epsilon_4^{b\dot{b}} + p_1 \cdot \epsilon_3^{a\dot{a}}\ p_2 \cdot \epsilon_4^{b\dot{b}}\ \frac{p_3 \cdot p_4}{p_1 \cdot p_3\, p_1 \cdot p_4}, \tag{4.8}$$

is however difficult to characterize in words and seems rather ad-hoc.

Furthermore, the contracted, little group-invariant results don't lead to any interesting soft limit behavior even in four dimensions. Generically, a contraction of the amplitude (4.7) must be of the form

$$c_1\, p_1 \cdot p_2\, p_3 \cdot p_4 + c_2\, p_1 \cdot p_3\, p_1 \cdot p_4. \tag{4.9}$$

This is a result of its invariance under the exchange of the scalars 1 and 2, which forces the general form

$$\begin{aligned}
&\alpha\, (p_1 \cdot p_2)^2 + \beta \left((p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2 \right) \\
&+\gamma\, p_1 \cdot p_2\ \ (p_1 \cdot p_3 + p_1 \cdot p_4) + \delta\, p_1 \cdot p_3\, p_1 \cdot p_4 \\
&= (\alpha + \beta - \gamma)\, (p_1 \cdot p_2)^2 + (\delta - 2\beta)\, p_1 \cdot p_3\, p_1 \cdot p_4.
\end{aligned}$$

However, (4.9) has an $O\left(t^2\right)$ soft limit automatically (taking any of the momenta to 0).

The last reason why the "characterization" using the tensor (4.8) is likely a coincidence comes from expressing the λ_{Aa} and $\tilde{\lambda}_{Aa}$ in terms of the λ and $\tilde{\lambda}$ of four dimensions. This is possible, as discussed in [18], for a momentum with non-trivial components only in the first four dimensions, i.e. $p^\mu = (p^0, p^1, p^2, p^3, 0, 0)$. In the four-point case, all configurations can be rotated so that each momentum has this form.

A comparison of the six dimensional γ-matrices with the four dimensional $\sigma^\mu_{\alpha\dot{\alpha}}$ and $\bar{\sigma}^{\mu\dot{\alpha}\alpha}$, where the greek indices are utilized instead of the capital latin used in chapter 3 in order to avoid ambiguity, reveals that the six-dimensional bispinors can be expressed as

$$\begin{aligned}
p^{AB} &= \begin{pmatrix} 0 & \lambda_\alpha \tilde{\lambda}^{\dot{\alpha}} \\ -\lambda_\alpha \tilde{\lambda}^{\dot{\alpha}} & 0 \end{pmatrix} \\
p_{AB} &= \begin{pmatrix} 0 & \lambda^\alpha \tilde{\lambda}_{\dot{\alpha}} \\ -\lambda^\alpha \tilde{\lambda}_{\dot{\alpha}} & 0 \end{pmatrix}.
\end{aligned}$$

This allows various decompositions into spinors. One particularly convenient is the following:

$$\begin{aligned}
\lambda_1 &= \begin{pmatrix} 0 \\ \tilde{\lambda}^{\dot{\alpha}} \end{pmatrix} & \lambda_2 &= \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix} \\
\tilde{\lambda}_1 &= \begin{pmatrix} 0 \\ \tilde{\lambda}_{\dot{\alpha}} \end{pmatrix} & \tilde{\lambda}_2 &= \begin{pmatrix} -\lambda^\alpha \\ 0 \end{pmatrix}.
\end{aligned} \tag{4.10}$$

