Möbius ribbon and Klein's bottle could be discussed in the first chapter as examples of constructions involving the conflict between our intuitions and mathematics.

4.2 Great Semantical Paradoxes

The ages old Liar's Paradox is believed to be the most potent and most important of all possible paradoxes of our thought. Its very form shows that the dilemma is caused by self-reference of the proposition, which is also called self-reflexivity of an utterance. Such looping of an utterance may have the form of a single sentence and then it is a direct vicious circle. It can also have the form of two or more sentences and then it is an indirect vicious circle. Traditionally, the name "Liar's Paradox" has been used with respect to "direct" utterances. The other, "indirect" form, composed of two mutually related propositions, is known by the name of its medieval discoverer as "Buridan's Paradox". Buridan's idea can be generalized so that it refers to any finite number of mutually related propositions. Since Liar's Paradox leads to a situation in which two contradictory propositions imply one another, it is called an antinomy.

4.2.1 Liar Antinomy

Self-referential paradoxes point to an important property of all systems that include semantic terms, which refer to expressions of the language of the system, or terms, which allow to define such terms, and are governed by the laws and rules of classical logical calculus. As was shown by Alfred Tarski (1901–1983) in his 1933 work *The Concept of Truth in Languages of Deductive Sciences*,[8] it appears that such a system is contradictory. One of the methods for solving the problem was proposed by Stanislaw Leśniewski (1886–1939) and later developed by Tarski. It distinguishes between a language and a meta-language, i.e. a language, in which one speaks of the propositions of a language.[9] A similar opinion about these paradoxes was expressed by Ramsey, who is famous for the division, presented in his 1931 book *The foundations of mathematics and other logical essays*, of paradoxes into two groups that are now known as first and second Ramsey's groups. The first one includes so-called logical paradoxes, or set theory paradoxes. The other one contains paradoxes that involve the concepts of signification and reference. These dilemmas are usually called semantic ones,

[8] Tarski [32].

[9] Borkowski [4], p. 356.

occasionally syntactic or epistemological ones. The greatest semantic paradox, for some the greatest paradox ever, is the Liar Antinomy.[10]

The reputation of Ancient inhabitants of Crete seems to have been low. The principal reason of it was their alleged propensity for lying. An interesting explanation of their permanent lies can be found in the myth of Medea. Idomeneus, the unfortunate king of Crete, was asked by Medea, the daughter of the king of Colchis, and Thetis, the mistress of Zeus and Poseidon, to decide which of them is more beautiful. He found the beauty of Thetis more appealing, which caused a great anguish in Medea, who uttered the memorable words: "All Cretans always lie" and cast a spell on the inhabitants of the island that made it impossible for them to tell the truth. Medea's comment became better known when it was repeated by Epimenides, a famous Cretan from Knossos who lived in the sixth century BC. He was a hal-legendary theologian, seer and thaumaturgos. Diogenes Laertios puts him on the list of first *sophoi* together with Thales of Miletus, Solon of Athens, Chilo of Sparta and Pittacus of Mytilene.[11] Epimenides was an author of poetic and prose works.[12] However, he is best remembered for recalling Medea's comment. While the utterance of an inhabitant of Colchis, a far away land beyond the Black Sea, causes no serious logical problems, the very same comment (All Cretans always lie) uttered by a Cretan seems to be evidently false. If Cretans always lie, Epimenides, who says that all Cretans always lie, must lie too. Accordingly, the proposition "All Cretans always lie" cannot be the truth if it is uttered by a Cretan. It should be remembered that though Medea's comment became a logical problem when it was repeated by a Cretan, Epimenides himself may have not realized the paradox, for, as noticed by Bocheński, he did not analyse any logical paradoxes. Liar Antinomy was not known even by Plato, though he analysed a similar problem of self-referent propositions in his dialogue *Euthydemus*.[13] Nor did Aristotle notice the importance of the problem, even though he was wondering whether one can tell the truth while telling lies[14]: "The argument is similar for the problem whether the same man can at the same time say what is false and what is true; but it appears to be a troublesome question because it is not easy to

[10] Our opinion is completely different. The Liar Antinomy, a great paradox that it is, must cede its priority to such profound paradoxes as the heap, the bald, etc., which are discussed in the final chapter devoted to ontological paradoxes.

[11] Diogenes Laertios, *Lives and Opinions of Great Philosophers*, I, 13, pp. 13–14; I, 41–42, p. 31.

[12] Diogenes Laertios, *Lives and Opinions of Great Philosophers*, I, 111–112, p. 69.

[13] Plato, *Complete Works*, ed. J. M. Cooper, Hackett Publishing Company 1997, *Euthydemus*, 283E–286E. Cf. also Bocheński [3], p. 131.

[14] Aristotle, *Sophistical Refutations* (transl. W. A. Pickard-Cambridge), in: *The Complete Works of Aristotle. The Revised Oxford Translation,* ed. J. Barnes, Princeton University Press 1984, vol. I, 180b. Cf. also, Bocheński [3], p. 132.

see whether it is saying what is true or saying what is false which should be stated without qualification."[15]

Although a question close to Liar Antinomy was apparently considered by philosophers before Aristotle, the problem was recognized as a logical dilemma only by an Aristotle's slightly older contemporary, Eubulides (fourth century BC), who is now credited to be the discoverer of this unusual paradox.

Eubulides, known to be a fierce opponent of Aristotle, was not only the author of Liar Antinomy but also of a number of extremely important paradoxes of vagueness and other logical problems of lesser importance, such as *The Hidden One*, *Electra*, *Cuckold*.[16] Unfortunately, Aristotle must have ignored the great logical discoveries of Eubulides, for he dismissed Liar Antinomy and, regrettably, remained silent about the paradoxes of vagueness. Bocheński notes that the problem of Liar was later analyzed by Theophrastus of Eresos (ca. 370–287 BC) and Chrysippus of Soloi (ca. 277–208 BC). The former two books to the problem, the latter, twenty-eight. In his attempt to reconstruct Chrysippus' solution, Bocheński had to overcome the difficulty of the language and fragmentary character of the evidence, yet he managed to show that Chrysippus probably believed liar's proposition to be senseless.[17] If this analysis is correct, Chrysippus proposed a solution, which is now reckoned as an important one. We should also remember Philetas from Kos (ca. 340–285 BC), who died in despair unable to solve the paradox.[18]

The ill reputation of Cretans must have been both widespread and lasting, for its traces can be seen even in the New Testament, written several centuries later. In the epistle to Titus, St. Paul repeats an evidently common opinion[19]: "It was one of them, their very own prophet, who said, 'Cretans are always liars, vicious brutes, lazy gluttons' That testimony is true." Naturally, the poet he had in mind was Epimenides of Knossos.

