

The Birth of Set Theory

At Halle, Cantor settled down to the mediocre life of an academic at a second-tier institution. Here, no great ideas were discussed at the mathematics department meetings, and there were no wonderful colloquia with great speakers on exciting new research topics.

By this time, Cantor had married Vally Guttmann, a friend of his sister, who came from a Berlin Jewish family. The two had met in Berlin and were married a few years after moving together to Halle, in 1875. They began to raise a family on Cantor's modest government income in the provinces, which was significantly less generous than the compensation at the University of Berlin. But it was here in a small city in the German countryside that Cantor developed a complete mathematical theory all his own.

Mathematical research is best done within a community of good mathematicians. Research results can be shared and ideas exchanged, so that new theories can develop and thrive. Working in isolation is hard and slow going, and there are many blind alleys into which a mathematician can stray when

there is no possibility of sharing ideas with colleagues. But Georg Cantor was somehow able to produce one of the most amazing theories in the history of civilization—working alone.

Cantor brought with him to Halle some powerful, important ideas in mathematical analysis, which he learned in Berlin from Karl Weierstrass. At Weierstrass's course on function theory, one of the best mathematics courses ever given, Cantor had been exposed to unique concepts. Weierstrass expounded on the technique he had developed following Bolzano's ideas on limits and infinite sequences, and came up with an insightful definition of an old discovery: irrational numbers. The Bolzano-Weierstrass approach to irrational numbers was based on limits and on the property of spaces the two had independently discovered, which states that an infinite sequence in a bounded space a limit point in the space.

In the Bolzano-Weierstrass framework, built on the ideas of the ancient Greeks, we define an irrational number as a limit of rational numbers. The distance from the members of the sequence to the irrational number that acts as the limit point keeps getting smaller. This is a similar mechanism to the one intrinsic to Zeno's paradox about a person who can never leave a room. The person walks half the distance to the door, then half the remaining distance, and so on ad infinitum. Here, the door may be taken to represent some irrational number, a limit of an infinite sequence of rational-number steps.

While he was in Berlin, Cantor's work remained under the influence of the Weierstrass tradition. In Halle, he continued pursuing mathematical analysis along these same lines. Weierstrass, the old high school teacher whose brilliance

brought him a professorship, did not believe in publishing his results and did not even like to have his students take lecture notes at his course. One reason his work survives is that a Swedish student of Weierstrass, who later became an important mathematician in his own right and a good friend to Cantor, took careful notes and organized them once he was back in Stockholm. This student was Gösta Mittag-Leffler (1846–1927).

There is a persistent rumor among mathematicians that Mittag-Leffler is the cause for there not being a Nobel Prize in mathematics. It is said that Alfred Nobel developed an intense dislike for the mathematician, and to prevent the possibility that Mittag-Leffler would ever receive a Nobel Prize for his work in mathematics, Nobel decided that there should be no such prize. Mathematicians have taken their collective punishment (if indeed that is what it was) in stride, and to this day, the highest prize in mathematics is the Fields Medal, awarded in recognition of outstanding achievement in mathematics.

By all accounts, Gösta Mittag-Leffler was not only an outstanding mathematician, but also a decent human being. During Cantor's darkest hours, when no one would listen to his fantastic ideas about infinity, let alone publish them, Mittag-Leffler regularly published Cantor's work in his journal, *Acta Mathematica*. He had married a fabulously wealthy woman, and used her family's fortune to support mathematics. In the 1880s, he built a sumptuous villa in a suburb of Stockholm, which he bequeathed to mathematicians as an institute of mathematics. (Perhaps Nobel felt that Mittag-Leffler had done enough for mathematicians so they didn't

need a Nobel Prize.) Mittag-Leffler approached Georg Cantor early in his career, recognizing Cantor's great potential—something others had not fully appreciated—and had Cantor translate some of his early papers into French, bringing him recognition through their publication.

During his early years at Halle, Cantor had another good friend and benefactor, Richard Dedekind (1831–1916). Cantor needed *Acta Mathematica* as an outlet for his research activity because other doors had been closed to him since Kronecker and Kummer opposed his work. In Berlin, the only mathematician who supported Cantor's new work on infinity was Weierstrass. The admiration was mutual, and throughout his life, Cantor always spoke highly of Weierstrass and of his methods in mathematical analysis. A paper Cantor wrote in 1877 was almost rejected for publication in a journal edited by Kronecker. What saved the day for Cantor was Dedekind's intervention on his behalf.