The evaluation of the six-dimensional polarization vectors in terms of the λ_α and $\tilde{\lambda}_{\dot{\alpha}}$ is then a straighforward task. Using the definitions of the polarization vectors (1.8) and (4.6) and the explicit form of the γ-matrices 4 and 5,

$$\sigma^4 = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} \qquad \sigma^5 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix},$$

it's possible to establish for $\mu = 0, 1, 2, 3$

$$\begin{aligned} \epsilon^\mu_{1\dot{1}} &= -\frac{1}{2\sqrt{2}}\left(-\tilde{\lambda}^{\dot{\alpha}}\tilde{\sigma}^{\mu\alpha}_{\dot{\alpha}}\chi_\alpha\right)\frac{\epsilon^{21}\langle q_1|p_{\dot{1}}]}{p\cdot q} \\ &= -\frac{1}{\sqrt{2}}\frac{[p|\tilde{\sigma}|q\rangle\,[q|p]}{[p|q]\,\langle p|q\rangle} \\ &= -\epsilon_+. \end{aligned}$$

The other two components vanish, $\epsilon^4_{1\dot{1}} = \epsilon^5_{1\dot{1}} = 0$. Analogically,

$$\begin{aligned} \epsilon^\mu_{2\dot{2}} &= (\epsilon_-, 0, 0) \\ \epsilon^\mu_{1\dot{2}} &= \left(0,0,0,0,\frac{i}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) \\ \epsilon^\mu_{2\dot{1}} &= \left(0,0,0,0,\frac{i}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right). \end{aligned}$$

With these results in mind, we can see where the cancelation of the contraction of the amplitude (4.7) with the tensor (4.8) comes from in DBI. Noticing that $\mathcal{M}^{DBI}_{1\dot{1}1\dot{1}} = \mathcal{M}^{DBI}_{2\dot{2}2\dot{2}} = 0$, the only nontrivial contributions (up to a swap of the indices of photons 3 and 4) to the contraction are

$$\begin{aligned} \mathcal{M}^{DBI}_{1\dot{2}2\dot{1}}T^{1\dot{2}2\dot{1}} &= \mathcal{M}^{DBI}_{1\dot{2}2\dot{1}}\;\epsilon_3^{1\dot{2}}\cdot\epsilon_4^{2\dot{1}} = -2B\;p_1\cdot p_3\;p_1\cdot p_4 \\ \mathcal{M}^{DBI}_{1\dot{1}2\dot{2}}T^{1\dot{1}2\dot{2}} &= \mathcal{M}^{DBI}_{1\dot{1}2\dot{2}}\;p_1\cdot\epsilon_3^{1\dot{1}}\;p_2\cdot\epsilon_4^{2\dot{2}}\;\frac{p_3\cdot p_4}{p_1\cdot p_3\;p_1\cdot p_4} = 2B\;p_1\cdot p_3\;p_1\cdot p_4. \end{aligned}$$

In other words, the cancelation is not a result of the tensor picking out the right 4d helicity components of the amplitude. This would be the case if it had non-zero components only where $\mathcal{M}^{DBI}$ vanishes. This, along with the other presented reasons, strongly suggests that the cancelation was a coincidence and moreover that a characterization of a subset of dimensionally reduced theories by the behavior of the unambigious contractions of the amplitudes is unlikely.

4.3 Soft limits in 6 dimensions

Defining the soft limit in four dimensions took some care - not all prescriptions lead to the same results and little group transformations can easily change the situation. In six dimensions, the problem is amplified by the existence of two pairs of spinors, each with its own little group transformations.

As a first hint of what the results could be, it's possible to attempt to lift the limiting procedure from 4 to 6 dimensions in the case of the four-point amplitude. Using the spinors in the form (4.10) (and the polarization vectors it leads to) and the prescription (3.7), it's straightforward to establish that $\mathcal{M}_{DBI}$ scales as $O\left(t^2\right)$ for all non-zero components. The $\mathcal{M}^{DBI}_{1\dot{1}2\dot{2}}$ and $\mathcal{M}^{DBI}_{2\dot{2}1\dot{1}}$ components correspond to the results established for $\mathcal{M}^{DBI}_{+-}$ in chapter 3, while the limiting behavior of $\mathcal{M}^{DBI}_{1\dot{2}2\dot{1}}$ and $\mathcal{M}^{DBI}_{2\dot{1}1\dot{2}}$ is a consequence of $\epsilon_{1\dot{2}} \cdot \epsilon_{2\dot{1}} = \text{const.}$. All the remaining components vanish.

