FACULTY
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HABILITATION THESIS

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Representations of Distributive Algebraic Lattices

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To My Parents

Contents

Introduction

All monoids in the thesis are supposed to be commutative. The *stable equivalence* on a monoid $\boldsymbol{M}$, denoted by $\sim_s$, is the least congruence on $\boldsymbol{M}$ such that the quotient $\boldsymbol{M}_s := \boldsymbol{M}/\sim_s$ is cancellative. The congruence is defined by $x \sim_s y$ if there exists $z \in \boldsymbol{M}$ such that $x + z = y + z$, for all $x, y \in \boldsymbol{M}$. The correspondence $\boldsymbol{M} \mapsto \boldsymbol{M}_s$ extends canonically to a functor that we denote by $(-)_s$.

$$
\begin{tikzcd}
\textbf{Directed Abelian groups} \arrow[d, "(-)^+"] \arrow[dddd, bend left=60, "\mathrm{Id}"] \\
\textbf{Cancellative monoids} \\
\textbf{Monoids} \arrow[u, "(-)_s"'] \arrow[uu, bend left=60, "(-)_*"] \arrow[d, "\nabla"] \arrow[dd, bend right=60, "\mathrm{Id}"'] \\
\langle\vee,\mathbf{0}\rangle\textbf{-semilattices} \arrow[d, bend left, "\mathrm{Id}"] \\
\textbf{Algebraic lattices} \arrow[u, bend left, "(-)_c"]
\end{tikzcd}
$$

FIGURE 1. Partially ordered Abelian groups, monoids, and algebraic lattices

There is an universal map $(-)_*\colon \boldsymbol{M} \to \boldsymbol{M}_*$ sending monoids to Abelian groups. Moreover the algebraic order on a monoid $\boldsymbol{M}$ induces a partial order on the target Abelian group $\boldsymbol{M}_*$; such that the image of the monoid corresponds to the positive cone of $\boldsymbol{M}_*$. The construction of the partially ordered Abelian group $\boldsymbol{M}_*$ for a given monoid $\boldsymbol{M}$ is an analogy of the construction of the field of fractions of a given commutative ring. We consider the set of formal differences between pairs of elements from $\boldsymbol{M}$ and an equivalence relation, say $\sim_*$, on them. The equivalence is given by $x - y \sim_* z - u$ provided that there is $w \in \boldsymbol{M}$ such that $x + u + w = z + y + w$. The map $(-)_*$ is dermined by $x \mapsto [x - 0]_{\sim_*}$, $x \in \boldsymbol{M}$. Again, the correspondence is canonically functorial. Notice that the partially ordered Abelian group $\boldsymbol{M}_*$ is directed, that is, it is, as a group, generated by the positive cone. It is straightforward to see that this is equivalent to the partial order on $\boldsymbol{M}_*$ being upwards directed.

Let $\boldsymbol{G}^+ := \{p \in \boldsymbol{G} \mid 0 \leq p\}$ denote the positive cone of a partially ordered Abelian group $\boldsymbol{G}$. Observing that an order preserving homomorphism $\boldsymbol{G} \to \boldsymbol{H}$ maps the positive cone $\boldsymbol{G}^+$ of $\boldsymbol{G}$ into the positive cone $\boldsymbol{H}^+$ of $\boldsymbol{H}$, we see that there is a functor $(-)^+$ from the category of partially ordered Abelian groups to monoids. Moreover, the composition $(-)^+ \circ (-)_*$ is naturally equivalent to the functor $\sim_s$.

We denote by $\asymp$ the least congruences on $\boldsymbol{M}$ such that $\boldsymbol{M}/{\asymp}$ is a $\langle\vee,\mathbf{0}\rangle$-semilattice and we set $\nabla(\boldsymbol{M}) := \boldsymbol{M}/{\asymp}$. As in the previous cases, the correspondence $\boldsymbol{M} \to \nabla(\boldsymbol{M})$ extends a functor.

The ideal lattice $\mathrm{Id}(\boldsymbol{\mathcal{S}})$ of a $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}$ is an algebraic lattice and, conversely, compact elements of an algebraic lattice $\boldsymbol{\mathcal{L}}$ form a $\langle\vee,\mathbf{0}\rangle$-semilattice, denoted by $\boldsymbol{\mathcal{L}}_c$. Both the correspondences extend to functors that are inverse to each other (up to obvious natural equivalences).

Here are more *ideal-type* functors to consider. Firstly, the functor that assigns to a monoid $\boldsymbol{M}$ the algebraic lattice $\mathrm{Id}(\boldsymbol{M})$ of all o-ideals of $\boldsymbol{M}$. Secondly, the functor $\boldsymbol{G} \mapsto \mathrm{Id}(\boldsymbol{G}^+)$ which assigns to a directed Abelian group the algebraic lattice of all convex subgroups of $\boldsymbol{G}$.

All the introduced functors are depicted in Figure 1. Note that the diagram of functors is commutative (up to natural equivalences).

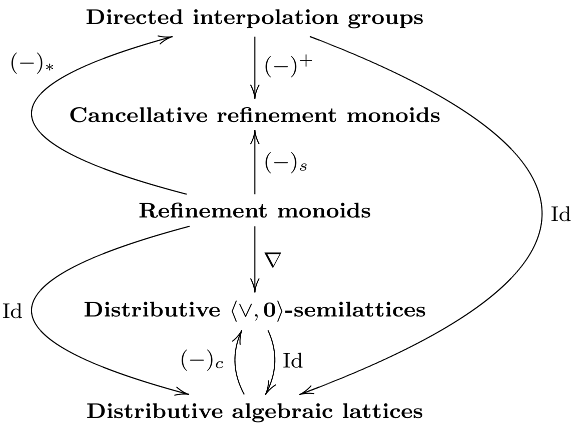

FIGURE 2. Directed interpolation groups, refinement monoids, and distributive algebraic lattices

We will be interested in structures that are mapped by the ideal functor Id to algebraic lattices that are distributive. Starting from the bottom of Figure 2, these are distributive $\langle\vee,\mathbf{0}\rangle$-semilattice (cf. [**27**, Section II.5]). Indeed, a $\langle\vee,\mathbf{0}\rangle$-semilattice is distributive if and only if $\mathrm{Id}(\boldsymbol{\mathcal{S}})$ is an algebraic distributive lattice. Next we consider the class of refinement monoids, i.e, the conical monoids that satisfy the Riesz refinement property. The maximal semilattice quotient $\nabla(\boldsymbol{M})$ of a refinement monoid $\boldsymbol{M}$ is a distributive $\langle\vee,\mathbf{0}\rangle$-semilattice and the lattice $\mathrm{Id}(\boldsymbol{M})$ of all o-ideals of $\boldsymbol{M}$ is distributive (cf. [**25**, lemma 2.4]. Finally, a directed Abelian group $\boldsymbol{G}$ is an interpolation group if and only if the positive cone $\boldsymbol{G}^+$ is a refinement monoid [**21**, Prop. 2.1]. In particular, the lattice $\mathrm{Id}(\boldsymbol{G})$ of all ideals (i.e, convex subgroups) of a directed interpolation group is again distributive.

There are more structures in the picture as we tried to depict in Figure 3. Given a ring $\boldsymbol{R}$, we denote by $\boldsymbol{V}(\boldsymbol{R})$ the monoid of all isomorphism classes

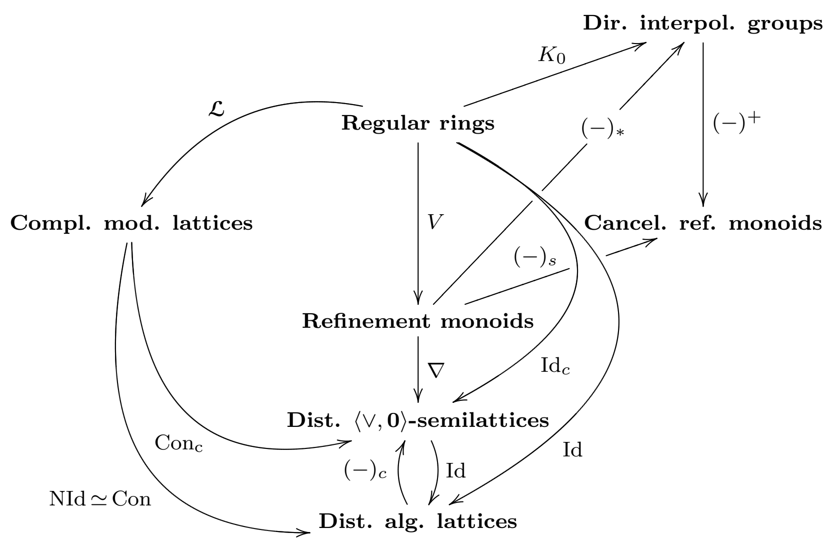

FIGURE 3. Regular rings, refinement monoids, and distributive $\langle\vee,\mathbf{0}\rangle$-semilattices

of finitely generated projective right $\boldsymbol{R}$-modules with addition derived from direct sums. If the ring $\boldsymbol{R}$ is (Von-Neumann) regular, the monoid $\boldsymbol{V}(\boldsymbol{R})$ satisfies the Riesz refinement property (see [**22**, Corollary 2.7]). The partially ordered Abelian group $\boldsymbol{V}(\boldsymbol{R})_*$, denoted by $K_0(\boldsymbol{R})$, is called the *Grothendieck group* of $\boldsymbol{R}$. When we limit ourselves unital rings, it is appropriate to assign to a ring $\boldsymbol{R}$ a partially ordered Abelian group $K_0(\boldsymbol{R})$ with an order-unit corresponding to the isomorphism class $[\,\boldsymbol{R}\,]$ and study the category of partially ordered Abelian groups with order units (cf. [**22**, Chapter 15]). If the ring $\boldsymbol{R}$ is regular, then $K_0(\boldsymbol{R})$ is a directed interpolation group.

We denote by $\mathcal{L}(\boldsymbol{R})$ the $\langle\vee,\mathbf{0}\rangle$-semilattice of all right finitely generated ideals of a ring $\boldsymbol{R}$. For a regular ring, the $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{L}(\boldsymbol{R})$ is closed under finite meets, therefore $\mathcal{L}(\boldsymbol{R})$ forms a lattice [**22**, Theorem 2.3]. Moreover, the lattice $\mathcal{L}(\boldsymbol{R})$ is modular and sectionally complemented (complemented if $\boldsymbol{R}$ is with an unit element).

Congruences of sectionally complemented modular lattices correspond to their neutral ideals (see [**27**, Section III.3.10]). In particular, if $\boldsymbol{R}$ is a regular ring, then the lattice $\mathrm{Con}(\mathcal{L}(\boldsymbol{R}))$ is isomorphic to the lattice $\mathrm{NId}(\mathcal{L}(\boldsymbol{R}))$ of all neutral ideals of $\mathcal{L}(\boldsymbol{R})$. By [**78**, Lemma 4.2], an ideal of the lattice $\mathcal{L}(\boldsymbol{R})$ (for a regular ring $\boldsymbol{R}$) is neutral if and only if it contains with each $a\boldsymbol{R}$ all principal ideals $b\boldsymbol{R}$ with $b\boldsymbol{R} \simeq a\boldsymbol{R}$. It follows that $\mathrm{Con}(\mathcal{L}(\boldsymbol{R})) \simeq \mathrm{NId}(\mathcal{L}(\boldsymbol{R})) \simeq \mathrm{Id}(\boldsymbol{R})$ (see [**78**, Lemma 4.3]), and so, the lattice $\mathrm{Id}(\boldsymbol{R})$ of two-sided ideals of a regular ring $\boldsymbol{R}$ is distributive. Moreover, combining [**78**, Corollary 4.4 and Proposition 4.6] we get the isomorphisms $\mathrm{Con}_c(\mathcal{L}(\boldsymbol{R})) \simeq$

$\nabla(\boldsymbol{V}(\boldsymbol{R})) \simeq \mathrm{Id}_c(\boldsymbol{R})$ of distributive $\langle\vee,\mathbf{0}\rangle$-semilattices, for every regular ring $\boldsymbol{R}$.

We have seen that a distributive algebraic lattice that is isomorphic to the lattice of two-sided ideals of a regular ring is at the same time isomorphic to the congruence lattice of a modular sectionally complemented lattice. This brings a connection with the *Congruence lattice problem*, whether every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice. The conjecture has an interesting history (see [**86**]) and remained open four over sixty years until the counter-example was found by F. Wehrung [**83**]. We will discuss the Congruence Lattice Problem in detail in Chapter 3.

In this thesis we study various *representation problems*, namely for distributive algebraic lattices (resp. corresponding distributive $\langle\vee,\mathbf{0}\rangle$-semilattices), refinement monoids, or directed Abelian groups. For example, we ask whether a given distributive algebraic lattice (or any algebraic lattice with particular properties) is isomorphic to a lattice of all two sided ideals of a regular ring, respectively, as a lattice of all compact subgroups of a directed Abelian group. We might also restrict to some class of regular rings as, for example, locally matricial algebras, or to some class of directed Abelian groups, for example, dimension groups.

A more complex question is when we seek for a functorial solution, that is, when we ask not only for representing a single object but for lifting particular diagrams. Given a diagram $\Delta\colon \mathbf{J}\to\mathbf{C}$ and a functor $\Psi\colon \mathbf{B}\to\mathbf{C}$, a *lifting* of Δ with respect to Ψ is a functor $\Phi\colon \mathbf{J}\to\mathbf{B}$ such that the composition $\Psi\circ\Phi$ is naturally equivalent to Δ.

The thesis consists of six chapters, each based on a single paper and related to a particular realization or lifting problem.

Chapter 1 is based on the paper [**66**]:

> *Liftings of distributive lattices by locally matricial algebras with respect to the* Id_c *functor*, Algebra Universalis **55** (2006), 239 – 257.

In the paper we study liftings with respect to the functor Id_c from the category of locally matricial algebras to the category of distributive $\langle\vee,\mathbf{0}\rangle$-semilattices. The problem goes back to [**9**]. In the unpublished notes G. Bergman proved that

- every countable distributive $\langle\vee,\mathbf{0}\rangle$-semilattice,
- every strongly distributive $\langle\vee,\mathbf{0}\rangle$-semilattice (i.e., a $\langle\vee,\mathbf{0}\rangle$-semilattice of all compact elements of the lattice of all hereditary subsets of a poset),

are isomorphic to the $\langle\vee,\mathbf{0}\rangle$-semilattices of all finitely generated two-sided ideals of locally matricial algebras. In [**64**] we developed a new construction and besides reproving the Bergman's results we have realized every distributive $\langle\vee,\mathbf{0}\rangle$-semilattices that is closed under finite meets, and so it forms a distributive lattice, as the $\langle\vee,\mathbf{0}\rangle$-semilattice of all finitely generated two-sided ideals of a locally matricial. In the presented paper [**66**] we simplify the construction from [**64**] and study possibilities of functorial solutions of the problem. We construct

- a simple finite subcategory $\mathbf{D}_{\bullet\!-}$ of the category **DLat** of all distributive $\langle \mathbf{0}, \mathbf{1}\rangle$-lattices,
- a subcategory $\mathbf{D}_{\curlywedge}$ of **DLat** corresponding to a partially ordered proper class, which cannot be lifted with respect to the Id_c functor.

On the positive side we prove that every diagram in **DLat** indexed by a partially ordered set and the subcategory $\mathbf{DLat_m}$ of **DLat** whose objects are all distributive $\langle \mathbf{0}, \mathbf{1}\rangle$-lattices and whose morphisms are $\langle \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$-embeddings can be lifted with respect to the Id_c functor.

Let us mention some applications of the results:

- The realization of distributive $\langle \vee, \mathbf{0}\rangle$-semilattices closed under finite meets by $\langle \vee, \mathbf{0}\rangle$-semilattices of all finitely generated ideals of locally matricial algebras answers the Γ-invariant realization problem from [**17**]. Given an uncountable cardinal κ we let $\boldsymbol{\mathcal{B}}_\kappa := \mathcal{P}(\kappa)/\operatorname{club}_\kappa$ denote the Boolean algebra of all subsets of κ modulo the filter $\operatorname{club}_\kappa$ generated by all closed unbounded subsets of κ. A $\mathbf{0}$-lattice $\boldsymbol{\mathcal{L}}$ is strongly dense if the poset of its non-zero elements contains a cofinal strictly decreasing chain. The dimension of a strongly dense $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$ is the minimum length of a cofinal strictly decreasing chain in $\boldsymbol{\mathcal{L}}$. Given a strongly dense modular $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$ of an uncountable dimension κ with a cofinal strictly decreasing chain $\mathcal{A} = \langle \boldsymbol{a}_\alpha \mid \alpha < \kappa\rangle$, we set

$$E(\mathcal{A}) := \{\alpha < \kappa \mid \exists \beta \in (\alpha, \kappa] \colon \boldsymbol{a}_\alpha \text{ is not complemented over } \boldsymbol{a}_\beta\},$$

  where $\boldsymbol{a}_\alpha$ is complemented over $\boldsymbol{a}_\beta$ if there exists $\boldsymbol{b} \in \boldsymbol{\mathcal{L}}$ such that $\boldsymbol{a}_\alpha \wedge \boldsymbol{b} = \boldsymbol{a}_\beta$ and $\boldsymbol{a}_\alpha \vee \boldsymbol{b} = \mathbf{1}$. The Γ-invariant of the $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$ is the block $\overline{E(\mathcal{A})} \in \boldsymbol{\mathcal{B}}_\kappa$. The block does not depend on the choice of the cofinal strictly decreasing chain $\mathcal{A}$ (cf. [**17**]). According to [**17**, Theorem 1.3], there is a distributive strongly dense $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}_{\overline{E}}$ of dimension κ with a Γ-invariant $\overline{E}$, for every $\overline{E} \in \boldsymbol{\mathcal{B}}_\kappa$. Passing to the ideal lattice $\mathrm{Id}(\boldsymbol{\mathcal{L}}_{\overline{E}})$, we get a distributive algebraic strongly dense $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice of dimension κ with the Γ-invariant $\overline{E}$. Applying [**64**, Theorem 4.7] or Theorem 7.1 from Chapter 1, we conclude that the lattice $\mathrm{Id}(\boldsymbol{\mathcal{L}}_{\overline{E}})$ is isomorphic to the lattice of all two-sided of a locally-matricial $\Bbbk$-algebra $\boldsymbol{R}$, where the field $\Bbbk$ can be chosen arbitrarily. Then $\boldsymbol{S} := \boldsymbol{R} \otimes_{\Bbbk} \boldsymbol{R}^{\mathrm{op}}$, where $\boldsymbol{R}^{\mathrm{op}}$ denotes the opposite ring to $\boldsymbol{R}$, is again a locally matricial $\Bbbk$-algebra, due to [**17**, Lemma 2.1]. The original $\Bbbk$-algebra $\boldsymbol{R}$ is naturally a right $\boldsymbol{S}$-module wia the multiplication given by $a \cdot (b \otimes c) = cab$. Observing that two-sided ideals of the $\Bbbk$-algebra $\boldsymbol{R}$ bijectively correspond to submodules of the right $\boldsymbol{S}$-module $\boldsymbol{R}$, we conclude that each algebraic distributive lattice that is realized as the lattice of two-sided ideals of a locally matricial algebra is realized as a submodule lattice of a module over a locally matricial algebra. In particular, all Γ-invariants are realized.

- The other application of the result is related to the Congruence Lattice Problem. In [**69**] E. T. Schmidt proved that every distributive $\mathbf{0}$-lattice is an image of a generalized Boolean lattice under a distributive $\langle\vee,\mathbf{0}\rangle$-homomorphism, and consequently, it is isomorphic to $\mathrm{Con_c}(\boldsymbol{\mathcal{L}})$ for a lattice $\boldsymbol{\mathcal{L}}$. Later, in [**72**] (see [**71**] for an earlier weaker result), E. T. Schmidt proved that every finite distributive lattice is the congruence lattice of a complemented modular lattice. Applying our construction, we infer that every distributive $\langle\mathbf{0},\mathbf{1}\rangle$-lattice is isomorphic to $\mathrm{Con_c}(\boldsymbol{\mathcal{L}}(\boldsymbol{R}))$ for a locally matricial algebra $\boldsymbol{R}$, hence its ideal lattice is representable as the congruence lattice of a complemented modular lattice. The unit element is not essential in the construction, and so we can easily get every distributive $\mathbf{0}$-lattice is isomorphic to the $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathrm{Con_c}(\boldsymbol{\mathcal{L}})$ for a sectionally complemented modular lattice $\boldsymbol{\mathcal{L}}$. This gives the result first obtained by P. Pudlák [**61**]. The Pudlák's approach provides a functorial solution and his results are directly (and independently) extended by Theorem 7.1.

Let us note that a different approach to the representations of distributive $\mathbf{0}$-lattices as $\mathrm{Id}_c(\boldsymbol{R})$ of locally matricial algebras $\boldsymbol{R}$, similar to the Bergman's constructins [**9**], is in [**57**] by M. Ploščica.

Chapter 2 is based on the paper [**68**]:

> *Distributive congruence lattices of congruence-permutable algebras*, Journal of Algebra **311** (2007), 96 – 116.

The paper is a joint work with Jiří Tůma and Friedrich Wehrung. It closely follows and extends results from [**60**] and [**74**]. In the earlier paper [**78**] F. Wehrung defined the *congruence splitting property* of lattices. The class of congruence splitting lattices (i.e. lattices satisfying the congruence splitting property) is closed under direct limits and it contains all sectionally complemented, all relatively complemented lattices, and all atomistic lattices. The distributive $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathbf{S}_\kappa$ (for $\kappa\geq\aleph_2$) constructed in [**77**] is not isomorphic to the $\langle\vee,\mathbf{0}\rangle$-semilattice of all compact congruences of any congruence splitting lattice. Since relatively complemented lattices are congruence splitting, the $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathbf{S}_\kappa$ (for $\kappa\geq\aleph_2$) is not isomorphic to $\mathrm{Con_c}(\boldsymbol{\mathcal{L}}(\boldsymbol{R}))$ (and, consequently, to $\mathrm{Id}_c(\boldsymbol{R})$) for any regular ring $\boldsymbol{R}$.

It was in [**78**], where a *uniform refinement property* was used for the first time. This is an infinite system of join-semilattice (or monoid) equations based on the Riesz refinement property that are satisfied for a certain class of join-semilattices, the $\langle\vee,\mathbf{0}\rangle$-semilattices of compact congruences of congruence splitting lattices in this case, and that do not hold for some $\langle\vee,\mathbf{0}\rangle$-semilattice, here $\mathbf{S}_\kappa$. Similar strategy was applied in [**60**], [**74**], and also in our paper [**68**].

The observation that congruence splitting lattices have permutable congruences lays behind [**74**]. Applying a variant of the uniform refinement property, J. Tůma and F. Wehrung proved that $\mathrm{Con_c}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$, where $\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa)$ denotes the free lattice in a non-distributive lattice variety $\mathcal{V}$ with $\kappa \geq \aleph_2$ generators, is not isomorphic to the $\langle \vee, \mathbf{0} \rangle$-semilattice of all compact congruences of any lattice with almost permutable congruences.

In the presented paper we show, using yet another modification of the uniform refinement property, that the $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathrm{Con_c}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$ is not isomorphic to the $\langle \vee, \mathbf{0} \rangle$-semilattice of all compact congruences of any algebra with almost permutable congruences. In particular, the algebraic distributive lattice $\mathrm{Con}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$ is isomorphic neither to the normal subgroup lattice of a group, nor to the submodule lattice of a module, nor the lattice of convex subgroups of a lattice-ordered group. These three cases are discussed separately and in the first two of them, the cardinal bound $\aleph_2$ (for the set of compact elements of the algebraic distributive lattice) is proved to be optimal. The negative result is obtained by proving that the algebraic distributive lattice $\mathrm{Con}(\boldsymbol{\mathcal{F}}_{\mathcal{V}}(\kappa))$ is not the range of any distance satisfying the V-condition of type 3/2.

We also study the functorial solution of the problem. We consider the category $\mathcal{D}$ of all surjective distances with morphisms being pairs of one-to-one maps and the forgetful functor Π from $\mathcal{D}$ to the category of $\langle \vee, \mathbf{0} \rangle$-semilattice with $\langle \vee, \mathbf{0} \rangle$-embeddings. On one side, we prove that the restriction of the functor Π to the V-distances of type 2 (i.e, the distances satisfying the V-condition of type 2) has a left inverse. On the other hand we find an unliftable cube by V-ditances of type 3/2. Similar examples are studied in [**74**]. The mysterious connection between sizes of counter-examples for representation problems and dimensions of unliftable cubes was later ingeniously explained by P. Gillibert and F. Wehrung, see [**38**].

Chapter 3 is based on the paper [**67**]:

> *Free trees and the optimal bound in Wehrung's theorem*,
> Fund. Math. **198** (2008), 217 – 228.

Following G. Birkhoff and O. Frink [**11**], the congruence lattice of a lattice is algebraic and due to N. Funayama and T. Nakayama [**20**] it is distributive. In early forties P. Dilworth observed that every finite distributive lattice is representable as a congruence lattice of a finite lattice and conjectured that every algebraic distributive lattice is isomorphic to the congruence lattice of a lattice. The conjecture, named as the *Congruence Lattice Problem*, shortly **CLP**, turned to be a prominent open problem of the lattice theory for over sixty years.