Cantor first met Dedekind while vacationing in Switzerland in 1872. Dedekind was then a professor of mathematics at the Polytechnic Institute at Brunswick. The two became good friends. Dedekind was the last student of Gauss, and was born in Brunswick—Gauss's birthplace. As a young student, Richard Dedekind was interested in physics and chemistry, but when he enrolled at the University of Göttingen in 1850, he developed a love of mathematics. In 1852, at the age of 21, Dedekind received his Ph.D. in mathematics with a dissertation on integrals, written under Gauss's supervision. Dedekind taught at the Brunswick Polytechnic for fifty years, and apparently did not aspire to be promoted and move to a better school.

Dedekind's greatest contribution to mathematics was in the area of irrational numbers and their definition. Dedekind invented the concept of a *cut*. A cut on the number line separates all the rational numbers below a given number from all the rational numbers above it. If the cut itself does not define a rational number, then the number is irrational. For example, the square root of two, an irrational number, is defined by the cut that separates all the rational numbers whose squares are greater than two from all the rational numbers whose squares are less than two. Dedekind's work with irrational numbers, and hence with infinities, made him a natural ally of Cantor.

Dedekind lived so long that a newspaper article erroneously listed his death years before his actual passing. The obituary amused Dedekind so much that he wrote a letter to the paper describing how he had spent the day on which he supposedly passed away. "I passed this day in perfect health and enjoyed a very stimulating conversation with my honored friend Georg Cantor of Halle," he wrote.

What Dedekind did not tell the newspaper was that for the seventeen years prior to 1899, he had not heard a word from his "honored friend." The reasons for the break in communications were as complex as Cantor's personality. The two mathematicians were very close in the early days, and were bound together by personal empathy as well as mathematics. Both mathematicians were pioneers researching irrational numbers and infinity. Dedekind cuts were one approach to the problem of irrational numbers, and Cantor's wide array of methods of infinity were another. It was only natural that Georg Cantor, who felt lonely at Halle, was

eager for a position in Berlin or another major center; or, failing that, that a good mathematician—such as his friend Dedekind—join the faculty at Halle.

When a position of professor at a German university became available, a rigid bureaucratic procedure was followed to fill it. The faculty members at the institution with the expected vacancy had to draw up a list of candidates for the new opening. The list was rank-ordered by the faculty and submitted to the German Ministry of Education. The ministry then considered the list, and, assuming it approved the order of the candidates, offered the position to the first person named. If that person accepted the position, the process was over. But if the person declined, the ministry moved to the next person on the list. If all named persons declined, the faculty at the institution was asked to draw up a new list.

Georg Cantor was the professor of mathematics at Halle, and once a vacancy for a second professorship materialized, he was responsible for drawing up the list of candidates. Cantor, with the agreement of others at Halle, placed Dedekind at the top of the list, and sent it off to the Education Ministry. Politely, Dedekind declined the offer. As for his reasons for declining, Dedekind listed financial considerations—the Brunswick position allowed him more financial freedom. Cantor was crushed by his friend's decision, and for the next seventeen years the two mathematicians did not exchange a single letter. Perhaps this was the first hint of the emotional instability that plagued Cantor in later years. In the meantime, Cantor founded the mathematical theory of sets.