Regrettably, Eubulides's formulation of Liar Antinomy has not survived to our times. It should be noted that antinomic character of liar's proposition could not be limited by Eubulides to the form that is associated with Epimenides. It is easy to notice that if a Cretan says that all Cretans lie, it is not a dilemma. To put it precisely, Epimenides's statement should be formulated: "All Cretans always lie". Thus it is always understood. Then it contains two general quantifiers: one points to all Cretans, the other, to all moments in which any Cretan utters a proposition or to all propositions uttered by any Cretan. This means that Epimenides's proposition predicates about every proposition uttered by any Cretan. Assuming the truth

[15] As can be seen, there is no visible reference to the Liar's Paradox here. Moreover, Bocheński rightly observes that one can hardly accept that Aristotle presented any solution of the paradox there. Bocheński [3], p. 132.

[16] Diogenes Laertios, *Lives and Opinions of Great Philosophers*, II, 108, p. 137.

[17] Bocheński [3], pp. 132–133.

[18] Bocheński [3], p. 131.

[19] *The Bible*, *Titus*, 1, 12–13.

of Epimenides's proposition we come to a conclusion that it must be false. Consequently, the truth of the proposition implies its falsity, which, in accordance with the law of classical logic $(p \to \neg p) \to \neg p$, means that the proposition is false. What is of key importance here is that the assumption of its falsity does not imply the truth of the proposition. Actually, if Epimenides's statement: "All Cretans always lie" is false, it does not mean that the very proposition is true. The falsity of the proposition implies only that there is a proposition uttered by a Cretan that is true. Our analysis shows then that Epimenides's proposition is a false one and so we are not faced with any dilemma here. Accordingly, one cannot claim that the famous Liar Antinomy has the form given to it by Epimenides.

There are many ways in which Liar Antinomy can be formulated. Each of them serves the same purpose: the proposition should predicate about its own falsity or, putting it more precisely, truthlessness. It is often believed that the falsity in liar's proposition is related to the principle of bivalence: *every sentence is true or false*, so the falsity of the proposition is identical with its truthlessness.[20] To tell the truth, if a proposition is false, it is difficult to think that it is not truthless. Having one logical value, it cannot have another one. However, to eliminate every shadow of doubt one should use such a version of liar's proposition which predicates about its own truthlessness: if a proposition is truthless, it is obvious that it is not true. Naturally, it is even better if such a proposition predicates about itself being not true.

One of the easier ways of expressing antinomic self-reference, and a relatively precise one, is that which uses the name given to a proposition; in our case it is "L" (for 'liar').

Liar Antinomy
L. Proposition marked as L is not true.

Let us assume that proposition L is true. This means that it is as it predicates, so it is not true. Accordingly, the proposition is not true. However, this is the very state of affairs it predicates about. Accordingly, the proposition is true, because things are precisely as it predicates. This way we get the famous dilemma: the proposition is true when and only when it is not true. The form of the paradox is such that two mutually exclusive possibilities imply one another. For this reason Liar's Paradox is called an antinomy. Formally, if $L \to \neg L$ and $\neg L \to L$, so:

$$L \leftrightarrow \neg L.$$

[20] Martin proves that both the sentence asserting its own falseness and the sentence asserting its truthlessness merit being called Liar sentences, since they lead to a contradiction with the Tarski convention (reminded later). Moreover, he also proves that a sentence predicating about its own falseness is a Liar sentence no matter whether we assume a two-valued logic or not. Such a sentence is sometimes called *ordinary Liar*, while the sentence predicating its truthlessness is called *Strengthened Liar*. Martin [20], pp. 1–2.

Liar Antinomy is inseparably bound with Tarski's definition of truth, also called *Tarski biconditionals*. In his famous 1933 book, *The Concept of Truth in Languages of Deductive Science*, a fundamental work in the history of logic,[21] Tarski proposed his definition of truth. Let us remind that according to this definition[22]: "It snows" is a true proposition if and only if it snows. Generally, a proposition p is true if it is as it predicates[23]:

$$\underline{T}.\quad v(p) = 1 \text{ if and only if } p$$

The original form of the definition is the following[24]:

$$x \text{ is a true proposition if and only if } p.$$

In this formula, the symbol "p" is substituted for any proposition, while "x" is substituted for a particular name of that proposition.[25]

The relation between Liar Antinomy and Tarski's definition $\underline{T}$ is so important that some solutions of the antinomy involve substituting another definition of truth for definition $\underline{T}$. Feferman lists three classes of possible solutions for Liar Antinomy. The first one involves language limitation. Here we find Tarski's idea of distinguishing separate language orders. The second class contains logic limiting solutions, i.e. approaches in which some other logic, which uses truth-value gaps, e.g. paraconsistent or three-valued logic, is substituted for the classical one. Finally, the third class includes solutions that limit Tarski's definition of truth.[26] Another classification of possible solutions of Liar Antinomy is presented by Robert L. Martin. He distinguishes four diagnoses of the problem[27]:

Diagnosis 1. Both L and $\underline{T}$ seem correct, as genuine consequences of our intuitive semantic concepts of truth, reference, etc. The incompatibility resulting from the set $\{L, \underline{T}\}$ is also correct. Such incompatibility proves only that it is not possible to join the two propositions in a non-contradictory opinion. The solution of Liar Antinomy should involve substituting L' for L or $\underline{T}'$ for $\underline{T}$. At the same time, L' or $\underline{T}'$ should be the result of a "rational reconstruction" of L or $\underline{T}$, respectively. Moreover, the new conjunction, i.e. L and $\underline{T}'$, or L' and $\underline{T}$, or L' and $\underline{T}'$, would have to be non-contradictory.