In the general, non-DBI, case, the amplitude scales as $O\left(t^1\right)$, $O\left(t^2\right)$ or $O\left(t^3\right)$ depending on the particular component. For example, $\mathcal{M}_{1\dot{1}1\dot{1}} \leftrightarrow \mathcal{M}_{++}$ scales as $O\left(t^1\right)$, while $\mathcal{M}_{2\dot{2}2\dot{2}} \leftrightarrow \mathcal{M}_{--}$ as $O\left(t^3\right)$. These results were to be expected as $\mathcal{M}_{a\dot{a}b\dot{b}}$ contains, in a sense, all the four-dimensional helicities and, as discussed in 3, for some of them the prescription used is not the right choice. However, the $O\left(t^2\right)$ limit uniform over all components would be a good characterization of DBI.

In principle, it would be possible to define the limiting process for a general n-point amplitude by rotating the configuration so that the momentum to be scaled and two other momenta to be used for compensation are of the form $p^\mu = \left(p^0, p^1, p^2, p^3, 0, 0\right)$, utilizing the decomposition (4.10) and following the same limiting prescription as in 4d. Informally speaking, this is not a very covariant procedure and regardless of whether it gives the right results or not (which is unknown to the author), this direction was not pursued further.

The correctness of the limiting procedure (3.7) relied on the 4-dimensional Shouten identity (1.9). Its 6-dimensional analogue

$$\langle 1_a 2_b 3_c 4_d \rangle 5_e + \langle 2_b 3_c 4_d 5_e \rangle 1_a + \langle 3_c 4_d 5_e 1_a \rangle 2_b + \langle 4_d 5_e 1_a 2_b \rangle 3_c + \langle 5_e 1_a 2_b 3_c \rangle 4_d = 0 \tag{4.11}$$

can be used to define a similar procedure, which requires 4 momenta to compensate for the fifth momentum which is scaled to 0 (the procedure is therefore useless for 4-point amplitudes):

$$\begin{aligned}
\hat{s}_1^A &= (1-\epsilon)\, s_1^A = t s_1^A \\
\hat{i}_2^A &= i_2^A + \epsilon \frac{\langle s_1 j_1 k_1 l_1 \rangle}{\langle i_1 j_1 k_1 l_1 \rangle} s_2^A \\
\hat{j}_2^A &= j_2^A + \epsilon \frac{\langle k_1 l_1 i_1 s_1 \rangle}{\langle i_1 j_1 k_1 l_1 \rangle} s_2^A \\
\hat{k}_2^A &= k_2^A + \epsilon \frac{\langle l_1 i_1 s_1 j_1 \rangle}{\langle i_1 j_1 k_1 l_1 \rangle} s_2^A \\
\hat{l}_2^A &= l_2^A + \epsilon \frac{\langle i_1 s_1 j_1 k_1 \rangle}{\langle i_1 j_1 k_1 l_1 \rangle} s_2^A .
\end{aligned} \tag{4.12}$$

The conservation of momentum is easily verified, $\hat{s}_1^A s_2^B + i_1^A \hat{i}_2^B + j_1^A \hat{j}_2^B + k_1^A \hat{k}_2^B + l_1^A \hat{l}_2^B$ fits precisely into the Shouten identity.

However, unlike in the 4-dimensional case, this description is not sufficient. There are the $\tilde{\lambda}$ spinors left to deal with. Recalling (4.4), we see that the $\hat{p}_{AB}$ matrix is completely determined by the above procedure. However, its decomposition into $\tilde{\lambda}$ is far from unique. Each prescription can be modified by arbitrary little group transformations.

This is the root of all problems. Other than requiring the relation between $\hat{\tilde{\lambda}}$ and $\hat{p}_{AB}$ to be continuous, there is no way to specify what the right prescription is. First of all, note that simply taking "the conjugate" of (4.12) (i.e. swapping λ for $\tilde{\lambda}$ and the angle brackets for square brackets) does not, in general, lead to a $\hat{p}_{AB}$ which would satisfy $\hat{p}_{AB} = \frac{1}{2}\epsilon_{ABCD}\hat{p}^{CD}$, with $\hat{p}^{CD}$ obtained by using (4.12). This can be easily seen as the $\tilde{\lambda}$ were not uniquely defined before starting the limiting procedure and the prescription itself is "not covariant with respect to the little group indices".