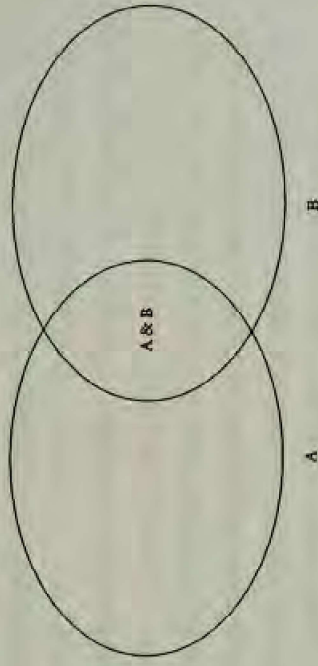
Georg Cantor came to the idea of infinity—actual infinity rather than the potential infinity of limits mathematicians had used for centuries—not from considering numbers directly, but from considering *sets*. Cantor started by looking at numbers, points on the line, that *converged* to a limit point. A limit point of a set is a point that is arbitrarily close to members of the set. He came to this line of thinking from the Weierstrassian idea of defining irrational numbers as limits of rational sequences. Then Cantor decided to look at the *set* of limit points of a given set of points. For example, the set of irrational numbers in an interval is the set of limit points of rational numbers in the interval. He then asked himself the question: “What would happen if I now looked at the set of all limit points of the set of limit points?” And he continued his reasoning onwards in this way. Defining a set P , he called the set of its limit points the *derived set*, P' . Now the set of limit points of the limit points was denoted P'' . From there he got P''' , P'''' , and then P''''' , . . . , $P^{(\infty)}$, Cantor was interested in the question: “When do the derived sets, if ever, become empty?” That is, was there a case in which no more limit points could be extracted? But Cantor was intrigued by the mere process of continuing to construct the derived sets, starting with infinite sets and continuing through more infinite sets. Ultimately, this conundrum led him to the key issues of his research—his life’s work on the nature of infinity itself. Cantor’s monumental work on infinity began with an investigation of the nature of sets. In addition to being the first person in history to deal with actual infinity, Cantor also became known as the father of the theory of sets.

To be sure, the elementary theory of sets had its origins long before Cantor. Any time we classify things we use set theory, in a rudimentary way. In his classic book *Naïve Set Theory*, Paul R. Halmos wrote: “A pack of wolves, a bunch of grapes, or a flock of pigeons are all examples of sets of things. The mathematical concept of a set can be used as the foundation for all known mathematics.”¹² And in fact, what we call today the *foundations* of mathematics is an area that includes set theory and logic. From things such as grapes or wolves—elements in a collection—the entire edifice of modern mathematics can be constructed. The great Italian mathematician Giuseppe Peano (1858–1932) defined the concept of a *number* using set theory in an ingenious way, as we will soon see.

Starting from the concept of a set, a collection of items or people, other sets can be constructed by using set operations. These operations correspond to the words *and*, *or*, and *not*. The union of two sets is the set whose members belong to one set *or* the other (or both). The intersection of two sets is the set of all elements that are members of one set *and* the other. And the complement of a set consists of all points that do *not* belong to the original set. Using the *or*, *and*, and *not* operations, we can define an interesting rule for sets that is useful in computer science (in the context of yes-no Boolean operations). The rule is due to Augustus De Morgan (1806–1871), who was one of the founders of the British Association for the Advancement of Science. The rule says:

$$\text{Not}(A \text{ or } B) = (\text{not } A) \text{ and } (\text{not } B)$$

This is demonstrated in the picture below. Look at the figure and convince yourself that the area that is outside the union of A , B is indeed the area that is *both* outside A and outside B .



One of the key elements of set theory is the famous *empty set* or *null set*: the set containing no elements at all. The empty set is everywhere—it is a subset of every set. Why? By contradiction: for the statement not to be true, we would have to exhibit a point that belongs to the empty set but does not belong to a given set, A . But since the empty set has no elements, no such element can be exhibited, so the statement is true.

Set theory leads invariably to great paradoxes. Since the foundations of mathematics consist of the theory of sets along with elements of mathematical logic, the paradoxes of set theory make the entire foundations of mathematics problematic. We may find it hard to believe that an elegant and seemingly very simple system of numbers and operations such as addition and multiplication—elements so intuitive that children learn them in school—should be fraught with

holes and logical hurdles. But such is mathematics. And when infinity is added to the mixture, the pitfalls multiply.

When Georg Cantor started developing his own set theory, he extended the results to infinite sets. Today, all of set theory deals with infinite sets. In developing his theory, Cantor made implicit use of a set of axioms.

Axioms are necessary, since they give mathematics a starting point. Without some sort of an axiom scheme, it would be impossible to construct any consistent theory with logic guiding its development and results. Among the mathematicians who put forward axiom schemes was Ernst Zermelo (1871–1956). In 1904, Zermelo constructed an alternative axiomatic system for set theory. This system later became known as the Zermelo-Fraenkel set theory (ZF), adding the name of the logician Abraham Fraenkel (1891–1965). But the ZF system, like its predecessors, was not to be free of paradoxes.