Diagnosis 2. It is incorrect to claim that L and $\underline{T}$ are contradictory, because of the equivocal character of the concept "true". In order to show that s is not true we cannot use definition $\underline{T}$, since the sense of the word "true" in the proposition stating that s is not true is different from the one we find in $\underline{T}$. A variant of this solution is indexing the predicate of truthfulness with an assumption that the change of extension of the predicate produced by the indicated level does not result in the change of meaning of the predicate.[28]

Diagnosis 3. Convention $\underline{T}$ is incorrect, since the predicate "is true" is only partly defined, as there are propositions, which cannot be given any logical value. In such cases, we speak of *truth-value gaps*. By assumption, $\underline{T}$ refers to all propositions. Yet, some of them have no logical value. This means that $\underline{T}$ alone is an example of a proposition, which has no logical value, when we put a proposition with no logical value in place of p. Naturally, this proposition is the liar sentence, for otherwise $\underline{T}$ could be applied to liar's proposition, and that would cause a contradiction. However, there is a trap in this approach, called *revenge problem*. It follows from it that L is not true, for it could only be true by virtue of $\underline{T}$. Accordingly, it is precisely as predicated by proposition L, i.e. L is true by virtue of $\underline{T}$—contradiction. According to Martin, every doctrine of truth-value gaps refers to the revenge problem.

Diagnosis 4. Proposition L is incorrect, because it is a proposition of natural language, and natural languages have no concept of truth. The only languages that have concepts of truth, falsity, un-truth and un-falsity are these, which possess the following reservation: propositions that include the above mentioned truth concepts do not belong to those languages. Thus, there is a separation of subjective language from the language about subjective language.

4.2.1.1 Tarski's Proposal

Tarski himself supported the fourth type of solutions. He presented his proposal in the same work in which he gave the above mentioned definition of truth. The test of correctness for the definition is Liar Antinomy, the analysis of which made Tarski introduce the above mentioned restriction[29]: "For every formalized language we can construct a formally correct and materially right metalingual definition of a true proposition by means of general logical expressions, expressions of a language itself, and morphological terms of the language only, on condition, however, that the metalanguage is of a higher level than the language that is analyzed. If the level of the metalanguage is equal, at best, to the level of the language, such a definition cannot be constructed." Alonzo Church (b. 1903), in his 1976 work *Comparison of Russell's Resolution of the Semantical Antinomies with That of Tarski* showed that

[21] Tarski [32].

[22] Tarski [32], p. 19.

[23] Tarski [32], pp. 17–19. Tarski limited the application of this definition to the languages, in which the objective level is separated from the upper level. Tarski [32], pp. 165–166.

[24] Naturally, the sentence $\underline{T}$ is understood as generally quantified with respect to p. Thus, $\underline{T}$ has actually the form: $\forall p\ (v(p) = 1 \text{ if and only if } p)$.

[25] Tarski [32], p. 18. This form of the definition is known as "Tarski biconditionals".

[26] Feferman [9], pp. 240–241.

[27] Martin [20], pp. 3–4.

[28] Every object, predication of which produces a true sentence, belongs to *extension* of a predicate. Every object, predication of which produces a false sentence, belongs to *anti-extension* of a predicate. If a predicate is an n-argument one, its extension contains ordered ns, predication of which produces true sentences, and its anti-extension contains ordered ns, predication of which produces false sentences. In this chapter, extension is called positive extension, while anti-extension is called negative extension.

[29] Tarski [32], p. 165.

the original form of the branched types' theory, proposed by Bertrand Arthur William Russell (1872–1970) in 1908,[30] that liberates the set theory from Russell's Antinomy, is a special case of Tarski's proposal to divide language into levels.[31]

What follows from Tarski's analyses is that the concept of truth is inexpressible in a natural language, understood as universal, i.e. containing languages of all possible levels beside the main one. Accordingly, talking about truth in a natural language seems senseless.[32] Consequently, there cannot be a predicate of the language that could make it possible for a sentence, in which it is the main functor, to express its own truth. This opinion is shared e.g. by Prior in *Correspondence Theory of Truth* (1967) and Wallace in *On the Frame of Reference* (1972).[33] Such a conclusion, however, is strongly non-intuitive. Following Tarski's opinion, to operate with a concept of truth in a natural language, one has to divide a natural language into levels of language, separated through applicability of definition $\underline{T}$. Other proposals suggest weakening of convention $\underline{T}$. They include also the ones, which assume truth-value gaps.

4.2.1.2 Parsons' Proposal

Tarski's idea to take into account existence of languages with various levels inspired Charles Parsons, who presented his solution of Liar Antinomy in a 1974 paper *The Liar Paradox*.[34] Martin rightly notes that Parson's approach represents another view. Indeed, the elimination of contradiction between L and $\underline{T}$ is achieved for the price of accepting the hierarchical form of proposition L. This new, more complex form of L makes it possible to distinguish a proposition and a statement expressed by the proposition. Definition $\underline{T}$ remains in the same form, but it takes into account the 'two-level' character of the proposition. Adding definition $\underline{T}$ to so reformed liar's proposition does not lead to contradiction, for it turns out that in its new form liar's proposition does not express any statement. Thus it is impossible to apply definition $\underline{T}$ here.

Parsons starts the presentation of his approach with the truth analysis of two propositions that are versions of liar's proposition:

(1P) The sentence written in the upper left-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971 expresses a false proposition.
(2P) The sentence written in the upper right-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971 does not express a true proposition.

[30] Russell [30].
[31] Church [7], pp. 301–302.
[32] Field [10], p. 374.
[33] Prior [26] and Wallace [33].
[34] Parsons (1974).

Let us assume that precisely two sentences were written at that time on the blackboard in question: (1P) in the upper left-hand corner and (2P) in the upper right-hand corner. Moreover, let "A" be the abbreviation for (1P), and "B" for (2P). Then:

(3P) $A =$ the sentence written in the upper left-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971.
(4P) B $=$ the sentence written in the upper right-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971.

