JEM013 - Game Theory

Seminar 4

Repeated games

Sophio Togonidze
Sophotogonidze@gmail.com

Repeated game:

- In ongoing interactions, the promise of future rewards and thread of future punishments, *may* sometime provide incentive for good behaviour today

Repeated game:

- In ongoing interactions, the promise of future rewards and thread of future punishments, *may* sometime provide incentive for good behaviour today

- In repeated games the same stage game (strategic form game) is played for some duration of T periods.

Repeated game:

- In ongoing interactions, the promise of future rewards and thread of future punishments, *may* sometime provide incentive for good behaviour today

- In repeated games the same stage game (strategic form game) is played for some duration of T periods.

- At each period, the outcomes of all past periods are observed by all players.

Repeated game:

• In ongoing interactions, the promise of future rewards and thread of future punishments, *may* sometime provide incentive for good behaviour today

• In repeated games the same stage game (strategic form game) is played for some duration of T periods.

• At each period, the outcomes of all past periods are observed by all players.

• Future payoffs are discounted proportionately at some rate $\delta \in (0, 1]$, called the discount factor.

Repeated game:

- In ongoing interactions, the promise of future rewards and thread of future punishments, *may* sometime provide incentive for good behaviour today

- In repeated games the same stage game (strategic form game) is played for some duration of T periods.

- At each period, the outcomes of all past periods are observed by all players.

- Future payoffs are discounted proportionately at some rate $\delta \in (0, 1]$, called the discount factor.

- The overall payoff is the sum of discounted payoffs at each period.

Repeated game:

- In ongoing interactions, the promise of future rewards and thread of future punishments, *may* sometime provide incentive for good behaviour today

- In repeated games the same stage game (strategic form game) is played for some duration of T periods.

- At each period, the outcomes of all past periods are observed by all players.

- Future payoffs are discounted proportionately at some rate $\delta \in (0, 1]$, called the discount factor.

- The overall payoff is the sum of discounted payoffs at each period.

- Repeated play of the same strategic game *can* introduces new equilibria by allowing players to condition their actions on the way their opponents played in the previous periods.

Repeated games

- Stage game G={S_1, …,S_N; U_1,…U_N}
- Repeated game G(T, δ)

T-number of repetitions (periods)

Integer - finitely repeated games

∞ - infinitely repeated games

δ- discount factor $\delta \in (0,1]$

- *after* observing strategy choices in all previous periods player k's payoff is

$$V_k = \sum_{t=1}^{T} \delta^{t-1} U_k\left(S_1(t),..,S_N(t)\right)$$

- present value

- average payoff $\pi_k = (1-\delta)V_k$ T→∞

Note: $\sum_{t=1}^{\infty} \delta^{t-1}\pi = \frac{\pi}{1-\delta}$ $\sum_{t=2}^{\infty} \delta^{t-1}\pi = \frac{\delta\pi}{1-\delta}$

Strategies in repeated game:

- choose stage-game strategies in the 1st period
- in following periods, choose stage-game strategies as a function of the strategies in the previous periods.

Prisoner's dilemma (PD) played twice, no discounting

Stage game :

Repeated game:
G(T=2, δ=1)

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect
a_2, b_2 - cooperate

Prisoner's dilemma (PD) played twice, no discounting

Stage game :