Even before Cantor weighed in with the overwhelming power of actual infinity, the system was plagued with inconsistencies. Sets were contemplated that were simply too huge, too incomprehensible to the limited human mind, to be amenable to taming by axiomatization.

But people tried nonetheless. And while the serious paradoxes of set theory would continue to beset humanity to our own time, the theory of sets was still useful in defining the concept of a number, as was brilliantly done by Giuseppe Peano. Starting in the 1880s, Peano began to study how numbers could be defined from the concept of a set using only set operations. He was helped in his work by a school of mathematics he had founded at the University of Turin,

including many gifted mathematicians such as Cesare Burali-Forti (1861-1931) and others.

Peano's foundations underlie all of mathematics. From them we can define not only the natural numbers but also the rational numbers, the real numbers, including irrational ones, complex numbers, and all of arithmetic. Peano's elegant derivation of a number system from sets is demonstrated below.

Peano defined zero as the empty set. One was then defined as the set containing the empty set. Two was the set that contained the empty set and the set containing the empty set. The process was then assumed to continue ad infinitum, defining every whole number. In terms of notation, the Peano number system is as follows:

$$\emptyset \{\emptyset\} \{\emptyset, \{\emptyset\}\} \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$$

(for the numbers 0, 1, 2, and 3. The rest follow by simple extension of the same principle).

So while paradoxes plagued and continue to plague the theory of sets, the discipline survives and continues to form the foundation for the entire field of mathematics. Cantor's genius was that he was able to arrive at actual infinity, and learn important truths about it, starting with sets. The sets he first defined were the powers, P , beginning with a set and then deriving the set of its limit points. Here, a set must necessarily be infinite. An infinite sequence in a bounded space, as Bolzano and Weierstrass have taught us, will have a limit point. When there are many sequences converging to many points, we may define the *set* of limit points of the original set. Then Cantor considered the limit points of this new set,

and so on. Soon enough, he had in his hand an infinite collection of sets with infinitely many points.

At this point, Cantor was about to outdo Peano. In the next stage of his research, Cantor defined a whole new world of numbers. These were not ordinary numbers (numbers we now call *ordinals*), as Peano had defined them. All of Cantor's numbers were beyond the finite world. They were called *transfinite*. These numbers were extensions of the concept of a number to the unknown, mysterious world of actual infinity. But among other perils on his road to the secret garden, Cantor would encounter a formidable principle with bizarre, disturbing consequences: the axiom of choice.

portable with the concept that there could be different *kinds* of infinity: one larger than the other—something most of us find too incredible to contemplate. Having made the psychological breakthrough, Cantor needed a language to describe these infinities. He wanted to give his infinities names.

Cantor called his infinities *transfinite numbers*, with the exception of the Absolute, which he viewed as beyond description. Until now, we've only seen two kinds of infinities: that of the integers, rational numbers, and algebraic numbers on the one hand, and the greater infinity of the transcendental numbers (and hence the entire real line containing them) on the other. Cantor knew one other order of infinity greater than the first two. This was the infinity of all functions, continuous and discontinuous ones, defined on the real line.²¹ Cantor now needed to find a symbol to represent these varying orders of infinity he had discovered, so he could distinguish them from one another—and from any new ones he might encounter in his search.

At first, Cantor made an assumption about his transfinite numbers. He proposed the existence of a number that was infinite and yet the *smallest number greater than all the finite numbers*. Cantor reasoned as follows. We can generate the successive natural numbers by adding the number one: $3=2+1$, $4=3+1$, $5=4+1$, and so on. While there is no *greatest* number—you can always add one to any number and get a larger one—there is still the possibility of the existence of a number *larger than all the finite numbers*. Cantor named his first transfinite number ω , the Greek letter omega. If the number one was the “alpha,” the first number, then the least infi-

nite number greater than all the finite numbers was the “omega.” Cantor then supposed that the principle of generation of numbers extended naturally to the transfinite numbers, allowing him to define the transfinite numbers $\omega+1$, $\omega+2$, . . . 2ω , . . . ω^2 , . . . ω^ω , . . . , and so on. He now had an infinity of infinite numbers.²²

Cantor soon went further, realizing that in order to develop his theory of the infinite he would need new definitions, and a new notation. Cantor was now in a position to extend the concept of a number using the ideas of set theory.