Considering conditions for truth Parsons analyses the first possibility, assuming that '"p" expresses a true proposition $\leftrightarrow p$' and '"p" expresses a false proposition $\leftrightarrow \neg p$'. Then, by virtue of standard propositional logic we can infer a following proposition from these assumptions: 'p' expresses a true proposition $\vee$ 'p' expresses a false proposition. Thus we get: 'p' expresses a proposition. This means that every proposition expresses a statement, which is in contradiction with Parson's idea. Consequently, he proposes replacing the earlier assumptions with new ones, which describe truth in such a way that not every proposition has to express a statement:

(5P) $\forall x$ ((x is a proposition $\wedge$ 'p' expresses x) $\rightarrow$ x is true $\leftrightarrow p$); and

(6P) $\forall x$ ((x is a proposition $\wedge$ 'p' expresses x) $\rightarrow$ x is false $\leftrightarrow \neg p$).

Let us assume now that x is a statement and that A expresses x. Directly from (5P) we get:

x is true $\leftrightarrow$ the sentence written in the upper left-hand corner of the blackboard in Room 913-D South Laboratory, The Rockefeller University, at 3:15 p.m. on December 16, 1971 expresses a false proposition.

With respect to (3P) we get:

(7P) x is true $\leftrightarrow A$ expresses a false proposition.

Let us assume that x is not true. Then,

$$\exists x\ (x \text{ is a sentence} \wedge \neg(x \text{ is true}) \wedge \text{'}A\text{' expresses } x),$$

so A expresses a false proposition. Consequently, by virtue of (7P), x is true. Since x is arbitrary:

(8P) $\forall x$ (x is a proposition $\wedge$ 'A' expresses x) $\rightarrow$ (x is true); hence

(9P) $\neg\exists x$ (x is a proposition $\wedge$ 'A' expresses x) $\wedge \neg$(x is true).

Let us assume now that y is a proposition and, moreover, that A expresses y. By virtue of (8P), y is true. Yet we get from (7P) that A expresses a false proposition, which contradicts (9P). This means that it has been proved that no proposition is expressed by A. Consequently, (1P) does not express any proposition. Similarly, (2P) does not express any proposition. This results in falsity of antecedents of main implications in conditions (5P) and (6P). Thus it is impossible to infer any contradiction. Liar Antinomy disappears.[35]

4.2.1.3 Burge's Proposal

The approach proposed by Burge, in his 1979 paper *Semantical Paradox*,[36] resembles Parson's proposal and is another attempt that can be seen as belonging to the second category. The key role here is also played by Tarski's idea of hierarchisation that separates the levels of truthfulness in the language with respect to the language order. Burge, however, does not apply it to liar sentence, but to the definition of truth $\underline{T}$. Truth is expressed through a class of indexed truth predicates T_i. Naturally, accepting such assumptions makes it impossible to infer a contradiction from liar sentence and Tarski's convention. Indeed, liar sentence rejects its truthfulness but this truthfulness must have a particular level k. Accordingly, the sentence is not *true*$_k$. At the same time, applying definition $\underline{T}$ to that very sentence of the liar makes it possible to state that it is true on the level $k+1$, i.e. liar sentence is *true*$_{k+1}$.[37] Moreover, Burge shows differences between his proposal and the one that uses truth-value gaps. In his case, no correctly formed sentence can have logical value in the absolute sense. True, a sentence can be neither true$_i$, nor false$_i$, but there must be such a k for which the sentence is either true$_k$, or false$_k$.[38] According to Martin, such a dependence excludes the possibility of a revenge problem.[39] Burge uses the concept of *pathologicality*: a sentence is *pathological* when it is intuitively empty or leads to a paradox.[40] Naturally, pathologicality of a sentence is relative to a given level k, which results in *pathologicality*$_k$ or *nonpathologicality*$_k$. A sentence may be pathological$_k$ and then it is false$_k$, at the same being *nonpathological*$_n$ for some $n > k$, and so a particular true$_n$. Burge introduces pathologicality in three ways, proposing more or less formal definitions of the concept and thus develops three constructions, which overcome the Liar Antinomy.[41] It is worth noting that Burge's understanding of pathologicality of a sentence preserves the law of excluded middle in every

[35] Parsons (1974), pp. 15–16.
[36] Burge [5].
[37] Burge [5], pp. 93–95.
[38] Burge [5], pp. 96–97.
[39] Martin [20], p. 7.
[40] Burge [5], p. 102.
[41] Burge [5], pp. 101–105.

construction, for every level i—for every i every sentence is either true$_i$, or false$_i$. What is especially interesting is Burge's final remark, where he admits his considerations assume an idealization through ignoring the phenomenon of vagueness when assessing the logical value of sentences.[42] It is clear that in many non-mathematical cases valuation of a sentence faces an insurmountable problem of vagueness. Burge's accurate remark becomes obvious in the light of the analyses of the phenomenon of vagueness, presented in the last chapter. This question apparently more serious than Liar Antinomy is given due analysis in the chapter *Ontogical Paradoxes*.

4.2.1.4 Martin's and Woodruff's Proposal

The third position is represented by the proposal of Martin and Woodruuff. Their approach, presented in a 1975 paper *"True-in-L" in L*,[43] makes use of Kleene's three valued logic.[44] The third value, traditionally marked by u (*unknown*), is essentially either truth or falsity, but it is unknown for us, which of them it actually is, so it cannot be substituted by either truth or falsity. It can be seen then that using u value makes it possible to express the truth-value gap. The formal language of S is the traditional language of logic of predicates with a difference consisting in individual variables being relative to a finite number of sorts, expressed with indices 1, …, k. Consequently, the domain of interpretation is a sum of domains, each of which corresponds to another sort of variables: $U = U_1 \cup \ldots \cup U_k$; for any $i \in \{1, \ldots, k\}, U_1 \neq \emptyset$. Moreover, language S contains a truth predicate T, which plays the key role in specifying partial order on the class of valuations interpreting language S. Valuations ascribing one of three logical values to sentences $P(a_1, \ldots, a_n)$, where P is a n-argument predicate, are partially ordered in such a way that $v_1 < v_2$, if a following condition is met: if v_1 ascribed the value of truth (falsity) to a sentence $T(a_1, \ldots, a_n)$, then v_2 ascribed to the same sentence $T(a_1, \ldots, a_n)$ the value of truth (falsity), too. Let us assume now that for a valuation v of the domain D and for some $j \in \{1, \ldots, k\}$, $S = U_j$. Then, v *partially represents truth for a language* S if and only if two conditions are met for some sentence $A \in S$: (1) if $vT(A) = 1$, then $v(A) = 1$, and (2) if $vT(A) = 0$, then $v(A) = 0$. In Martin's and Woodruff's construction, there is a possibility, then, that on a certain level sentences with a truth predicate T have no logical value of truth or falsity, so they have the value u. In turn, sentences that do not contain this predicate can be valued according to intuitions—some can be accepted as true,