The direction pursued in this work is to define a canonical way of decomposing the rank-2 antisymmetric matrix $\hat{p}_{AB}$ into the spinors $\tilde{\lambda}$. It was established only "so that it works algebraically" and consequently it has no physical meaning or interpretation. The procedure, guaranteed to work on almost all (i.e. except a set of measure zero) antisymmetric 4×4 rank-2 matrices A to decompose them into an antisymmetric product $u \otimes v - v \otimes u$ (the notation has been chosen for legibility, but one can substitute the objects of interest, p and $\tilde{\lambda}$), is as follows.

First, choose a fixed random vector r with four components. Let

$$\begin{aligned} \tilde{v}^T &= r^T \cdot A \\ \tilde{u} &= A \cdot \tilde{v} \\ \tilde{A} &= \tilde{u} \otimes \tilde{v} - \tilde{v} \otimes \tilde{u}. \end{aligned}$$

Assuming that $A = u \otimes v - v \otimes u$ for u and v mutually orthogonal (which is not an additional requirement, as it's always possible to orthogonalize the vectors), $\tilde{A}$ is proportional to A:

$$\begin{aligned} \tilde{v}^T &= r^T \cdot (u \otimes v - v \otimes u) = (r \cdot u)\, v - (r \cdot v)\, u \\ \tilde{u} &= (u \otimes v - v \otimes u)\left((r \cdot u)\, v - (r \cdot v)\, u\right) = (r \cdot u)\, v^2 u + (r \cdot v)\, u^2 v \\ \tilde{A} &= \left((r \cdot u)^2 v^2 + (r \cdot v)^2 u^2\right) A. \end{aligned}$$

The last remaining step is to normalize $\tilde{u}$ and $\tilde{v}$ so that $\tilde{A} = A$. The whole procedure might sometimes fail (e.g. when $r \cdot A = 0$ or when $u^2 = v^2 = 0$, which is possible since they might be complex), but these cases are not generic and can be neglected, especially since any numerical exploration of soft limits should be made on random configurations of momenta in order to figure out the behavior common to all configurations rather than a few artificially chosen ones. Furthermore, even these edge cases could be taken care of by a few minor modifications - generating a new random vector if $r \cdot A = 0$ and rotating the configuration randomly if $u^2 = v^2 = 0$.

Holding the randomly initialized vector r fixed throughout the limiting process leads to a well-defined continuous λ to $\tilde{\lambda}$ relation. In our numerical experiments, a prescription equivalent to (4.12) was utilized and the $\tilde{\lambda}$ spinors modified using the above-described procedure. Further details concerning the generation of random configurations of momenta can be found in appendix A. The six-point amplitudes were computed using various values for the parameters of the interaction Lagrangian (2.2). The results were not consistent with the "uniform $O(t^2)$ limit iff DBI" hypothesis. Namely, the various components of the amplitude showed $O(t^2)$ and $O(t^3)$ behavior in the DBI-case and $O(t^1)$ to $O(t^5)$ in the general case.

While this would be a good characterization of DBI, it's somewhat unsatisfactory. It's reasonable to expect that if a correct canonical soft-limit prescription existed, it would lead to the uniform $O(t^2)$ limit in DBI. Furthermore, the results are difficult to verify, as the six-point case suffers from a combinatorial explosion of different terms, rendering explicit analytical checks unfeasible and making the possibility of mistakes (on the part of the author) in numerical simulation impossible to eliminate.

In summary, this work has not succeeded in defining a satisfactory procedure for soft limits in six dimensions, however, as the argumentation has shown, it's entirely plausible that such a prescription does not exist. There does not seem to be a canonical way of relating the $\tilde{\lambda}$ spinors to the λs and consequently no right way to relate their limiting prescriptions. Unlike the four-dimensional case, it's not clear what it means not to make unnecessary little group transformations. Further work would be needed to establish these (null) results in a rigorous manner.