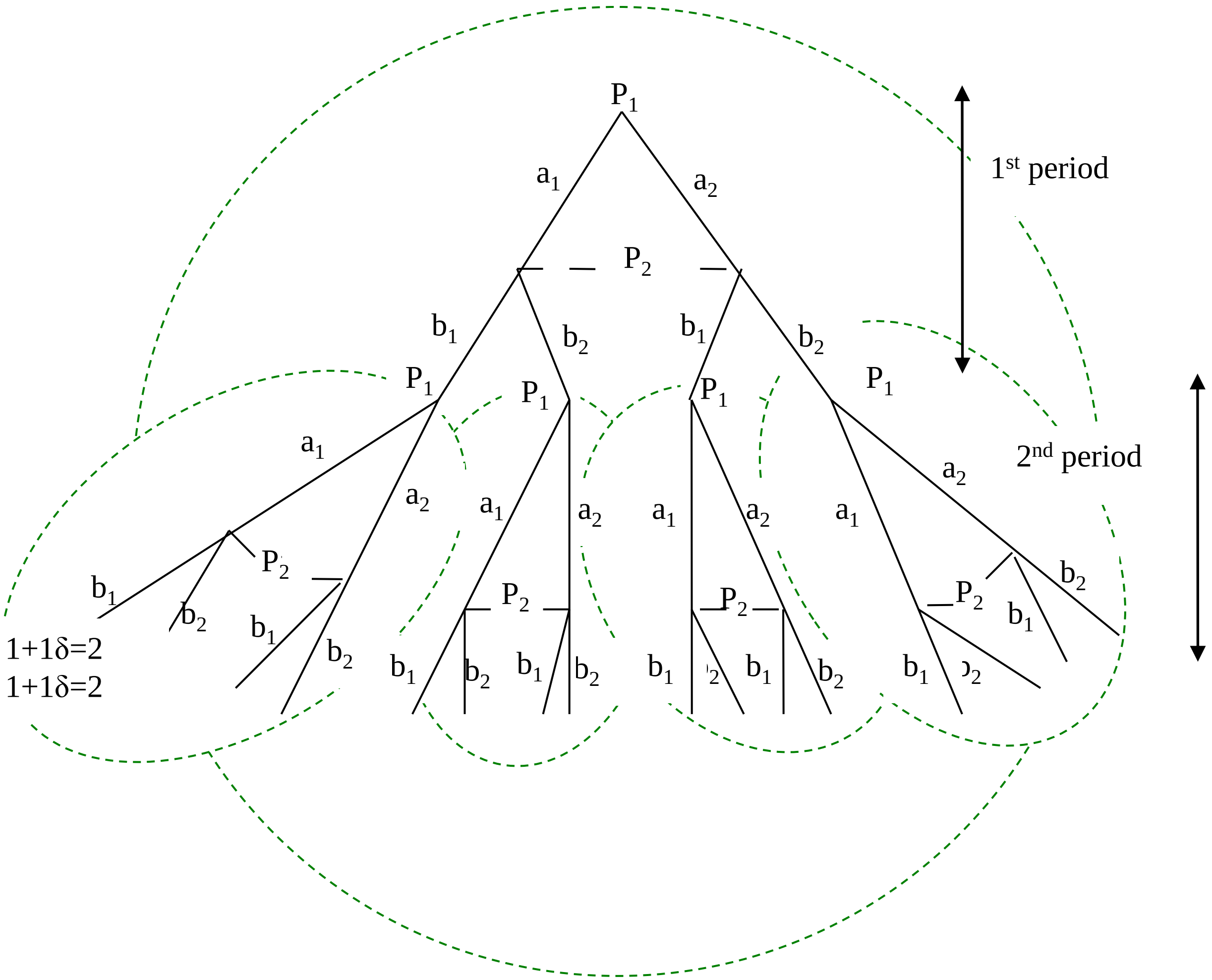
		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect

a_2, b_2 - cooperate

Repeated game:

$G(T=2, \delta=1)$



◯ - 5 subgames

Prisoner’s dilemma (PD) played twice, no discounting

Stage game

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect
a_2, b_2 - cooperate

Nodes :

a_1 a_2 - 1st period
a_1 a_2 - 2nd period
a_1 a_2
a_1 a_2
a_1 a_2

: Repeated game:
$G(T=2, \delta=1)$

Strategies in repeated game:

P_1’s strategies:

Type A:

$a_1a_1a_1 \rightarrow$ t=1 play a_1
t=2 play a_1 if P_2 played b_1 at t=1
play a_1 if P_2 played b_2 at t=1

$a_1a_1a_2 \rightarrow$ t=1 play a_1
t=2 play a_1 if P_2 played b_1 at t=1
play a_2 if P_2 played b_2 at t=1

:

:

Total **8 strategies** ($=2 \cdot 2^2$) of **Type A**

Total **32 strategies** ($=2 \cdot 2^4$) of **Type B**

Prisoner's dilemma (PD) played twice, no discounting

Stage game

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect
a_2, b_2 - cooperate

: Repeated game:
G(T=2, δ=1)

SPNE

- Find all NE of the stage game

-unique NE (a_1,b_1), payoffs (1,1)

⇓

-unique SPNE: play NE every period

Note: *Important assumptions!*

☐ *finite repetitions*
☐ *unique NE of the stage game*

Discount factor doesn't matter in this case

- At the last stage – always NE!

-at 2nd stage : (a_1,b_1), payoffs (1,1)

Prisoner's dilemma (PD) played twice, no discounting

=> by BI → the 1st stage game:

		P_2	
		b_1	b_2
P_1	a_1	1 ; 1	5 ; 0
	a_2	0 ; 5	4 ; 4

Just add payoffs of the 2nd stage

Repeated game:
G(T=2, δ=1)

SPNE

- Find all NE of the stage game

-unique NE (a_1,b_1), payoffs (1,1)

⇓

-unique SPNE: play NE every period

Note: *Important assumptions!*

☐ *finite repetitions*
☐ *unique NE of the stage game*

Discount factor doesn't matter in this case

- At the last stage – always NE!

-at 2nd stage : (a_1,b_1), payoffs (1,1)

Prisoner's dilemma (PD) played twice, no discounting

=> by BI → the 1st stage game:

$$\begin{array}{cc|c|c|}
 & & \multicolumn{2}{c}{P_2} \\
 & & b_1 & b_2 \\ \hline
P_1 & a_1 & \underline{1} ; \underline{1} & \underline{5} ; 0 \\ \hline
 & a_2 & 0 ; \underline{5} & 4 ; 4 \\ \hline
\end{array}$$

Just add payoffs of the 2nd stage

NE: (a_1,b_1); payoffs (1,1)

SPNE: $(a_1a_1a_1a_1a_1, b_1b_1b_1b_1b_1)$ → play (a_1,b_1) every stage; payoffs (1,1)

No cooperation in finitely repeated PD.

Repeated game:
G(T=2, δ=1)

SPNE

•Find all NE of the stage game

-unique NE (a_1,b_1), payoffs (1,1)

-unique SPNE: play NE every period

Note: *Important assumptions!*

- ☐ *finite repetitions*
- ☐ *unique NE of the stage game*

Discount factor doesn't matter in this case

•At the last stage – always NE!
-at 2nd stage : (a_1,b_1), payoffs (1,1)

Prisoner’s dilemma (PD) played twice, no discounting

=> by BI → the 1st stage game:

		P_2	
		b_1	b_2
P_1	a_1	1 ; 1	5 ; 0
	a_2	0 ; 5	4 ; 4

Just add payoffs of the 2nd stage

NE: (a_1,b_1); payoffs (1,1)

SPNE: $(a_1a_1a_1a_1a_1, b_1b_1b_1b_1b_1)$ →
play (a_1,b_1) every stage; payoffs (1,1)

No cooperation in finitely repeated PD.

We can get more cooperation in infinitely repeated games.

Repeated game:
G(T=2, δ=1)

SPNE

- Find all NE of the stage game

-unique NE (a_1,b_1), payoffs (1,1)

⇩

-unique SPNE: play NE every period

Note: *Important assumptions!*

☐ *finite repetitions*
☐ *unique NE of the stage game*

Discount factor doesn’t matter for this case

- At the last stage – always NE!

-at 2nd stage : (a_1,b_1), payoffs (1,1)

Prisoner's dilemma (PD) infinitely repeated , with discounting $\delta<1$

Stage game

:

Repeated game:

$G(T=\infty,\ \delta\epsilon(0;1))$

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect

a_2, b_2 - cooperate

Prisoner’s dilemma (PD) infinitely repeated , with discounting $\delta<1$

Stage game :

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect
a_2, b_2 - cooperate

Repeated game:
$G(T=\infty, \delta\epsilon(0;1))$

All subgames in $G(T=\infty, \delta\epsilon(0;1))$ are themselves $G(T=\infty, \delta\epsilon(0;1))$games.