[42] Burge [5], pp. 114–115.
[43] Martin and Woodruuff [21].
[44] As is known, in this logic, apart from truth (1) and falsity (0) there is the third logical value (u), which intuitively corresponds to the unknown logical value. Thus, if $v(p) = 1$ and $v(q) = u$, then the logical value of the conjunction $p \wedge q$ (i.e., $v(p \wedge q) = u$) is unknown, since everything depends on what is the unknown to us value of the sentence q: if q is true, then the conjunction is true, but if q is false, then the conjunction is false.

some other as false, as long as they meet conditions of Kleene's table of logical values. At the same time, the higher is the level, greater is the number of sentences with a logical value different from u. Thanks to the partial order specified on the set of valuations, Martin and Woodruff can use Kuratowski-Zorn Lemma on existence of maximal element in partially ordered set, in which every chain has an upper bound. This maximal level is the aim of the whole construction for no longer specifies only a partial representation of truth, but a complete one, called *representation*, which entails simultaneous meeting of two conditions: (1) $vT(A) = 1$ iff $v(A) = 1$, and (2) $vT(A) = 0$ iff $v(A) = 0$.

Naturally, a reasoning that leads to a contradiction, which is characteristic for the Liar Antinomy, cannot be performed on the language level S, i.e. the jth level of Martin's and Woodruff's model interpreting truth. On the jth level, sentences containing predicate T are valued in u, i.e. they are neither true nor false.[45]

4.2.1.5 Kripke's Proposal

Kripke's work, the *Outline of a Theory of Truth*,[46] also published in1975, presents the same approach to the liar sentence and definition $\underline{T}$ as the proposal of Martin and Woodruff, even though the two papers were written independently. It may accepted that Kripke's proposal, as well as those of Skyrms, Herzberger, or Gupta, which are modeled on it, belong to the third approach.

With a given domain D, Kripke interprets one-argument predicate $P(x)$ with help of a pair of disjunct sets (E_1,E_2), being extension and anti-extension of predicate P, respectively.[47] Predicating P of an object from E_1 is a true sentence, predicating it of an object E_2 is a false sentence, and predicating it of an object from outside $E_1 \cup E_2$ is an unspecified sentence.[48] The distribution of logical values is exactly the same as in Kleene's three valued logic. Moreover, Kripke proposes to extend the language S with help of a new, one-argument predicate T, whose interpretation is, naturally, a pair R_1,R_2), which is its extension and anti-extension. (Naturally, outside the set $R_1 \cup R_2$ the predicate T is unspecified. Let S be the interpretation of the language S, and $S(R_1,R_2)$, the interpretation of the language S extended with T. The new interpretation is an extension of the old one in the sense that (R_1,R_2) is an interpretation of T while the remaining predicates have the same interpretations as in S. Let R_1' be the set of true sentences, and R_2' a set of false sentences in $S(R_1,R_2)$. If T is understood as expressing the truth of sentences from the language S, which already contains the predicate T,

[45] Martin [20], p. 7.

[46] Kripke [16].

[47] If P is an unspecified predicate, neither its extension nor anti-extension is a set. As can be seen, Kripke tacitly assumes this idealization, mentioned by Parsons. According to this idealization, extension and anti-extension is assigned in a specified way in case of every predicate, also the unspecified one—cf. *Parsons's Proposal* above.

[48] Kripke [16], p. 64.

then $R_1 = R_1'$ and $R_2 = R_2'$. This means that for any sentence A of the language S: 1. A satisfies T iff A is true and 2. A falsifies T iff A is false. The pair (R_1,R_2), which meets his condition, is called by Kripke a *fixed point*. In the next step, Kripke proves the existence of fixed points by constructing one of them. Using Kuratowski-Zorn Lemma he proves that every fixed point can be extended to a maximal fixed point.

Similarly as in Martin and Woodruff, at the first stage of Kripke's construction, i.e. its first interpretation, the predicate T is completely unspecified—its extension and anti-extension is an empty set. Each consecutive extension of the interpretation somehow narrows down the description of the predicate. Each consecutive interpretation extends not only the set of true sentences but also the set of false sentences. The end of those extensions is determined by the maximal fixed point. Naturally, the solution of the Liar Antinomy is the same as in Martin's and Woodruff's construction. Kripke shows also the mutual equivalence between Tarski's language hierarchy and his own model.

Kripke's idea to use partial valuations was applied by Brian Skyrms in his 1982 paper *Intensional Aspects of Semantical Self-Reference*, published in 1984.[49] Another form of Kripke's model is the construction by Herzberger. He presented his attempt to solve the Liar's Paradox in a 1982 paper *Notes on Naive Semantics*.[50] Yet another version of Kripke's approach is the proposal of Anil Gupta, analysed below. We remind this construction because of its classical character, since it is based on traditional models for the first order logic.

4.2.1.6 Gupta's Proposal

In his 1982 paper *Truth and Paradox*,[51] Anil Gupta gave up the inductive way of specifying the predicate, which expresses the truth of a sentence, characteristic of the two above constructions and proposed a variant of the rule for establishing the truth of sentences remaining within the standard set by Tarski, who related truth to the levels of language. Gupta based his approach on four assumptions:

1. The language has, aside from the concept of truth, none of the other complicating factors: indexicals, vagueness, ambiguity, intensional constructions, truth-value gaps, etc. Gupta assumes that the language he considers is a classical first-order quantificational language.
2. The theory of meaning of a class of sentences gives the assertibility conditions of sentences belonging to that class. This means that for every sentence of the class the theory tells us the conditions under which the sentence is correctly assertible and when it is not correctly assertible. In other words, the theory of meaning provides truth conditions for sentences. Gupta stresses here that the

[49] Martin (1982), pp. 119–131.

[50] Herzberger [15].

[51] Gupta [14].

assumption has nothing to do with the argument between realists with anti-realists concerning the status of semantics.
3. Truth is (expressed by) a predicate.
4. The only objects truth can be predicated about are sentences.