The cardinality of a set is a measure of the number of elements the set contains. For a finite set, the cardinality is simply the number of elements in the set. A set containing three dogs has cardinal number 3. A set of 106 people in a movie theater has cardinal number 106. What about the set of all integers? The set of all rational numbers? What is the cardinality of an infinite set? Cantor wanted to be able to define the cardinal number of an infinite set. At first, he used his ω to denote the cardinal number of a countable set such as the set of all integers. He also used the usual sign we often use to denote infinity: ∞ . But soon he decided that the cardinal numbers needed new symbols. Cantor decided to name his infinities—his transfinite cardinal numbers—using the Hebrew letter aleph, $\aleph$. Why did he choose the aleph?

In the 1880s, when Cantor's first results were published, there was strong resistance to his ideas from mathematicians. Kronecker was the champion of the anti-Cantor camp, but other mathematicians were uncomfortable with the concept of an actual infinity as well. When this idea was further developed to include varying orders of actual infinity in Cantor's

published work, these traditionalists became even more edgy, and a number of mathematicians began to fulminate against his heresy. But support for Cantor's work came from an unexpected source: the pope.

Pope Leo XIII made a strong attempt during his papacy to understand science and to bridge the apparent gap between scientific discovery and Church dogma. From the time he was elected Pope in 1878, Leo XIII inspired a transformation that resulted in a more enlightened atmosphere in the Church. The Holy See now encouraged and supported investigations of the laws of nature. Mathematics, and the role of infinity in particular, as seen in the light of Augustine's writings, gained importance during this period as a way to understand God.

The German priest Constantin Gutberlet held unusual views about infinity. Like Cantor, he believed that actual infinity could be contemplated by the human mind and that such contemplation could help believers get close to the Divine.

Gutberlet came under attack by other theologians and, in his defense, quoted Cantor's papers on infinity in his letters to rival theologians. Thus began Cantor's lengthy correspondence with the Christian clergy about the meaning of infinity and God. But Cantor had believed in the strong connection between infinity and God even before he began the epistolary exchanges with theologians in the late 1870s. Cantor had been brought up in a religious household. He always believed in God, and in later years—when under attack by mathematicians—used to say that God Himself had revealed to him the existence of the transfinite numbers. He knew the numbers were real because "God had told me so," and thus

he required no further proof. But which God told him about the transfinite numbers?

Ivor Grattan-Guinness, a British historian of mathematics, argued in his paper, "Towards a Biography of Georg Cantor," that Cantor was definitely *not* Jewish.²³ Other biographers such as Boyer and Bell wrote that Cantor came from a Jewish background, and the letter from Cantor's brother reprinted in Charraud's book describes the family as being of Jewish descent on both sides. Grattan-Guinness, who seems to have had a desire to cast Cantor without any Jewish blood, still mentioned that the family was descended from Spain or Portugal several generations back, and that Cantor's wife, Vally, came from a Berlin Jewish family.

We know that the converted Jews of Spain and Portugal practiced their religion in secrecy for generations. The Marranos, forcibly-converted Jews of Spain and Portugal, became outwardly Christian, but at home practiced Judaism furtively, or at least followed some Jewish customs. Most of them knew Hebrew, and all knew their heritage. In private letters, such as the one written by Cantor's brother, they could be open with one another and freely discuss their families' traditions and provenance.

Grattan-Guinness suggested that the name Cantor was not Jewish and was derived from the Latin word for singer. It is true that the root of the word *cantor* comes from the verb to sing, but the particular form *cantor* rather than *chanteur* or *cantante* could only mean one kind of singer, the singer in the synagogue—the cantor.

The Spanish and Portuguese conversos left the Iberian peninsula in the fifteenth and sixteenth centuries and traveled

north, first to Amsterdam and then farther east to Germany and Baltic Russia. It was probably in this way that Cantor's ancestors came to St. Petersburg, Denmark, and Germany. They kept their Jewish family name and their traditions, including that of marrying other Jews, as Georg Cantor had in fact done. While practicing Christianity on the outside, they were nonetheless well-versed in Judaism, and it was only through such a tradition that Cantor could learn the Hebrew alphabet. No other peoples in Europe studied Hebrew. They would have had no reason or inclination to do so.