Conclusion

As demonstrated in this work, the spinor-helicity formalism is a powerful addition to the QFT toolbox. Far from being a finished article, the formalism and its descendents are proving to be a significant source of progress in the field. Their use on the study of effective field theories from the point of view of soft limits is an important part of completing our knowledge about QFT in general. In this work, the focus was on a particular class of dimensionally reduced vector field theories.

Parts of this work effectively amount to an useful guide on computing amplitudes in four-dimensional theories with vectors and scalars. Examples of explicit calculations are hard to come by, especially in the modern topics, so our hope is that this part of the work will find its purpose. The treatment of six-dimensional spinor-helicity formalism was mostly centered on attempts to find a characterization of DBI utilizing either the amplitudes or their unambigious contractions. The question of soft limits was discussed, presenting a clear picture in 4 dimensions and outlining the difficulties in the six-dimensional case. Whether or not these problems can be overcome is an open question, but there are good reasons to be sceptical.

All in all, if there is one point to be taken from this work, it's the efficiency and splendour of the spinor-helicity formalism. The insights it brought to light about the structure of Quantum Field Theories are of fundamental nature and hard to overstate. Furthermore, it's not unreasonable to believe that this might not be the end of the story.

A. Generating random configurations of momenta

For the numerical study of soft limits, a way of generating momentum configurations is needed. Hand-made configurations might not be representative of the whole situation, while there is little doubt that random configurations will capture all the important aspects.

In four dimensions, the process was already described in [4]. A similar version is presented here for completeness. First, spinors λ_i and $\tilde{\mu}_i$ are generated randomly (according to any desired continuous distribution), including their imaginary part, for each of the n particles. Spinors $\tilde{\lambda}$ are then computed as

$$\tilde{\lambda}_i = \langle \lambda_{i-2} | \lambda_{i-1} \rangle \tilde{\mu}_{i-1} + \langle \lambda_{i+1} | \lambda_{i-1} \rangle \tilde{\mu}_i + \langle \lambda_{i+1} | \lambda_{i+2} \rangle \tilde{\mu}_{i+1},$$

where the subscripts are understood mod n. The momentum conservation in the configuration defined by $\lambda_i, \tilde{\lambda}_i$ is straightforward to verify using the Shouten identity (1.9).

In the six-dimensional case, the key ingredient is once again the Shouten identity (4.11). Proceeding in analogy with the above, λ_{1i} and μ_i are randomly generated and λ_{2i} computed as

$$\begin{aligned}\lambda_{2i} = {} & \langle \lambda_{1(i-4)} \lambda_{1(i-3)} \lambda_{1(i-2)} \lambda_{1(i-1)} \rangle \mu_{i-2} + \langle \lambda_{1(i+1)} \lambda_{1(i-3)} \lambda_{1(i-2)} \lambda_{1(i-1)} \rangle \mu_{i-1} \\ & + \langle \lambda_{1(i+1)} \lambda_{1(i+2)} \lambda_{1(i-2)} \lambda_{1(i-1)} \rangle \mu_i + \langle \lambda_{1(i+1)} \lambda_{1(i+2)} \lambda_{1(i+3)} \lambda_{1(i-1)} \rangle \mu_{i+1} \\ & + \langle \lambda_{1(i+1)} \lambda_{1(i+2)} \lambda_{1(i+3)} \lambda_{1(i+4)} \rangle \mu_{i+2}.\end{aligned}$$

Note that this time, in order to be able to use the Shouten identity, at least five momenta are needed. To generate 4-particle configurations in 6 dimensions, it's possible to generate a configuration in 4d and lift it by adding two zero components (possibly rotating afterwards). The configuration defined by $\lambda_{1i}, \lambda_{2i}$ satisfies the momentum conservation. The "conjugate" spinors $\tilde{\lambda}$ can then be computed using the process described in section 4.3.

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