The assumptions accepted here define the space, in which Gupta intends to specify the truth conditions for sentences of language S. He notes that in case of sentences without the truth functor T these conditions are defined by standard, classical models. Naturally, it is the sentences with the truth functor T that are important for Gupta's considerations of the problem. He notices, however, that not all sentences containing the predicate T are problematic, i.e. paradoxical. An example of a sentence causing no interpretation problems is, e.g. the Cretan's sentence that all Cretans lie.[52] As we have already shown, this sentence is false. Similarly, a sentence asserting its truth raises hardly any doubts. So Gupta narrows down his search for an adequate way of interpreting sentences to the ones he himself defines as paradoxical. He assumes that language S equipped with a tool for creating names for all sentences. Then, if "b" is a name of a sentence "b is not true", then by virtue of Tarski's convention we obtain:

"b is not true" is true if and only if b is not true,

which clearly leads to a contradiction if only we use the classical interpretation of language S. From this point of view, Tarski's approach, which evidently is an inspiration for the following construction, seems justified.[53]

Gupta uses the analysis of the problem of self-reference for an introduction to his solution of Liar Antinomy. His language S includes, beside the predicate T, two parentheses 'and', the application of which is limited to closed formulae, i.e. sentences. So, if $(\forall x)(Fx \rightarrow Tx)$ is a sentence of language S, then '$(\forall x)(Fx \rightarrow Tx)$' is a name of that sentence.[54] Let $M = \langle D, I \rangle$ be any model of language S. Naturally, D is the domain of the model M, and I its interpretative function. Moreover, set Zd of sentences in language S is included in D and[55]:

(i) I assigns to a quotation name 'A' the sentence A.
(ii) If a is not a quotation name than $I(a) \notin Zd$.
(iii) If F is an n-place predicate and $d_i \in Zd(1 \leq i \leq n)$, then $\langle d_1, \ldots, d_j, \ldots, d_n \rangle \in I(F)$ iff for all $d_i' \in Zd, \langle d_1, \ldots, d_i', \ldots, d_n \rangle \in I(F)$.
(iv) If f is an n-place function symbol then the range of $I(f)$ does not contain any sentences. Further if $d_i, d_i' \in Zd(1 \leq i \leq n)$ then, $I(f)(d_1, \ldots, d_i, \ldots, d_n) = I(f)\big(d_1, \ldots, d_i', \ldots, d_n\big)$.

[52] Gupta [14], p. 181.
[53] Gupta [14], pp. 181–183.
[54] Gupta himself admits that in this way he makes use of symbols characteristic for the way of distinguishing the expressions of the language and those of metalanguage [14], p. 184f.
[55] Gupta [14], pp. 180–181.

Gupta shows that the model which satisfies such conditions may be extended to the so-called *standard model*, in which Tarski's definition no longer leads to paradoxical consequences. $M' = \langle D', I' \rangle$ is a *standard extension* of the model M if and only if $D = D'$ and I' is a function I, with the only difference being that I' coordinates a subset of set D to the predicate T. We shall say then that M' is a model generated by M and $I'(T)$, symbolically $M' = M + I'(T)$. Later, Gupta proposes that the name "standard model" mean a standard extension of a model of the language S.[56]

In his construction of a standard model for a given model of language S that satisfies conditions (i)–(iv), Gupta uses the concept of a "set of true sentences of language S at ordinal level α for the set U", in short $Tr(\alpha, U)$. To define it, he uses transfinite recursion:

Let M be a model of language S, satisfying conditions (i)–(iv)*. Moreover, let $U \subseteq D$. Then,*

(Tr i) *If $\alpha = 0$ then $Tr(\alpha, U) = U$.*
(Tr ii) *If $\alpha = \beta + 1$ then $Tr(\alpha, U)$ is a set of sentences true in the standard model $M + Tr(\beta, U)$.*
(Tr iii) *If α is a limit ordinal then* $Tr(\alpha, U) = \{d : \exists \beta < \alpha (d \in \cap_{\beta \leq \gamma < \alpha} Tr(\gamma, U))\}$.

One can clearly see that the solution of Liar Antinomy is found in the introduction of successive levels of true sentences: a sentence asserting the truth of a sentence from level α belongs to the set of true sentences from levels higher than α. Thus only the standard model eliminates the truth-value gaps of a given model of language S. It is no wonder, then, that for the model M of language S and for $U \subseteq D$, $M + Tr(\omega, U)$ is that standard model of language S, in which all substitutions of Tarski's definition are true, and so they no longer lead to paradoxical consequences.[57] If b is a name of a sentence $Tb \vee \neg Tb$, then, according to Gupta's concept of truthfulness of sentences, the substitution of Tarski's definition $T(\text{'}Tb \vee \neg Tb\text{'}) \leftrightarrow Tb \vee \neg Tb$ is a true sentence, and so the sentence $Tb \vee \neg Tb$ is also true.[58] Then there is no paradox, for the sentence $Tb \vee \neg Tb$ is true not being false.

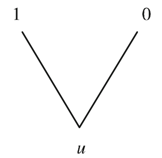
4.2.1.7 Feferman's Proposal

In his 1982 paper *Towards Useful Type-Free Theories, I*,[59] Feferman presented an approach based on the idea of truth-value gaps, which, however, in no way realized Tarski's idea of language levels hierarchy and in this sense was different from the ones presented above. Equivalence between semantic paradoxes, especially

[56] Gupta notices that a possibility of extending the model M to a standard model is guaranteed also by conditions weaker than (i)–(iv).
[57] Gupta [14], p. 186.
[58] Gupta [14], p. 191.
[59] Feferman [9].

Liar Antinomy, and paradoxes of na set theory, especially Russell's Antinomy, is a well known fact. Feferman rightly observes that Russell's type theory has its semantic equivalent in Tarski's postulate that language be divided into successive orders. This postulate is realized in many ways; the more important of them have been cited above. Feferman's intention is to obtain such a solution that could be a semantic equivalent of those approaches in set theory which depend on its non-contradictory axiomatic character. The key concept of the new approach is the so-called *partial predicate*.