No last stage => we cannot use BI.

Prisoner’s dilemma (PD) infinitely repeated , with discounting δ<1

Stage game

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect
a_2, b_2 - cooperate

:

Repeated game:
G(T=∞, δϵ(0;1))

All subgames in G(T=∞, δϵ(0;1)) are themselves G(T=∞, δϵ(0;1))games.

No last stage => we cannot use BI.

Construct trigger strategy:

Def:
Designate some profile of stage-game strategies (s_1, .., s_N) to play each period. If only player k deviates from s_k in some period t, play k’s lowest-payoff stage-game NE from t+1 on. Otherwise players continue playing the strategies (s_1, .., s_N) each period.

Prisoner's dilemma (PD) infinitely repeated , with discounting $\delta<1$

Stage game

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1} ; \underline{1}$	$\underline{5} ; 0$
	a_2	$0 ; \underline{5}$	$4 ; 4$

a_1, b_1 –defect
a_2, b_2 - cooperate

:

Repeated game:

$G(T=\infty, \delta \epsilon (0;1))$

All subgames in $G(T=\infty, \delta \epsilon (0;1))$ are themselves $G(T=\infty, \delta \epsilon (0;1))$games.

No last stage => we cannot use BI.

Construct trigger strategy:

Def:
Designate some profile of stage-game strategies (s_1, .., s_N) to play each period. If only player k deviates from s_k in some period t, play k's lowest-payoff stage-game NE from t+1 on. Otherwise players continue playing the strategies (s_1, .., s_N) each period.

Trigger strategy:

t=1 play (a_2, b_2)
t>1 play (a_2, b_2) if at (t-1) (a_2, b_2) was played,
otherwise play (a_1, b_1).

Switch to NE.

Prisoner’s dilemma (PD) infinitely repeated , with discounting δ<1

Stage game

		P_2	
		b_1	b_2
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0
	a_2	0 ; $\underline{5}$	4 ; 4

a_1, b_1 –defect
a_2, b_2 - cooperate

Trigger strategy:

t=1 play (a_2, b_2)
t>1 play (a_2, b_2) if at (t-1) (a_2, b_2) was played,
otherwise play (a_1, b_1).

: Repeated game:
G(T=∞, δϵ(0;1))

Check if it is SPNE: Calculate a present value for playing (a_2, b_2) forever and compare it with a present value for one stage deviation and playing (a_1, b_1) from now on.

(a_2, b_2) forever ≥ the best one stage deviation + (a_1, b_1) forever

$$4/(1-\delta) \geq 5+1\delta/(1-\delta)$$
$$4 \geq 5-5\delta+\delta$$

=> δ* =1/4 - critical value of δ for which trigger strategy is SPNE

=> In infinitely repeated PD there are 2 types of SPNE:
- Play (a_1, b_1) each period (any δϵ(0;1))
- Play the trigger strategy (δ ≥ δ*)

Game 4.3

Stage game P_2 :

Repeated game:
G(T=2, δϵ(0;1))

		b_1	b_2	b_3
P_1	a_1	1 ; 1	5 ; 0	0 ;-2
	a_2	0 ; 5	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	3 ; 3

Game 4.3

Stage game P_2 :

Repeated game:
$G(T=2, \delta \epsilon (0;1))$

		b_1	b_2	b_3
P_1	a_1	1 ; 1	5 ; 0	0 ;-2
	a_2	0 ; 5	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	3 ; 3

2 pure strategy NE:
- (a_1,b_1), payoffs (1,1)
- (a_3,b_3), payoffs (3,3)

In this game there is a SPNE, where (a_2,b_2) is played at the 1st period, although (a_2,b_2) is not NE.

Conditional strategy:

t=1 play (a_2, b_2)
t=2 play (a_3, b_3) if at t=1 (a_2, b_2) was played,
otherwise play (a_1, b_1).

Game 4.3

Stage game :

P_2

		b_1	b_2	b_3
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

Conditional strategy:

t=1 play (a_2, b_2)

t=2 play (a_3, b_3) if at t=1 (a_2, b_2) was played,
otherwise play (a_1, b_1).

Repeated game:
$G(T=2, \delta\epsilon(0;1))$

Check if it is SPNE:

The best deviation: P_1: $a_2 \rightarrow a_1$
$(4 \rightarrow 5)$
P_2: $b_2 \rightarrow b_1$
$(4 \rightarrow 5)$

	No deviation		Deviation
P_1:	$4+3\delta$	$\geq$	$5+\delta$
P_2:	$4+3\delta$	$\geq$	$5+\delta$

=> $\delta \geq 1/2$ - critical value of δ for which the conditional strategy is SPNE

Other SPNE?

Any combination of the stage game NE is always SPNE

Game 4.3

Stage game

		P_2		
		b_1	b_2	b_3
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

:

Repeated game:
G(T=3, δϵ(0;1))

Conditional strategy:

t=1 play (a_2, b_2)
t=2 play (a_2, b_2) if at t=1 (a_2, b_2) was played,
otherwise play (a_1, b_1).
t=3 play (a_3, b_3) if at t=2 (a_2, b_2) was played,
otherwise play (a_1, b_1).

Game 4.3

Stage game

:

		P_2		
		b_1	b_2	b_3
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

Conditional strategy:

t=1 play (a_2, b_2)

t=2 play (a_2, b_2) if at t=1 (a_2, b_2) was played,
otherwise play (a_1, b_1).

t=3 play (a_3, b_3) if at t=2 (a_2, b_2) was played,
otherwise play (a_1, b_1).

Repeated game:
$G(T=3, \delta \epsilon (0;1))$

Find the critical value of δ
for which this conditional strategy is SPNE.

- Last stage t=3 no deviation (2NE)

- t=2

No deviation	Deviation
$P_{1,} P_2$: $4+4\delta+3\delta^2 \geq$	$4+5\delta+\delta^2$

$\Rightarrow \delta \geq 1/2$ - (same as the case with T=2)

- t=1

No deviation	Deviation
$P_{1,} P_2$: $4+4\delta+3\delta^2 \geq$	$5+\delta+\delta^2$

$2\delta^2 + 3\delta - 1 \geq 0$
$\Rightarrow \quad \delta \geq 0.281$

- Put both conditions together => $\delta \geq 1/2$.

Game 4.3

Stage game :

			P_2	
		b_1	b_2	b_3
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

Repeated game:
G(T=2, δϵ(0;1))

Is it possible to construct a conditional strategy, where (a_2, b_2) is played at the 1st period at G(T=2,δ=0.45)?

Conditional strategy:

t=1 play (a_2, b_2)
t=2 play (a_3, b_3) if at t=1 (a_2, b_2) was played,
otherwise play (a_1, b_1).

δ≥1/2 - critical value of δ for which the conditional strategy is SPNE

Game 4.3

Stage game

P_2

P_1

	b_1	b_2	b_3
a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

:

Repeated game:
$G(T=2, \delta\epsilon(0;1))$

Is it possible to construct a conditional strategy, where (a_2, b_2) is played at the 1st period at $G(T=2,\delta=0.45)$?

With pure strategy punishment – no,
with mixed strategy punishment – yes.

Conditional strategy:

t=1 play (a_2, b_2)
t=2 play (a_3, b_3) if at t=1 (a_2, b_2) was played,
otherwise play (a_1, b_1).

$\delta\geq 1/2$ - critical value of δ for which the conditional strategy is SPNE

Game 4.3

Stage game

			P_2	
		b_1	b_2	b_3
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

Conditional strategy:

t=1 play (a_2, b_2)
t=2 play (a_3, b_3) if at t=1 (a_2, b_2) was played,
otherwise play (0.5 a_1+0.5 a_3; 0.5 b_1+0.5 b_3)

:

Repeated game:
$G(T=2, \delta \epsilon (0;1))$

Is it possible to construct a conditional strategy, where (a_2, b_2) is played at the 1st period at $G(T=2,\delta=0.45)$?

	No deviation		Deviation
P_1, P_2:	$4+3\delta$	$\geq$	$5+2\delta$

=> $\delta=1$ critical value of δ for which the conditional strategy is SPNE

=> for $\delta=0.45$ this conditional strategy is not SPNE

Game 4.3

Stage game

:

Repeated game:
G(T=∞, δϵ(0;1))

Find the critical value of δ for which this trigger strategy is SPNE.

		P_2		
		b_1	b_2	b_3
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

Trigger strategy::

t=1 play (a_2, b_2)
t>1 play (a_2, b_2) if at (t-1) (a_2, b_2) was played,
otherwise play (a_1, b_1) forever.

Game 4.3

Stage game :

			P_2	
		b_1	b_2	b_3
P_1	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0
	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$

Trigger strategy::

t=1 play (a_2, b_2)
t>1 play (a_2, b_2) if at (t-1) (a_2, b_2) was played,
otherwise play (a_1, b_1) forever.

Repeated game:
G(T=∞, δϵ(0;1))

Find the critical value of δ for which this trigger strategy is SPNE.

No deviation   Deviation

P_1, P_2: 4/(1-δ) ≥ 5+δ/(1-δ)

=> δ≥1/4 - critical value of δ for which the trigger strategy is SPNE

=> we get better prediction for cooperation than in finitely repeated game.

Homework 4.1

Stage game P_2 :

Repeated game:
G(T=2, δϵ(0;1))

		b_1	b_2	b_3	b_4	b_5
	a_1	$\underline{1}$; $\underline{1}$	$\underline{5}$; 0	0 ;-2	0 ; 0	0 ; 0
	a_2	0 ; $\underline{5}$	4 ; 4	-1 ; 0	0 ; 0	0 ; 0
P_1	a_3	-2 ; 0	0 ;-1	$\underline{3}$; $\underline{3}$	0 ; 0	0 ; 0
	a_4	0 ; 0	0 ; 0	0 ; 0	$\underline{5}$; $\underline{1}$	0 ; 0
	a_5	0 ; 0	0 ; 0	0 ; 0	0 ; 0	$\underline{1}$; $\underline{5}$

Find SPNE where (a_2,b_2) is played at the 1st period. SPNE should sustain renegotiation.
Find the critical value of δ.

Homework 4.2

Stage game

		P_2		
		b_1	b_2	b_3
P_1	a_1	10;10	2 ;12	0 ;13
	a_2	12; 2	5 ; 5	0 ; 0
	a_3	13; 0	0 ; 0	1 ; 1

:

Repeated game:
G(T=2, $\delta \epsilon (0;1]$)

Find all pure strategy SPNE.

Homework 4.3

Stage game

:

		P_2		
		b_1	b_2	b_3
P_1	a_1	10;10	2 ;12	0 ;13
	a_2	12; 2	5 ; 5	0 ; 0
	a_3	13; 0	0 ; 0	1 ; 1

Repeated game:

$G(T=\infty, \delta\epsilon(0;1))$

Find all pure strategy SPNE.

Homework 4.4

Stage game

P_2

		b_1	b_2	b_3
P_1	a_1	10;10	2 ;12	-1;13
	a_2	12; 2	5 ; 6	0 ; 0
	a_3	12; -1	0 ; 0	1 ; 1

:

Repeated game:

G(T=2, δϵ(0;1))

Find a ***pure-strategy*** *subgame-perfect equilibrium of the repeated game in which the players play* ***(a_1,b_1) in the first round*** *of the repeated game. Find the critical discount factor to support this strategy as a subgame-perfect equilibrium. Is it possible to support this strategy when δ= 0.59?*

Is it possible to support ***(a_1,b_1) in the first round*** *of the repeated game using a conditional strategy equilibrium with a* ***mixed-strategy*** *punishment when δ= 0.59?*