The Iberian Jews also knew the Kabbalah. Jewish mysticism was born in Spain, and most of the important rabbis of the Kabbalah lived and worked in that country. As the Spanish and Portuguese Jews traveled and dispersed throughout northern Europe, they carried with them the tradition of Jewish mystical studies. The idea of the Ein Sof, the infinitude of God, was an important part of their communal hidden tradition. Georg Cantor had to have been aware of the role of the letter aleph as a symbol of God and His infinity. He told his colleagues and friends that he was proud of his choice of the letter aleph to symbolize the transfinite numbers, since aleph was the first letter of the Hebrew alphabet and he saw in the transfinite numbers a new beginning in mathematics: the beginning of the actual infinite.

But Cantor did not need to have had a thorough knowledge of Kabbalah to encounter the concept of infinity and its associated symbol, the aleph, within Jewish tradition. All he had to have been acquainted with was the most basic of Jewish prayers, *Adon Olam* (Master of the Universe). For a verse of this prayer, recited by Jews several times daily, is that God

rules the universe *beli reshit, beli tachlit*: with no beginning, with no end. Hence the concept of infinity was known to anyone with a Jewish background, including the converted Jews of Iberia.

Cantor hypothesized the existence of a sequence of alephs. He named the lowest order of infinity, the infinity of the integers and the rational (as well as algebraic) numbers, aleph-zero. It is written as $\aleph_0$. Cantor believed that the alephs continued. He knew that the irrational numbers, in particular the transcendental ones, were more numerous than the rationals—they could not be put into a one-to-one correspondence with the rationals—and so there had to be a higher aleph to describe them. And the set of all functions on the real line had a still higher order of infinity, so a higher aleph would be required to describe them. But Cantor did not know whether there were any distinct orders of infinity, other alephs, *between* the aleph of the rationals and that of the irrationals, and between the one of the irrationals and that of the functions. This question would occupy much of the rest of his life.

At any rate, Cantor put forward the hypothesis that there was a series of alephs to describe higher and higher orders of infinity: $\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, \aleph_5, \aleph_6, \aleph_7, \dots$

While he believed that larger and larger alephs existed, without knowing their exact placement, he did know something about the way the alephs interacted. Cantor discovered transfinite arithmetic. Here are some of the rules of this new mathematics of infinity that Cantor uncovered:

$$\aleph_0 + 1 = \aleph_0.$$

If we add one to the number representing the infinitely many integers, we still get that infinite number of integers (or rational or algebraic numbers). Adding one to the lowest order of infinity still leaves that infinity alone—it cannot take it to a higher level. Similarly, adding any finite number to the first order of infinity leaves that infinity the same:

$$\aleph_0 + n = \aleph_0.$$

Now, from Galileo's work showing that there are as many integers as there are squared integers (there is a one-to-one correspondence between the two sets), we know that adding two aleph-zeros to one another still leaves us with aleph-zero (adding the two infinite sets of the odd numbers to the even numbers gives us all the integers—a set whose cardinality is still aleph-zero). A similar argument could be that by adding all the integers (aleph-zero of them) to all the fractions (aleph-zero as well) we get all the rationals (still aleph-zero).

So we have:

$$\aleph_0 + \aleph_0 = \aleph_0.$$

And similarly:

$$\aleph_0 \times n = \aleph_0,$$

where n is any number. Also:

$$\aleph_0 \times \aleph_0 = \aleph_0.$$

However, as we will see in the next chapter, the operation of exponentiation raises the cardinality of a set. Cantor was elated upon discovering the amazing laws of transfinite arithmetic. He now had a sequence of transfinite numbers, fol-

lowing their own logic and their own laws of arithmetic. If only he could show other mathematicians the truth and beauty of his divine theory, he would be a happy man. But, so far, it was only some theologians who appreciated his work. They understood that the mathematician had given them an elegant framework for understanding the immensity of God. Cantor had built a divine temple, level upon level of infinite numbers, the alephs. It would be hard to miss the uncanny similarity between Cantor's structure of alephs, each grounded on a previous one, and a visual image of the Kabbalah popular during that period. This is the picture of the nested circles of the Ein Sof—infinities also represented by alephs:

