JEM013 - Game Theory

Seminar 1

Backward induction, Rationalizability

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Readings

Main textbooks:

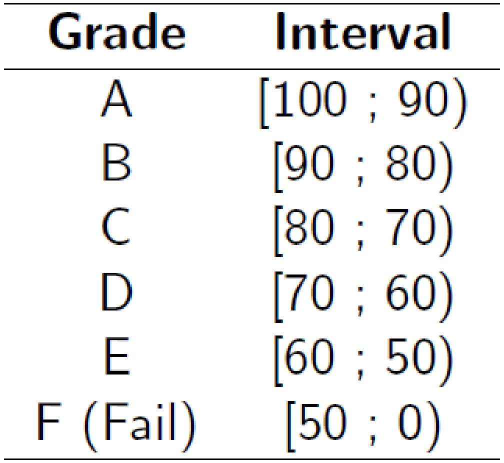
- (G) R. Gibbons: Game Theory for Applied Economists, Princeton University Press, 1992.

Additional Textbooks:

- (M) A.Mas-Colell, M. Whinston, and J. Green: Microeconomic Theory, Oxford University Press, 1995.
- (BP) P. Belleflamme and M. Peitz. "Industrial Organization. Markets and Strategies." Cambridge University Press. 2015
- (S) O. Shy "Industrial Organization. Theory and Applications" MIT Press. 1995

Requirements and Grading

- The students are evaluated according to written exams.
- 4 exams available to take for this GT class
- To pass the course, students are required to score at least 50 % from the exam.



Grade	Interval
A	[100 ; 90)
B	[90 ; 80)
C	[80 ; 70)
D	[70 ; 60)
E	[60 ; 50)
F (Fail)	[50 ; 0)

Exam Dates

- Exam 1: March 6, 2023, 8:00- 9:20, room 109
- Exam 2: March 20, 2023, 8:00- 9:20, room 109
- Exam 3: April 3, 2023, 8:00- 9:20, room 109
- Exam 4: April 25, 2023, 8:00- 9:20, room 109

You need to register on Moodle2:
https://dl2.cuni.cz/

What is a game?

*“In brief, a game is being played whenever human beings **interact**.” (Ken Binmore)*

What is Game Theory?

*“Game theory is concerned with the actions of decision makers who are **conscious** that their actions affect each other. ” (Eric Rasmusen)*

*“It is mostly about what happens when people **interact in a rational manner**. ”*
(Ken Binmore)

“Essential Elements“

Players: **Players** are the individuals who make decisions. Each player's goal is to maximize his utility by choice of actions.

Actions: An **action** or move by player ***i***, denoted $\mathbf{a_i}$ is a choice he can make.

Player i's **action set**, $\mathbf{A_i = \{a_i\}}$, is the entire set of actions available to him.

An **action combination (profile)** is an ordered set $\mathbf{a = \{a_i\}}$; (i = 1,..., n) of one action for each of the n players in the game.

Payoffs: By player i's **payoff** $\mathbf{u_i(s_1,..,s_n)}$ is the utility player ***i*** receives after the game has been played out

The assumptions

1. **ALL payoff consequences**

All aspects of preferences are included into payoffs

2. **Full Rationality**

Players have *rational preferences*
Players aim to *maximize their payoffs*
Players are *perfect calculators*

3. **Common knowledge**

Information is *common knowledge*.
If it is known to all the players.
If each player knows that all the players know it.
If each player knows that all the players know that all the players know it,
...
and so forth and infinitum.

Information

Information

Complete information

Information

Complete information

Each player *knows* about *all* other players'
payoffs, available *actions* and *information*
AND
it is *Common knowledge*

Information

Complete information

Incomplete information

Each player *knows* about *all* other players' *payoffs*, available *actions* and *information*
AND
it is *Common knowledge*

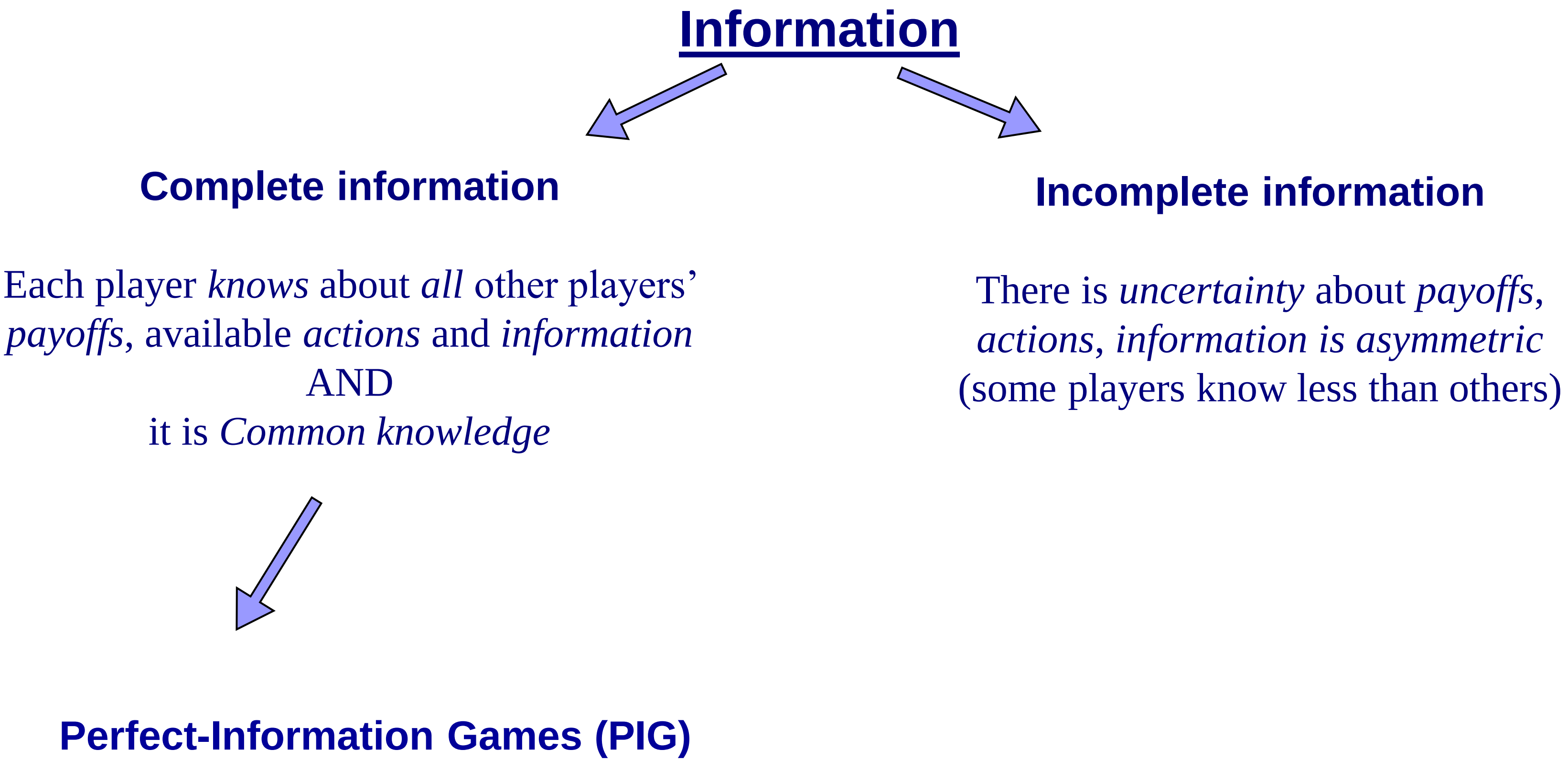
Information

Complete information

Each player *knows* about *all* other players' *payoffs*, available *actions* and *information*
AND
it is *Common knowledge*

Incomplete information

There is *uncertainty* about *payoffs*, *actions*, *information is asymmetric*
(some players know less than others)



Information

Complete information

Each player *knows* about *all* other players' *payoffs*, available *actions* and *information*
AND
it is *Common knowledge*

Incomplete information

There is *uncertainty* about *payoffs*, *actions*, *information is asymmetric* (some players know less than others)

Perfect-Information Games (PIG)

Information

Complete information

Each player *knows* about *all* other players' *payoffs*, available *actions* and *information*
AND
it is *Common knowledge*

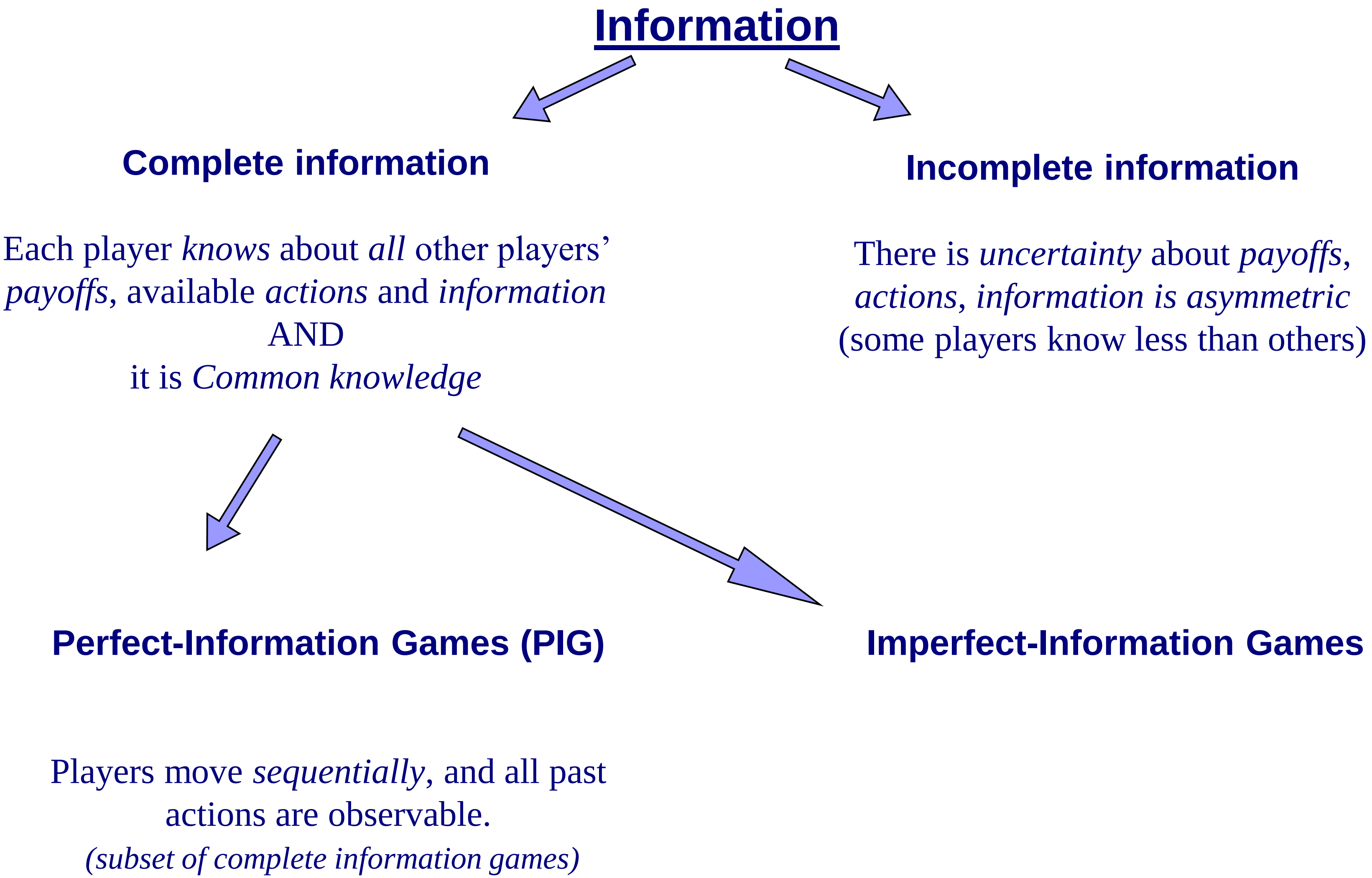
Perfect-Information Games (PIG)

Players move *sequentially*, and all past actions are observable.
(subset of complete information games)

Incomplete information

There is *uncertainty* about *payoffs*, *actions*, *information is asymmetric* (some players know less than others)

Information



Complete information

Each player *knows* about *all* other players' *payoffs*, available *actions* and *information*
AND
it is *Common knowledge*

Incomplete information

There is *uncertainty* about *payoffs*, *actions*, *information is asymmetric* (some players know less than others)

Perfect-Information Games (PIG)

Players move *sequentially*, and all past actions are observable.
(subset of complete information games)

Imperfect-Information Games

Information

Complete information

Each player *knows* about *all* other players' *payoffs*, available *actions* and *information*
AND
it is *Common knowledge*

Incomplete information

There is *uncertainty* about *payoffs, actions, information is asymmetric* (some players know less than others)

Perfect-Information Games (PIG)

Players move *sequentially*, and all past actions are observable.
(subset of complete information games)

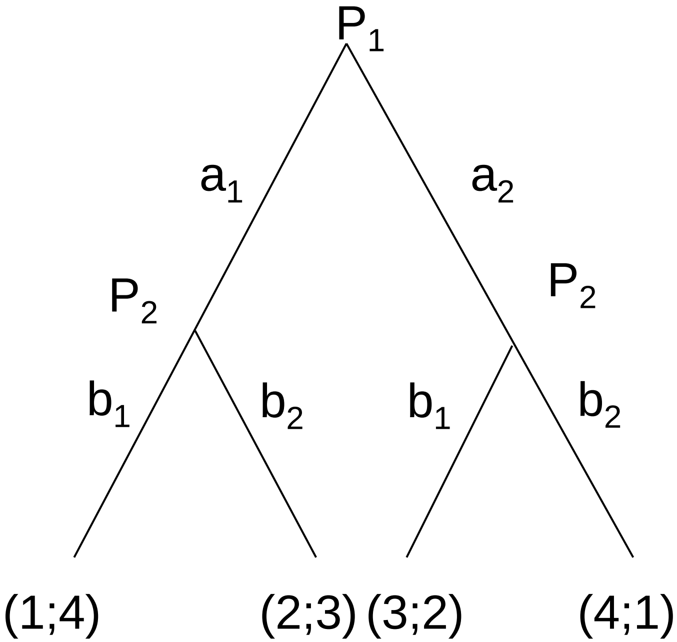
Imperfect-Information Games

Players move *simultaneously*, or *do not observe actions* of other players.
(subset of complete information games)

Usual form of presentation

Perfect-Information Games

Extensive Form- Game Tree



The **extensive form** is a description of a game consisting of

(1) A configuration of nodes and branches running without any closed loops from a single starting node to its end nodes.

(2) An indication of which node belongs to which player.

(3) The probabilities that Nature uses to choose different branches at its nodes.

(4) The information sets into which each player's nodes are divided.

(5) The payoffs for each player at each end node.

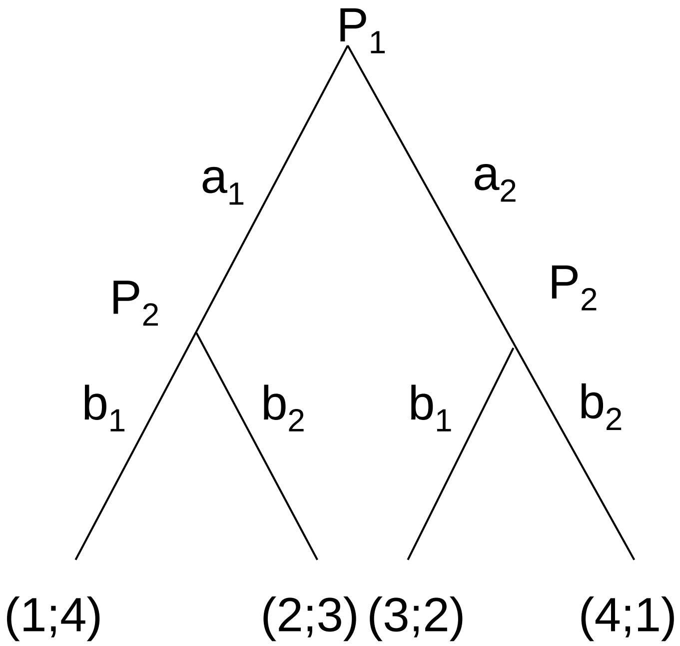
Usual form of presentation

Perfect-Information Games

Imperfect-Information Games

Extensive Form- Game Tree

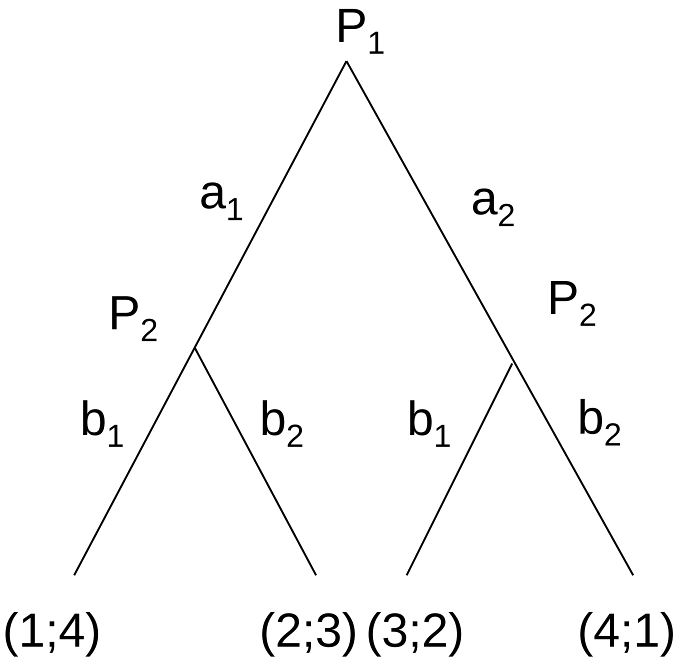
Normal (strategic) Form- Payoff Matrix



Usual form of presentation

Perfect-Information Games

Extensive Form- Game Tree



Imperfect-Information Games

Normal (Strategic) Form- Payoff Matrix

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{P_2} \\ & & b_1 & b_2 \\ \cline{3-4} & a_1 & 1 ; 4 & 2 ; 3 \\ \cline{3-4} P_1 & a_2 & 3 ; 2 & 4 ; 1 \\ \cline{3-4} \end{array}$$

Usual form of presentation

Imperfect-Information Games

Normal (Strategic) Form- Payoff Matrix

P_2

		b_1	b_2
P_1	a_1	1 ; 4	2 ; 3
	a_2	3 ; 2	4 ; 1

The **Normal Form** (or Strategic form) consists of:

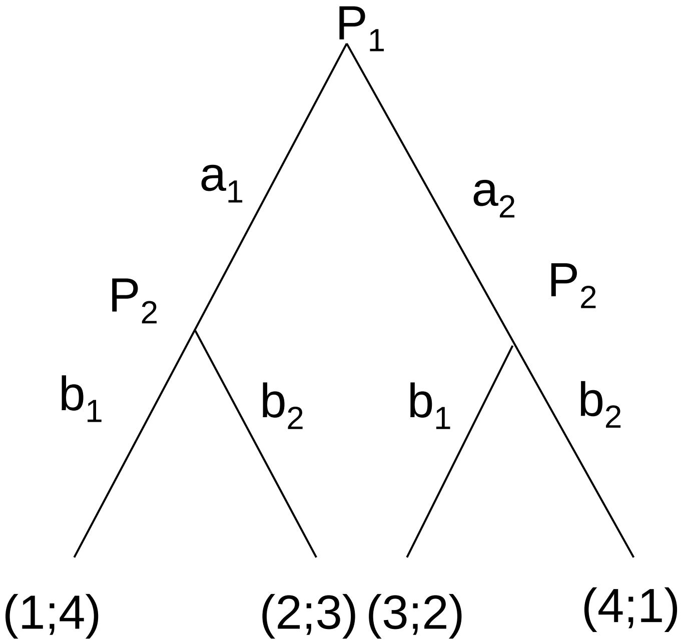
1. All possible strategy profiles $(s_1, s_2, \ldots s_p)$
2. Payoff functions mapping s_i into the payoff n-vector u_i, $(i = 1, 2, .. p)$,

where n is the number of players and p is the number of strategy profiles.

Strategies

Perfect-Information Games

Complete Contingent Plans



P_1's strategies: a_1, a_2

P_2's strategies: b_1b_1, b_1b_2, b_2b_1, b_2b_2

Imperfect-Information Games

Actions

P_2

		b_1	b_2
P_1	a_1	1 ; 4	2 ; 3
	a_2	3 ; 2	4 ; 1

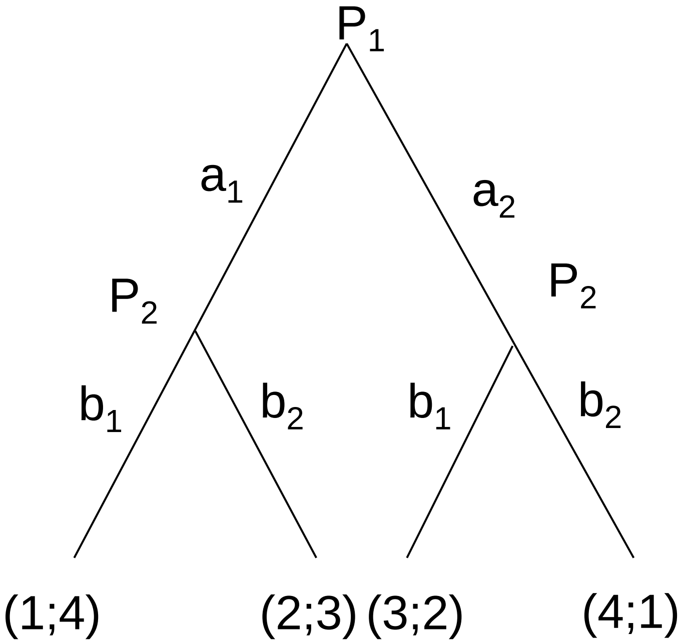
P_1's strategies: a_1, a_2

P_2's strategies: b_1, b_2

Perfect-Information Games

Extensive Form- Game Tree

Normal (strategic) form- Payoff Matrix?

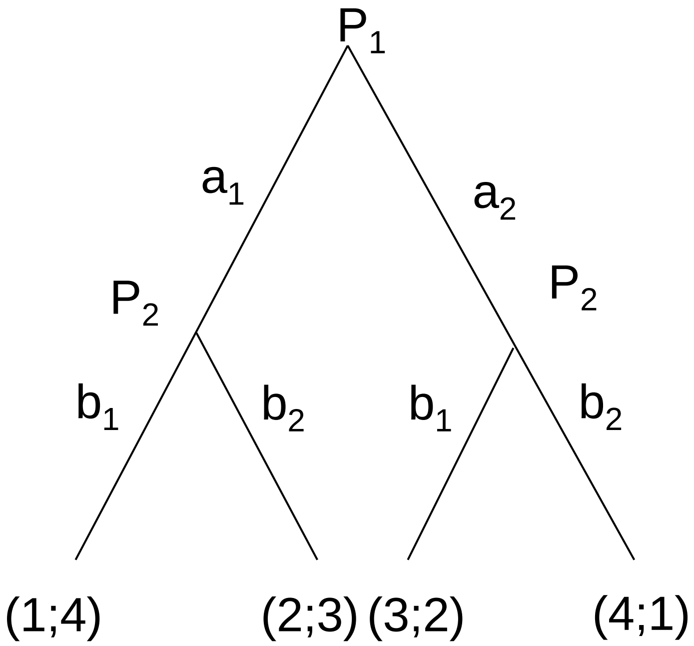


P_1's strategies: a_1, a_2

P_2's strategies: b_1b_1, b_1b_2, b_2b_1, b_2b_2

Perfect-Information Games

Extensive Form- Game Tree



P_1's strategies: a_1, a_2

P_2's strategies: b_1b_1, b_1b_2,
b_2b_1, b_2b_2

Normal (Strategic) Form- Payoff Matrix?

		P_2			
		b_1b_1	b_1b_2	b_2b_1	b_2b_2
P_1	a_1	1;4	1;4	2;3	2;3
	a_2	3;2	4;1	3;2	4;1

Imperfect-Information Games

Extensive Form- Game Tree ?

Normal (strategic) Form- Payoff Matrix

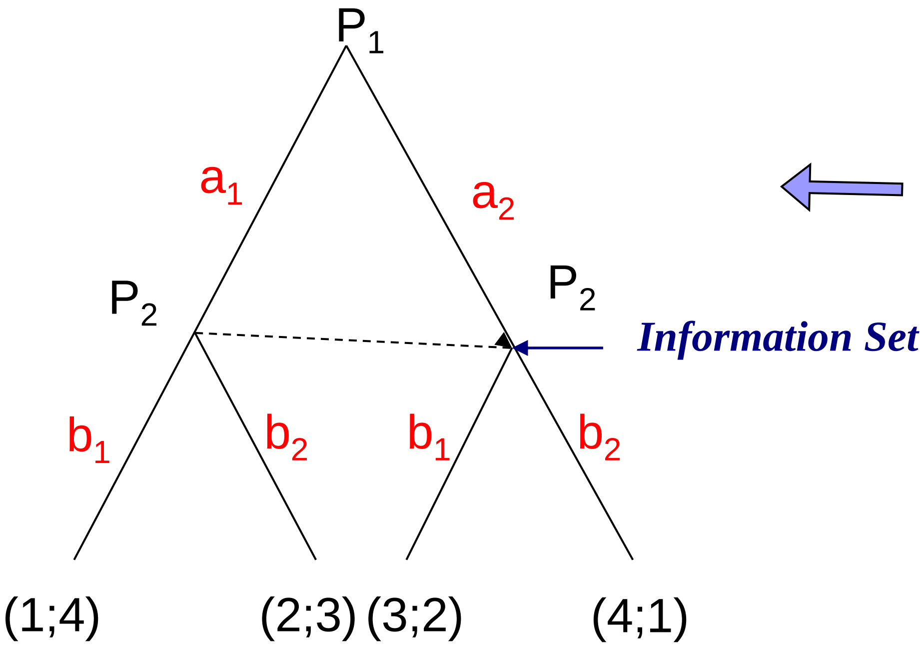
$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{P_2} \\ & & b_1 & b_2 \\ \hline & a_1 & 1 ; 4 & 2 ; 3 \\ \cline{3-4} P_1 & & & \\ & a_2 & 3 ; 2 & 4 ; 1 \\ \hline \end{array}$$

P_1's strategies: a_1, a_2

P_2's strategies: b_1, b_2

Imperfect-Information Games

Extensive Form- Game Tree ?



Normal (strategic) Form- Payoff Matrix

		P_2	
		b_1	b_2
P_1	a_1	1 ; 4	2 ; 3
	a_2	3 ; 2	4 ; 1

P_1's strategies: a_1, a_2

P_2's strategies: b_1, b_2

*Player i's **information set** at any particular point of the game is the set of different nodes in the game tree that he knows might be the actual node, but between which he cannot distinguish by direct observation.*

Strategies (cont.)

Player i's **strategy** s_i is a *rule* that tells him which action to choose at each instant of the game, given his information set.

Player i's **strategy set** or strategy space $S_i = \{s_i\}$ is the set of strategies available to him.

A **strategy combination (profile)** $s = (s_1,\ldots,s_n)$ is an ordered set consisting of one strategy for each of the n players in the game.

There are 2 DIFFERENT definitions of a strategy:

A **strategy** for a player is a full description of what he will do in every possible contingency that:

A) can possibly arise given his own plans;

B) can possibly arise in the game;

Strategies (cont.)

There are 2 DIFFERENT definitions of a strategy:

A **strategy** for a player is a full description of what he will do in every possible contingency that:

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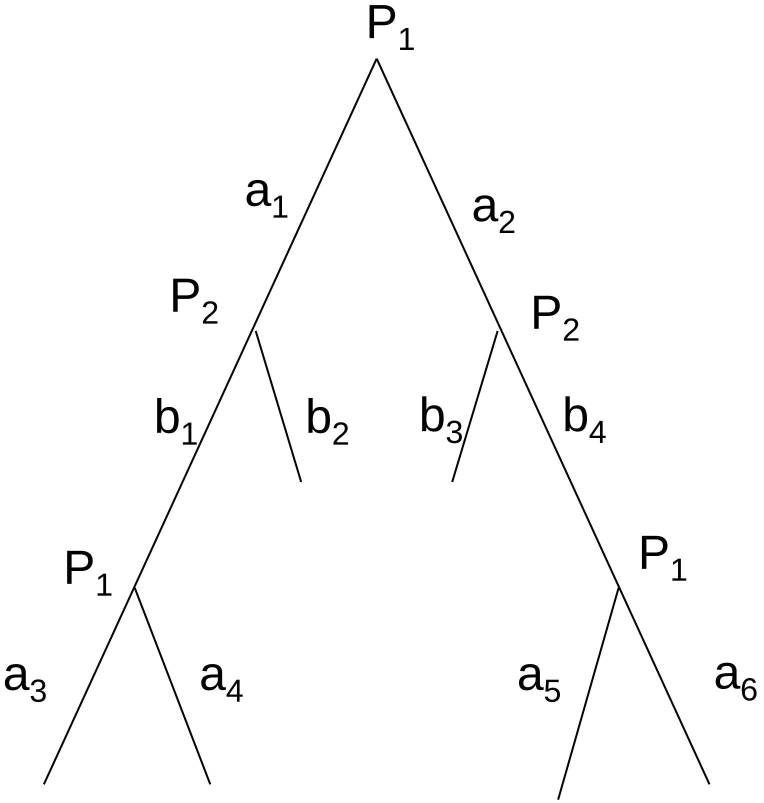
B) Can possibly arise in the game;



Type A:

P_1: a_1a_3
a_1a_4
a_2a_5
a_2a_6

Type A:

P_2: b_1b_3
b_1b_4
b_2b_3
b_2b_4

Type B:

P_1: $a_1a_3a_5$
$a_1a_3a_6$
$a_1a_4a_5$
$a_1a_4a_6$
$a_2a_3a_5$
$a_2a_3a_6$
$a_2a_4a_5$
$a_2a_4a_6$

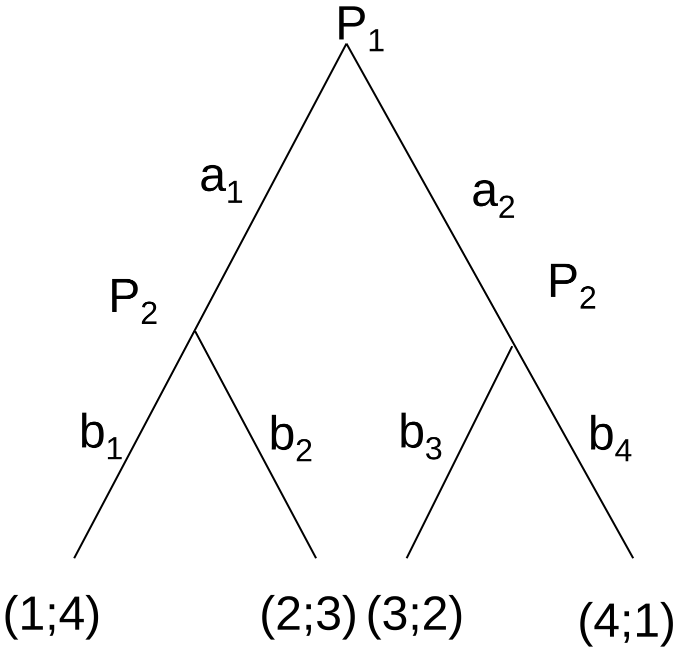
Type B:

P_2: b_1b_3
b_1b_4
b_2b_3
b_2b_4

Solution concept

Perfect-Information Games

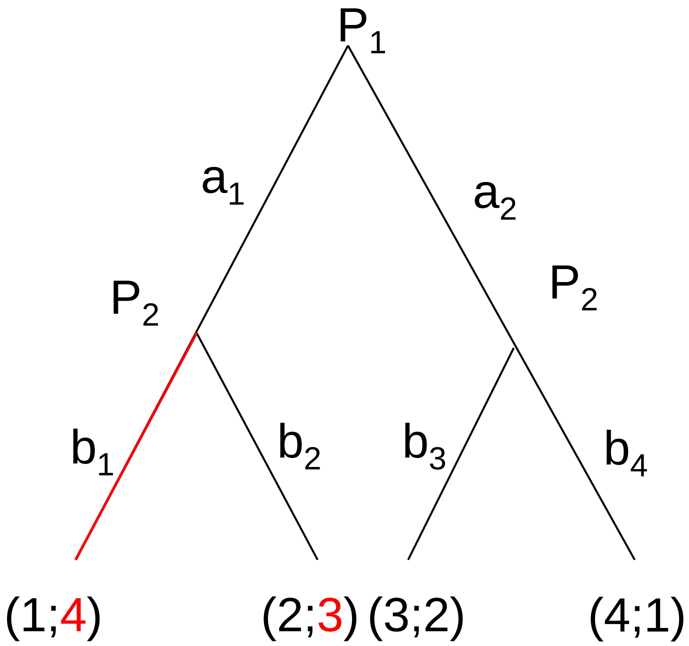
Backward induction



Solution concept

Perfect-Information Games

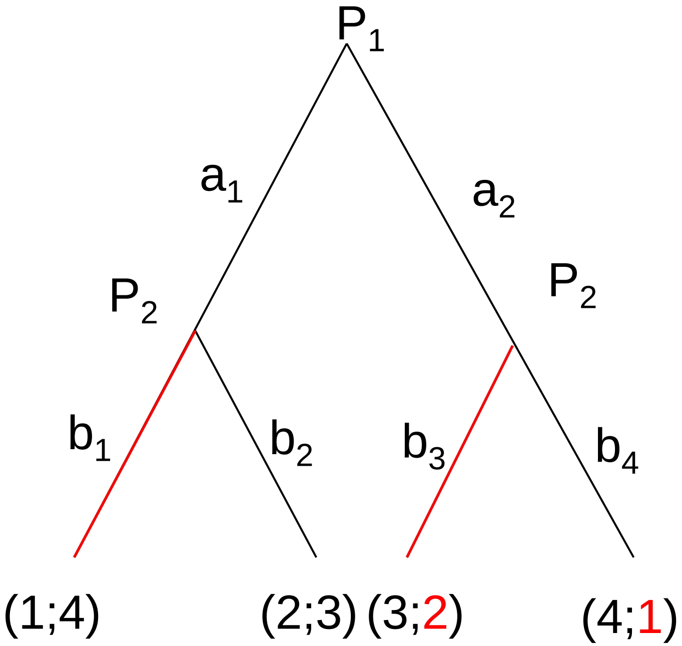
Backward induction



Solution concept

Perfect-Information Games

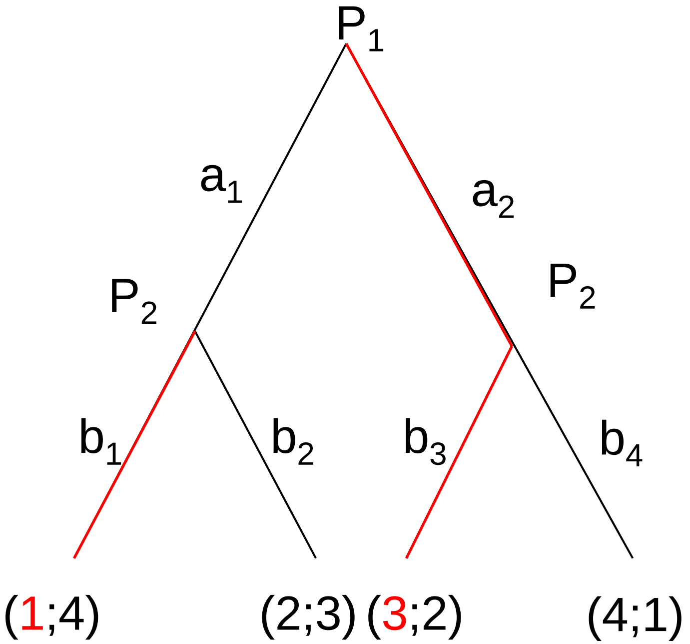
Backward induction



Solution concept

Perfect-Information Games

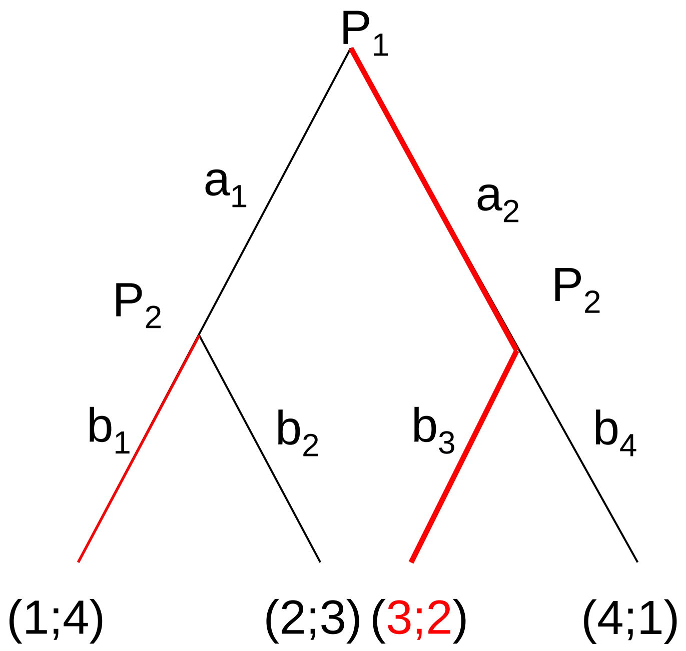
Backward induction



Solution concept

Perfect-Information Games

Backward induction



BI outcome: $P_1 \rightarrow a_2$; $P_2 \rightarrow b_3$
BI strategies profile: $(a_2; b_1 b_3)$
Payoffs: (3;2)

Backward induction:

1. Determine the optimal actions at final decision nodes.
2. Given that these will be the actions taken at the final decision nodes, determine the optimal actions at the next-to-last decision nodes, and so on backward through the game tree.
3. Contineous path of the optimal actions from the starting node to the end node shows BI outcome of the game.

Outcome*- prediction of Players' actual behavior*
Strategies **-** *prediction of Players' complete contingent plans*

Solution concept

Players iteratively delete strictly dominated strategies.

A strategy is **strictly dominated** if there is some alternative strategy, that yields a greater payoff regardless of what the other players do.

Imperfect-Information Games

Iterated strict dominance (ISD)

		P_2	
		b_1	b_2
P_1	a_1	1 ; 4	2 ; 3
	a_2	3 ; 2	4 ; 1

Solution concept

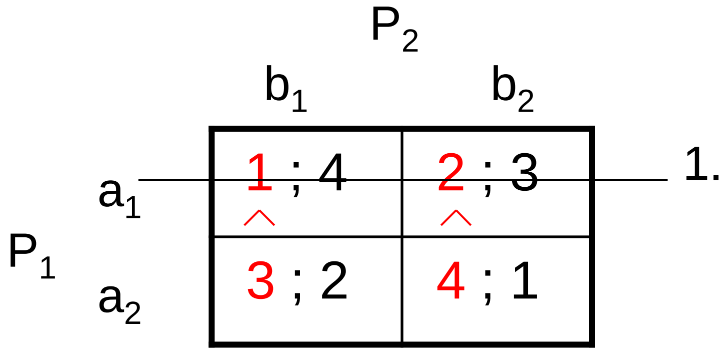
Imperfect-Information Games

Players iteratively delete strictly dominated strategies.

A strategy is **strictly dominated** if there is some alternative strategy, that yields a greater payoff regardless of what the other players do.

1. a_1 is strictly dominated (SDed) by a_2 .

Iterated strict dominance (ISD)



		P_2		
		b_1	b_2	
P_1	~~a_1~~	~~1 ; 4~~ ^	~~2 ; 3~~ ^	1.
	a_2	3 ; 2	4 ; 1	

Solution concept

Imperfect-Information Games

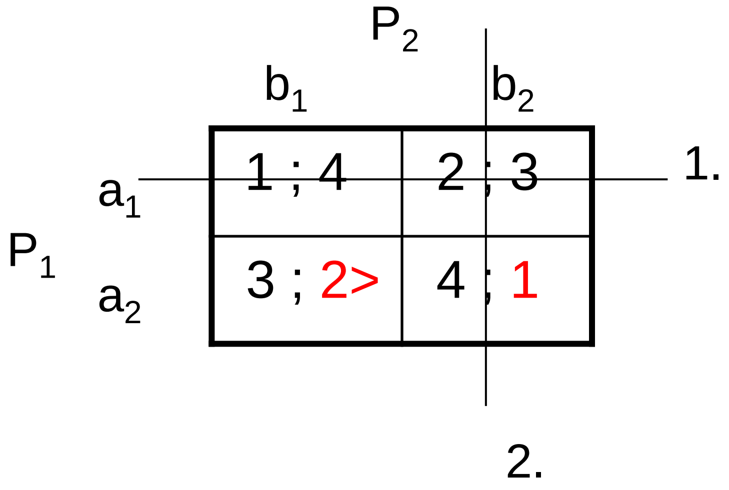
Iterated strict dominance (ISD)

Players iteratively delete strictly dominated strategies.

A strategy is **strictly dominated** if there is some alternative strategy, that yields a greater payoff regardless of what the other players do.

1. a_1 is strictly dominated (SDed) by a_2 .

2. b_2 is strictly dominated (SDed) by b_1 .



		P_2	
		b_1	b_2
P_1	a_1	1 ; 4	2 ; 3
	a_2	3 ; 2>	4 ; 1

Solution concept

Imperfect-Information Games

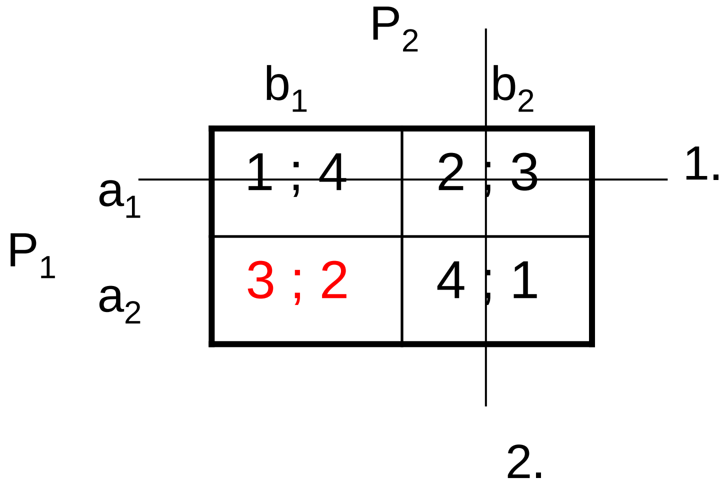
Players iteratively delete strictly dominated strategies.

A strategy is **strictly dominated** if there is some alternative strategy, that yields a greater payoff regardless of what the other players do.

1. a_1 is strictly dominated (SDed) by a_2 .

2. b_2 is strictly dominated (SDed) by b_1 .

Iterated strict dominance (ISD)



		P_2	
		b_1	b_2
P_1	a_1	1 ; 4	2 ; 3
	a_2	3 ; 2	4 ; 1

Outcome: $(a_2; b_1)$
Payoffs: (3;2)

Solution concept

Imperfect-Information Games

Players iteratively delete strategies, which are never best response (but not necessary are strictly dominated by some other strategy) using common knowledge of rationality:

Each player is rational.
Each players knows that each player is rational.
Each... that each player is rational,
Each... and infinitum

Rationalizability

P_2

		b_1	b_2
	a_1	3;0	0;0
P_1	a_2	1;0	1;0
	a_3	0;0	3;0

Solution concept

Imperfect-Information Games

Rationalizability

There is no strategy which is strictly dominated by some other strategy

a_1 is optimal if $P_2 \rightarrow b_1$
a_3 is optimal if $P_2 \rightarrow b_2$

Are there ANY beliefs that a_2 is optimal?

		P_2	
		b_1	b_2
P_1	a_1	$\underline{3};\underline{0}$	$0;\underline{0}$
	a_2	$1;\underline{0}$	$1;\underline{0}$
	a_3	$0;\underline{0}$	$\underline{3};\underline{0}$

Solution concept

Imperfect-Information Games

Rationalizability

Expected utility:

$EU_1(a_1)=3q$
$EU_1(a_2)=1$
$EU_1(a_3)=3-3q$

$EU_1(a_1)=EU_1(a_2) \Rightarrow 3q=1 \Rightarrow \mathbf{q=1/3}$
$EU_1(a_3)=EU_1(a_2) \Rightarrow 3-3q=1 \Rightarrow \mathbf{q=2/3}$

if $q>1/3$ $a_1>a_2$
if $q<2/3$ $a_3>a_2$

$\Rightarrow$ a_2 is never best response
$\Rightarrow$ **a_2 is not rationalizable**

P_2

P_1

	b_1	b_2
a_1	$\underline{3};\underline{0}$	$0;\underline{0}$
~~a_2~~	~~$1;\underline{0}$~~	~~$1;\underline{0}$~~
a_3	$0;\underline{0}$	$\underline{3};\underline{0}$
	q	$1-q$

Solution concept

Imperfect-Information Games

Rationalizability

		P_2	
		b_1	b_2
	a_1	$\underline{3}$;$\underline{0}$	0;$\underline{0}$
P_1	~~a_2~~	~~1;$\underline{0}$~~	~~1;$\underline{0}$~~
	a_3	0;$\underline{0}$	$\underline{3}$;$\underline{0}$
		q	1-q

Expected utility:

$EU_1(a_1)=3q$
$EU_1(a_2)=1$
$EU_1(a3)=3-3q$

$EU_1(a_1)=EU_1(a_2) \Rightarrow 3q=1 \Rightarrow \mathbf{q=1/3}$
$EU_1(a_3)=EU_1(a_2) \Rightarrow 3-3q=1 \Rightarrow \mathbf{q=2/3}$

if $q>1/3$ $a_1>a_2$
if $q<2/3$ $a_3>a_2$

$\Rightarrow$ a_2 is never best response
$\Rightarrow$ **a_2 is not rationalizable**

This conclusion depends on payoffs!

Solution concept

There is no strategy which is strictly dominated by some other strategy

a_1 is optimal if $P_2 \rightarrow b_1$
a_3 is optimal if $P_2 \rightarrow b_2$

Are there ANY beliefs that a_2 is optimal?

Imperfect-Information Games

Rationalizability

P_2

		b_1	b_2
	a_1	$\underline{3};\underline{0}$	$0;\underline{0}$
P_1	a_2	$2;\underline{0}$	$2;\underline{0}$
	a_3	$0;\underline{0}$	$\underline{3};\underline{0}$

Solution concept

Imperfect-Information Games

Rationalizability

$$\begin{array}{cc|c|c|} & & \multicolumn{2}{c}{P_2} \\ & & b_1 & b_2 \\ \hline & a_1 & \underline{3};\underline{0} & 0;\underline{0} \\ \hline P_1 & a_2 & 2;\underline{0} & 2;\underline{0} \\ \hline & a_3 & 0;\underline{0} & \underline{3};\underline{0} \\ \hline & & q & 1-q \end{array}$$

Expected utility:

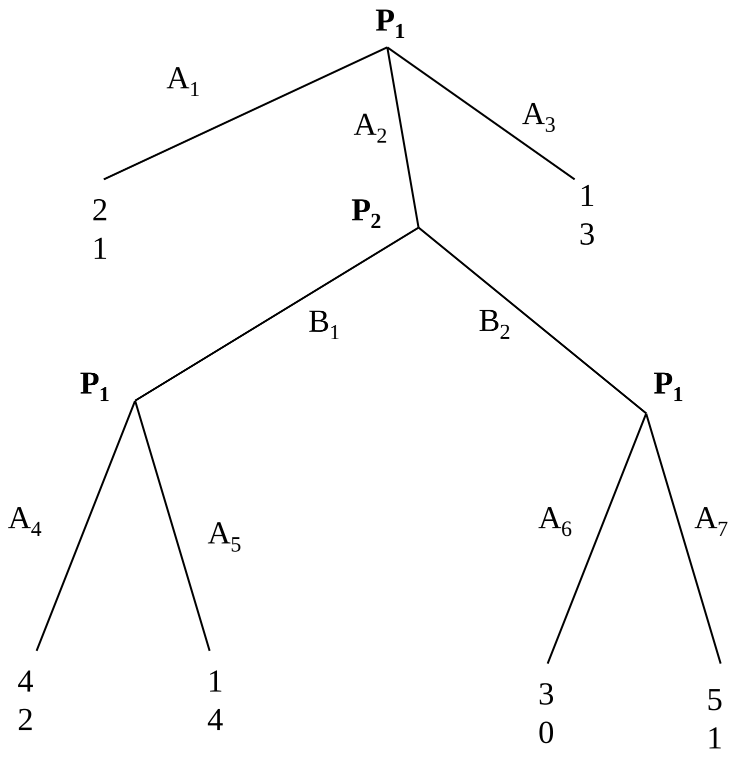
$EU_1(a_1)=3q$
$EU_1(a_2)=2$
$EU_1(a3)=3-3q$

$EU_1(a_1)=EU_1(a_2) \Rightarrow 3q=2 \Rightarrow \mathbf{q=2/3}$
$EU_1(a_3)=EU_1(a_2) \Rightarrow 3-3q=2 \Rightarrow \mathbf{q=1/3}$

if $q>2/3$ $a_1>a_2$
if $q<1/3$ $a_3>a_2$

$\Rightarrow$ a_2 is optimal if $2/3>q>1/3$
$\Rightarrow$ **a_2 is rationalizable**

Homework 1.1: Consider the game



1.1.1 **Find backward-induction outcome** of this game. Write the payoffs which players receive at the end of the game.

1.1.2 **Write** down all possible **strategies of type A** for both players

1.1.3 **Write** down all possible **strategies of type B** for both players.

1.1.4 Which strategies (of type B) are consistent with backwards induction in this game?

1.1.5 **Write this game in normal (strategic) form** using the strategies of type A.

1.1.6 **List**, for each player, the **set of strategies consistent with rationalizability** (When you eliminate any strategy, explain clearly why you eliminate it.)

Homework 1.2: Consider the game

		P_2			
		b_1	b_2	b_3	b_4
P_1	a_1	6; 1	1; 2	5; 4	-1; -2
	a_2	5; 5	4; 2	2; 1	0; -1
	a_3	1; 2	0; 6	2; 3	2; -3
	a_4	0;-1	-4; 0	0; 1	-6; 0

1.2.1 **List**, for each player, the **set of strategies consistent with rationalizability** (When you eliminate any strategy, explain clearly why you eliminate it.)

1.2.2 **Write** this game in the **extensive form (game tree).**