Let M be a set. $R^{\sim}$ is a partial, k-argument predicate on M, for $1 \leq k \leq \omega$, if it is a partial function mapping M^k into the set $\{1,0\}$ two logical values: truth and falsity. The introduction of u, the third logical value of unspecifiedness makes it possible to identify this partial function with a complete function mapping M^k into the set $\{1, u, 0\}$. In both cases, Feferman uses the same symbol "$R^{\sim}$". Partial ordering of the three logical values



makes it possible to use this order for specifying the order of predicates: $R_1^{\sim} \leq R_2^{\sim}$ if and only if any $m_1, \ldots, m_k \in M, R_1^{\sim}(m_1, \ldots, m_k) \leq R_2^{\sim}(m_1, \ldots, m_k)$. Then, $R_1^{\sim}$ is a subfunction of function $R_2^{\sim}$. Naturally, if $R_1^{\sim} \leq R_2^{\sim}$ and $R_2^{\sim} \leq R_1^{\sim}$, then $R_1^{\sim} = R_2^{\sim}$. Now let $M_0 = (M, \ldots)$ be an ordinary structure interpreting language S_0; the structure is ordinary because its function is not partial. $M = (M_0, R_1^{\sim}, \ldots, R_n^{\sim}, \ldots)$ is then a partial structure, because of the partial character of predicates $R_1^{\sim}, \ldots, R_n^{\sim}, \ldots$ Feferman limits his further considerations to simple structures $M = (M_0, R^{\sim})$ with one partial predicate $R^{\sim}$. $(M_0, R_1^{\sim}) = M_1 \leq M_2 = (M_0, R_2^{\sim})$, if $R_1^{\sim} \leq R_2^{\sim}$.

Let K be a class of structures $M = (M_0, R^{\sim})$, ordered by the relation $\leq$, for a certain structure M_0. Every structure $M = (M_0, R^{\sim})$ is coordinated by the operator Γ with a new structure $\Gamma(M) = (M_0, \Gamma(R^{\sim}))$. Γ is a monotonic operator: $M_1 \leq M_2$ implies $\Gamma(M_1) \leq \Gamma(M_2)$. Later, Feferman proves a theorem, according to which for any monotonic operator Γ and any structure $M \leq \Gamma(M)$ there is a smallest structure M^*, such that $M \leq M^*$ and $\Gamma(M^*) = M^*$.[60]

Feferman's intentions are clear. The accepted ordering of three logical values makes every partial predicate $R^{\sim}$ a 'sub'-predicate, maybe also a partial one, which is formed from $R^{\sim}$ when at least one instance of the value u is substituted with the value 1 or 0. Such a gradual specification of the partial predicate is secured by the monotonic operator Γ. A sequence of partial structures that are

[60] Feferman [9], pp. 250–252.

formed in this way is closed by a structure, which cannot be specified by operator Γ, since it is already completely specified: its predicate is no longer a partial predicate. Naturally, this more general theory is applicable to structures, in which the partial, gradually specified predicate is the truth predicate. Thus, its originality notwithstanding, Feferman's proposal is apparently situated in the within the series of solutions originated by Martin and Woodruff, and Kripke.

Most of constructions presented above accept an assumption that the liar sentence has no logical value of truth or falsity and ascribe to it some third logical value, thus treating the liar sentence L as a logical one. There is, however, an opinion denying that sentence L has the logical value of truth or falsity, or any other, and altogether refuses to accept L as a sentence in logical sense. In Poland, this standpoint is presented for instance by Gumański in his 1992 paper *Logical and semantical antinomies*.[61] Regrettably, such standpoint practically stops, or at least limits, some research approaches to the Liar's Paradox, especially the ones focused on understanding truth and inference.

Outside Martin's classification, we find the proposal of Graham Priest, which can be classified in Feferman's second group of solutions, which weaken logic. Weakening of logic in such a way that it makes it possible to accept contradictions at the same time avoiding making the consequences of accepted sentences trivial is a fundamental idea of contradiction tolerating logics, known as paraconsistent logics.

4.2.1.8 Priest's Proposal

Martin's classification of four types of solutions for Liar Antinomy does not include *dialetheism*, an opinion, which accepts the existence of sentences that true and false at the same time.[62] As we have shown above, for the liar sentence L the equality $L \leftrightarrow \neg L$ is true. Yet, when the principle of identity is applied to the liar sentence, $L \leftrightarrow L$, we get $L \leftrightarrow L \wedge \neg L$. Consequently, the liar sentence is false, for it is equivalent to a contradictory sentence. However, the principle of identity can be applied to the negation of the liar sentence: $\neg L \leftrightarrow \neg L$. This means that we have another equivalence: $\neg L \leftrightarrow L \wedge \neg L$, which proves the negation of the liar sentence. Consequently, the liar sentence L is true and false at the same time and so is its negation. Dialetheism's creator and advocate is Graham Priest.[63]

[61] Gumański [13].

[62] Priest, e.g. [23–25].

[63] This reasoning does not have to be accepted. $L \leftrightarrow \neg L$ is, namely, a conjunction of two implications: $L \rightarrow \neg L$ and $\neg L \rightarrow L$. From the first implication follows the rejection of the truthfulness of L, from the other, the rejection of the truthfulness of $\neg L$. Such an interpretation leads to the justification of a proposition that the liar sentence L is an example of a sentence that has no logical value, so it illustrates the truth value gap. Yet, in the light of the first reasoning, which proves that L is a sentence that is true and false at the same time, it is clear that the argument leading to a conclusion that L has no logical value or is not based on the classical logic.

Naturally, though such conclusion was reached in accordance with the rules of classical logic, it is impossible to reconcile the stance that accepts it with any logic that upholds the principle of contradiction and Duns Scotus's Law. For this reason dialetheism is found in paraconsistent logics, which suspend the principle of contradiction and weaken Duns Scotus's Law thus tolerating contradiction. Paraconsistent theories are not trivial even when they include a sentence and its negation. Liar Antinomy has a special importance for dialetheists, because the liar sentence is apparently the only example of a sentence that is true and false at the same time. Naturally, this view goes far beyond the conclusions of the liar sentence, for the only thing that can be inferred about it is that the sentence is not true if it is true and vice versa. From this point of view, dialetheism, seen as a solution of Liar Antinomy, seems to go too far. What is worse, it is difficult to find another, truly philosophical justification of it apart from the liar sentence.[64]

Though technically advanced, the above proposals do not seem specially attractive, mostly for the reason of their artificial and non-intuitive character. Can the introduction, more or less manifest, of language hierarchies explain this problem, admittedly great yet surprisingly simple in its structure? Isn't the truth only one? Can introduction of a third logical value, no matter how we understand it, be any solution if also the following sentence, analogous to the liar's one, is paradoxical: "This sentence is false or has a third logical value"?[65] Finally, is acknowledging that there are sentences in logical sense, e.g. the liar's one, which have the logical value of truth and falsity at the same time, not an excessively bold move? Can the sense of classical logic, extraordinary and exceptional as it is, be actually undermined in such a simple way?

The next proposal, presented by Barwise and Etchemendy in their famous 1987 book *The Liar. An Essay on Truth and Circularity*,[66] gives a new solution to Liar Antinomy, different from all other ones.

4.2.1.9 Barwise's and Etchemendy's Proposal

This approach belongs to the so-called situation theory, which takes into account the context of utterances. According to Barwise and Etchemendy, neither the self-reference of the utterance, nor improper understanding of truth is the source of Liar Antinomy, but neglecting the context.

[64] Sometimes it is pointed out that such justifications are provided by sentences with vague terms. This opinion, however, seems difficult to accept. Cf. this chapter.

[65] If this sentence is true, it is false or has the third logical value, so it is untrue. If it is false, then things are as it predicates, so it is true and thus not false. If it has the third logical value, things are as it predicates, so it is true and thus does not have the third logical value. Naturally, this reasoning is yet another versions of the Revenge Paradox, mentioned above.

[66] Barwise and Etchemendy [2].

In their analysis of the liar sentence, Barwise and Etchemendy start with the traditional distinction between a sentence and a proposition.[67] As a sequence of symbols, graphic or acoustic, a sentence is neither true nor false. Sentences do not fall under the criterion of truth and falsity, just like material objects, e.g. a pencil, table, etc. Accordingly, if a person a says: $L =$ "What I am stating here is not true", the expression "What I am stating here" refers to a proposition stated by the sentence. Thus the sentence L uttered by a refers to a proposition p. This means that by uttering sentence L, the person a states a proposition "p is not true". However, since the proposition stated by a is marked with a symbol "p", p and "p is not true" are the same:

$$p = [p \text{ is not true}] \tag{4.1}$$

In the situation theory, assessing the truth of the proposition is dependent on the context of an utterance. Often a person uttering a proposition is unaware that his utterance refers to the context of the situation, which determines the truthfulness of the proposition. Still, regardless of the person's awareness, the context is the key to the truthfulness of the proposition he utters. Let us mark the context of uttering sentence L by the person a by a symbol "c". The utterance: "What I am stating here" refers to a proposition that p is true in context c. In formal notation of situation theory we put it: $c| = p$. Since it has been noticed that the utterance "What I am stating here" refers also to a proposition p, then p must be the same as $c| = p$. Thus we get another equality:

$$p = [c| = p] \tag{4.2}$$

Let us consider two cases now: one when proposition p is true and another one when proposition p is not true.

1. Let us assume that proposition p is true. Then, p is this very true context, which states the truth of the proposition p, so we get:

   $$c = p. \tag{4.3}$$

   Yet, in the light of (4.1), proposition p is at the same time the falsehood of proposition p. This means that:

   $$c| = [p \text{ is not true}], \tag{4.4}$$

   which gives us a contradiction. Indeed, by virtue of (4.3), p is a true proposition in the context c, yet it follows from (4.4) that p is not true in the same context c. To explain the problem Devlin uses the example of seasons in various parts of the globe: true, it is summer and it is not at the same time, but summer is in the USA, while it is not in Australia. So, the difference of

[67] We summarise Barwise's and Etchemendy's proposal following its clear and elegant presentation by Devlin [8], pp. 338–343.

context, in this case the USA or Australia, makes a given proposition acceptable in one context and inadmissible in another. It is not so, however, when the proposition analyzed above in the very same context turned out to be true and not true. It remains to analyze the other instance of falseness of proposition p.

2. Let us assume that p is not true. According to an earlier assumption, c is the context of the utterance of person a. Thus we obtain a condition (4.4), which is contradictory with respect to equality (4.1). The conclusion is easy: c cannot be the context of person's a false proposition p. Devlin supports the recognition of this conclusion with the following example: if someone says that June is a winter month in country X, then the USA cannot be country X. He cautions us, however, not to jump to the conclusion that Australia is country X; what is only certain is that the USA cannot be country X.

Finishing his presentation of Barwise's and Etchemendy's solution of Liar Antinomy, Devlin states that the antinomy disappears when we pay proper attention to the context. When person a says: "What I am stating here is false", he is uttering a proposition implicitly referring to a certain context c, i.e. to the context in which the sentence was uttered. If the proposition is true, it is true in context c, which, as it turns out, leads to a contradiction. Consequently, the proposition must be false. Yet c cannot be the context, in which it is stated that the proposition is false, since the supposition that c is this context also leads to contradiction. Person a who says (in context c) "What I am stating here is false", utters a false statement. The fact that it is false, however, cannot be stated in the same context c. This seems a weird conclusion; similarly, it is most unusual for someone to utter the original sentence, on which the liar's "paradox" is based.[68] The weirdness of this conclusion, noticed by Devlin, does not have to be explained by the strangeness of the liar sentence. The sentence is strange indeed. The approach to it, however, even though it is illustrated with appropriate and persuasive examples does not manage to keep the analogy to the problem of the liar sentence until the final conclusion. What is unusual is that falsity of proposition p cannot be stated in the context c, while the fact that June is not a winter month can actually be stated in the USA, i.e. in the context, in which the proposition: "June is a winter month" is false. This means that although the postulate to take into account the context of an utterance in order to properly assess the truthfulness of the proposition stated in it is well justified, its application by Barwise and Etchemendy is rather faulty in the case of the liar sentence.

The opinion that context o fan utterance influence the logical evaluation of a proposition seems to be well justified. Moreover, one can suspect that failing to take this context into account should make it impossible to assess the logical value of a proposition. Our own proposal of a solution for the Liar Antinomy seems to justify both claims. Since our approach involves a fairly simple analysis on the level of sentence calculus, the distinction between a sentence and a proposition stated in it is not necessary.

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[68] Devlin [8], p. 342.