Many partial results was obtained, see [**27**, Appendix C] and the survey paper [**75**] until a counter-example was constructed by F. Wehrung [**83**]. The Wehrung's counter-example has $\aleph_{\omega+1}$ compact elements. In Chapter 3 we improve the size of the counter-example construcitng a distributive $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-semilattice of size $\aleph_2$ such that is not the range of a weakly distributive

$\langle\vee,\mathbf{0}\rangle$-homomorphism from $\operatorname{Con}_c \boldsymbol{A}$ with 1 in its image, for any algebra $\boldsymbol{A}$ with either a congruence-compatible structure of a $\langle\vee,\mathbf{1}\rangle$-semilattice or a congruence-compatible structure of a lattice. In particular, our $\langle\vee,\mathbf{0}\rangle$-semilattice is not isomorphic to the $\langle\vee,\mathbf{0}\rangle$-semilattice of compact congruences of any lattice. Thus we provide a conter-example to **CLP** of the lowest possible cardinality. The main ingredient of our proof is the modification of Kuratowski's Free Set Theorem, which involves what we call *free trees.*

- Chapter 4 is based on the paper [**65**]:

  Countable chains of distributive lattices as maximal semilattice quotients of positive cones of dimension groups, Comment. Math. Univ. Carolin. **47** (2006), 11 – 20.

The Grothendieck group $K_0(\boldsymbol{R})$ of a regular ring $\boldsymbol{R}$ is a directed preordered Abelian group with interpolation. If the ring $\boldsymbol{R}$ is unit-regular, then $K_0(\boldsymbol{R})$ is partially ordered and the positive cone $K_0^+(\boldsymbol{R})$ corresponds to the monoid $\boldsymbol{V}(\boldsymbol{R})$ of isomorphism classes of finitely generated projective right $\boldsymbol{R}$-modules.

Recall that a partially ordered Abelian group $\boldsymbol{G}$ is *unperforated* if $np \geq 0$ implies that $p \geq 0$ for all $p \in \boldsymbol{G}$. A dimension group is an unperforated directed partially ordered Abelian group with interpolation. A simplicial directed Abelian group is a free abelian group of a finite rank n with a basis, say, $p_1, \dots, p_n$ with the positive cone $\mathbb{Z}^+p_1 \times \cdots \times \mathbb{Z}^+p_n$. Dimension groups are exactly direct limits of simplicial directed Abelian groups in the category of pre-ordered Abelian groups (with order-preserving group homomorphisms) [**16**, Theorem 2.2].

Let us fix a field $\mathbb{F}$. Locally matricial $\mathbb{F}$-algebras are unit-regular and their Grothendieck groups are dimension groups. Following [**22**, Chapter 15], we call direct limits of countable chains of matricial $\mathbb{F}$-algebras *ultramatricial*, and countable dimension groups *ultrasimplicial.* By [**22**, Theorem 15.24], every ultrasimplicial group appears as the Grothendieck group of an ultramatricial $\mathbb{F}$-algebra and the ultramatricial $\mathbb{F}$-algebra is determined by its Grothendieck group up to the Morita-equivalence [**22**, Corollary 15.27]. The first part of this correspondence extends to dimension groups of size $\aleph_1$, due to [**24**]. In particular, every dimension group of size at most $\aleph_1$ is represented as the Grothendieck group of a locally matricial $\mathbb{F}$-algebra. On the other hand, Grothendieck groups of size $\aleph_1$ do not determine the locally matricial algebras up to the Morita equivalence as in the countable case (see [**22**, Example 15.28]). In [**77**] there is constructed a dimension group of size $\aleph_2$ that is not isomorphic to the Grothendieck group of any regular ring.

As depicted in Figure 1, if $\boldsymbol{R}$ is an unit-regular ring, we have the isomorphisms $\operatorname{Id}(K_0(\boldsymbol{R})) \simeq \operatorname{Id}(\boldsymbol{R})$. The question, whether every distributive $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathbb{S}$ is isomorphic to $\nabla(\boldsymbol{G}^+)$ for some dimension group $\boldsymbol{G}$ was stated as [**37**, Problem 1]. We solved this problem in [**63**], where we constructed a counter-example of size $\aleph_2$. Since every countable distributive $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathbb{S}$ is isomorphic to the maximal semilattice quotient of the positive cone of a dimension group (see [**37**, Theorem 5.2]), only the case of cardinality $\aleph_1$ remained open. This was resolved by F. Wehrung [**80**], who constructed a distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice $\mathbf{S}_{\omega_1}$ of size $\aleph_1$ that is not isomorphic to $\nabla(\boldsymbol{M})$ for any Riesz monoid with an order-unit of finite stable rank. This readily implies that the $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice $\mathbf{S}_{\omega_1}$ is not realized as the maximal semilattice quotient of the positive cone of any dimension group. As in some previously discussed constructions, he found a variant of the uniform refinement property, here denoted by $\mathbf{URP}_{\mathrm{sr}}$, that holds in any Riesz monoid $\boldsymbol{M}$ with order-unit of finite stable rank but that is not satisfied by $\mathbf{S}_{\omega_1}$.

It follows from [**80**, Corollary 7.2] that every direct limit of a countable sequence of distributive lattices and $\langle\vee,\mathbf{0}\rangle$-homomorphisms satisfies $\mathbf{URP}_{\mathrm{sr}}$ and it was stated as [**80**, Problem 1], whether such a direct limit is isomorphic to $\nabla(\boldsymbol{G}^+)$ for a dimension group $\boldsymbol{G}$. Recall that every distributive $\langle\vee,\mathbf{0}\rangle$-semilattice closed under finite meets is isomorphic to $\mathrm{Id}_c(\boldsymbol{R})$ for a locally-matricial algebra $\boldsymbol{R}$ and consequently to $\nabla(K_0(\boldsymbol{R})^+)$ for the dimension group $K_0(\boldsymbol{R})$ due to [**64**]. In Chapter 4 we give a negative answer to this question by constructing an increasing countable chain of Boolean join-semilattices, with all inclusion maps being $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-homomorphisms, whose union cannot be represented as the maximal semilattice quotient of the positive cone of any dimension group. Furthermore, we construct a similar example with a countable chain of strongly distributive bounded join-semilattices.

Chapter 5 is based on the paper [**62**]:

> *On the construction and the realization of wild monoids*,
> to appear in Archivum Mathematicum (Brno).

Many still open problems about the structure of regular rings have reformulations in terms of the corresponding monoids $\boldsymbol{V}(\boldsymbol{R})$ of isomorphism classe of finitely generated projective right $\boldsymbol{R}$-modules. Let us say that a monoid $\boldsymbol{M}$ is *realizable* (by a regular ring $\boldsymbol{R}$) if $\boldsymbol{M}\simeq\boldsymbol{V}(\boldsymbol{R})$. According to [**22**, Theorem 2.8], all such monoids are refinement monoids. The fundamental problem by K. R. Goodearl [**23**] asks which refinement monoids are realizable. By [**77**] there are non-realizable refinement monoids of cardinality $\aleph_2$ but there is not yet known a non-realizable refinement monoid of size $<\aleph_2$. Particularly interesting question is whether all countable refinement monoids admit realization, indeed, the answer would shed light on a number of related problems regarding regular rings or C^*-algebras.

Some comprehensive positive results were obtained so far, namely the realization of monoids of row finite quivers [**4**, Theorems 4.2 and 4.4] and the realization of finitely generated primitive monoids with all primes free [**3**, Theorem 2.2]. These realizations are obtained via direct limit construction and the monoids can be realized by regular $\mathbb{F}$-algebras over an arbitrary field $\mathbb{F}$. On the other hand there are countable refinement monoids realizable by regular $\mathbb{F}$-algebras over a countable field $\mathbb{F}$ but not over any uncountable field (see [**2**, Sec. 4]).

Many positive realization results (in general context) are obtained by direct limit construction from diagrams of finitely generated (or even finite) objects, e.g., every distributive $\langle\vee, \mathbf{0}\rangle$-semilattice is a direct limit of finite distributive $\langle\vee, \mathbf{0}\rangle$-semilattices (cf. [**61**, Fact 4 on p. 100]). This is not the case of refinement monoids. Following [**5**] we call a refinement monoid *time* provided that it is a direct limit of finitely generated refinement monoids and *wild* otherwise. The existence of wild refinement monoids indicates that the Goodeatl's fundamental problem is essentially distinct from the other, seemingly similar, realization problems.

An prominent example of a wild refinement monoid is due to G. Bergman and K. R. Goodearl [**22**, Examples 4.26 and 5.10]. We study the example, develop elementary methods of computing the monoids $\boldsymbol{V}(\boldsymbol{R})$ for directly-finite regular rings $\boldsymbol{R}$, and construct a class of directly finite non-cancellative refinement (therefore wild) monoids realizable by regular algebras over an arbitrary field.

Chapter 6 is based on the paper [**50**]:

> *A maximal Boolean sublattice that is not the range of a Banaschewski function*, to appear in Algebra Universalis.

This paper is a joint work with Samuel Mokriš.

A Banaschewki function on a bounded lattice $\boldsymbol{\mathcal{L}}$ is a map $\beta\colon \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}}$ such that $\boldsymbol{a} \leq \boldsymbol{b}$ implies $\beta(\boldsymbol{b}) \leq \beta(\boldsymbol{a})$ and $\mathbf{1} = \boldsymbol{a} \oplus \beta(\boldsymbol{a})$, for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{\mathcal{L}}$. The terminology is motivated by the early result of B. Banaschewski that the subspace lattice of a vector space admits such a map. Simultaneously we can define a Banaschewski function on a ring $\boldsymbol{R}$ as a map $f\colon \boldsymbol{R} \to \mathrm{Idem}(\boldsymbol{R})$ such that $a\boldsymbol{R} = f(a)\boldsymbol{R}$ and $a\boldsymbol{R} \subseteq b\boldsymbol{R}$ implies that $f(a) \trianglelefteq f(b)$, for all $a, b \in \boldsymbol{R}$. (Here $e \trianglelefteq f$ means that $e = ef = fe$, for all $e, f \in \mathrm{Idem}(\boldsymbol{R})$.) A connection between these two notions of the Banaschewski function is established by [**84**, Lemma 3.5]: An unital regular ring $\boldsymbol{R}$ admits a Banaschewski function if and only if the complemented modular lattice $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ does.

A notion replacing Banachewski function for lattices without a maximal element is a *Banaschewski measure* [**84**, Definition 5.5]. Every countable sectionally complemented lattice has a Banaschewski measure due to [**84**, Corollary 5.6].

Yet another notion related to the Banaschewski function and the Banaschewski measure is a Banschewski trace [**84**, Definition 5.1]. In [**84**, Section 6] F. Wehrung discovered a close connection between exitence of Banschewski traces (resp. Banschewski measures) and coorinatizability of sectionally complemented modular lattices. This connection is applied in [**85**] in order to construct a non-cordinatizable sectionally complemented modular lattice of size $\aleph_1$ with a large 4-frame. The example shows that the variant of the *Jónson's coordinatization theorem* that states that sectionally complemented modular lattices $\boldsymbol{\mathcal{L}}$ with large n-frames, for $n \geq 4$, and with a contable cofinal chain is coordinatizable (see [**45**]) does not hold for larger cardinalities.

We study ranges of Banaschewski functions on countable complemented modular lattices. According to [**84**, Theorem 4.1 and Corollary 4.8], a countable complemented modular lattice $\mathcal{L}$ has a Banaschewski function with a Boolean range and all the Boolean ranges of Banschewski functions on the lattice $\mathcal{L}$ are isomorphic maximal Boolean sublattices of $\mathcal{L}$. In [**84**, Problem 2] it is asked whether every maximal Boolean sublattice of a countable complemented modular lattice $\mathcal{L}$ appears as a range of some Banaschewski function and whether the maximal Boolean sublattices of $\mathcal{L}$ are isomorphic. We construct a countable complemented modular lattice $\mathcal{S}$ with two non-isomorphic maximal Boolean sublattices $\mathcal{H}$ and $\mathcal{G}$ and we represent the lattice $\mathcal{H}$ as the range of a Banaschewski function on $\mathcal{S}$. Furthermore, we prove that the lattice $\mathcal{S}$ is coordinatizable, in spite of not containing a 3-frame. We show that the lattices $\mathcal{H}$ and $\mathcal{G}$ correspond to maximal Abelian (regular) subalgebras of the regular algebra $\boldsymbol{S}$ realizing the lattice $\mathcal{S}$.

Basic concepts

We summarize basic concepts, more specific notions will be introduced in each chapter.

Set theoretic notions. We will use the standard set theoretic notation and terminology. We denote by $\mathcal{P}(W)$ the set of all subsets of a set W. Furthermore, we denote by $[\,W\,]^{<\omega}$ the set of all finite subsets of W and by $[\,W\,]^n$ the set of all n-element subset of W. We use $|W|$ to denote the cardinality of the set W.

Given a map $f\colon U \to V$, we will use the same notation $f\colon \mathcal{P}(U) \to \mathcal{P}(V)$ for the map sending $X \mapsto f(X) = \{f(x) \mid x \in X\}$, for all $X \subseteq U$, and we denote by $f^{-1}\colon \mathcal{P}(V) \to \mathcal{P}(U)$ the map defined by $Y \mapsto \{u \in U \mid f(u) \in Y\}$, for all $Y \subseteq V$.

We denote by **On** the class of all ordinal numbers. We identify each ordinal number with the set of its predecessors, in particular, $n := \{0, \dots, n-1\}$, for each finite ordinal (i.e, non-negative integer) n. We denote by ω the first infinite ordinal, and by ω_n the first ordinal of size $\aleph_n$, for every positive integer n. As in [**83**], we put $\varepsilon(n) := n \mod 2$, for every integer n.

We denote by $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$, the set of all, all non-negative, and all positive integers, respectively. We use $\mathbb{Q}$, and $\mathbb{Q}^+$, to denote the set of all rational numbers, and the set of all positive rational numbers, respectively. We denote by $\mathbb{R}$ the field of real numbers and by $\mathbb{R}_+ := \{r \in \mathbb{R} \mid 0 \leq r\}$ the set of all non-negative reals.

Partially ordered sets. Let $\langle P, \leq \rangle$ be a partially ordered set. A subset D of P is *dense in* P provided that for every $p \in P$, there is $d \in D$ with $d \leq p$. Given $p, q \in P$, we write $p \perp q$ if there is no element of P smaller than both the elements p and q. A partially ordered set P is *upwards directed* provided that for each finite $F \subseteq P$, there is $p \in P$ such that $f \leq p$ for all $f \in F$.

A subset H of a partially ordered set P is called *hereditary* (or *lower*) provided that $p \leq h$ implies that $p \in H$, for all $h \in H$ and $p \in P$. Dually, a subset C of P is called *co-hereditary* (or *upper*) provided that $c \leq p$ implies $p \in C$ for all $c \in C$ and $p \in P$.

For a subset X of a partially ordered set P, we denote by $\downarrow_P(X)$, (resp. $\uparrow_P(X)$), the least hereditary, (resp. co-hereditary), subset of P containing X. For a singleton set $\{x\} = X \subseteq P$ we write $\downarrow_P(x)$, (resp. $\uparrow_P(x)$), instead of $\downarrow_P(X)$, (resp. $\uparrow_P(X)$).

Let $\langle Q, \leq \rangle$ be a partial ordered set and $P \subseteq Q$. We denote by $\operatorname{Her}(P, \leq)$ the lattice (necessarily distributive) of all hereditary subsets of P. We will use the notation $\operatorname{Her}(P)$ when the order $\leq$ is understood.

Category theory. Given a category $\mathbf{C}$ and objects $a, b \in \mathbf{C}$, we denote by $\hom_{\mathbf{C}}(a, b)$ the collection of all morphisms from a to b. The identity morphism $a \to a$ in the category $\mathbf{C}$ is denoted by $\mathbf{1}_a$.

As in [**75**, Section 5], a *diagram* in a category $\mathbf{C}$ is a functor $\Delta\colon \mathbf{J} \to \mathbf{C}$, where $\mathbf{J}$ is a small category. Often the category $\mathbf{J}$ will correspond to a partially ordered set.

Let $\Delta\colon \mathbf{J} \to \mathbf{C}$ and $\Phi\colon \mathbf{B} \to \mathbf{C}$ be functors. We say that a functor $\Psi\colon \mathbf{J} \to \mathbf{B}$ *lifts* Δ *with respect to* Φ provided that the composition $\Phi \circ \Psi$ is naturally equivalent to the functor Θ (See Figure 4). In particular, if Δ is an inclusion functor, we say that Ψ *lifts* $\mathbf{J}$ with respect to Φ. Note that our definition of the "lifting of functors" corresponds to the definition of the *lifting of diagrams* in [**75**, p. 455].

$$\begin{tikzcd}
\mathbf{J} \arrow[rr, "\Psi"] \arrow[dr, "\Delta"'] & \arrow[d, phantom, "\simeq"] & \mathbf{B} \arrow[dl, "\Phi"] \\
& \mathbf{C} &
\end{tikzcd}$$

FIGURE 4. The lifting of a functor

Recall that a *directed system* in a category $\mathbf{C}$ is a diagram $\Delta\colon P \to \mathbf{C}$, where P is (a category represented by) an upwards directed poset. We will call a colimit of a directed system a *direct limit*. We say that a functor $\Phi\colon \mathbf{C} \to \mathbf{D}$ *preserves direct limits* provided that Φ maps the direct limit of a directed system Δ in $\mathbf{C}$ to the direct limit of $\Phi \circ \Delta$ in $\mathbf{D}$ (see [**10**, Definition 7.8.1]).

Congruences and universal algebra. For a subset F of an algebra $\boldsymbol{A}$, we denote by $\Theta_{\boldsymbol{A}}(F)$, or by $\Theta(F)$ if $\boldsymbol{A}$ is understood, the least congruence of $\boldsymbol{A}$ that identifies all elements of the set F. In particular, given elements $\mathbf{x}, \mathbf{y} \in \boldsymbol{A}$, we denote by $\Theta_{\boldsymbol{A}}(\mathbf{x}, \mathbf{y})$ (or by $\Theta(\mathbf{x}, \mathbf{y})$ if $\boldsymbol{A}$ is understood) the least congruence of $\boldsymbol{A}$ identifying $\mathbf{x}$ and $\mathbf{y}$. Furthermore, in case $\boldsymbol{A}$ is a lattice, we put $\Theta^{+}_{\boldsymbol{A}}(\mathbf{x}, \mathbf{y}) := \Theta_{\boldsymbol{A}}(\mathbf{x} \wedge \mathbf{y}, \mathbf{x})$. We denote by $\operatorname{Con} \boldsymbol{A}$ the lattice of all congruences of $\boldsymbol{A}$ and by $\operatorname{Con_c} \boldsymbol{A}$ the $\langle \vee, \mathbf{0} \rangle$-semilattice of all compact (i.e., finitely generated) congruences of $\boldsymbol{A}$.

Let $\boldsymbol{A}$ be an algebra and $\Theta \in \operatorname{Con} \boldsymbol{A}$. We say that an n-ary operation ϕ on $\boldsymbol{A}$ is Θ-*compatible* if $\langle \mathbf{x}_i, \mathbf{y}_i \rangle \in \Theta$, for all $i = 0, \dots, n-1$, implies that

$$\langle \phi(\mathbf{x}_0, \dots, \mathbf{x}_{n-1}), \phi(\mathbf{y}_0, \dots, \mathbf{y}_{n-1}) \rangle \in \Theta,$$

for all $\mathbf{x}_i, \mathbf{y}_i \in \boldsymbol{A}$, $i = 0, \dots, n-1$. We say that the n-ary operation ϕ is *congruence compatible* provided that ϕ is Θ-compatible for all $\Theta \in \operatorname{Con} \boldsymbol{A}$ (cf. [**58, 83**]). In particular, a semilattice operation $\vee$, resp. $\wedge$, on $\boldsymbol{A}$ is congruence-compatible if $\langle \mathbf{x}, \mathbf{y} \rangle \in \Theta$ implies that $\langle \mathbf{x} \vee \mathbf{z}, \mathbf{y} \vee \mathbf{z} \rangle \in \Theta$, resp. $\langle \mathbf{x} \wedge \mathbf{z}, \mathbf{y} \wedge \mathbf{z} \rangle \in \Theta$, for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \boldsymbol{A}$ and all $\Theta \in \operatorname{Con} \boldsymbol{A}$.

We say that the algebra $\boldsymbol{A}$ has *permutable congruences* provided that $\Theta \vee \Phi = \Theta \circ \Phi = \Phi \circ \Theta$, for all $\Theta, \Phi \in \operatorname{Con} \boldsymbol{A}$.

Lattices and join-semilattices. A nonzero element $\boldsymbol{x}$ of a join-semilattice $\boldsymbol{\mathcal{S}}$ is called *join-irreducible* if $\boldsymbol{x} = \boldsymbol{y} \vee \boldsymbol{z}$ implies that $\boldsymbol{x} = \boldsymbol{y}$ or $\boldsymbol{x} = \boldsymbol{z}$ for all $\boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{\mathcal{S}}$. We denote by $J(\boldsymbol{\mathcal{S}})$ the partially ordered set of all join-irreducible elements of a join-semilattice $\boldsymbol{\mathcal{S}}$.

A $\langle \vee, \mathbf{0} \rangle$-semilattice $\boldsymbol{\mathcal{S}}$ is *distributive* if for every $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \boldsymbol{\mathcal{S}}$ satisfying $\boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}$, there are $\boldsymbol{a}' \leq \boldsymbol{a}$ and $\boldsymbol{b}' \leq \boldsymbol{b}$ such that $\boldsymbol{a}' \vee \boldsymbol{b}' = \boldsymbol{c}$. A distributive $\langle \vee, \mathbf{0} \rangle$-semilattice in which every element is a finite join of join-irreducible elements will be called *strongly distributive*.

A join-homomorphism $\boldsymbol{h} \colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ is called *weakly distributive* at $\boldsymbol{x} \in \boldsymbol{\mathcal{S}}$, if for all $\boldsymbol{y}_0, \boldsymbol{y}_1 \in \boldsymbol{\mathcal{T}}$ such that $\boldsymbol{h}(\boldsymbol{x}) \leq \boldsymbol{y}_0 \vee \boldsymbol{y}_1$, there are $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \boldsymbol{\mathcal{S}}$ such that $\boldsymbol{x} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1$ and $\boldsymbol{h}(\boldsymbol{x}_i) \leq \boldsymbol{y}_i$, for all $i < 2$ (see [**83**]). The homomorphism $\boldsymbol{h}$ is *weakly distributive* if it is weakly distributive at every element of $\boldsymbol{\mathcal{S}}$.

Commutative monoids. All monoids are supposed to be commutative and they will be written additively. A monoid $\boldsymbol{M}$ is equipped with the *algebraic preordering*: $x \leq_{\boldsymbol{M}} y$ provided that there is $z \in \boldsymbol{M}$ such that $x + z = y$, for all $x, y \in \boldsymbol{M}$. We denote by $\equiv_{\boldsymbol{M}}$ the equivalence relation induced by the algebraic preordering $\leq_{\boldsymbol{M}}$; that is, $x \equiv_{\boldsymbol{M}} y$ provided that $x \leq_{\boldsymbol{M}} y$ and $y \leq_{\boldsymbol{M}} x$, for all $x, y \in \boldsymbol{M}$. We might drop the subscript $_{\boldsymbol{M}}$ when the monoid $\boldsymbol{M}$ is understood.

The class of all $\langle \vee, \mathbf{0} \rangle$-semilattices coincides with the class of all commutative monoids with all elements idempotent. On the other hand, for every commutative monoid $\boldsymbol{M}$, there exists a least congruence, denoted by $\asymp$, on $\boldsymbol{M}$ such that $\boldsymbol{M}/{\asymp}$ is a $\langle \vee, \mathbf{0} \rangle$-semilattice (see [**25**]). We set $\nabla(\boldsymbol{M}) := \boldsymbol{M}/{\asymp}$. We call the monoid $\nabla(\boldsymbol{M})$ the *maximal semilattice quotient* of $\boldsymbol{M}$. The correspondence $\boldsymbol{M} \to \nabla(\boldsymbol{M})$ naturally extends to a direct limits preserving functor from the category of all commutative monoids to the category of all $\langle \vee, \mathbf{0} \rangle$-semilattices [**25**]. Given $x \in \boldsymbol{M}$, we denote by $\boldsymbol{x}$ the corresponding element in $\nabla(\boldsymbol{M})$.

A monoid $\boldsymbol{M}$ is *conical* provided that $x + y = 0 \implies x = y = 0$, for all $x, y \in \boldsymbol{M}$. A monoid $\boldsymbol{M}$ satisfies the *Riesz refinement property* provided that whenever $x_1 + x_2 = y_1 + y_2$ in $\boldsymbol{M}$, there are elements $z_{ij} \in \boldsymbol{M}$, $i, j = 1, 2$, such that

$$x_i = z_{i1} + z_{i2} \text{ and } y_j = z_{1j} + z_{2j}, \tag{0.1}$$

for all $i, j \in 1, 2$. A *refinement monoid* is a conical monoid satisfying the Riesz refinement property.

We say that $\boldsymbol{M}$ is *a Riesz monoid* provided that for all $x, y, z \in \boldsymbol{M}$ such that $x \leq y + z$, there are $y' \leq y$ and $z' \leq z$ in $\boldsymbol{M}$ satisfying $x = y' + z'$. Every commutative monoid satisfying the Riesz refinement property is a Riesz monoid while the converse is not true in general. Note that for join-semilattices, i.e., monoids in which every element is an idempotent, these two properties coincide. Observe also that $\langle \vee, \mathbf{0} \rangle$-semilattices satisfying the refinement property are exactly *distributive* $\langle \vee, \mathbf{0} \rangle$-semilattices (cf. [**27**, Section II.5]).

A monoid $\boldsymbol{M}$ satisfies the *interpolation property* provided that for all $x_i, y_j \in \boldsymbol{M}$, $i, j = 1, 2$, with $x_i \leq_{\boldsymbol{M}} y_j$, for all $i, j \in \{1, 2\}$, there is $z \in \boldsymbol{M}$ such that $x_i \leq_{\boldsymbol{M}} z \leq_{\boldsymbol{M}} y_j$, for all $i, j \in 1, 2$. A cancellative conical monoid is a refinement monoid if and only if it satisfies the interpolation property [**21**, Proposition 2.1]. In general, there are refinement monoids that do not satisfy the interpolation property (cf. [**51**] and Chapter 4, Section 4).

Rings and modules. A ring $\boldsymbol{R}$ is *(von Neumann) regular*[1] provided that for every $a \in \boldsymbol{R}$ there is $b \in \boldsymbol{R}$ such that $aba = a$. There are many characterizations of regular rings. Probably the most prominent one is that a ring $\boldsymbol{R}$ is regular if and only if each right (resp. left) finitely generated ideal of $\boldsymbol{R}$ is generated by an idempotent [**22**, Theorem 1.1].

An ideal $\boldsymbol{I}$ of a ring $\boldsymbol{R}$ is *regular* if for each element $a \in \boldsymbol{I}$, there is $b \in \boldsymbol{I}$ with $a = aba$. By [**22**, Lemma 1.3], an ideal of a regular ring is regular; in fact, a ring $\boldsymbol{R}$ is regular if and only if both $\boldsymbol{R}/\boldsymbol{I}$ and $\boldsymbol{I}$ are regular, for every ideal $\boldsymbol{I}$ of $\boldsymbol{R}$.

An *Abelian regular ring* is a ring $\boldsymbol{R}$ whose all idempotents are central. For various characterizations of Abelian regular rings see [**22**, Theorem 3.2]. A *maximal Abelian regular subalgebra* of a regular algebra $\boldsymbol{R}$ is an Abelian regular subalgebra of $\boldsymbol{R}$ that is not properly contained in any Abelian regular subalgebra of the ring $\boldsymbol{R}$.

Given a ring $\boldsymbol{R}$, we denote by $\mathrm{FP}(\boldsymbol{R})$ the class of all finitely generated projective right $\boldsymbol{R}$-modules. Given $\boldsymbol{R}$-modules $\boldsymbol{A}$ and $\boldsymbol{B}$, the notation $\boldsymbol{A} \leq \boldsymbol{B}$ means that $\boldsymbol{A}$ is a submodule of $\boldsymbol{B}$ and $\boldsymbol{A} \lesssim \boldsymbol{B}$ denotes that the module $\boldsymbol{A}$ is isomorphic to a submodule of $\boldsymbol{B}$. We will use the notation $\boldsymbol{A} \leq^{\oplus} \boldsymbol{B}$, resp. $\boldsymbol{A} \lesssim^{\oplus} \boldsymbol{B}$, to denote that $\boldsymbol{A}$ is a direct summand of $\boldsymbol{B}$, resp. that $\boldsymbol{A}$ is isomorphic to a direct summand of $\boldsymbol{B}$.

An element e of a ring $\boldsymbol{R}$ is an *idempotent* if $e = ee$. We denote by $\mathrm{Idem}(\boldsymbol{R})$ the set of all idempotents in the ring $\boldsymbol{R}$. Idempotents e and f are *orthogonal* provided that $ef = fe = 0$.

Given a ring $\boldsymbol{R}$ and right $\boldsymbol{R}$-modules $\boldsymbol{A}$ and $\boldsymbol{B}$, we denote by $\hom_{\boldsymbol{R}}(\boldsymbol{A}, \boldsymbol{B})$ the set of all $\boldsymbol{R}$-linear maps $\boldsymbol{A} \to \boldsymbol{B}$. We denote by $\mathbf{0}$ the zero monoid, the zero module, the zero vector space, depending on the context.

[1] It is common to shorten the title by dropping *von Neumann* and call the von Neumann regular rings just *regular* (cf. [**22**]). We will follow this custom.

CHAPTER 1

Lifting of distributive lattices by locally matricial algebras

1. Introduction

This chapter follows [**64**], where we have proved that every distributive $\langle \mathbf{0}, \mathbf{1} \rangle$-lattice is, as a join-semilattice, isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra. Having discussed this result with Friedrich Wehrung in a Summer School in Košická Belá, Slovakia, in 2003, we dealt with the question whether it can be solved functorially, that is, whether there is a functor from the category **DLat** of distributive lattices to the category of locally matricial algebras such that its composition with the functor Id_c, which assigns to a locally matricial algebra the lattice of its finitely generated ideals, is equivalent to the identity functor. It is easily rejected for the category of all distributive $\langle \mathbf{0}, \mathbf{1} \rangle$-lattices, however, it still can be true if we restrict ourselves to its suitable subcategories. One such restriction was made in [**82**], where F. Wehrung asked the following:

PROBLEM [**82**, Problem 3].. Let $\mathbb{F}$ be a field. Does there exist a functor Φ, from distributive $\langle \mathbf{0}, \mathbf{1} \rangle$-lattices with $\langle \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$-embeddings to locally matricial algebras over the filed $\mathbb{F}$ with $\mathbb{F}$-linear (unital) ring homomorphisms such that $\mathrm{Id}_c \circ \Phi$ is equivalent to the identity?

We are going to prove that such a functor Φ exists. Moreover, we prove that every diagram of the category of distributive lattices can be lifted with respect to the Id_c functor and we illustrate on simple examples that these results cannot be much improved. Our proofs are based on the result that a functor to **DLat** can be lifted with respect to the Id_c functor if and only if it can be lifted with respect to the functor $\mathcal{C} \colon \mathbf{Bases} \to \mathbf{DLat}$; objects of **Bases** are projections $\pi \colon X \to \boldsymbol{\mathcal{L}}$ from a set X on a distributive $\langle \mathbf{0}, \mathbf{1} \rangle$-lattice $\boldsymbol{\mathcal{L}}$ such that the pre-image of every element of $\boldsymbol{\mathcal{L}}$ is infinite and morphisms are commutative squares

$$F\colon \quad \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \pi_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_2} \\ \boldsymbol{\mathcal{L}}_1 & \xrightarrow[\boldsymbol{f}]{} & \boldsymbol{\mathcal{L}}_2 \end{array} \quad ,$$

where $\boldsymbol{f}$ is $\langle \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$-homomorphism, and $f \colon X_1 \to X_2$ is a map satisfying the property (5.1) below, and $\mathcal{C}$ denotes the *forgetful functor* which assigns to an object $\pi \colon X \to \boldsymbol{\mathcal{L}}$ the distributive $\langle \mathbf{0}, \mathbf{1} \rangle$-lattice $\boldsymbol{\mathcal{L}}$ and to a morphism $F = \langle f, \boldsymbol{f} \rangle$ the $\langle \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$-homomorphism $\boldsymbol{f}$ (Corollary 6.3). Proving the existence of a lifting of a given functor to the category **DLat** with respect to the functor $\mathcal{C}$ is much easier than proving the existence of its lifting with respect to the functor Id_c.

There has already appeared a number of papers related to the problem of the representation of distributive $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-semilattices as the semilattices of finitely generated ideals of a von Neumann ring, in particular, of a locally matricial algebra. Thus, G. M. Bergman [**9**] has proved that every distributive $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-semilattice which either is countable or corresponds to the semilattice of all compact hereditary subsets of a partially ordered set is isomorphic to the semilattice of locally matricial algebra. F. Wehrung proved that every distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice is isomorphic to the semilattice of finitely generated ideals of some von Neumann regular ring [**79**] but it follows from his results in [**81**] that we cannot require the ring to be unit regular, so not even locally matricial. Finally, the results in [**77, 78**] give an example of a distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice which is not isomorphic to the semilattice of finitely generated ideals of any von Neumann regular ring. In [**64**], we have proved that a distributive $\langle\mathbf{0},\mathbf{1}\rangle$-lattice is isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra. A different proof, based on similar methods as the Bergman's constructions in [**9**], is given by M. Ploščica in [**57**].

2. Notation and terminology

We will apply a specific construction of direct limits in a category $\mathbf{C}$ of algebras of a finitary type (cf. [**10**, Lemma 8.1.10]). Given an upwards directed partially ordered set P and a directed system $\mathcal{A} := \langle \boldsymbol{A}_p, f_{p,q} \mid p < q \text{ in } P\rangle$ in $\mathbf{C}$, we denote by A' the disjoint union of the underlying sets of the algebras $\boldsymbol{A}_p$. Given $p, q \in P$, $\mathbf{a} \in \boldsymbol{A}_p$, and $\mathbf{b} \in \boldsymbol{A}_q$, we write $\mathbf{a} \sim \mathbf{b}$ if there is $r \geq p, q$ in P such that the images of $\mathbf{a}$ and $\mathbf{b}$ in $\boldsymbol{A}_r$ coincide. Since the poset P is upwards directed, the relation $\sim$ is an equivalence on A'. We denote by $[\,\mathbf{a}\,]_\sim$ the block of the equivalence $\sim$ containing an element $\mathbf{a} \in A'$ and we use A to denote the set of all $\sim$-blocks in A'. For each $p \in P$, the correspondence $\mathbf{a} \mapsto [\,\mathbf{a}\,]_\sim$ defines a map $F_p\colon \boldsymbol{A}_p \to A$. The set A together with the maps F_p, $p \in P$, form a set-theoretic direct limit of the directed system $\mathcal{A}$. Since we deal with algebras of a finitary type, the operations on $\boldsymbol{A}_p$-s induce operations on A, and so we get an algebra $\boldsymbol{A}$, with the universe A, such that the maps F_p are homomorphisms and $\langle \boldsymbol{A}, F_p \mid p \in P\rangle$ is a direct limit of the directed system $\mathcal{A}$ in the category $\mathbf{C}$ (see the proof of [**10**, Lemma 8.1.10]). We will denote by $\varinjlim$ (Lim) this particular construction, while the direct limit in the categorical sense (determined up to isomorphisms) will be denoted by $\varinjlim$.

We will use the following notation:

- **DLat** := the category of all distributive bounded lattices (with $\langle\vee,\wedge,\mathbf{0},\mathbf{1}\rangle$-homomorphisms),
- **DSem** := the category of all distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattices (with $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-homomorphisms),
- **dsem** := the full subcategory of **DSem** of all finite distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattices,
- **bool** := the full subcategory of **dsem** of all finite Boolean $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattices.

Let $\mathbb{F}$ be a commutative field. We denote by $\mathbb{F}$-**Alg** the category of all unital associative algebras over the field $\mathbb{F}$. Let Id_c denote the functor from the category $\mathbb{F}$-**Alg** to the category of $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattices which assigns to a $\mathbb{F}$-algebra $\boldsymbol{R}$ the semilattice $\mathrm{Id}_c(\boldsymbol{R})$ of all finitely generated ideals of $\boldsymbol{R}$ and to a $\mathbb{F}$-linear ring homomorphism $\varphi\colon \boldsymbol{R} \to \boldsymbol{S}$ the $\langle \vee, \mathbf{0}, \mathbf{1}\rangle$-homomorphism $\mathrm{Id}_c(\varphi)\colon \mathrm{Id}_c(\boldsymbol{R}) \to \mathrm{Id}_c(\boldsymbol{S})$ given by $I \mapsto \boldsymbol{S}\varphi(I)\boldsymbol{S}$. It is straightforward to verify that the functor Id_c preserves direct limits.

A matricial $\mathbb{F}$-algebra is an $\mathbb{F}$-algebra of the form $\mathbb{M}_{t_1}(\mathbb{F}) \times \cdots \times \mathbb{M}_{t_n}(\mathbb{F})$, where $t_1, \dots, t_n$ are natural numbers and $\mathbb{M}_t(\mathbb{F})$ is the ring of all matrices of type $t \times t$ over the commutative field $\mathbb{F}$. A locally matricial $\mathbb{F}$-algebra is a direct limit of matricial $\mathbb{F}$-algebras. We denote by $\mathbb{F}$-**Loc** the category of locally matricial $\mathbb{F}$-algebras (with unital $\mathbb{F}$-linear ring homomorphisms), and by $\mathbb{F}$-**mat** the full subcategory of $\mathbb{F}$-**Loc** of all matricial $\mathbb{F}$-algebras.

3. The category aux revised

In this section we define an auxiliary category **aux**. Objects of **aux** are finite families $\boldsymbol{B} = \{B^i \mid i \in I\}$ of finite non-empty pairwise disjoint sets.

Let $\boldsymbol{B}_1 = \{B_1^i \mid i \in I_1\}$ and $\boldsymbol{B}_2 = \{B_2^j \mid j \in I_2\}$ be objects of **aux**. A *premorphism* from $\boldsymbol{B}_1$ to $\boldsymbol{B}_2$ consits from a set $\boldsymbol{h} = \{h^j \mid j \in I_2\}$ of bijections

$$h^j\colon \bigcup_{i \in I_1} \left(A^{j,i} \times B_1^i\right) \to B_2^j,$$

where $\boldsymbol{A} = \{A^{j,i} \mid i \in I_1 \text{ and } j \in I_2\}$ is a family of (possibly empty) finite sets. We denote the collection of all premorphisms from $\boldsymbol{B}_1$ to $\boldsymbol{B}_2$ by $\mathrm{pre}(\boldsymbol{B}_1, \boldsymbol{B}_2)$. Premorphisms $\boldsymbol{h}$ and $\widetilde{\boldsymbol{h}}$ from $\boldsymbol{B}_1$ to $\boldsymbol{B}_2$ are *equivalent*, which we denote by $\boldsymbol{h} \sim \widetilde{\boldsymbol{h}}$, provided that there exist maps $g^{j,i}\colon A^{j,i} \to \widetilde{A^{j,i}}$ such that

$$h^j(a,b) = \widetilde{h}^j(g^{j,i}(a), b)$$

for all $a \in A^{j,i}$ and $b \in B_1^i$, as Figure 1 displays.

$$\begin{tikzcd}
\bigcup_{i\in I_1}(A^{j,i} \times B_1^i) \arrow[d, "\bigcup_{i\in I_1}(g^{j,i}\times \mathbf{1}_{B_1^i})"'] \arrow[drr, "h_j"] & & \\
\bigcup_{i\in I_1}(\widetilde{A^{j,i}} \times B_1^i) \arrow[rr, "\widetilde{h_j}"'] & & B_2^j
\end{tikzcd}$$

FIGURE 1. The equivalence of premorphisms

It is easy to see that the relation $\sim$ is an equivalence on $\mathrm{pre}(\boldsymbol{B}_1, \boldsymbol{B}_2)$, in particular, all the maps $g^{j,i}$ are bijections.

We define morphisms in **aux** to be the blocks of the equivalence $\sim$. The symbol $[\boldsymbol{h}]$ denotes the block containing the premorphism $\boldsymbol{h}$. Given objects B_1, B_2, B_3 in **aux** and premorphisms $\boldsymbol{h}_1 \in \mathrm{pre}(\boldsymbol{B}_1, \boldsymbol{B}_2)$ and $\boldsymbol{h}_2 \in \mathrm{pre}(\boldsymbol{B}_2, \boldsymbol{B}_3)$, we put

$$A^{k,i} := \bigcup_{j \in I_2} (A_2^{k,i} \times A_1^{j,i})$$

and

$$h^k((a_2,a_1),b) := h_2^k(a_2, h_1^j(a_1,b))$$

for all $b \in B_1^i$, $a_1 \in A_1^{j,i}$, and $a_2 \in A_2^{k,j}$. The family $\boldsymbol{h} = \{h^k \mid k \in I_3\}$ forms a premorphism $\boldsymbol{h}$ from $\boldsymbol{B}_1$ to $\boldsymbol{B}_3$ which we denote by $\boldsymbol{h}_2 \circ \boldsymbol{h}_1$ and which we call the *composition of premorphisms* $\boldsymbol{h}_2$ and $\boldsymbol{h}_1$. It is proved in [**64**, Lemma 2.2.] that the equivalence class $\boldsymbol{h}$ does not depend on the choice of the representatives of the classes $\boldsymbol{h}_2$ and $\boldsymbol{h}_1$ and so we can define the composition of morphisms in **aux** by $[\,\boldsymbol{h}_2\,] \circ [\,\boldsymbol{h}_1\,] = [\,\boldsymbol{h}_2 \circ \boldsymbol{h}_1\,]$. The composition of premorphisms is depicted in Figure 2. In [**64**, Section 2] we have verified that **aux** is a category. Recall, that the identity morphism at an object $\boldsymbol{B} = \{B^i \mid i \in I\}$ in **aux** corresponds to the equivalence class of the collection of maps $h^i\colon \{i\} \times B^i \to B^i$, $(i,b) \mapsto b$.

$$\begin{tikzcd}
\bigcup_{j\in I_2}\Big(A_2^{k,j} \times \Big(\bigcup_{i\in I_1}(A_1^{j,i} \times B_1^i)\Big)\Big) \arrow[r, "\simeq"] \arrow[d, "\bigcup_{j\in I_2}\left(\mathbf{1}_{A_2^{k,j}} \times h_1^j\right)"'] & \bigcup_{i\in I_1}\Big(\underbrace{\bigcup_{j\in I_2}(A_2^{k,j} \times A_1^{j,i})}_{A^{k,i}} \times B_1^i\Big) \arrow[ddl, "h_k"'] \\
\bigcup_{j\in I_2}(A_2^{k,j} \times B_2^j) \arrow[d, "h_2^k"] & \\
B_3^k &
\end{tikzcd}$$

FIGURE 2. The composition of premorphisms

To every object $\boldsymbol{B} = \{B^i \mid i \in I\}$ of **aux**, we have assigned the Boolean semilattice $\Lambda(\boldsymbol{B}) := (\mathcal{P}(I), \cup)$ and given a morphism $[\,\boldsymbol{h}\,] \in \mathbf{aux}(B_1, B_2)$, the correspondence

$$J \mapsto \Big\{ j \in I_2 \mid \bigcup_{i\in J} A^{j,i} \neq \emptyset \Big\},$$

where $J \in \mathcal{P}(I_1)$, determines a $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-homomorphism $\Lambda([\,\boldsymbol{h}\,])\colon \Lambda(\boldsymbol{B}_1) \to \Lambda(\boldsymbol{B}_2)$. Thus we have defined a functor Λ from the category **aux** to the category **bool** of finite Boolean join-semilattices. Further, given a commutative field $\mathbb{F}$, we have defined a functor $\Delta\colon \mathbf{aux} \to \mathbb{F}\text{-}\mathbf{mat}$ so that there is a natural equivalence $\eta\colon \mathrm{Id}_c \circ \Delta \to \Lambda$. As the consequence of [**64**, Lemma 2.9], we get that

PROPOSITION 3.1. *The functor* $\Delta\colon \mathbf{aux} \to \mathbb{F}\text{-}\mathbf{mat}$ *lifts, via the natural equivalence* $\eta\colon \mathrm{Id}_c \circ \Delta \to \Lambda$*, the functor* Λ *with respect to* Id_c *(see Figure 3).*

$$\begin{tikzcd} \mathbf{aux} \arrow[rr, "\Delta"] \arrow[dr, "\Lambda"'] & & \mathbb{F}\text{-}\mathbf{mat} \arrow[dl, "\mathrm{Id}_c"] \arrow[dl, phantom, bend right, "\overset{\simeq}{\eta}"] \\ & \mathbf{bool} & \end{tikzcd}$$

FIGURE 3. The lifting of Λ by $\mathrm{Id}_c \circ \Delta$

4. The correspondence $\mathcal{B}\colon \mathbf{dsem} \to \mathbf{bool}$ revised

In [**64**, Section 1], we have defined a correspondence $\mathcal{B}\colon \mathbf{dsem} \to \mathbf{bool}$ as follows: For $\boldsymbol{\mathcal{S}} \in \mathbf{dsem}$ we define $\mathcal{B}(\boldsymbol{\mathcal{S}})$ to be the finite Boolean $\langle \emptyset, \cup \rangle$-semilattice $\mathcal{P}(J(\boldsymbol{\mathcal{S}}))$ (recall that $J(\boldsymbol{\mathcal{S}})$ denotes the set of join-irreducible elements of the $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-semilattice $\boldsymbol{\mathcal{S}}$. Given a homomorphism $f\colon \boldsymbol{\mathcal{S}}_1 \to \boldsymbol{\mathcal{S}}_2$ in **dsem**, we define its image $\mathcal{B}(f)\colon \mathcal{B}(\boldsymbol{\mathcal{S}}_1) \to \mathcal{B}(\boldsymbol{\mathcal{S}}_2)$ to be the map sending

$$X \mapsto \{ j \in J(\boldsymbol{\mathcal{S}}_2) \mid j \leq f(\bigvee X) \},$$

for all $X \subseteq J(\boldsymbol{\mathcal{S}}_1)$. The correspondence $\mathcal{B}$ preserves the composition of morphisms but the image of the identity morphism at $\boldsymbol{\mathcal{S}}$ may not be the identity morphism at $\mathcal{B}(\boldsymbol{\mathcal{S}})$. Indeed, $\mathcal{B}(\mathbf{1}_{\boldsymbol{\mathcal{S}}}) = \mathbf{1}_{\mathcal{B}(\boldsymbol{\mathcal{S}})}$ if and only if the $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-semilattice $\boldsymbol{\mathcal{S}}$ is Boolean.

For every $\boldsymbol{\mathcal{S}} \in \mathbf{dsem}$ denote by $u_{\boldsymbol{\mathcal{S}}}$ and $v_{\boldsymbol{\mathcal{S}}}$ the $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-homomorphisms defined by

$$\begin{aligned} u_{\boldsymbol{\mathcal{S}}} : \mathcal{B}(\boldsymbol{\mathcal{S}}) &\to \boldsymbol{\mathcal{S}} & \qquad v_{\boldsymbol{\mathcal{S}}} : \boldsymbol{\mathcal{S}} &\to \mathcal{B}(\boldsymbol{\mathcal{S}}) \\ X &\mapsto \bigvee X & x &\mapsto \{ j \in J(\boldsymbol{\mathcal{S}}) \mid j \leq x \}. \end{aligned} \tag{4.1}$$

Observe that for every $\boldsymbol{\mathcal{S}} \in \mathbf{dsem}$, $u_{\boldsymbol{\mathcal{S}}} \circ v_{\boldsymbol{\mathcal{S}}} = \mathbf{1}_{\boldsymbol{\mathcal{S}}}$, and for every homomorphism $f\colon \boldsymbol{\mathcal{S}}_1 \to \boldsymbol{\mathcal{S}}_2$ in **dsem**, $v_{\boldsymbol{\mathcal{S}}_2} \circ f \circ u_{\boldsymbol{\mathcal{S}}_1} = \mathcal{B}(f)$, that is, the following diagrams

$$\begin{tikzcd} \boldsymbol{\mathcal{S}} \arrow[rr, "\mathbf{1}_{\boldsymbol{\mathcal{S}}}"] \arrow[dr, "v_{\boldsymbol{\mathcal{S}}}"'] & & \boldsymbol{\mathcal{S}} \\ & \mathcal{B}(\boldsymbol{\mathcal{S}}) \arrow[ur, "u_{\boldsymbol{\mathcal{S}}}"'] & \end{tikzcd} \qquad \begin{tikzcd} \boldsymbol{\mathcal{S}}_1 \arrow[r, "f"] & \boldsymbol{\mathcal{S}}_2 \arrow[d, "v_{\boldsymbol{\mathcal{S}}_2}"] \\ \mathcal{B}(\boldsymbol{\mathcal{S}}_1) \arrow[u, "u_{\boldsymbol{\mathcal{S}}_1}"] \arrow[r, "\mathcal{B}(f)"'] & \mathcal{B}(\boldsymbol{\mathcal{S}}_2) \end{tikzcd} \tag{4.2}$$

commute.

LEMMA 4.1. *Let P be an upwards directed partially ordered set without maximal elements and $\langle \boldsymbol{\mathcal{S}}_p, f_{p,q} \mid p < q \text{ in } P \rangle$ a directed system in* **dsem**. *If*

$$\langle \boldsymbol{\mathcal{S}}, F_\rho \mid p \in P \rangle = \varinjlim \langle \boldsymbol{\mathcal{S}}_p, f_{p,q} \mid p < q \text{ in } P \rangle,$$

then

$$\langle \boldsymbol{\mathcal{S}}, F_\rho \circ u_{\boldsymbol{\mathcal{S}}_p} \mid p \in P \rangle = \varinjlim \langle \mathcal{B}(\boldsymbol{\mathcal{S}}_p), \mathcal{B}(f_{p,q}) \mid p < q \text{ in } P \rangle. \tag{4.3}$$

PROOF. We denote by Q the Cartesian product $P \times \{0, 1\}$ partially ordered by $\langle p, i \rangle < \langle q, j \rangle$ if and only if $p < q$. For every $p \in P$ we put $\boldsymbol{\mathcal{S}}_{\langle p,0 \rangle} := \boldsymbol{\mathcal{S}}_p$ and $\boldsymbol{\mathcal{S}}_{\langle p,1 \rangle} := \mathcal{B}(\boldsymbol{\mathcal{S}}_p)$. For every ordered pair $p < q$ in P we set

$f_{\langle p,0\rangle,\langle q,0\rangle} := f_{p,q}$, $f_{\langle p,0\rangle,\langle q,1\rangle} := v_q \circ f_{p,q}$, $f_{\langle p,1\rangle,\langle q,0\rangle} := f_{p,q} \circ u_p$, and $f_{\langle p,1\rangle,\langle q,1\rangle} := \mathcal{B}(f_{p,q})$. It follows directly from the commutativity of diagrams (4.2) that

$$\langle \mathbf{S}_{\langle p,i\rangle}, f_{\langle p,i\rangle,\langle q,j\rangle} \mid \langle p,i\rangle < \langle q,j\rangle \text{ in } Q\rangle$$

is a directed system in **dsem**.

We set $f_{\langle p,0\rangle} := F_\rho \colon \mathbf{S}_p \to \mathbf{S}$ and $f_{\langle p,1\rangle} := F_\rho \circ u_p \colon \mathcal{B}(\mathbf{S}_p) \to \mathbf{S}$, for all $p \in P$. We get readily from the definitions that

$$f_{\langle p,0\rangle} = F_\rho = f_q \circ f_{p,q} = f_{\langle q,0\rangle} \circ f_{\langle p,0\rangle,\langle q,0\rangle},$$

for all $p < q$ in P. We conclude that

$$\langle \mathbf{S}, f_{\langle p,0\rangle} \mid p \in P\rangle = \varinjlim \langle \mathbf{S}_{\langle p,0\rangle}, f_{\langle p,0\rangle,\langle q,0\rangle} \mid p < q \text{ in } P\rangle$$

due to the assumption of the lemma.

Given $p < q$ in P, we compute using (4.2) that

$$\begin{aligned}
f_{\langle p,0\rangle} &= F_\rho = f_q \circ f_{p,q} = f_q \circ u_q \circ v_q \circ f_{p,q} = f_{\langle q,1\rangle} \circ f_{\langle p,0\rangle,\langle q,1\rangle},\\
f_{\langle p,1\rangle} &= F_\rho \circ u_p = f_q \circ f_{p,q} \circ u_p = f_{\langle q,0\rangle} \circ f_{\langle p,1\rangle,\langle q,0\rangle},\\
f_{\langle p,1\rangle} &= f_q \circ f_{p,q} \circ u_p = f_q \circ u_q \circ v_q \circ f_{p,q} \circ u_p = f_q \circ u_q \circ \mathcal{B}(f_{p,q})\\
&= f_{\langle q,1\rangle} \circ f_{\langle p,1\rangle,\langle q,1\rangle}.
\end{aligned}$$

Since P has no maximal element, $P \times \{0\}$ is cofinal in $Q = P \times \{0,1\}$, hence

$$\langle \mathbf{S}, f_{\langle p,i\rangle} \mid p \in P \text{ and } i \in \{0,1\}\rangle = \varinjlim \langle \mathbf{S}_{\langle p,i\rangle}, f_{\langle p,i\rangle,\langle q,j\rangle} \mid \langle p,i\rangle < \langle q,j\rangle \text{ in } Q\rangle.$$

Since $P \times \{1\}$ are cofinal in $Q = P \times \{0,1\}$, we conclude that

$$\langle \mathbf{S}, f_{\langle p,1\rangle} \mid p \in P\rangle = \varinjlim \langle \mathbf{S}_{\langle p,1\rangle}, f_{\langle p,1\rangle,\langle q,1\rangle} \mid p < q \text{ in } P\rangle,$$

which is (4.3). ☐

Lemma 4.1 coincides with [**64**, Proposition 1.1]. Its proof is straightforward but it requires numbers of tedious verifications. Therefore we present another shorter proof here. The proof is based only on the commutativity of diagrams (4.2), and so it can be generalized for a similar situation in an arbitrary category. However, we shall need it only in the presented form.

The proof of the following lemma is simple and we leave it to the reader.

LEMMA 4.2. *Let* $\mathbf{C}$ *be a category with direct limits. Let* P *and* Q *be upwards directed partially ordered sets,* $\langle A_p, f_{p,q} \mid p < q \text{ in } P\rangle$*, and* $\langle B_p, g_{p,q} \mid p < q \text{ in } Q\rangle$ *directed systems in* $\mathbf{C}$*, and* $\langle A, F_\rho \mid p \in P\rangle$*,* $\langle B, g_q \mid q \in Q\rangle$ *their direct limits, respectively. Suppose that for every* $p \in P$*, there exists* $p^\star \in Q$ *and a homomorphism* $h_p \colon A_p \to B_{p^\star}$ *such that if* $p < q$ *in* P *and* $p^\star$*,* $q^\star < r$ *in* Q*, then* $q_{p^\star,r} \circ h_p = q_{q^\star,r} \circ h_q \circ f_{p,q}$*. Then there exists a unique homomorphism* $h \colon A \to B$ *such that* $h \circ F_\rho = g_{p^\star} \circ h_p$*, for all* $p \in P$*.*

5. Representation of distributive lattices revised

We denote by **Bases** the category whose objects are projections $\pi\colon X \to \boldsymbol{\mathcal{L}}$ of a set X onto a distributive $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$, and whose morphisms are commutative diagrams

$$F\colon \quad \begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \pi_2} \\ \boldsymbol{\mathcal{L}}_1 & \xrightarrow[\boldsymbol{f}]{} & \boldsymbol{\mathcal{L}}_2 \end{array}\ ,$$

where $\pi_1\colon X_1 \to \boldsymbol{\mathcal{L}}_1$ and $\pi_2\colon X_2 \to \boldsymbol{\mathcal{L}}_2$ are objects of the category **Bases**, $\boldsymbol{f}\colon \boldsymbol{\mathcal{L}}_1 \to \boldsymbol{\mathcal{L}}_2$ is a $\langle \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$-homomorphism, and $f\colon X_1 \to X_2$ is a map satisfying

$$f(x) = f(y) \implies \pi_2(f(x)) = \pi_2(f(y)) = 0, \tag{5.1}$$

for all $x \neq y$ in X_1, with the obvious composition of morphisms and identities.

As before, we denote by $\mathcal{C}\colon \mathbf{Bases} \to \mathbf{DSem}$ the *forgetful functor*, which assigns to an object $\pi\colon X \to \boldsymbol{\mathcal{L}}$ the lattice $\boldsymbol{\mathcal{L}}$ and to a morphism $F = \langle f, \boldsymbol{f}\rangle$ the $\langle \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$-homomorphism $\boldsymbol{f}$.

Further we denote by **bases** the full subcategory of the category **Bases** whose objects are projections of a finite set on a finite distributive lattice.

We shall now define a functor $[\,\Xi\,]$ from the category **bases** to the category **aux**: Given an object $\pi\colon X \to \boldsymbol{\mathcal{L}}$ in the category **bases** and an element $a \in \boldsymbol{\mathcal{L}}$, we set

$$a^\pi := \{x \in X \mid \pi(x) \geq a\}.$$

and given a morphism $F = \langle f, \boldsymbol{f}\rangle$ in $\hom_{\mathbf{bases}}(\pi_1, \pi_2)$ and an element $a \in \boldsymbol{\mathcal{L}}_1$, we define $a^F := \boldsymbol{f}(a^{\pi_1})$.

Given a set X, we denote by $\mathrm{To}(X)$ the set of all total orders on X. Given a total order $\boldsymbol{\alpha} \in \mathrm{To}(X)$, we denote by $\mathrm{Her}(X, \boldsymbol{\alpha})$ the set of all hereditary subsets (including the empty set) of X with respect to $\boldsymbol{\alpha}$. Given a subset $Y \subseteq X$ and $\boldsymbol{\alpha} \in \mathrm{To}(X)$, we denote by $\boldsymbol{\alpha} \restriction Y$ the restriction of the total order $\boldsymbol{\alpha}$ to Y.

Let X be a finite n-element set, $\boldsymbol{\alpha}\colon \alpha_0 < \cdots < \alpha_{n-1}$, $\boldsymbol{\beta}\colon \beta_0 < \cdots < \beta_{n-1}$ total orders on X, and $Y \subseteq X$. We write $\boldsymbol{\alpha} \sim_Y \boldsymbol{\beta}$ provided that for every $i < n$, $\alpha_i \neq \beta_i$ implies that α_i, $\beta_i \in Y$. Thus we have defined the equivalence relation $\sim_Y$ on the set $\mathrm{To}(X)$. We denote by $[\,\boldsymbol{\alpha}\,]_Y$ the equivalence class containing the total order $\boldsymbol{\alpha} \in \mathrm{To}(X)$.

Let X_1, X_2 be finite sets and $f\colon X_1 \to X_2$ a one-to-one map. Given

$$\boldsymbol{\alpha}\colon \alpha_0 < \cdots < \alpha_{n-1} \in \mathrm{To}(X_1)$$

we denote by $f\boldsymbol{\alpha}$ the total order

$$f\boldsymbol{\alpha}\colon f(\alpha_0) < \cdots < f(\alpha_{n-1}) \in \mathrm{To}(f(X_1)).$$

DEFINITION 5.1. Let $\pi\colon X \to \boldsymbol{\mathcal{L}}$ be an object of the category **bases**. For every $a \in J(\boldsymbol{\mathcal{L}})$ we set

$$\Xi(\pi)^a := \{\boldsymbol{\alpha} \in \mathrm{To}(a^\pi) \mid b^\pi \notin \mathrm{Her}(a^\pi, \boldsymbol{\alpha}) \text{ for all } b \in J(\boldsymbol{\mathcal{L}}) \text{ with } a < b\},$$

and $[\,\Xi\,](\pi) := \{\Xi(\pi)^a \mid a \in J(\boldsymbol{\mathcal{L}})\}$.

Let

$$\begin{tikzcd} F\colon & X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ & \boldsymbol{\mathcal{L}}_1 \arrow[r, "\boldsymbol{f}"'] & \boldsymbol{\mathcal{L}}_2 \end{tikzcd}$$

be a morphism in **bases**, $a \in J(\boldsymbol{\mathcal{L}}_1)$, and $b \in J(\boldsymbol{\mathcal{L}}_2)$. If $f(a) \ngeq b$, we put $\operatorname{dom}\Xi(F)^{a,b} := \emptyset$. If $\boldsymbol{f}(a) \geq b$, then $a^F \subseteq b^{\pi_2}$, and we denote by $\operatorname{dom}\Xi(F)^{a,b}$ the set of all classes $\left[\,\boldsymbol{\beta}'\,\right]_{a^F}$, where $\boldsymbol{\gamma} \in \mathrm{To}(b^{\pi_2})$, satisfying the following properties:

(Ξ_1) $a^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1), \boldsymbol{\gamma})$;
(Ξ_2) $c^{\pi_2} \notin \mathrm{Her}(b^{\pi_2}, \boldsymbol{\gamma})$, for every $c \in J(\boldsymbol{\mathcal{L}}_2)$ satisfying $b < c \leq \boldsymbol{f}(a)$.

Observe that the validity of (Ξ_1), (Ξ_2) does not depend on the choice of the representative of the class $[\,\boldsymbol{\gamma}\,]_{a^F}$. As in [**64**], our construction makes use of the following well-known property of lattice homomorphisms (cf. [**49**, Exercise 2.63.10]).

LEMMA 5.2. *Let $\boldsymbol{\mathcal{L}}_1$ and $\boldsymbol{\mathcal{L}}_2$ be finite distributive lattices and $\boldsymbol{f}\colon \boldsymbol{\mathcal{L}}_1 \to \boldsymbol{\mathcal{L}}_2$ a $\langle \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$-homomorphism. Then for each $b \in J(\boldsymbol{\mathcal{L}}_2)$ there is $c \in J(\boldsymbol{\mathcal{L}}_1)$ such that $\boldsymbol{f}^{-1}(\uparrow_{\boldsymbol{\mathcal{L}}_2}(b)) = \uparrow_{\boldsymbol{\mathcal{L}}_1}(c)$.*

COROLLARY 5.3. *Let $\pi_i\colon X_i \to \boldsymbol{\mathcal{L}}_i$, $i = 1, 2$, be objects in* **bases**, *$F = \langle f, \boldsymbol{f}\rangle\colon \pi_1 \to \pi_2$ a morphism in* **bases**, *and $b \in J(\boldsymbol{\mathcal{L}}_2)$. Then $f^{-1}(b^{\pi_2}) = c^{\pi_1}$, for some $c \in J(\boldsymbol{\mathcal{L}}_1)$.*

LEMMA 5.4. *Let*

$$\begin{tikzcd} F\colon & X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ & \boldsymbol{\mathcal{L}}_1 \arrow[r, "\boldsymbol{f}"'] & \boldsymbol{\mathcal{L}}_2 \end{tikzcd}$$

be a morphism in **bases** *and $b \in J(\boldsymbol{\mathcal{L}}_2)$. Then the correspondence*

$$\langle [\,\boldsymbol{\beta}\,]_{a^F}, \boldsymbol{\alpha}\rangle \mapsto \boldsymbol{\gamma},$$

where $\boldsymbol{\gamma} \in \mathrm{To}(b^{\pi_2})$ is such that $\boldsymbol{\gamma} \sim_{a^F} \boldsymbol{\beta}$ and $\boldsymbol{\gamma} \restriction a^F = f\boldsymbol{\alpha}$, defines a map

$$\Xi(F)^b\colon \bigcup_{a \in J(\boldsymbol{\mathcal{L}}_1)} \Big(\operatorname{dom}\Xi(F)^{a,b} \times \Xi(\pi_1)^a\Big) \to \Xi(\pi_2)^b. \tag{5.2}$$

PROOF. Let $a \in J(\boldsymbol{\mathcal{L}}_1)$ and $b \in J(\boldsymbol{\mathcal{L}}_2)$. If $\boldsymbol{f}(a) \ngeq b$, then $\operatorname{dom}\Xi(F)^{a,b} = \emptyset$. Suppose that $\boldsymbol{f}(a) \geq b$, and let $[\boldsymbol{\delta}]_{a^F} \in \operatorname{dom}\Xi(F)^{a,b}$ and $\boldsymbol{\alpha} \in \Xi(\pi_1)^a$. It follows from (5.1) that $f \restriction a^{\pi_1}$ is one-to-one, and so we can define $f\boldsymbol{\alpha}$.

According to [**64**, Lemma 4.1], there is a unique $\boldsymbol{\gamma} \in \mathrm{To}(b^{\pi_2})$ satisfying $\boldsymbol{\gamma} \sim_{a^F} \boldsymbol{\delta}$ and $\boldsymbol{\gamma} \restriction a^F = f\boldsymbol{\alpha}$.

We prove that $\boldsymbol{\gamma} \in \Xi(\pi_2)^b$. Towards a contradiction suppose that there is $d \in J(\boldsymbol{\mathcal{L}}_2)$ such that $b < d$ and $d^{\pi_2} \in \mathrm{Her}(b^{\pi_2}, \boldsymbol{\gamma})$. It follows from Corollary 5.3 that there is $c \in J(\boldsymbol{\mathcal{L}}_1)$ satisfying $f^{-1}(d^{\pi_2}) = c^{\pi_1}$. Therefore $c^F = d^{\pi_2} \cap f(X_1)$, and so $c^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1), \boldsymbol{\gamma})$. Applying (Ξ_1) we get that $a^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1)), \boldsymbol{\delta})$, hence $a^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1)), \boldsymbol{\gamma})$. It follows that either $c^F \subsetneq a^F$ or $a^F \subseteq c^F$, hence either $a < c$ or $c \leq a$ in $\boldsymbol{\mathcal{L}}_1$.

If $a < c$, then $\boldsymbol{\gamma} \restriction a^F = f\boldsymbol{\alpha}$ and $c^F \in \mathrm{Her}((b^{\pi_2} \cap f(X_1)), \boldsymbol{\gamma})$. This implies that $c^{\pi_1} \in \mathrm{Her}(\boldsymbol{\alpha})$, which is in a contradiction with $\boldsymbol{\alpha} \in \Xi(\pi_1)^a$. If $c \leq a$, then $d \leq \boldsymbol{f}(a)$, that is, $a^F \subseteq d^{\pi_2}$. By our assumption $d^{\pi_2} \in \mathrm{Her}(b^{\pi_2}, \boldsymbol{\gamma})$. Since $a^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1), \boldsymbol{\delta})$ and $\boldsymbol{\gamma} \restriction_{a^F} \boldsymbol{\delta}$, we conclude that $d^{\pi_2} \in \mathrm{Her}(b_2^{\pi}, \boldsymbol{\delta})$. This contradicts (Ξ_2), since then $b < d \leq \boldsymbol{f}(a)$ and $d^{\pi_2} \in \mathrm{Her}(b_2^{\pi}, \boldsymbol{\delta})$. $\square$

LEMMA 5.5. *The map $\Xi(F)^b$, defined by* (5.2)*, is a bijection.*

PROOF. First we prove that $\Xi(F)^b$ is onto. Let $\boldsymbol{\beta}$ be an arbitrary element of $\Xi(\pi_2)^b$. By Corollary 5.3, there exists $c \in J(\boldsymbol{\mathcal{L}}_1)$ with $f^{-1}(b^{\pi_2}) = c^{\pi_1}$. It follows that $b^{\pi_2} \cap f(X_1) = c^F$. Since the set

$$\mathcal{A} := \{a' \in J(\boldsymbol{\mathcal{L}}_1) \mid {a'}^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1), \boldsymbol{\beta})\}$$

contains c, it is non-empty. It is easy to see that the set $\{{a'}^F \mid a' \in \mathcal{A}\}$ is totally ordered by inclusion. We denote by a the element of $\mathcal{A}$ corresponding to the minimum a^F and we denote by $\boldsymbol{\alpha}$ the total order of a^{π_1} such that $f\boldsymbol{\alpha} = \boldsymbol{\beta} \restriction a^F$. Observe that $\boldsymbol{\alpha} \in \Xi(\pi_1)^a$.

We prove that $[\boldsymbol{\beta}]_{a^F} \in \operatorname{dom} \Xi(F)^{a,b}$. Since $a \in \mathcal{A}$, we have that $a^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1), \boldsymbol{\beta})$. Let $d \in J(\boldsymbol{\mathcal{L}}_2)$ satisfy $b < d < \boldsymbol{f}(a)$. Then, since $\boldsymbol{\beta} \in \Xi(\pi_2)^b$, we conclude that $d^{\pi_2} \notin \mathrm{Her}(b^{\pi_2}, \boldsymbol{\beta})$.

Finally we prove that the map $\Xi(F)^b$ is one-to-one. Let $\boldsymbol{\beta} \in \operatorname{dom} \Xi(\pi_2)^b$ and let $a \in J(\boldsymbol{\mathcal{L}}_1)$, $\boldsymbol{\alpha} \in \Xi(\pi_1)^a$ be as in the previous paragraph. Suppose that

$$\Xi(F)^b(\langle [\bar{\boldsymbol{\beta}}]_{\bar{a}^F}, \bar{\boldsymbol{\alpha}} \rangle) = \boldsymbol{\beta},$$

for some $\bar{a} \in J(\boldsymbol{\mathcal{L}}_1)$, $\bar{\boldsymbol{\alpha}} \in \Xi(\pi_1)^{\bar{a}}$, and $[\bar{\boldsymbol{\beta}}]_{\bar{a}^F} \in \operatorname{dom} \Xi(F)^{\bar{a},b}$. From property (Ξ_1) we infer that $\bar{a}^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1)), \bar{\boldsymbol{\beta}})$. Since $\bar{\boldsymbol{\beta}} \sim_{\bar{a}^F} \boldsymbol{\beta}$, we have that $\bar{a}^F \in \mathrm{Her}(b^{\pi_2} \cap f(X_1), \boldsymbol{\beta})$, and so $\bar{a} \in \mathcal{A}$. By the definition, $f\bar{\boldsymbol{\alpha}} = \boldsymbol{\beta} \restriction \bar{a}^F$, and so it follows from the properties of $\Xi(F)^{\bar{a}}$ that $\bar{a}^F$ is a minimal element of the set $\{{a'}^F \mid a' \in \mathcal{A}\}$, totally ordered by inclusion. We conclude that $\bar{a} = a$. Now it is easy to see that $\bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha}$ and $[\bar{\boldsymbol{\beta}}]_{\bar{a}^F} = [\bar{\boldsymbol{\beta}}]_{a^F} = [\boldsymbol{\beta}]_{a^F}$. $\square$

COROLLARY 5.6. *Let F be a morphism in the category* **bases**. *Then $\Xi(F)$ is a premorphism in the category* **aux**.

DEFINITION 5.7. We say that a morphism

$$\begin{tikzcd} F\colon & X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ & \boldsymbol{\mathcal{L}}_1 \arrow[r, "\boldsymbol{f}"'] & \boldsymbol{\mathcal{L}}_2 \end{tikzcd}$$

in **bases** is *efficient* if for every $b \in J(\boldsymbol{\mathcal{L}}_2)$, there exists $x \in X_2 \setminus f(X_1)$ with $\pi_2(x) = b$.

LEMMA 5.8. *Let*

$$\begin{tikzcd} F\colon & X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ & \boldsymbol{\mathcal{L}}_1 \arrow[r, "\boldsymbol{f}"'] & \boldsymbol{\mathcal{L}}_2 \end{tikzcd}$$

be an efficient morphism in **bases**. *Then* $\operatorname{dom} \Xi(F)^{a,b} \neq \emptyset$ *if and only if* $b \leq \boldsymbol{f}(a)$*, for all* $a \in J(\boldsymbol{\mathcal{L}}_1)$*,* $b \in J(\boldsymbol{\mathcal{L}}_2)$*.*

PROOF. The implication "$\Leftarrow$" follows directly from the definition. We prove the opposite one. Let $a \in J(\boldsymbol{\mathcal{L}}_1)$ and $b \in J(\boldsymbol{\mathcal{L}}_2)$ satisfy $b \leq \boldsymbol{f}(a)$. Since the morphism F is efficient, there is $x \in X_2 \setminus f(X_1)$ with $\pi_2(x) = b$. Let $\boldsymbol{\alpha} \in \Xi(\pi_1)^a$. We choose $\boldsymbol{\beta}\colon \beta_0 < \cdots < \beta_n \in \operatorname{To}(b^{\pi_2})$ such that $x = \beta_0$ and $\boldsymbol{\alpha}^F \in \operatorname{Her}(b^{\pi_2} \cap f(X_1), \boldsymbol{\beta})$. It is straightforward that $[\,\boldsymbol{\beta}\,]_{a^F} \in \operatorname{dom} \Xi(F)^{a,b}$ and $\Xi(F)^b(\langle [\,\boldsymbol{\beta}\,]_{a^F}\,, \boldsymbol{\alpha}\rangle) = \boldsymbol{\beta}$. $\square$

COROLLARY 5.9. *If* $F = \langle f, \boldsymbol{f}\rangle\colon \pi_1 \to \pi_2$ *is an efficient morphism, then* $\Lambda([\,\Xi(F)\,]) = \mathcal{B}(f)$*.*

LEMMA 5.10. *Let*

$$\begin{tikzcd} F\colon & X_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & X_2 \arrow[d, "\pi_2"] \\ & \boldsymbol{\mathcal{L}}_1 \arrow[r, "\boldsymbol{f}"'] & \boldsymbol{\mathcal{L}}_2 \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} G\colon & X_2 \arrow[r, "g"] \arrow[d, "\pi_2"'] & X_3 \arrow[d, "\pi_3"] \\ & \boldsymbol{\mathcal{L}}_2 \arrow[r, "\boldsymbol{g}"'] & \boldsymbol{\mathcal{L}}_3 \end{tikzcd}$$

be morphisms in **bases**. *Then* $[\,\Xi(G \circ F)\,] = [\,\Xi(G)\,] \circ [\,\Xi(F)\,]$.

PROOF. According to the definition of the composition of premorphisms in the category **aux**,

$$\operatorname{dom}(\Xi(G) \circ \Xi(F))^{a,c} = \bigcup_{b \in J(\boldsymbol{\mathcal{L}}_2)} \operatorname{dom} \Xi(G)^{b,c} \times \operatorname{dom} \Xi(F)^{a,b},$$

for all $a \in J(\boldsymbol{\mathcal{L}}_1), c \in J(\boldsymbol{\mathcal{L}}_3)$. The composition $(\Xi(G) \circ \Xi(F))^c$ is a map defined by the correspondence

$$\langle\langle [\,\boldsymbol{\gamma}'\,]_{b^G}\,, [\,\boldsymbol{\beta}'\,]_{a^F}\rangle, \boldsymbol{\alpha}\rangle \mapsto \Xi(G)^c\left([\,\boldsymbol{\gamma}'\,]_{b^G}\,, \Xi(F)^b(\langle [\,\boldsymbol{\beta}'\,]_{a^F}\,, \boldsymbol{\alpha}\rangle)\right).$$

In order to prove that $[\,\Xi(G)\circ\Xi(F)\,] = [\,\Xi(G\circ F)\,]$, we define maps $g^{a,c}$ from $\operatorname{dom}(\Xi(G)\circ\Xi(F))^{a,c}$ to $\operatorname{dom}\Xi(G\circ F)^{a,c}$ by

$$g^{a,c}\left(\langle[\,\boldsymbol{\gamma}'\,]_{b^G}, [\,\boldsymbol{\beta}'\,]_{a^F}\rangle\right) = [\,\boldsymbol{\gamma}''\,]_{a^{G\circ F}},$$

where $\boldsymbol{\gamma}'' \in \operatorname{To}(c^{\pi_3})$ satisfies $\boldsymbol{\gamma}'' \sim_{b^G} \boldsymbol{\gamma}'$ and $\boldsymbol{\gamma}'' \restriction b^G \sim_{a^{G\circ F}} g\boldsymbol{\beta}'$. Again, it is easily seen that these properties determine $\boldsymbol{\gamma}''$ uniquely.

Let $a \in J(\boldsymbol{\mathcal{L}}_1)$, $b \in J(\boldsymbol{\mathcal{L}}_2)$, and $c \in J(\boldsymbol{\mathcal{L}}_3))$ satisfy $\boldsymbol{f}(a) \geq b$ and $\boldsymbol{g}(b) \geq c$. Let $[\,\boldsymbol{\gamma}'\,]_{b^G} \in \operatorname{dom}\Xi(G)^{b,c}$, $[\,\boldsymbol{\beta}'\,]_{a^F} \in \operatorname{dom}\Xi(F)^{a,b}$, and $\boldsymbol{\alpha} \in \Xi(\pi_1)^a$. We verify that

$$\begin{aligned}(\Xi(G)\circ\Xi(F))^c\left(\left([\,\boldsymbol{\gamma}'\,]_{b^G}, [\,\boldsymbol{\beta}'\,]_{a^F}\right), \boldsymbol{\alpha}\right) &=\\ \Xi(G\circ F)^{a,c}\left(g^{a,c}\left([\,\boldsymbol{\gamma}'\,]_{b^G}, [\,\boldsymbol{\beta}'\,]_{a^F}\right), \boldsymbol{\alpha}\right)&.\end{aligned} \tag{5.3}$$

First we evaluate the left hand side of the equality (5.3):

$$\begin{aligned}&(\Xi(G)\circ\Xi(F))^c\left(\left([\,\boldsymbol{\gamma}'\,]_{b^G}, [\,\boldsymbol{\beta}'\,]_{a^F}\right), \boldsymbol{\alpha}\right) =\\ &\Xi(G)^c\Big([\,\boldsymbol{\gamma}'\,]_{b^G}, \Xi(F)^b([\,\boldsymbol{\beta}'\,]_{a^F}, \boldsymbol{\alpha})\Big) =\\ &\Xi(G)^c([\,\boldsymbol{\gamma}'\,]_{b^G}, \boldsymbol{\beta}),\end{aligned}$$

where $\boldsymbol{\beta} \in \operatorname{To}(b^{\pi_2})$ satisfies $\boldsymbol{\beta} \sim_{a^F} \boldsymbol{\beta}'$ and $\boldsymbol{\beta} \restriction a^F = f\boldsymbol{\alpha}$, and

$$\Xi(G)^c([\,\boldsymbol{\gamma}'\,]_{b^G}, \boldsymbol{\beta}) = \boldsymbol{\gamma},$$

where $\boldsymbol{\gamma} \in \operatorname{To}(c^{\pi_3})$ satisfies $\boldsymbol{\gamma} \sim_{b^G} \boldsymbol{\gamma}'$, and $\boldsymbol{\gamma} \restriction b^G = g\boldsymbol{\beta}$.

Now we compute the right hand side of (5.3):

$$\begin{aligned}&\Xi(G\circ F)^{a,c}\left(g^{a,c}\left([\,\boldsymbol{\gamma}'\,]_{b^G}, [\,\boldsymbol{\beta}'\,]_{a^F}\right), \boldsymbol{\alpha}\right) =\\ &\Xi(G\circ F)^{a,c}\left([\,\boldsymbol{\gamma}''\,]_{a^{G\circ F}}, \boldsymbol{\alpha}\right),\end{aligned}$$

where $\boldsymbol{\gamma}'' \in \operatorname{To}(c^{\pi_3})$ satisfies $\boldsymbol{\gamma}'' \sim_{b^G} \boldsymbol{\gamma}'$ and $\boldsymbol{\gamma}'' \restriction b^G \sim_{a^{G\circ F}} g\boldsymbol{\beta}'$, and

$$\Xi(G\circ F)^{a,c}\left([\,\boldsymbol{\gamma}''\,]_{a^{G\circ F}}, \boldsymbol{\alpha}\right) = \delta,$$

where $\delta \in \operatorname{To}(c^{\pi_3})$ satisfies $\delta \sim_{a^{G\circ F}} \boldsymbol{\gamma}''$ and $(g\circ f)\boldsymbol{\alpha} = \delta \restriction a^{G\circ F}$.

It remains to compare $\boldsymbol{\gamma}$ and δ. Since $\boldsymbol{f}(a) \geq b$, we have that $a^{G\circ F} \subseteq b^G$. The equality $\boldsymbol{\beta} \restriction a^F = f\boldsymbol{\alpha}$ implies $g\boldsymbol{\beta} \restriction a^{G\circ F} = (g\circ f)\boldsymbol{\alpha}$, and since $\boldsymbol{\gamma} \restriction b^G = g\boldsymbol{\beta}$, we infer that $\boldsymbol{\gamma} \restriction a^{G\circ F} = (\boldsymbol{\gamma} \restriction b^G) \restriction a^{G\circ F} = (g\circ f)\boldsymbol{\alpha}$. Now $\boldsymbol{\gamma} \restriction b^G = g\boldsymbol{\beta}$ and $\boldsymbol{\beta} \sim_{a^F} \boldsymbol{\beta}'$, thus $\boldsymbol{\gamma} \restriction b^G \sim_{a^{G\circ F}} g\boldsymbol{\beta}'$, and since $\boldsymbol{\gamma} \sim_{b^G} \boldsymbol{\gamma}'$, we conclude that $\boldsymbol{\gamma} \sim_{a^{G\circ F}} \boldsymbol{\gamma}''$. This together with the equality $(g\circ f)\alpha = \boldsymbol{\gamma} \restriction a^{G\circ F}$ implies that $\delta = \boldsymbol{\gamma}$. $\square$

Lemma 5.11. *The equality* $[\,\Xi(\mathbf{1}_\pi)\,] = \mathbf{1}_{\Xi(\pi)}$ *holds true for all* $\pi \in \mathbf{bases}$.

Proof. Let

$$\mathbf{1}_\pi\colon\quad \begin{tikzcd} X \arrow[r, "\mathbf{1}_X"] \arrow[d, "\pi"'] & X \arrow[d, "\pi"] \\ \boldsymbol{\mathcal{L}} \arrow[r, "\mathbf{1}_{\boldsymbol{X}}"'] & \boldsymbol{\mathcal{L}} \end{tikzcd}$$

be the identity morphism at an object π in **bases**. Note that by the definition of morphisms in the category **bases**, the equality $\mathbf{1}_{\boldsymbol{X}} = \mathbf{1}_{\boldsymbol{\mathcal{L}}}$ holds true. Let $a, b \in J(\boldsymbol{\mathcal{L}})$. If $a \ngeq b$, then $\operatorname{dom}\Xi(\mathbf{1}_\pi)^{a,b} = \emptyset$ by the definition. If $a \geq b$, then $\operatorname{dom}\Xi(\mathbf{1}_\pi)^{a,b}$ is a set of all $\left[\boldsymbol{\beta}'\right]_{a^{\mathbf{1}_\pi}}$ satisfying $a^\pi = a^{\mathbf{1}_\pi} \in \operatorname{Her}(X, \boldsymbol{\beta}')$ and $c^\pi \notin \operatorname{Her}(X, \boldsymbol{\beta}')$, for all $c \in J(\boldsymbol{\mathcal{L}})$ satisfying $b < c \leq a$. It follows that $a = b$, hence $\boldsymbol{\beta}' \sim_{a^{\mathbf{1}_\pi}} \boldsymbol{\beta}''$ for all $\boldsymbol{\beta}', \boldsymbol{\beta}'' \in \operatorname{To}(b^\pi)$, whence $\operatorname{dom}\Xi(\mathbf{1}_\pi)^{a,b}$ is a one-element set. This proves that $[\,\Xi(\mathbf{1}_\pi)\,] = \mathbf{1}_{\Xi(\pi)}$. □

COROLLARY 5.12. *Let π be an object of a category* **bases** *and F a morphism in* **bases**. *We set $[\,\Xi\,](\pi) := \Xi(\pi)$ and $[\,\Xi\,](F) := [\,\Xi(F)\,]$. Then the assignment $[\,\Xi\,]$ forms a functor* **bases** $\to$ **aux**.

The situation we have got at the moment is illustrated in Figure 4. The arrow $\mathcal{B}\colon \mathbf{dsem} \to \mathbf{bool}$ is dotted since $\mathcal{B}$ is not a functor; it only preserves the composition of morphisms. The trapezium on the left is not commutative but it commutes if we restrict ourselves to efficient morphisms.

$$\begin{tikzcd}
\mathbf{bases} \arrow[r, "{[\,\Xi\,]}"] \arrow[d, "\mathcal{C}"'] & \mathbf{aux} \arrow[rr, "\Delta"] \arrow[dr, "\Lambda"'] & \arrow[d, phantom, "\overset{\simeq}{\eta}"] & \mathbb{F}\text{-}\mathbf{mat} \arrow[dl, "\mathrm{Id}_c"] \\
\mathbf{dsem} \arrow[rr, dotted, "\mathcal{B}"'] & & \mathbf{bool} &
\end{tikzcd}$$

FIGURE 4

6. Lifting of the functor $\mathcal{C}$ with respect to Id_c

Given an object $\pi\colon X \to \boldsymbol{\mathcal{L}}$ in **Bases**, we set $\operatorname{Fin}(\pi) := \{\rho \in \mathbf{bases} \mid \rho \subseteq \pi\}$, i.e, $\operatorname{Fin}(\pi)$ denotes the set of all "finite" sub objects of π. Given $\pi\colon X \to \boldsymbol{\mathcal{L}} \in \mathbf{Bases}$, and objects $\rho \subseteq \sigma$ in $\operatorname{Fin}(\pi)$, we denote by $\iota_{\rho,\sigma}$ the inclusion morphism from ρ to σ. Similarly, given $\pi\colon X \to \boldsymbol{\mathcal{L}} \in \mathbf{Bases}$, and $\rho \in \operatorname{Fin}(\pi)$, we denote by $\iota_{\rho,\pi}$ the inclusion morphism $\rho \hookrightarrow \pi$. Observe that a composition of efficient morphisms is again efficient. Therefore we can define an order relation "$\sqsubseteq$" on the set $\operatorname{Fin}(\pi)$ as $\rho \sqsubseteq \sigma$ if $\rho \subseteq \sigma$ and the inclusion morphism $\iota_{\rho,\sigma}$ is efficient.

Given an object $\pi \in \mathbf{bases}$, we put $u_\pi := u_{\mathcal{C}\pi}$ and $v_\pi := v_{\mathcal{C}\pi}$. Observe that u_π and v_π are the morphisms defined by (4.1).

We denote by $\boldsymbol{m}$ the composition $\Delta \circ [\,\Xi\,] : \mathbf{bases} \to \mathbb{F}\text{-}\mathbf{mat}$. Recall that $[\,\Xi\,]$ is a functor due to Corollary 5.12. Now we show how to extend the functor $\boldsymbol{m}$ to a functor $\boldsymbol{M}\colon \mathbf{Bases} \to \mathbb{F}\text{-}\mathbf{Loc}$.

Let $\pi\colon X \to \boldsymbol{\mathcal{L}}$ be an object in **Bases**. Then the set $\operatorname{Fin}(\pi)$, partially ordered by the relation $\sqsubseteq$, is upwards directed, and $\langle \pi, \iota_{\rho,\pi} \mid \rho \in \operatorname{Fin}(\pi)\rangle$ is a direct limit of the directed system $\langle \rho, \iota_{\rho,\sigma} \mid \rho \sqsubseteq \sigma \text{ in } \operatorname{Fin}(\pi)\rangle$ in the category **Bases**. We set $\boldsymbol{M}(\pi) := \varinjlim \langle \boldsymbol{m}(\rho), \boldsymbol{m}(\iota_{\rho,\sigma}) \mid \rho \sqsubseteq \sigma \text{ in } \operatorname{Fin}(\pi)\rangle$ and we define $\boldsymbol{M}(\iota_{\rho,\pi})$ to be the corresponding limit morphisms.

Let $F\colon \pi_1 \to \pi_2$ be a morphism in **Bases**. For every $\rho \in \mathrm{Fin}(\pi_1)$ we select $\rho^\star \in \mathrm{Fin}(\pi_2)$ so that the image of $F \restriction \rho$ is contained in $\rho^\star$, and denote by F_ρ the morphism in $\hom_{\mathbf{bases}}(\rho, \rho^\star)$ that coincides with the restriction $F \restriction \rho$. It is straightforward that if $\rho \sqsubseteq \sigma$ in $\mathrm{Fin}(\pi_1)$ and $\rho^\star, \sigma^\star \sqsubseteq \tau$ in $\mathrm{Fin}(\pi_2)$, then

$$\iota_{\rho^\star,\tau} \circ F_\rho = \iota_{\sigma^\star,\tau} \circ F_\sigma \circ \iota_{\rho,\sigma}. \tag{6.1}$$

Thus $\boldsymbol{M}(\iota_{\rho^\star,\tau}) \circ \boldsymbol{m}(F_\rho) = \boldsymbol{M}(\iota_{\sigma^\star,\tau}) \circ \boldsymbol{m}(F_\tau) \circ \boldsymbol{M}(\iota_{\rho,\sigma})$, and, by Lemma 4.2, there exists a unique $\mathbb{F}$-linear ring homomorphism $h\colon \boldsymbol{M}(\pi_1) \to \boldsymbol{M}(\pi_2)$ satisfying

$$h \circ \boldsymbol{M}(\iota_{\rho,\pi_1}) = \boldsymbol{M}(\iota_{\rho^\star,\pi_2}) \circ \boldsymbol{m}(F_\rho), \tag{6.2}$$

for every $\rho \in \mathrm{Fin}(\pi_1)$.

Lemma 6.1. *The map h does not depend on the choice of the elements $\rho^\star$.*

Proof. For every $\rho \in \mathrm{Fin}(\pi_1)$ we select another $\rho^{\star\star} \in \mathrm{Fin}(\pi_2)$ so that the image of $F \restriction \rho$ is contained in $\rho^{\star\star}$, and we denote by $F^\star_\rho$ the morphism in $\mathbf{Bases}(\rho, \rho^{\star\star})$ which coincides with the restriction $F \restriction \rho$. Then, as above, there exists a unique $\mathbb{F}$-linear ring homomorphism $h^\star$ such that

$$h^\star \circ \boldsymbol{M}(\iota_{\rho,\pi_1}) = \boldsymbol{M}(\iota_{\rho^{\star\star},\pi_2}) \circ \boldsymbol{m}(F^\star_\rho),$$

for every $\rho \in \mathrm{Fin}(\pi)$. Now, for each $\rho \in \mathrm{Fin}(\pi_1)$, we select $\rho^\dagger \in \mathrm{Fin}(\pi_2)$ such that $\rho^\star, \rho^{\star\star} \sqsubseteq \rho^\dagger$, and denote by $F^\dagger_\rho$ the morphism in $\mathbf{Bases}(\rho, \rho^\dagger)$ corresponding to the restriction $F \restriction \rho$. Since

$$\iota_{\rho^\star,\rho^\dagger} \circ F_\rho = F_{\rho^\dagger},$$

we have that

$$\boldsymbol{M}(\iota_{\rho^\star,\pi_2}) \circ \boldsymbol{m}(F_\rho) = \boldsymbol{M}(\iota_{\rho^\dagger,\pi_2}) \circ \boldsymbol{M}(\iota_{\rho^\star,\rho^\dagger}) \circ \boldsymbol{M}(F_\rho) = \boldsymbol{M}(\iota_{\rho^\dagger,\pi_2}) \circ \boldsymbol{m}(F^\dagger_\rho),$$

whence the map h satisfies the equality

$$h \circ \boldsymbol{M}(\iota_{\rho,\pi_1}) = \boldsymbol{M}(\iota_{\rho^\dagger,\pi_2}) \circ \boldsymbol{m}(F_{\rho^\dagger}),$$

for every $\rho \in \mathrm{Fin}(\pi_1)$. Similarly we get that $h^\star$ satisfies

$$h^\star \circ \boldsymbol{M}(\iota_{\rho,\pi_1}) = \boldsymbol{M}(\iota_{\rho^\dagger,\pi_2}) \circ \boldsymbol{m}(F_{\rho^\dagger}),$$

for all $\rho \in \mathrm{Fin}(\pi_1)$. From the unicity of such a map we deduce that $h = h^\star$. □

We set $\boldsymbol{m}(F) := h$. It is straightforward that $\boldsymbol{M}\colon \mathbf{Bases} \to \mathbb{F}\text{-}\mathbf{Loc}$ is a direct limits preserving functor.

We denote by ω-**Bases** the full subcategory of the category **Bases** whose objects are $\pi\colon X \to \boldsymbol{\mathcal{L}}$ such that $\pi^{-1}(\{a\})$ is infinite for every $a \in \boldsymbol{\mathcal{L}}$, and we set $\mathcal{C}^\dagger := \mathcal{C} \restriction \omega\text{-}\mathbf{Bases}$.

Proposition 6.2. *The functor $\boldsymbol{M}$ lifts the functor $\mathcal{C}^\dagger$ with respect to the functor* Id_c.

PROOF. We have defined $(\mathrm{Id}_c \circ \boldsymbol{M})(\rho) = (\mathrm{Id}_c \circ \Delta)([\,\Xi\,](\rho))$, for all $\rho \in$ **Bases**, and so $\eta_{[\,\Xi\,](\rho)}$ is an isomorphism from $(\mathrm{Id}_c \circ \boldsymbol{M})(\rho)$ to $\mathcal{B}(\mathcal{C}(\rho))$. We abbreviate the notation putting $\eta_\rho := \eta_{[\,\Xi\,](\rho)}$. Let $F\colon \pi_1 \to \pi_2$ be a morphism in **Bases**. By Corollary 5.9 we have that $\mathcal{B}(\mathcal{C}(F)) = \Lambda([\,\Xi(F)\,])$. Since $\eta\colon \mathrm{Id}_c \circ [\,\Xi\,] \to \Lambda$ is a natural equivalence, we conclude that

$$\mathcal{B}(\mathcal{C}(F)) = (\eta_{\pi_2} \circ (\mathrm{Id}_c \circ \boldsymbol{M})(F) \circ \eta_{\pi_1}^{-1}. \tag{6.3}$$

Let $\pi\colon X \to \boldsymbol{\mathcal{L}}$ be an object in **Bases**. By the definition

$$\langle \boldsymbol{M}(\rho), \boldsymbol{M}(\iota_{\rho,\pi}) \rangle_{\rho \in \mathrm{Fin}(\pi)} = \varinjlim \langle \boldsymbol{M}(\rho), \boldsymbol{M}(\iota_{\rho,\sigma}) \rangle_{\rho \sqsubseteq \sigma \text{ in } \mathrm{Fin}(\pi)}.$$

Since the functor Id_c preserves direct limits, we infer that that

$$\langle \mathrm{Id}_c\, \boldsymbol{M}(\pi), \mathrm{Id}_c\, \boldsymbol{M}(\iota_{\rho,\pi}) \rangle_{\rho \in \mathrm{Fin}(\pi)}$$

is the direct limit

$$\varinjlim \langle (\mathrm{Id}_c \circ \boldsymbol{M})(\rho), (\mathrm{Id}_c \circ \boldsymbol{M})(\iota_{\rho,\sigma}) \mid \rho \sqsubseteq \sigma \text{ in } \mathrm{Fin}(\pi) \rangle.$$

It follows from (6.3) that the directed system

$$\langle (\mathrm{Id}_c \circ \boldsymbol{M})(\rho), (\mathrm{Id}_c \circ \boldsymbol{M})(\iota_{\rho,\sigma}) \mid \rho \subsetneq \sigma \text{ in } \mathrm{Fin}(\pi) \rangle$$

is isomorphic, via the isomorphisms $\{\eta_\rho \mid \rho \in \mathrm{Fin}(\pi)\}$, to the directed system

$$\langle \mathcal{B}(\mathcal{C}(\rho)), \mathcal{B}(\mathcal{C}(\iota_{\rho,\sigma})) \mid \rho \sqsubseteq \sigma \text{ in } \mathrm{Fin}(\pi) \rangle.$$

Since $\pi \in$ **Bases**, we infer that the partially ordered set $\mathrm{Fin}(\pi)$ has no maximal elements. By Lemma 4.1 we get that

$$\langle \mathcal{C}(\pi), \mathcal{C}(\iota_{\rho,\pi}) \circ u_\rho \mid \rho \in \mathrm{Fin}(\pi) \rangle$$

is the direct limit

$$\varinjlim \langle \mathcal{B}(\mathcal{C}(\rho)), \mathcal{B}(\mathcal{C}(\iota_{\rho,\sigma})) \mid \rho \sqsubseteq \sigma \text{ in } \mathrm{Fin}(\pi) \rangle.$$

The isomorphisms $\{\eta_\rho \mid \rho \in \mathrm{Fin}(\pi)\}$ induce a unique isomorphism

$$\eta_\pi\colon (\mathrm{Id}_c \circ \boldsymbol{M})(\pi) \to \mathcal{C}(\pi)$$

such that

$$\eta_\pi \circ (\mathrm{Id}_c \circ \boldsymbol{M})(\iota_{\rho,\pi}) = \mathcal{C}(\iota_{\rho,\pi}) \circ u_\rho \circ \eta_\rho. \tag{6.4}$$

for all $\rho \in \mathrm{Fin}(\pi)$.

Let $F\colon \pi_1 \to \pi_2$ be a morphism in **Bases**. As above we select objects $\rho^\star \in \mathrm{Fin}(\pi_2)$ and we define the morphisms F_ρ, for all $\rho \in \mathrm{Fin}(\pi_1)$. It follows from (6.1) that

$$\mathcal{B}(\mathcal{C}(\iota_{\rho^\star,\tau})) \circ \mathcal{B}(\mathcal{C}(F_\rho)) = \mathcal{B}(\mathcal{C}(\iota_{\sigma^\star,\tau})) \circ \mathcal{B}(\mathcal{C}(F_\sigma)) \circ \mathcal{B}(\mathcal{C}(\iota_{\rho,\sigma})),$$

for all $\rho \subseteq \sigma$ in $\mathrm{Fin}(\pi_1)$ and all $\tau \in \mathrm{Fin}(\pi_2)$ with $\rho^\star, \sigma^\star \sqsubseteq \tau$. Applying Lemma 4.2, we get that there is a unique homomorphism $h\colon \mathcal{C}(\pi_1) \to \mathcal{C}(\pi_2)$ such that

$$h \circ \mathcal{C}(\iota_{\rho,\pi_1}) \circ u_\rho = \mathcal{C}(\iota_{\rho^\star,\pi_2}) \circ u_{\rho^\star} \circ \mathcal{B}\mathcal{C}(F_\rho), \tag{6.5}$$

for all $\rho \in \mathrm{Fin}(\pi)$. Now, it follows from (6.2) that

$$(\mathrm{Id}_c \circ \boldsymbol{M})(F) \circ (\mathrm{Id}_c \circ \boldsymbol{M})(\iota_{\rho,\pi}) = (\mathrm{Id}_c \circ \boldsymbol{M})(\iota_{\rho^\star,\pi_2}) \circ (\mathrm{Id}_c \circ \boldsymbol{M})(F_\rho). \tag{6.6}$$

Applying (6.4) we derive from (6.6) that

$$(\mathrm{Id}_c \circ \boldsymbol{M})(F)\circ\eta_{\pi_1}^{-1}\circ\mathcal{C}(\iota_{\rho,\pi_1})\circ u_\rho\circ\eta_\rho = \eta_{\pi_2}^{-1}\circ\mathcal{C}(\iota_{\rho^\star,\pi_2})\circ u_{\rho^\star}\circ\mathcal{B}(\mathcal{C}(F_\rho))\circ\eta_\rho. \tag{6.7}$$

Composing (6.7) with η_{π_2} from the left and with η_p^{-1} from the right, we get that

$$\eta_{\pi_2}\circ(\mathrm{Id}_c \circ\boldsymbol{M})(F)\circ\eta_{\pi_1}^{-1}\circ\mathcal{C}(\iota_{\rho,\pi_1})\circ u_\rho = \mathcal{C}(\iota_{\rho^\star,\pi_2})\circ u_{\rho^\star}\circ\mathcal{B}(\mathcal{C}(F_\rho)),$$

which, according to Lemma 4.2, implies that $\eta_{\pi_2}\circ(\mathrm{Id}_c \circ\boldsymbol{M})(F)\circ\eta_{\pi_1}^{-1} = h$.

Finally, since by the definition $F\circ\iota_{\rho,\pi_1} = \iota_{\rho^\star,\pi_2}\circ F_\rho$, for all $\rho\in\mathrm{Fin}(\pi)$, we have that

$$\mathcal{C}(F)\circ\mathcal{C}(\iota_{\rho,\pi_1})\circ u_\rho = \mathcal{C}(\iota_{\rho^\star,\pi_2})\circ u_{\rho^\star}\circ\mathcal{B}(\mathcal{C}(F_\rho)).$$

We conclude that $\eta_{\pi_2}\circ(\mathrm{Id}_c \circ\boldsymbol{M})(F)\circ\eta_{\pi_1}^{-1} = h = \mathcal{C}(F)$, which proves the proposition. □

COROLLARY 6.3. *Let $\Phi\colon \mathbf{C}\to\mathbf{DSem}$ be a functor whose image is in* **DLat**. *The functor Φ can be lifted with respect to the functor $\mathcal{C}^\dagger$ if and only if it can be lifted with respect to the functor* Id_c.

PROOF. ($\Rightarrow$) Let $\Psi\colon\mathbf{C}\to\mathbf{Bases}$ be a functor that lifts Φ with respect to to the functor $\mathcal{C}^\dagger$. According to Proposition 6.2, we have that

$$\mathrm{Id}_c \circ\boldsymbol{M}\circ\Psi\simeq\mathcal{C}^\dagger\circ\Psi\simeq\Phi,$$

whence the composition $\boldsymbol{M}\circ\Psi$ lifts the functor Φ with respect to the functor Id_c.

($\Leftarrow$) Suppose that a functor Θ lifts the functor Φ with respect to the functor Id_c. Given a ring (in particular a locally matricial $\mathbb{F}$-algebra) $\boldsymbol{R}$, we denote by $\boldsymbol{R}a\boldsymbol{R}$ the two sided ideal of $\boldsymbol{R}$ generated by an element $a\in\boldsymbol{R}$. We define a functor $\Psi\colon\mathbf{C}\to\mathbf{Bases}$ as follows:

- Given an object $c\in\mathbf{C}$, we set $\Psi(c)\colon\Theta(c)\times\omega\to(\mathrm{Id}_c \circ\Theta)(c)$ to be the map given by $\langle a,n\rangle\mapsto\boldsymbol{R}a\boldsymbol{R}$;
- Given a morphism $f\colon c_1\to c_2$ in $\mathbf{C}$, we define $\Psi(f)$ to be the map given by the commutative diagram

$$\Psi(f)\colon\quad \begin{array}{ccc} \Theta(c_1)\times\omega & \xrightarrow{\Theta(f)\times\omega} & \Theta(c_2)\times\omega \\ {\scriptstyle\Psi(c_1)}\big\downarrow & & \big\downarrow{\scriptstyle\Psi(c_2)} \\ (\mathrm{Id}_c \circ\Theta)(c_1) & \xrightarrow[(\mathrm{Id}_c \circ\Theta)(f)]{} & (\mathrm{Id}_c \circ\Theta)(c_2) \end{array}$$

It is straightforward that the functor Ψ lifts the functor Φ with respect to the functor $\mathcal{C}^\dagger$. □

7. Existence and non-existence of liftings

We denote by $\mathbf{DLat_m}$ the category whose objects are distributive $\langle\mathbf{0},\mathbf{1}\rangle$-lattices and whose morphisms are one-to-one $\langle\vee,\wedge,\mathbf{0},\mathbf{1}\rangle$-homomorphisms. We apply Corollary 6.3 to prove that the category $\mathbf{DLat_m}$ as well as every diagram in **DLat** has a lifting with respect to the functor Id_c. We will start with the category $\mathbf{DLat_m}$.

THEOREM 7.1. *The category* $\mathbf{DLat_m}$ *has a lifting with respect to the functor* Id_c.

PROOF. In accordance with Corollary 6.3 it suffices to find a functor Φ of the category $\mathbf{DLat_m}$ with respect to the functor $\mathcal{C}^\dagger$. It is easy, we only have to guarantee that its image is in **Bases**. Let X be an infinite set. Given a distributive $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice $\boldsymbol{\mathcal{L}}$, we define $\Phi(\boldsymbol{\mathcal{L}})$ to be the map $\boldsymbol{\mathcal{L}} \times X \to \boldsymbol{\mathcal{L}}$ defined by $\langle a, x\rangle \mapsto a$. Given a $\langle \vee, \wedge, \mathbf{0}, \mathbf{1}\rangle$-embedding $f\colon \boldsymbol{\mathcal{L}}_1 \to \boldsymbol{\mathcal{L}}_2$, we define $\Phi(f)$ to be the morphism

$$\Phi(f)\colon\quad \begin{array}{ccc} \boldsymbol{\mathcal{L}}_1 \times X & \xrightarrow{f \times \mathbf{1}_X} & \boldsymbol{\mathcal{L}}_2 \times X \\ {\scriptstyle \Phi(\mathcal{L}_1)}\downarrow & & \downarrow{\scriptstyle \Phi(\mathcal{L}_2)} \\ \boldsymbol{\mathcal{L}}_1 & \xrightarrow[f]{} & \boldsymbol{\mathcal{L}}_2 \end{array}$$

in the category ω-**Bases**. $\square$

As opposed to Theorem 7.1, even a simple finite subcategory of the category **DLat** cannot be lifted with respect to Id_c, which is demonstrated in Example 7.1. Before that, we define the notion of a $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice homomorphism that *separates zero*.

DEFINITION 7.2. We say that a $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice homomorphism $f\colon \boldsymbol{\mathcal{L}}_1 \to \boldsymbol{\mathcal{L}}_2$ *separates zero* provided that $f(a) > 0$ for all $0 \neq a \in \boldsymbol{\mathcal{L}}_1$.

Observe that if $\mathrm{Id}_c(\varphi)$ separates zero for $\varphi\colon \boldsymbol{R}_1 \to \boldsymbol{R}_2$ in $\mathbb{F}$-**Alg**, then the $\mathbb{F}$-algebra homomorphism φ is one-to-one.

For an ordinal number α we denote by $\mathfrak{C}_\alpha$ a well-ordered chain of all ordinals $< \alpha$. Given ordinal numbers α and β, we denote by $f_{\alpha,\beta}\colon \mathfrak{C}_\alpha \to \mathfrak{C}_\beta$ the zero separating $\langle \mathbf{0}, \mathbf{1}\rangle$-lattice-homomorphism

$$f_{\alpha,\beta}(\gamma) = \begin{cases} 1 & : \gamma > 0, \\ 0 & : \gamma = 0, \end{cases}$$

for all $\gamma \in \mathfrak{C}_\alpha$. Let $\mathbf{D}_{\bullet\!-}$ be the diagram displayed in Figure 5.

$$\begin{tikzcd} \mathfrak{C}_3 \arrow[r, bend left, "f_{3,3}"] \arrow[r, bend right, "\mathbf{1}_{\mathfrak{C}_3}"'] & \mathfrak{C}_3 \arrow[r, "f_{3,2}"] & \mathfrak{C}_2 \end{tikzcd}$$

FIGURE 5. The diagram $\mathbf{D}_{\bullet\!-}$

EXAMPLE 7.1. There is no lifting of the diagram $\mathbf{D}_{\bullet\!-}$ with respect to the functor Id_c.

Proof. Assume that there is a lifting Φ of $\mathbf{D}_{\bullet\!-}$ with respect to the functor Id_c. Since $f_{3,2}$ separates zero, $\Phi(f_{3,2})$ is one-to-one. It follows that

$$\Phi(f_{3,2}\circ f_{3,3}) = \Phi(f_{3,2})\circ\Phi(f_{3,3}) \neq \Phi(f_{3,2})\circ\Phi(\mathbf{1}_{C_3}) = \Phi(f_{3,2}\circ\mathbf{1}_{C_3}) = \Phi(f_{3,2}),$$

while $f_{3,2}\circ f_{3,3} = f_{3,2}$. This is a contradiction. $\square$

Theorem 7.3. *Let J be a partially ordered set and $\mathbf{D}\colon J \to \mathbf{DLat}$ a diagram in $\mathbf{DLat}$. Then the diagram $\mathbf{D}$ has a lifting with respect to the functor Id_c.*

Proof. Again, by Corollary 6.3, it suffice to find a lifting $\mathbf{E}$ of $\mathbf{D}$ with respect to $\mathcal{C}$. Let $\{X_j \mid j \in J\}$ be a collection of infinite pairwise disjoint sets. For every $j \in J$ we set

$$Y_j := \bigcup_{i\leq j \text{ in } J} \mathbf{D}(i)\times X_i,$$

and we let $\mathbf{E}(j)\colon Y_j \to \mathbf{D}(j)$ be the map sending $\langle a, x\rangle \in \mathbf{D}(i)\times X_i$ to $\mathbf{D}(i \to j)(a)$ (observe that the map is onto because the maps $\mathbf{D}(j)\times X_j \to \mathbf{D}(j)$ are onto). Given $j \leq k$ in J, we let $\mathbf{E}(j \to k)$ to be the morphism

$$\mathbf{E}(i\to j)\colon\quad \begin{tikzcd} Y_j \arrow[r, "\subseteq"] \arrow[d, "\mathbf{E}(j)"'] & Y_k \arrow[d, "\mathbf{E}(k)"] \\ \mathbf{D}(j) \arrow[r, "\mathbf{D}(j\to k)"'] & \mathbf{D}(k) \end{tikzcd}$$

in ω-**Bases**. $\square$

The last example represents the diagram (viewed as a subcategory) $\mathbf{D}_{\curlywedge}$ of the category $\mathbf{DLat}$ corresponding to the partially ordered class (α runs through the class of all ordinals) that cannot be lifted with respect to Id_c (see Figure 6) .

$$\begin{tikzcd}[column sep=small] & & \mathfrak{C}_2 & & & \\ \mathfrak{C}_2 \arrow[urr] & \mathfrak{C}_3 \arrow[ur] & \dots & \mathfrak{C}_\alpha \arrow[ul] & \dots & (2 \leq \alpha \in \mathbf{On}) \end{tikzcd}$$

Figure 6. The subcategory $\mathbf{D}_{\curlywedge}$

Example 7.2. We denote by $\mathbf{D}_{\curlywedge}$ a subcategory of $\mathbf{DLat}$ whose objects are lattices $\{\mathfrak{C}_\alpha \mid 2 \leq \alpha \in \mathbf{On}\}$. Non-identity morphisms in $\mathbf{D}_{\curlywedge}$ are $\{f_{\alpha,2} \mid \alpha \in \mathbf{On}\}$. The subcategory $\mathbf{D}_{\curlywedge}$ has no lifting with respect to the functor Id_c.

Proof. Assume that there is a lifting Φ of $\mathbf{D}_{\curlywedge}$ with respect to the functor Id_c. Assume the contrary. Let α be an ordinal number whose cardinality is bigger than $|\Phi(\mathfrak{C}_2)|$. Since $f_{\alpha,2}$ separates zero, $\Phi(f_{\alpha,2})\colon \Phi(\mathfrak{C}_\alpha) \to \Phi(\mathfrak{C}_2)$ is an embedding. This contradicts that $|\Phi(\mathfrak{C}_\alpha)| \geq |\alpha| > |\Phi(\mathfrak{C}_2)|$. $\square$

CHAPTER 2

Distributive congruence lattices of congruence-permutable algebras

1. Introduction

Representing algebraic lattices as congruence lattices of algebras often gives rise to very hard open problems. The most well-known of those problems, the Congruence Lattice Problem, usually abbreviated **CLP**, asks whether every distributive algebraic lattice is isomorphic to the congruence lattice of some lattice, see the survey paper [**75**]. This problem has been solved by the third author[1] in [**83**]. For algebraic lattices that are not necessarily distributive, there are several deep results, one of the most remarkable, due to W.A. Lampe [**47**], stating that every algebraic lattice with compact unit is isomorphic to the congruence lattice of some groupoid. This result is further extended to join-complete, unit-preserving, compactness preserving maps between two algebraic lattices [**48**]. Although some of our methods are formally related to Lampe's, for example the proof of Theorem 9.1 via Proposition 4.6, we shall be concerned only about distributive algebraic lattices. This topic contains some not so well-known but also unsolved problems, as, for example, whether every distributive algebraic lattice is isomorphic to the congruence lattice of an algebra in some congruence-distributive variety.

If one drops congruence-distributivity, then one would expect the problems to become easier. Consider, for example, the two following problems:

CGP: Is every distributive algebraic lattice isomorphic to the normal subgroup lattice of some group?

CMP: Is every distributive algebraic lattice isomorphic to the submodule lattice of some module?

The problem **CGP** was originally posed for finite distributive (semi)lattices by E. T. Schmidt as [**70**, Problem 5]. A positive solution was provided by H. L. Silcock, who proved in particular that every finite distributive lattice $\mathfrak{D}$ is isomorphic to the normal subgroup lattice of some finite group $\boldsymbol{G}$ (see [**73**]). Later P. P. Pálfy proved that $\boldsymbol{G}$ may be taken finite solvable (see [**55**]). However, the general question seemed open until now. Similarly, the statement of **CMP** has been communicated to the authors by Jan Trlifaj, and nothing seemed to be known about the general case.

A common feature of the varieties of all groups and of all modules over a given ring is that they are congruence-permutable, for example, any two congruences of a group are permutable. Thus both **CGP** and **CMP** are, in some sense, particular instances of the following question:

CPP: Is every distributive algebraic lattice isomorphic to the congruence lattice of some algebra with permuting congruences? [**70**, Problem 3]

Although the exact formulation of [**70**, Problem 3] asked whether every Arguesian algebraic lattice is isomorphic to the congruence lattice of an algebra

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with permutable congruences, it was mentioned there that even the distributive case was open. Meanwhile, the Arguesian case was solved negatively by M. D. Haiman in [**40, 39**], however, the distributive case remained open.

Recall that an algebra $\boldsymbol{A}$ has almost permutable congruences (see [**74**]), if $\Theta\vee\Phi = \Theta\circ\Phi\cup\Phi\circ\Theta$, for all congruences $\Theta, \Phi \in \operatorname{Con} \boldsymbol{A}$ (where the notation $\Theta\circ\Phi$ stands for the usual composition of relations). The three-element chain is an easy example of a lattice with almost permutable congruences but not with permutable congruences. On the other hand, it is not difficult to verify that every congruence almost permutable variety of algebras is congruence permutable. The last two authors of the present paper[2] obtained in [**74**] negative congruence representation results of distributive semilattices by lattices with almost permutable congruences, but nothing was said there about arbitrary algebras with permutable congruences. Furthermore, our attempts based on the "uniform refinement properties" introduced in that paper failed, as these properties turned out to be quite lattice-specific.

In the present paper, we introduce a general framework that makes it possible to extend the methods of [**74**] to arbitrary algebras, and thus solving **CPP**—and, in fact, its generalization to algebras with almost permutable congruences — negatively. Hence, both **CGP** and **CMP** also have negative solutions. In fact, the negative solution obtained in **CGP** for groups extends to loops, as the variety of all loops is also congruence-permutable. Another byproduct is that we also get a negative solution for the corresponding problem for lattice-ordered groups.

Our counterexample is the same as in [**60**] and in [**74**], namely the congruence lattice of a free lattice with at least $\aleph_2$ generators in any non-distributive variety of lattices. We also show that the size $\aleph_2$ is optimal, by showing that every distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the submodule lattice of some module, and also to the normal subgroup lattice of some locally finite group, see Theorems 6.1 and 7.3. We also prove that every distributive algebraic lattice with at most countably many compact elements is isomorphic to the ℓ-ideal lattice of some lattice-ordered group, see Theorem 8.3.

In order to reach our negative results, the main ideas are the following.

(i) Forget about the algebraic structure, just keep the partition lattice representation.

(ii) State a weaker "uniform refinement property" that settles the negative result.

For Point (1), we are looking for a very special sort of lattice homomorphism of a given lattice into some partition lattice, namely, the sort that is induced, as in Proposition 3.2, by a semilattice-valued distance, see Definition 3.1. For a $\langle\vee, \mathbf{0}\rangle$-semilattice $\boldsymbol{S}$ and a set X, an $\boldsymbol{S}$-valued distance on X is a map $\delta\colon X \times X \to \boldsymbol{S}$ satisfying the three usual statements characterizing distances (see Definition 3.1). Every such δ induces a map φ from

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$\mathcal{S}$ to the partition lattice of X (see Proposition 3.2), and if δ satisfies the so-called *V-condition*, then φ is a join-homomorphism. Furthermore, the V-condition of type n says that the equivalences in the range of φ are pairwise $(n+1)$-permutable. Those "distances" have been introduced by B. Jónsson for providing a simple proof of Whitman's Theorem that every lattice can be embedded into some partition lattice, see [**43**] or [**27**, Theorems IV.4.4 and IV.4.8].

While it is difficult to find a suitable notion of morphism between partition lattices, it is easy to do such a thing with our distances, see Definition 3.1. This makes it possible to define what it means for a commutative diagram of $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{S}$ to have a lifting, modulo the forgetful functor, by distances. In particular, we prove, in Theorem 9.2, that the cube $\mathcal{D}_{ac}$ considered in [**74**, Section 7] does not have a lifting by any diagram of V-distances "of type 3/2", that is, the equivalences in the ranges of the corresponding partition lattice representations cannot all be almost permutable. This result had been obtained only for lattices in [**74**].

The original proof of Theorem 9.2 was our main inspiration for getting a weaker "uniformrefinement property", that we denote here by $\mathbf{WURP}^{=}$ (see Definition 4.1). First, we prove that if $\delta\colon X\times X\to\mathcal{S}$ is an $\mathcal{S}$-valued V-distance of type 3/2 with range generating $\mathcal{S}$, then the $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{S}$satisfies $\mathbf{WURP}^{=}$(see Theorem 4.3). Next, we prove that for any free lattice $\boldsymbol{\mathcal{F}}$ with at least $\aleph_2$ generators in any non-distributive variety of lattices, the compact congruence semilattice $\mathrm{Con_c}\,\boldsymbol{\mathcal{F}}$ does not satisfy $\mathbf{WURP}^{=}$(see Corollary 5.8). Therefore, $\mathrm{Con}\,\boldsymbol{\mathcal{F}}$ is not isomorphic to $\mathrm{Con}\,\boldsymbol{A}$, for any algebra $\boldsymbol{A}$ with almost permutable congruences (see Corollary 5.7).

On the positive side, we explain why all previous attempts at finding similar negative results for representations of type 2 (and above) failed. We prove, in particular, that for every distributive $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{S}$, there exists a surjective V-distance $\delta_{\mathcal{S}}\colon X_{\mathcal{S}}\times X_{\mathcal{S}}\twoheadrightarrow\mathcal{S}$ of type 2, which, moreover, depends functorially on $\mathcal{S}$(see Theorem 9.1). In particular, the diagram $\mathcal{D}_{\bowtie}$ considered in [**76**], which is not liftable, with respect to the congruence lattice functor, in any variety whose congruence lattices satisfy a nontrivial identity, is nevertheless liftable by V-distancesof type 2.

2. Preliminaries

The following statement of infinite combinatorics is due to C. Kuratowski [**46**].

THE KURATOWSKI'S FREE SET THEOREM [**46**]. Let n be a positive integer and X a set. Then $|X|\geq n$ if and only if for every map $\phi\colon\ [\,X\,]^n\to[\,X\,]^{<\omega}$, there exists $U\in[\,X\,]^{n+1}$ such that $u\notin\phi(U\setminus\{u\})$, for any $u\in U$.

As in [**60, 78**], only the case $n=2$ will be used.

3. V-distances of type n

Definition 3.1. Let $\boldsymbol{S}$ be a $\langle\vee,\mathbf{0}\rangle$-semilattice and let X be a set. A map $\delta\colon X\times X\to\boldsymbol{S}$ is an *$\boldsymbol{S}$-valued distance* on X, if the following statements hold:

(i) $\delta(x,x)=\mathbf{0}$, for all $x\in X$.
(ii) $\delta(x,y)=\delta(y,x)$, for all $x,y\in X$.
(iii) $\delta(x,z)\leq\delta(x,y)\vee\delta(y,z)$, for all $x,y,z\in X$.

The *kernel* of δ is defined as $\{\langle x,y\rangle\in X\times X\mid\delta(x,y)=\mathbf{0}\}$. The *V-condition* on δ is the following condition:

> For all $x,y\in X$ and all $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{S}$ such that $\delta(x,y)\leq\boldsymbol{a}\vee\boldsymbol{b}$, there are $n\in\omega\setminus\{0\}$ and $x=z_0,z_1,\dots,z_{n+1}=y$ such that for all $i\leq n$, $\delta(z_i,z_{i+1})\leq\boldsymbol{a}$ in case i is even, while $\delta(z_i,z_{i+1})\leq\boldsymbol{b}$ in case i is odd.

In case n is the same for all $x,y,\boldsymbol{a},\boldsymbol{b}$, we say that the distance δ satisfies the *V-condition of type n*, or is a *V-distance of type n*.

We say that δ satisfies the *V-condition of type* $3/2$, or is a *V-distance of type* $3/2$, if for all $x,y\in X$ and all $\boldsymbol{a},\boldsymbol{b}\in\boldsymbol{S}$ such that $\delta(x,y)\leq\boldsymbol{a}\vee\boldsymbol{b}$, there exists $z\in X$ such that either ($\delta(x,z)\leq\boldsymbol{a}$ and $\delta(z,y)\leq\boldsymbol{b}$) or ($\delta(x,z)\leq\boldsymbol{b}$ and $\delta(z,y)\leq\boldsymbol{a}$).

We say that a *morphism* from $\lambda\colon X\times X\to\boldsymbol{S}$ to $\mu\colon Y\times Y\to\boldsymbol{T}$ is a pair $\langle f,\boldsymbol{f}\rangle$, where $\boldsymbol{f}\colon\boldsymbol{S}\to\boldsymbol{T}$ is a $\langle\vee,\mathbf{0}\rangle$-homomorphism and $f\colon X\to Y$ is a map such that $\boldsymbol{f}(\lambda(x,y))=\mu(f(x),f(y))$, for all $x,y\in X$. The forgetful functor sends $\lambda\colon X\times X\to\boldsymbol{S}$ to $\boldsymbol{S}$ and $\langle f,\boldsymbol{f}\rangle$ to $\boldsymbol{f}$.

Denote by $\operatorname{Eq}X$ the lattice of all equivalence relations on a set X. For a positive integer n, we say as usual that $\alpha_0,\alpha_1\in\operatorname{Eq}X$ are *$(n{+}1)$-permutable*, if the compositions $\alpha_i\circ\alpha_{1-i}\circ\alpha_i\circ\cdots\circ\alpha_{i+n \mod 2}$ of length $n+1$, are for $i=0,1$ equal. In particular, 2-permutable is the same as permutable. With every distance is associated a homomorphism to some $\operatorname{Eq}X$, as follows.

Proposition 3.2. *Let $\boldsymbol{S}$ be a $\langle\vee,\mathbf{0}\rangle$-semilattice and let $\delta\colon X\times X\to\boldsymbol{S}$ be an $\boldsymbol{S}$-valued distance. Then one can define a map $\varphi\colon\boldsymbol{S}\to\operatorname{Eq}X$ by the rule $\varphi(a)=\{\langle x,y\rangle\in X\times X\mid\delta(x,y)\leq\boldsymbol{a}\}$, for all $\boldsymbol{a}\in\boldsymbol{S}$. Furthermore,*

(i) *The map φ preserves all existing meets.*
(ii) *If δ satisfies the V-condition, then φ is a join-homomorphism.*
(iii) *If the range of δ join-generates $\boldsymbol{S}$, then φ is an order-embedding.*
(iv) *If the distance δ satisfies the V-condition of type n, then all equivalences in the range of φ are pairwise $(n+1)$-permutable.*

Any algebra gives rise to a natural distance, namely the map $\langle\mathbf{x},\mathbf{y}\rangle\mapsto\Theta(\mathbf{x},\mathbf{y})$ giving the principal congruences.

Proposition 3.3. *Let n be a positive integer and let $\boldsymbol{A}$ be an algebra with $(n+1)$-permutable congruences. Then the semilattice $\operatorname{Con_c}\boldsymbol{A}$ of compact congruences of $\boldsymbol{A}$ is join-generated by the range of a V-distance of type n.*

Proof. Let $\delta\colon\boldsymbol{A}\times\boldsymbol{A}\to\operatorname{Con_c}\boldsymbol{A}$ be defined by $\delta(\mathbf{x},\mathbf{y})=\Theta_{\boldsymbol{A}}(\mathbf{x},\mathbf{y})$, the principal congruence generated by $\langle\mathbf{x},\mathbf{y}\rangle$, for all $\mathbf{x},\mathbf{y}\in\boldsymbol{A}$. The assumption that $\boldsymbol{A}$ has $(n+1)$-permutable congruences means exactly that δ is a V-distance of type n. $\square$

Of course, $\boldsymbol{A}$ has almost permutable congruences if and only if the canonical distance $\theta_{\boldsymbol{A}} \colon \boldsymbol{A} \times \boldsymbol{A} \to \mathrm{Con_c}\,\boldsymbol{A}$ satisfies the V-condition of type $3/2$. We shall focus attention on three often encountered varieties all members of which have permutable (i.e., 2-permutable) congruences:

- The variety of all right modules over a given ring $\boldsymbol{R}$. The congruence lattice of a right module $\boldsymbol{M}$ is canonically isomorphic to the submodule lattice $\mathrm{Sub}\,\boldsymbol{M}$ of $\boldsymbol{M}$. We shall denote by $\mathrm{Sub}_c\,\boldsymbol{M}$ the $\langle\vee,\mathbf{0}\rangle$-semilattice of all finitely generated submodules of $\boldsymbol{M}$.
- The variety of all groups. The congruence lattice of a group $\boldsymbol{G}$ is canonically isomorphic to the normal subgroup lattice $\mathrm{Sub}\,\boldsymbol{G}$ of $\boldsymbol{G}$. We shall denote by $\mathrm{Sub}_c\,\boldsymbol{G}$ the $\langle\vee,\mathbf{0}\rangle$-semilattice of all finitely generated normal subgroups of $\boldsymbol{G}$.
- The variety of all ℓ-groups (i.e., lattice-ordered groups), see [**1**]. The congruence lattice of an ℓ-group $\boldsymbol{G}$ is canonically isomorphic to the lattice $\mathrm{Id}^{\ell}\,\boldsymbol{G}$ of all convex normal subgroups, or ℓ-ideals, of $\boldsymbol{G}$. We shall denote by $\mathrm{Id}^{\ell}_c\,\boldsymbol{G}$ the $\langle\vee,\mathbf{0}\rangle$-semilattice of all finitely generated ℓ-ideals of $\boldsymbol{G}$. Hence we obtain immediately the following result.

Corollary 3.4.

(i) *Let $\boldsymbol{M}$ be a right module over any ring $\boldsymbol{R}$. Then* $\mathrm{Sub}_c\,\boldsymbol{M}$ *is join-generated by the range of a V-distance of type* 1 *on $\boldsymbol{M}$.*

(ii) *Let $\boldsymbol{G}$ be a group. Then* $\mathrm{Sub}_c\,\boldsymbol{G}$ *is join-generated by the range of a V-distance of type* 1 *on $\boldsymbol{G}$.*

(iii) *Let $\boldsymbol{G}$ be an ℓ-group. Then* $\mathrm{Id}^{\ell}_c\,\boldsymbol{G}$ *is join-generated by the range of a V-distance of type* 1 *on $\boldsymbol{G}$.*

The V-distances corresponding to (i), (ii), and (iii) above are, respectively, given by $\delta(\mathbf{x},\mathbf{y}) = (\mathbf{x}-\mathbf{y})\boldsymbol{R}$, $\delta(\mathbf{x},\mathbf{y}) = [\mathbf{x}\mathbf{y}^{-1}]$ (the normal subgroup of $\boldsymbol{G}$ generated by $\mathbf{x}\mathbf{y}^{-1}$), and $\delta(\mathbf{x},\mathbf{y}) = \boldsymbol{G}(\mathbf{x}\mathbf{y}^{-1})$ (the ℓ-ideal of $\boldsymbol{G}$ generated by $\mathbf{x}\mathbf{y}^{-1}$).

The assignments $\boldsymbol{M} \mapsto \mathrm{Sub}_c\,\boldsymbol{M}$, $\boldsymbol{G} \mapsto \mathrm{Sub}_c\,\boldsymbol{G}$, and $\boldsymbol{G} \mapsto \mathrm{Id}^{\ell}_c\,\boldsymbol{G}$ can be canonically extended to direct limits preserving functors to the category $\mathbf{Sem_0}$ of all $\langle\vee,\mathbf{0}\rangle$-semilattices with $\langle\vee,\mathbf{0}\rangle$-homomorphisms.

4. An even weaker uniform refinement property

The following infinitary axiom $\mathbf{WURP}^{=}$ is a weakening of all the various "uniform refinement properties" considered in [**60, 74, 78**]. Furthermore, the proof that follows, aimed at obtaining Theorem 5.6, is very similar to the proofs of [**60**, Theorem 3.3] and [**74**, Theorem 2.1].

Definition 4.1. Let $\boldsymbol{e}$ be an element in a $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}$. We say that $\boldsymbol{\mathcal{S}}$ satisfies $\mathbf{WURP}^{=}(\boldsymbol{e})$, if there exists a positive integer m such that for all families $\langle \boldsymbol{a}_i \mid i \in I \rangle$ and $\langle \boldsymbol{b}_i \mid i \in I \rangle$ of elements of $\boldsymbol{S}$ such that $\boldsymbol{e} \leq \boldsymbol{a}_i \vee \boldsymbol{b}_i$ for all $i \in I$, there are an m-sequence $\langle I_u \mid u < m \rangle$ of subsets of I such that $\bigcup_{u<m} I_u = I$ and a family $\langle \boldsymbol{c}_{i,j} \mid \langle i, j \rangle \in I \times I \rangle$ of elements of $\boldsymbol{S}$ such that the following statements hold:

(i) $\boldsymbol{c}_{i,j} \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\boldsymbol{c}_{i,j} \leq \boldsymbol{b}_i \vee \boldsymbol{b}_j$, for all $u < m$ and all $i, j \in I_u$.
(ii) $\boldsymbol{e} \leq \boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j}$, for all $u < m$ and all $i, j \in I_u$.
(iii) $\boldsymbol{c}_{i,k} \leq \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$, for all $i, j, k \in I$.

Say that $\boldsymbol{S}$ satisfies $\mathbf{WURP}^{=}$, if $\boldsymbol{S}$ satisfies $\mathbf{WURP}^{=}(\boldsymbol{e})$ for all $\boldsymbol{e} \in \boldsymbol{S}$.

The following easy lemma is instrumental in the proof of Corollary 5.7.

LEMMA 4.2. *Let $\boldsymbol{S}$ and $\boldsymbol{T}$ be $\langle \vee, \mathbf{0} \rangle$-semilattices, let $\boldsymbol{f} \colon \boldsymbol{S} \to \boldsymbol{T}$ be a weakly distributive $\langle \vee, \mathbf{0} \rangle$-homomorphism, and let $\boldsymbol{e} \in \boldsymbol{S}$. If $\boldsymbol{S}$ satisfies $\boldsymbol{WURP}^{=}(\boldsymbol{e})$, then $\boldsymbol{T}$ satisfies $\boldsymbol{WURP}^{=}(\boldsymbol{f}(\boldsymbol{e}))$.*

THEOREM 4.3. *Let $\boldsymbol{S}$ be a $\langle \vee, \mathbf{0} \rangle$-semilattice and $\delta \colon X \times X \to \boldsymbol{S}$ a V-distance of type $3/2$ with range join-generating $\boldsymbol{S}$. Then $\boldsymbol{S}$ satisfies $\boldsymbol{WURP}^{=}$.*

PROOF. Let $\boldsymbol{e} \in \boldsymbol{S}$. As $\boldsymbol{S}$ is join-generated by the range of δ, there are a positive integer n and elements $x_\ell, y_\ell \in X$, for $\ell < n$, such that $\boldsymbol{e} = \bigvee\{\delta(x_\ell, y_\ell) \mid \ell < n\}$. For all $i \in I$ and all $\ell < n$, from $\delta(x_\ell, y_\ell) \leq \boldsymbol{a}_i \vee \boldsymbol{b}_i$ and the assumption on δ it follows that there exists $z_{i,\ell} \in X$ such that

$$\begin{aligned} &\text{either } \delta(x_\ell, z_{i,\ell}) \leq \boldsymbol{a}_i \text{ and } \delta(z_{i,\ell}, y_\ell) \leq \boldsymbol{b}_i \\ &\quad\text{or } \delta(x_\ell, z_{i,\ell}) \leq \boldsymbol{b}_i \text{ and } \delta(z_{i,\ell}, y_\ell) \leq \boldsymbol{a}_i. \end{aligned} \tag{4.1}$$

For all $i \in I$ and all $\ell < n$, denote by $P(i, \ell)$ and $Q(i, \ell)$ the following statements:

$$P(i,\ell)\colon \delta(x_\ell, z_{i,\ell}) \leq \boldsymbol{a}_i \text{ and } \delta(z_{i,\ell}, y_\ell) \leq \boldsymbol{b}_i;$$
$$Q(i,\ell)\colon \delta(x_\ell, z_{i,\ell}) \leq \boldsymbol{b}_i \text{ and } \delta(z_{i,\ell}, y_\ell) \leq \boldsymbol{a}_i.$$

We shall prove that $m = 2n$ is a suitable choice for witnessing $\mathbf{WURP}^{=}(\boldsymbol{e})$. We put

$$I_u := \{i \in I \mid (\forall \ell \in u) P(i, \ell) \text{ and } (\forall \ell \in n \setminus u) Q(i, \ell)\}, \text{ for all } u \in \mathcal{P}(n).$$

We claim that $I = \bigcup\{I_u \mid u \in \mathcal{P}(n)\}$. Indeed, let $i \in I$, and put $u = \{\ell < n \mid P(i, \ell)\}$. It follows from (4.1) that $Q(i, \ell)$ holds for all $\ell \in n \setminus u$, whence $i \in I_u$. Now we put

$$\boldsymbol{c}_{i,j} := \bigvee_{\ell<n} \delta(z_{i,\ell}, z_{j,\ell}),$$

for all $i, j \in$ and we prove that the family $\langle \boldsymbol{c}_{i,j} \mid \langle i, j \rangle \in I \times I \rangle$ satisfies the required conditions, with respect to the family $\langle I_u \mid u \in U \rangle$ of 2^n subsets of I. So, let $i, j, k \in I$. The inequality $\boldsymbol{c}_{i,k} \leq \boldsymbol{c}_{i,j} \vee \boldsymbol{c}_{j,k}$ holds trivially. Now suppose that $i, j \in I_u$, for some $u \in U$. Let $\ell < n$. If $\ell \in u$, then

$$\begin{aligned} \delta(z_{i,\ell}, z_{j,\ell}) &\leq \delta(z_{i,\ell}, x_\ell) \vee \delta(x_\ell, z_{j,\ell}) \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j, \\ \delta(x_\ell, y_\ell) &\leq \delta(x_\ell, z_{j,\ell}) \vee \delta(z_{j,\ell}, z_{i,\ell}) \vee \delta(z_{i,\ell}, y_\ell) \leq \boldsymbol{a}_j \vee \boldsymbol{c}_{i,j} \vee \boldsymbol{b}_i, \end{aligned}$$

while if $\ell \in n \setminus u$,

$$\delta(z_{i,\ell}, z_{j,\ell}) \leq \delta(z_{i,\ell}, y_\ell) \vee \delta(y_\ell, z_{j,\ell}) \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j,$$
$$\delta(x_\ell, y_\ell) \leq \delta(x_\ell, z_{i,\ell}) \vee \delta(z_{i,\ell}, z_{j,\ell}) \vee \delta(z_{j,\ell}, y_\ell) \leq \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j} \vee \boldsymbol{a}_j.$$

whence both inequalities $\delta(z_{i,\ell}, z_{j,\ell}) \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\delta(x_\ell, y_\ell) \leq \boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j}$ hold in any case. It follows that $\boldsymbol{c}_{i,j} \leq \boldsymbol{a}_i \vee \boldsymbol{a}_j$ and $\boldsymbol{e} \leq \boldsymbol{a}_j \vee \boldsymbol{b}_i \vee \boldsymbol{c}_{i,j}$. Exchanging x and y in the argument leading to the first inequality also yields that $\boldsymbol{c}_{i,j} \leq \boldsymbol{b}_i \vee \boldsymbol{b}_j$. □

COROLLARY 4.4. *Let $\boldsymbol{A}$ be an algebra with almost permutable congruences. Then* $\mathrm{Con_c}\,\boldsymbol{A}$ *satisfies* $\boldsymbol{WURP}^=$.

REMARK 4.5. In case the distance δ satisfies the V-condition of type 1, the statement $\mathbf{WURP}^=$ in Theorem 4.3 can be strengthened by taking $m = 1$ in Definition 4.1. Similarly, if $\boldsymbol{A}$ is an algebra with permutable congruences, then $\mathrm{Con_c}\,\boldsymbol{A}$ satisfies that strengthening of $\mathbf{WURP}^=$. In particular, as any group, resp. any module, has permutable congruences, both $\mathrm{Sub}_c\,\boldsymbol{G}$, for a group $\boldsymbol{G}$, and $\mathrm{Sub}_c\,\boldsymbol{M}$, for a module $\boldsymbol{M}$, satisfy the strengthening of $\mathbf{WURP}^=$ obtained by taking $m = 1$ in Definition 4.1.

As we shall see in Theorem 5.6, not every distributive $\langle \vee, \mathbf{0} \rangle$-semilattice can be join-generated by the range of a V-distance of type 3/2. The situation changes dramatically for type 2. It is proved in [**26**, Appendix 7] that any modular algebraic lattice is isomorphic to the congruence lattice of an algebra with 3-permutable congruences. This easily implies the following result; nevertheless, we provide a much more direct argument, which will be useful for the proof of Theorem 9.1.

PROPOSITION 4.6. *Any distributive $\langle \vee, \mathbf{0} \rangle$-semilattice is the range of some V-distance of type* 2.

PROOF. Let $\mathbb{S}$ be a distributive $\langle \vee, \mathbf{0} \rangle$-semilattice. We first observe that the map $\mu_{\mathbb{S}} \colon \mathbb{S} \times \mathbb{S} \to \mathbb{S}$ defined by the rule

$$\mu_{\mathbb{S}}(x, y) := \begin{cases} x \vee y, & \text{if } x \neq y, \\ \mathbf{0}, & \text{if } x = y \end{cases} \tag{4.2}$$

is a surjective $\mathbb{S}$-valued distance on $\mathbb{S}$. Now suppose that we are given a surjective $\mathbb{S}$-valued distance $\delta \colon X \times X \to \mathbb{S}$, and let $x, y \in X$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{S}$ be such that $\delta(x, y) \leq \boldsymbol{a} \vee \boldsymbol{b}$. Since $\mathbb{S}$ is distributive, there are $\boldsymbol{a}' \leq \boldsymbol{a}$ and $\boldsymbol{b}' \leq \boldsymbol{b}$ such that $\delta(x, y) = \boldsymbol{a}' \vee \boldsymbol{b}'$. We put $X' = X \cup \{u, v\}$, where u and v are two distinct outside points, and we extend δ to a distance δ' on X' by putting $\delta'(z, u) := \delta(z, x) \vee \boldsymbol{a}'$ and $\delta'(z, v) := \delta(z, y) \vee \boldsymbol{a}'$, for all $z \in X$, while $\delta'(u, v) = \boldsymbol{b}'$. It is straightforward to verify that δ' is an $\mathbb{S}$-valued distance on X' extending δ. Furthermore, $\delta'(x, u) = \boldsymbol{a}' \leq \boldsymbol{a}$, $\delta'(u, v) = \boldsymbol{b}' \leq \boldsymbol{b}$, and $\delta'(v, y) = \boldsymbol{a}' \leq \boldsymbol{a}$. Iterating this construction transfinitely, taking direct limits at limit stages, yields an $\mathbb{S}$-valued V-distance of type 2 extending δ. □

5. Failure of $\mathbf{WURP}^{=}$ in $\mathrm{Con_c}\,\mathcal{F}$, for $\mathcal{F}$ free bounded lattice

The main proof of the present section, that is, the proof of Theorem 5.6, follows the lines of the proofs of [**60**, Theorem 3.3] and [**74**, Corollary 2.1]. However, there are a few necessary changes, mainly due to the new "uniform refinement property" not being the same as the previously considered ones. As the new result extends to any algebra, and not only lattices (see Corollary 5.7), we feel that it is still worthwhile to show the main lines of the proof in some detail.

From now on until Lemma 5.5, we shall fix a non-distributive lattice variety $\mathcal{V}$. For every set X, denote by $\mathcal{B}_{\mathcal{V}}(X)$ (or $\mathcal{B}(X)$ in case $\mathcal{V}$ is understood) the bounded lattice in $\mathcal{V}$ freely generated by two-element chains $s_i < t_i$, for $i \in X$. Note that if Y is a subset of X, then there is a unique retraction from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$, sending each s_i to $\mathbf{0}$ and each t_i to $\mathbf{1}$, for every $i \in X \setminus Y$. Thus, we shall often identify $\mathcal{B}(Y)$ with the bounded sublattice of $\mathcal{B}(X)$ generated by all s_i and t_i ($i \in Y$). Moreover, the above mentioned retraction from $\mathcal{B}(X)$ onto $\mathcal{B}(Y)$ induces a retraction from $\mathrm{Con_c}\,\mathcal{B}(X)$ onto $\mathrm{Con_c}\,\mathcal{B}(Y)$. Hence, we shall also identify $\mathrm{Con_c}\,\mathcal{B}(Y)$ with the corresponding subsemilattice of $\mathrm{Con_c}\,\mathcal{B}(X)$.

Now we fix a set X such that $|X| \geq \aleph_2$. We denote, for all $i \in X$, by $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ the compact congruences of $\mathcal{B}(X)$ defined by

$$\boldsymbol{a}_i := \Theta(\mathbf{0}, s_i) \vee \Theta(t_i, \mathbf{1}); \quad \boldsymbol{b}_i := \Theta(s_i, t_i). \tag{5.1}$$

In particular, note that $\boldsymbol{a}_i \vee \boldsymbol{b}_i = \mathbf{1}$, the largest congruence of $\mathcal{B}(\mathrm{X})$.

Now, towards a contradiction, suppose that there are a positive integer n, a decomposition $X = \bigcup\{X_k \mid k < n\}$, and a family $\langle \boldsymbol{c}_{i,j} \mid \langle i,j\rangle \in X \times X\rangle$ of elements of $\mathrm{Con_c}\,\mathcal{B}(X)$ witnessing the statement that $\mathrm{Con_c}\,\mathcal{B}(X)$ satisfies $\mathbf{WURP}^{=}(\mathbf{1})$, where $\mathbf{1}$ denotes the largest congruence of $\mathcal{B}(X)$. We pick $k < n$ such that $|X_k| = |X|$. By "projecting everything on $\mathcal{B}(X_k)$" (as in [**74**, page 224]), we might assume that $X_k = X$.

Since the Con functor preserves direct limits, for all $i, j \in X$, there exists a finite subset $F(\{i,j\})$ of X such that both $\boldsymbol{c}_{i,j}$ and $\boldsymbol{c}_{j,i}$ belong to $\mathrm{Con_c}\,\mathcal{B}(F(\{i,j\}))$. By Kuratowski's Theorem, there are distinct elements $0, 1, 2$ of X such that $0 \notin F(\{1,2\})$, $1 \notin F(\{0,2\})$, and $2 \notin F(\{0,1\})$. Denote by $\pi\colon \mathcal{B}(X) \twoheadrightarrow \mathcal{B}(\{0,1,2\})$ the canonical retraction. For every $i \in \{0,1,2\}$, denote by i' and i'' the other two elements of $\{0,1,2\}$, arranged in such a way that $i' < i''$. We put $\boldsymbol{d}_i := (\mathrm{Con_c}\,\pi)(\boldsymbol{c}_{i',i''})$, for all $i \in \{0,1,2\}$.

Applying the semilattice homomorphism $\mathrm{Con_c}\,\pi$ to the inequalities satisfied by the elements $\boldsymbol{c}_{i,j}$ yields

$$\boldsymbol{d}_0 \subseteq \boldsymbol{a}_1 \vee \boldsymbol{a}_2, \boldsymbol{b}_1 \vee \boldsymbol{b}_2; \quad \boldsymbol{d}_1 \subseteq \boldsymbol{a}_0 \vee \boldsymbol{a}_2, \boldsymbol{b}_0 \vee \boldsymbol{b}_2; \quad \boldsymbol{d}_2 \subseteq \boldsymbol{a}_0 \vee \boldsymbol{a}_1, \boldsymbol{b}_0 \vee \boldsymbol{b}_1; \tag{5.2}$$

$$\boldsymbol{d}_0 \vee \boldsymbol{a}_2 \vee \boldsymbol{b}_1 = \boldsymbol{d}_1 \vee \boldsymbol{a}_2 \vee \boldsymbol{b}_0 = \boldsymbol{d}_2 \vee \boldsymbol{a}_1 \vee \boldsymbol{b}_0 = \mathbf{1}; \tag{5.3}$$

$$\boldsymbol{d}_1 \subseteq \boldsymbol{d}_0 \vee \boldsymbol{d}_2. \tag{5.4}$$

As in [**60**, Lemma 2.1], it is not hard to prove the following.

LEMMA 5.1. *The congruence* $\boldsymbol{d}_i$ *belongs to* $\operatorname{Con_c} \mathbf{B}(\{i', i''\})$, *for all* $i \in \{0, 1, 2\}$.

Since $\mathcal{V}$ is a non-distributive variety of lattices, it follows from a classical result of lattice theory that $\mathcal{V}$ contains as a member some lattice $\mathbf{M} \in \{\mathbf{M}_3, \mathbf{N}_5\}$. Decorate the lattice $\mathbf{M}$ with three 2-element chains $x_i < y_i$ (for $i \in \{0, 1, 2\}$) as in [**60**], which we illustrate on Figure 1.

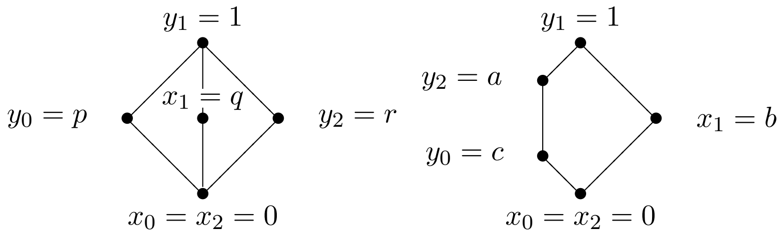

FIGURE 1. The decorations of $\mathbf{M}_3$ and $\mathbf{N}_5$.

The relevant properties of these decorations are summarized in the two following straightforward lemmas.

LEMMA 5.2. *The decorations defined above satisfy the following inequalities*

$$x_0 \wedge y_1 \leq x_1; \quad y_1 \leq x_1 \vee y_0; \quad x_1 \wedge y_0 \leq x_0; \quad y_0 \leq x_0 \vee y_1;$$
$$x_1 \wedge y_2 \leq x_2; \quad y_2 \leq x_2 \vee y_1; \quad x_2 \wedge y_1 \leq x_1; \quad y_1 \leq x_1 \vee y_2,$$

but $y_2 \nleq x_2 \vee y_0$.

LEMMA 5.3. *The sublattice of* $\mathbf{M}$ *generated by* $\{x'_i, x''_i, y'_i, y''_i\}$ *is distributive, for all* $i \in \{0, 1, 2\}$.

Now we shall denote by $\mathbf{D}$ be the free product (i.e., the coproduct) of two 2-element chains, say $u_0 < v_0$ and $u_1 < v_1$, in the variety of all distributive lattices. The lattice $\mathbf{D}$ is diagrammed on Figure 2.

The join-irreducible elements of $\mathbf{D}$ are $u_0, u_1, v_0, v_1, u'_0 := u_0 \wedge v_1, u'_1 := u_1 \wedge v_0$, and $w := v_0 \wedge v_1$. Since $\mathbf{D}$ is finite distributive, its congruence lattice is finite Boolean, with seven atoms $\boldsymbol{p} := \Theta_{\mathbf{D}}(p_*, p)$, for $p \in J(\mathbf{D})$ (where p_* denotes the unique lower cover of p in $\mathbf{D}$), that is,

$$\boldsymbol{u}_0 = \Theta^+_{\mathbf{D}}(u_0, v_1); \qquad \boldsymbol{u}_1 = \Theta^+_{\mathbf{D}}(u_1, v_0);$$
$$\boldsymbol{v}_0 = \Theta^+_{\mathbf{D}}(v_0, u_0 \vee v_1); \qquad \boldsymbol{v}_1 = \Theta^+_{\mathbf{D}}(v_1, u_1 \vee v_0);$$
$$\boldsymbol{u}'_0 = \Theta^+_{\mathbf{D}}(u_0 \wedge v_1, u_1); \qquad \boldsymbol{u}'_1 = \Theta^+_{\mathbf{D}}(u_1 \wedge v_0, u_0);$$
$$\boldsymbol{w} = \Theta_{\mathbf{D}}((u_0 \wedge v_1) \vee (u_1 \wedge v_0), v_0 \wedge v_1).$$

For all $i \in \{0, 1, 2\}$, let $\pi_i \colon \mathbf{B}(\{i', i''\}) \to \mathbf{D}$ be the unique lattice homomorphism sending $s_{i'} \mapsto u_0$, $t_{i'} \mapsto v_0$, $s_{i''} \mapsto u_1$, and $t_{i''} \mapsto v_1$. Furthermore, denote by $\rho \colon \mathbf{B}(\{0, 1, 2\}) \to \mathbf{M}$ the unique lattice homomorphism sending

$s_i \mapsto x_i$ and $t_i \mapsto y_i$ (for all $i \in \{0,1,2\}$); denote by ρ_i the restriction of ρ to $\mathcal{B}(\{i', i''\})$.

We shall restate [**60**, Lemma 3.1] here for convenience.

LEMMA 5.4. *Let $\mathcal{L}$ be any distributive lattice, let a, b, a', b' be elements of $\mathcal{L}$. Then the equality $\Theta^+_{\mathcal{L}}(a,b) \cap \Theta^+_{\mathcal{L}}(a',b') = \Theta^+_{\mathcal{L}}(a \wedge a', b \vee b')$ holds.*

Now we put $\boldsymbol{e}_i := (\mathrm{Con_c}\,\pi_i)(\boldsymbol{d}_i)$, for all $i \in \{0,1,2\}$.

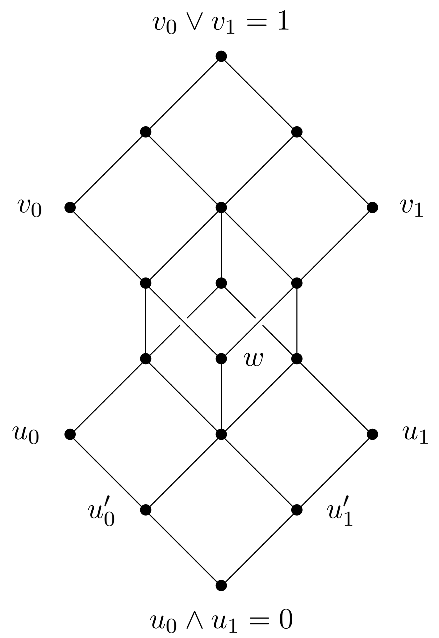

FIGURE 2. The distributive lattice $\mathcal{D}$.

LEMMA 5.5. *The containments $\boldsymbol{e}^- \subseteq \boldsymbol{e}_i \subseteq \boldsymbol{e}^+$ hold for all $i \in \{0,1,2\}$, where we put*

$$\boldsymbol{e}^- = \Theta^+_{\mathcal{D}}(u_0 \wedge v_1, u_1) \vee \Theta^+_{\mathcal{D}}(v_1, u_1 \vee v_0),$$
$$\boldsymbol{e}^+ = \Theta^+_{\mathcal{D}}(u_0 \wedge v_1, u_1) \vee \Theta^+_{\mathcal{D}}(v_1, u_1 \vee v_0) \vee \Theta^+_{\mathcal{D}}(u_1 \wedge v_0, u_0) \vee \Theta^+_{\mathcal{D}}(v_0, u_0 \vee v_1).$$

PROOF. Applying $\mathrm{Con_c}\,\pi_i$ to the inequalities (5.2) and (5.3) yields the following inequalities:

$$\boldsymbol{e}_i \subseteq \Theta(0,u_0) \vee \Theta(0,u_1) \vee \Theta(v_0,1) \vee \Theta(v_1,1), \tag{5.5}$$

$$\boldsymbol{e}_i \subseteq \Theta(u_0,v_0) \vee \Theta(u_1,v_1), \tag{5.6}$$

$$\boldsymbol{e}_i \vee \Theta(0,u_1) \vee \Theta(v_1,1) \vee \Theta(u_0,v_0) = \mathbf{1}. \tag{5.7}$$

By using Lemma 5.4 and the distributivity of $\mathrm{Con}\,\mathcal{D}$, we obtain, by meeting (5.5) and (5.6), the inequality $\boldsymbol{e}_i \subseteq \boldsymbol{e}^+$. On the other hand, by using (5.7) together with the equality

$$\Theta(0,u_1)\vee\Theta(v_1,1)\vee\Theta(u_0,v_0)=\boldsymbol{u}_0\vee\boldsymbol{u}_1\vee\boldsymbol{u}_1'\vee\boldsymbol{v}_0\vee\boldsymbol{w},$$

(see Figure 2), we obtain that $\boldsymbol{e}^-=\boldsymbol{u}_0'\vee\boldsymbol{v}_1\subseteq\boldsymbol{e}_i$. □

Now, for all $i\in\{0,1,2\}$, it follows from Lemma 5.3 that there exists a unique lattice homomorphism $\varphi_i\colon\boldsymbol{\mathcal{D}}\to\boldsymbol{\mathcal{M}}$ such that $\varphi_i\circ\pi_i=\rho_i$. Since $\mathrm{Con_c}$ is a functor, we get from this and from Lemma 5.5 that for all $i\in\{0,1,2\}$,

$$\begin{aligned}&(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_i)=(\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}_i)\subseteq(\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}^+)=\\&(\mathrm{Con_c}\,\varphi_i)(\Theta^+(u_0\wedge v_1,u_1)\vee\Theta^+(v_1,u_1\vee v_0)\vee\\&\Theta^+(u_1\wedge v_0,u_0)\vee\Theta^+(v_0,u_0\vee v_1))=\\&\Theta^+(x_{i'}\wedge y_{i''},x_{i''})\vee\Theta^+(y_{i''},x_{i''}\vee y_{i'})\vee\\&\Theta^+(x_{i''}\wedge y_{i'},x_{i'})\vee\Theta^+(y_{i'},x_{i'}\vee y_{i''}).\end{aligned}\tag{5.8}$$

while

$$\begin{aligned}(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_i)&=(\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}_i)\supseteq(\mathrm{Con_c}\,\varphi_i)(\boldsymbol{e}^-)\\&=(\mathrm{Con_c}\,\varphi_i)(\Theta^+(u_0\wedge v_1,u_1)\vee\Theta^+(v_1,u_1\vee v_0))\\&=\Theta^+(x_{i'}\wedge y_{i''},x_{i''})\vee\Theta^+(y_{i''},x_{i''}\vee y_{i'}).\end{aligned}\tag{5.9}$$

In particular, we obtain, using Lemma 5.2,

$$\begin{aligned}&(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_0)=\mathbf{0},\\&(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_2)=\mathbf{0},\\\text{while}\quad&(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_1)\supseteq\Theta^+(x_0\wedge y_2,x_2)\vee\Theta^+(y_2,x_2\vee y_0)\neq\mathbf{0}.\end{aligned}$$

On the other hand, by applying $\mathrm{Con_c}\,\rho$ to (5.4), we obtain that

$$(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_1)\subseteq(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_0)\vee(\mathrm{Con_c}\,\rho)(\boldsymbol{d}_2),$$

a contradiction. Therefore, we have proved the following theorem.

THEOREM 5.6. *Let $\mathcal{V}$ be any non-distributive variety of lattices, let X be any set such that $|X|\geq\aleph_2$. Denote by $\boldsymbol{\mathcal{B}}_{\mathcal{V}}(X)$ the free product in $\mathcal{V}$ of X copies of a two-element chain with a least and a largest element added. Then $\mathrm{Con_c}\,\boldsymbol{\mathcal{B}}_{\mathcal{V}}(X)$ does not satisfy* $\boldsymbol{WURP}^=$ *at its largest element.*

A "local" version of Theorem 5.6 is presented in Theorem 9.2.

Observe that $\mathrm{Con_c}\,\boldsymbol{\mathcal{B}}_{\mathcal{V}}(X)$, being the semilattice of compact congruences of a lattice, is distributive.

As in [**60**, Corollary 4.1], we obtain the following.

COROLLARY 5.7. *Let $\boldsymbol{\mathcal{L}}$ be any lattice that admits a lattice homomorphism onto a free bounded lattice in the variety generated by either $\boldsymbol{\mathcal{M}}_3$ or $\boldsymbol{\mathcal{N}}_5$ with $\aleph_2$ generators. Then $\mathrm{Con_c}\,\boldsymbol{\mathcal{L}}$ does not satisfy* $\boldsymbol{WURP}^=$. *In particular, there exists no V-distance of type $3/2$ with range join-generating $\mathrm{Con_c}\,\boldsymbol{\mathcal{L}}$. Hence there is no algebra $\boldsymbol{A}$ with almost permutable congruences such that $\mathrm{Con}\,\boldsymbol{\mathcal{L}}\simeq\mathrm{Con}\,\boldsymbol{A}$.*

PROOF. The first part of the proof goes like the proof of [**60**, Corollary 4.1], using Lemma 4.2. The rest of the conclusion follows from Theorem 4.3. $\square$

COROLLARY 5.8. *Let $\mathcal{V}$ be any non-distributive variety of lattices and let $\boldsymbol{\mathcal{F}}$ be any free (resp., free bounded) lattice with at least $\aleph_2$ generators in $\mathcal{V}$. Then there exists no $\mathcal{V}$-distance of type $3/2$ with range join-generating $\operatorname{Con_c}\boldsymbol{\mathcal{F}}$. In particular, there is no algebra $\boldsymbol{A}$ with almost permutable congruences such that $\operatorname{Con}\boldsymbol{\mathcal{F}} \simeq \operatorname{Con}\boldsymbol{A}$.*

By using Cor 3.4, we thus obtain the following.

COROLLARY 5.9. *Let $\mathcal{V}$ be a non-distributive variety of lattices, let $\boldsymbol{\mathcal{F}}$ be any free (resp., free bounded) lattice with at least $\aleph_2$ generators in $\mathcal{V}$, and put $\boldsymbol{\mathcal{D}} := \operatorname{Con}\boldsymbol{\mathcal{F}}$—a distributive, algebraic lattice with $\aleph_2$ compact elements. Then there is no module $\boldsymbol{M}$ (resp., no group $\boldsymbol{G}$, resp. no ℓ-group $\boldsymbol{G}$) such that $\operatorname{Sub}\boldsymbol{M} \simeq \boldsymbol{\mathcal{D}}$ (resp., $\operatorname{Sub}\boldsymbol{G} \simeq \boldsymbol{\mathcal{D}}$, $\operatorname{Id}^{\ell}\boldsymbol{G} \simeq \boldsymbol{\mathcal{D}}$).*

Hence, not every distributive algebraic lattice is isomorphic to the submodule lattice of some module, or to the normal subgroup lattice of some group. However, our proof of this negative result requires at least $\aleph_2$ compact elements. As we shall see in Sections 6 and 7, the $\aleph_2$ bound is, in both cases of modules and groups, optimal.

6. Representing distributive algebraic lattices with at most $\aleph_1$ compact elements as submodule lattices of modules

In this section we deal with congruence lattices of right modules over rings.

THEOREM 6.1. *Every distributive $\langle\vee, \mathbf{0}\rangle$-semilattice of size at most $\aleph_1$ is isomorphic to the submodule lattice of some right module.*

PROOF. Let $\boldsymbol{\mathcal{S}}$ be a distributive $\langle\vee, \mathbf{0}\rangle$-semilattice of size at most $\aleph_1$. If $\boldsymbol{\mathcal{S}}$ has a largest element, then it follows from the main result of [**79**] that $\boldsymbol{\mathcal{S}}$ is isomorphic to the semilattice $\operatorname{Id}_c(\boldsymbol{R})$ of all finitely generated two-sided ideals of some (unital) von Neumann regular ring $\boldsymbol{R}$.

In order to reduce ideals to submodules, we use a well-known trick. As $\boldsymbol{R}$ is a bimodule over itself, the tensor product $\overline{\boldsymbol{R}} = \boldsymbol{R}^{\mathrm{op}} \otimes \boldsymbol{R}$ can be endowed with a structure of (unital) ring, with multiplication satisfying $(a\otimes b) \cdot (a'\otimes b') = (a'a)\otimes(bb')$ (both $a'a$ and bb' are evaluated in $\boldsymbol{R}$). Then $\boldsymbol{R}$ is a right $\overline{\boldsymbol{R}}$-module, with scalar multiplication given by $x \cdot (a\otimes b) = axb$, and the submodules of $\boldsymbol{R}_{\overline{\boldsymbol{R}}}$ are exactly the two-sided ideals of $\boldsymbol{R}$. Hence, $\operatorname{Sub}_c \boldsymbol{R}_{\overline{\boldsymbol{R}}} = \operatorname{Id}_c(\boldsymbol{R}) \simeq \boldsymbol{\mathcal{S}}$.

In case $\boldsymbol{\mathcal{S}}$ has no unit, it is an ideal of the distributive $\langle\vee, \mathbf{0}, \mathbf{1}\rangle$-semilattice $\boldsymbol{\mathcal{S}}' = \boldsymbol{\mathcal{S}} \cup \{\mathbf{1}\}$ for a new largest element $\mathbf{1}$. By the previous paragraph, $\boldsymbol{\mathcal{S}}' \simeq \operatorname{Sub}_c \boldsymbol{M}$ for some right module $\boldsymbol{M}$, hence $\boldsymbol{\mathcal{S}} \simeq \operatorname{Sub}_c \boldsymbol{N}$ where $\boldsymbol{N}$ is the submodule of $\boldsymbol{M}$ consisting of those elements $x \in \boldsymbol{M}$ such that the submodule generated by x is sent to an element of $\boldsymbol{\mathcal{S}}$ by the isomorphism $\mathrm{Sub}_c\,\boldsymbol{M} \simeq \boldsymbol{\mathcal{S}}'$. $\square$

The commutative case is quite different. For example, for the ideal lattice of a commutative von Neumann regular ring $\boldsymbol{R}$ is finite, then, as it is distributive and complemented; it must be Boolean. In particular, the three-element chain is not isomorphic to the ideal lattice of any commutative von Neumann regular ring. Even if regularity is removed, not every finite distributive lattice is allowed. For example, one can prove the following result: A finite distributive lattice $\boldsymbol{\mathcal{D}}$ is isomorphic to the submodule lattice of a module over some commutative ring if and only if $\boldsymbol{\mathcal{D}}$ is isomorphic to the ideal lattice of some commutative ring, if and only if $\boldsymbol{\mathcal{D}}$ is a product of chains. In particular, the square $\mathbf{2} \times \mathbf{2}$ with a new bottom (resp., top) element added is not isomorphic to the submodule lattice of any module over a commutative ring.

7. Representing distributive algebraic lattices with at most $\aleph_1$ compact elements as normal subgroup lattices of groups

Every nonabelian simple group is "neutral" in the sense of [**19**]. Hence, the direction (1) $\Rightarrow$ (5) in [**19**, Theorem 8.5] yields the following well-known result, which holds despite the failure of congruence-distributivity in the variety of all groups.

LEMMA 7.1. *Let $n < \omega$ and $\langle \boldsymbol{G}_i \mid i < n\rangle$ be a finite sequence of simple non-abelian groups. Then the normal subgroups of $\prod_{i<n} \boldsymbol{G}_i$ are exactly the trivial ones, namely the products of the form $\prod_{i<n} \boldsymbol{H}_i$, where $\boldsymbol{H}_i$ is either $\boldsymbol{G}_i$ or $\{1_{\boldsymbol{G}_i}\}$, for all $i < n$. Consequently,* $\mathrm{Sub}\left(\prod_{i<n} \boldsymbol{G}_i\right) \simeq \mathbf{2}^n$.

We denote by $\mathcal{F}$ the class of all finite products of alternating groups of the form $\boldsymbol{A}_n$, for $n \geq 5$. For a group homomorphism $f\colon \boldsymbol{G} \to \boldsymbol{H}$, we denote by $\mathrm{Sub}\,f\colon \mathrm{Sub}\,\boldsymbol{G} \to \mathrm{Sub}\,\boldsymbol{H}$ the $\langle \vee, \mathbf{0}\rangle$-homomorphism that with any normal subgroup $\boldsymbol{X}$ of $\boldsymbol{G}$ associates the normal subgroup of $\boldsymbol{H}$ generated by $f(\boldsymbol{X})$. The following square amalgamation result is crucial. It is an analogue for groups of [**28**, Theorem 1] (for lattices) or [**79**, Theorem 4.2] (for regular algebras over a division ring).

LEMMA 7.2. *Let $\boldsymbol{G}_0, \boldsymbol{G}_1, \boldsymbol{G}_2$ be groups in $\mathcal{F}$ and let $f_1\colon \boldsymbol{G}_0 \to \boldsymbol{G}_1$ and $f_2\colon \boldsymbol{G}_0 \to \boldsymbol{G}_2$ be group homomorphisms. Let $\boldsymbol{\mathcal{B}}$ be a finite Boolean semilattice, and, for $i \in \{1, 2\}$, let $\boldsymbol{g}_i\colon \mathrm{Sub}\,\boldsymbol{G}_i \to \boldsymbol{\mathcal{B}}$ be $\langle \vee, \mathbf{0}\rangle$-homomorphisms such that*

$$\boldsymbol{g}_1 \circ \mathrm{Sub}\,f_1 = \boldsymbol{g}_2 \circ \mathrm{Sub}\,f_2. \tag{7.1}$$

Then there are a group $\boldsymbol{G}$ in $\mathcal{F}$, group homomorphisms $g_i\colon \boldsymbol{G}_i \to \boldsymbol{G}$, for $i \in \{1, 2\}$, and an isomorphism $\boldsymbol{\alpha}\colon \mathrm{Sub}\,\boldsymbol{G} \to \boldsymbol{\mathcal{B}}$ such that $g_1 \circ f_1 = g_2 \circ f_2$ and $\boldsymbol{\alpha} \circ \mathrm{Sub}\,g_i = \boldsymbol{g}_i$ for all $i \in \{1, 2\}$.

Outline of proof. We follow the lines of the proofs of [**28**, Theorem 1] or [**79**, Theorem 4.2]. First, by decomposing $\mathcal{B}$ as a finite power of $\mathbf{2}$, observing that $\mathcal{F}$ is closed under finite direct products, and using Lemma 7.1, we reduce to the case where $\mathcal{B} = \mathbf{2}$, the two-element chain. Next, denoting by $\boldsymbol{h}$ the $\langle\vee,\mathbf{0}\rangle$-homomorphism appearing on both sides of (7.1), we put $\boldsymbol{G}_0' = \{x \in \boldsymbol{G}_0 \mid \boldsymbol{h}([\,x\,]) = \mathbf{0}\}$ (where $[\,x\,]$ denotes, again, the normal subgroup generated by x), and, similarly, $\boldsymbol{G}_i' = \{x \in \boldsymbol{G}_i \mid \boldsymbol{g}_i([\,x\,]) = \mathbf{0}\}$, for $i \in \{1, 2\}$. So $\boldsymbol{G}_i'$ is a normal subgroup of $\boldsymbol{G}_i$, for all $i \in \{0, 1, 2\}$, and replacing $\boldsymbol{G}_i$ by $\boldsymbol{G}_i/\boldsymbol{G}_i'$ makes it possible to reduce to the case where both $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$ separate zero while both f_1 and f_2 are group embeddings.

Hence the problem that we must solve is the following: given group embeddings $f_i\colon \boldsymbol{G}_0 \to \boldsymbol{G}_i$, for $i \in \{1, 2\}$, we must find a finite, simple, non-abelian group $\boldsymbol{G}$ with group embeddings $g_i\colon \boldsymbol{G}_i \to \boldsymbol{G}$ such that $g_1 \circ f_1 = g_2 \circ f_2$. By the positive solution of the amalgamation problem for finite groups (see [**53**, Section 15]), followed by embedding the resulting group into some alternating group with index at least 5, this is possible. $\square$

Now every distributive $\langle\vee,\mathbf{0}\rangle$-semilattice of size at most $\aleph_1$ is the direct limit of some direct system of finite Boolean $\langle\vee,\mathbf{0}\rangle$-semilattices and $\langle\vee,\mathbf{0}\rangle$-homomorphisms; furthermore, we may assume that the indexing set of the direct system is a 2-ladder, that is, a lattice with zero where every interval is finite and every element has at most two immediate predecessors. Hence, by imitating the method of proof used in [**28**, Theorem 2] or [**79**, Theorem 5.2], it is not difficult to obtain the following result.

Theorem 7.3. *Every distributive $\langle\vee,\mathbf{0}\rangle$-semilattice of size at most $\aleph_1$ is isomorphic to the finitely generated normal subgroup semilattice of some group which is a direct limit of members of $\mathcal{F}$.*

Reformulating the result in terms of algebraic lattices rather than semilattices, together with the observation that all direct limits of groups in F are locally finite, gives the following.

Corollary 7.4. *Every distributive algebraic lattice with at most $\aleph_1$ compact elements is isomorphic to the normal subgroup lattice of some locally finite group.*

8. Representing distributive algebraic lattices with at most $\aleph_0$ compact elements as ℓ-ideal lattices of ℓ-groups

The variety of ℓ-groups is quite special, as it is both congruence-distributive and congruence-permutable. The following lemma does not extend to the commutative case (for example, $\mathbb{Z} \times \mathbb{Z}$ cannot be embedded into any simple commutative ℓ-group).

Lemma 8.1. *Every ℓ-group can be embedded into some simple ℓ-group.*

PROOF. It follows from [**56**, Corollary 5.2] that every ℓ-group $\boldsymbol{G}$ embeds into an ℓ-group $\boldsymbol{H}$ in which any two positive elements are conjugate. In particular, $\boldsymbol{H}$ is simple. $\square$

The following result is a "one-dimensional" analogue for ℓ-groups of Lemma 7.2.

LEMMA 8.2. *For any ℓ-group $\boldsymbol{G}$, any finite Boolean semilattice $\boldsymbol{\mathcal{B}}$, and any $\langle\vee,\mathbf{0}\rangle$-homomorphism $\boldsymbol{f}\colon \operatorname{Id}_c^\ell \boldsymbol{G} \to \boldsymbol{\mathcal{B}}$, there are an ℓ-group $\boldsymbol{H}$, an ℓ-homomorphism $f\colon \boldsymbol{G} \to \boldsymbol{H}$, and an isomorphism $\boldsymbol{\alpha}\colon \operatorname{Id}_c^\ell \boldsymbol{H} \to \boldsymbol{\mathcal{B}}$ such that $\boldsymbol{f} = \boldsymbol{\alpha} \circ \operatorname{Id}_c^\ell f$.*

PROOF. Suppose first that $\boldsymbol{\mathcal{B}} = \mathbf{2}$. Let $\boldsymbol{G}(x)$ denote the ℓ-ideal generated by x. Observing that $I := \{x \in \boldsymbol{G} \mid f(\boldsymbol{G}(x)) = 0\}$ is an ℓ-ideal of $\boldsymbol{G}$, we let $\boldsymbol{H}$ be any simple ℓ-group extending $\boldsymbol{G}/I$ (see Lemma 8.1), we let $f\colon \boldsymbol{G} \to \boldsymbol{H}$ be the composition of the canonical projection $\boldsymbol{G} \twoheadrightarrow \boldsymbol{G}/I$ with the inclusion map $\boldsymbol{G}/I \hookrightarrow \boldsymbol{H}$, and we let $\boldsymbol{\alpha}\colon \operatorname{Id}_c^\ell \boldsymbol{H} \to \mathbf{2}$ be the unique isomorphism.

Now suppose that $\boldsymbol{\mathcal{B}} = \mathbf{2}^n$, for a natural number n. For each $i < n$, we apply the result of the paragraph above to the i^{th} component $\boldsymbol{f}_i\colon \operatorname{Id}_c^\ell \boldsymbol{G} \to \mathbf{2}$ of $\boldsymbol{f}$, getting a simple ℓ-group $\boldsymbol{H}_i$, an ℓ-homomorphism $f_i\colon \boldsymbol{G} \to \boldsymbol{H}_i$, and the isomorphism $\boldsymbol{\alpha}_i\colon \operatorname{Id}_c^\ell \boldsymbol{H}_i \to \mathbf{2}$. Then we put $\boldsymbol{H} := \prod_{i<n} \boldsymbol{H}_i$, $f\colon x \mapsto \langle f_i(x) \mid i < n\rangle$, and we let $\boldsymbol{\alpha}\colon \operatorname{Id}_c^\ell \boldsymbol{H} \to \mathbf{2}^n$ be the canonical isomorphism. $\square$

THEOREM 8.3. *Every distributive at most countable $\langle\vee,\mathbf{0}\rangle$-semilattice is isomorphic to the semilattice of all finitely generated ℓ-ideals of some ℓ-group.*

Equivalently, every distributive algebraic lattice with (at most) countably many compact elements is isomorphic to the ℓ-ideal lattice of some ℓ-group.

PROOF. It follows from [**12**, Theorem 3.1] (see also [**25**, Theorem 6.6]) that every distributive at most countable $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}$ can be expressed as the direct limit of a sequence $\langle \boldsymbol{\mathcal{B}}_n \mid n < \omega\rangle$ of finite Boolean semilattices, with all transition maps $\boldsymbol{f}_n\colon \boldsymbol{\mathcal{B}}_n \to \boldsymbol{\mathcal{B}}_{n+1}$ and limiting maps $\boldsymbol{g}_n\colon \boldsymbol{\mathcal{B}}_n \to \boldsymbol{\mathcal{S}}$ being $\langle\vee,\mathbf{0}\rangle$-homomorphisms. We fix an ℓ-group $\boldsymbol{G}_0$ with an isomorphism $\boldsymbol{\alpha}_0\colon \operatorname{Id}_c^\ell \boldsymbol{G}_0 \twoheadrightarrow \boldsymbol{\mathcal{B}}_0$. Suppose having constructed an ℓ-group $\boldsymbol{G}_n$ with an isomorphism $\boldsymbol{\alpha}_n\colon \operatorname{Id}_c^\ell \boldsymbol{G}_n \to \boldsymbol{\mathcal{B}}_n$. Applying Lemma 8.2 to $\boldsymbol{f}_n \circ \boldsymbol{\alpha}_n$, we obtain an ℓ-group $\boldsymbol{G}_{n+1}$, an ℓ-homomorphism $f_n\colon \boldsymbol{G}_n \to \boldsymbol{G}_{n+1}$, and an isomorphism $\boldsymbol{\alpha}_{n+1}\colon \operatorname{Id}_c^\ell \boldsymbol{G}_{n+1} \to \boldsymbol{\mathcal{B}}n+1$ such that $\boldsymbol{f}_n \circ \boldsymbol{\alpha}_n = \boldsymbol{\alpha}_{n+1} \circ \operatorname{Id}_c^\ell f_n$. Defining $\boldsymbol{G}$ as the direct limit of the sequence

$$\boldsymbol{G}_0 \xrightarrow{f_0} \boldsymbol{G}_1 \xrightarrow{f_1} \boldsymbol{G}_2 \xrightarrow{f_2} \cdots\cdots ,$$

an elementary categorical argument yields an isomorphism from $\operatorname{Id}_c^\ell \boldsymbol{G}$ onto the direct limit $\boldsymbol{\mathcal{S}}$ of the sequence $\langle \boldsymbol{\mathcal{B}}_n \mid n < \omega\rangle$. $\square$

9. Functorial representation by V-distances of type 2

Observe that the argument of Proposition 4.6 is only a small modification (with a more simple-minded proof) of B. Jónsson's proof that every modular lattice has a type 2 representation, see [**43**] or [**27**, Theorem IV.4.8]. It follows from Corollary 5.7 that "type 2" cannot be improved to "type 1". In view of Proposition 3.2, this is somehow surprising, as every distributive lattice has an embedding with permutable congruences into some partition lattice. This illustrates the observation that one can get much more from a distance than from an embedding into a partition lattice. We shall now present a strengthening of Proposition 4.6 that shows that the construction can be made functorial. We introduce notations for the following categories:

(1) $\mathbf{DSem_m}$, the category of all distributive $\langle \vee, \mathbf{0} \rangle$-semilattices with $\langle \vee, \mathbf{0} \rangle$-embeddings.
(2) $\mathbf{Dist}$, the category of all surjective distances of the form $\delta\colon X \times X \twoheadrightarrow \mathcal{S}$ with kernel the identity and $\mathcal{S}$ a distributive $\langle \vee, \mathbf{0} \rangle$-semilattice, with morphisms (see Definition 3.1) of the form

$$\langle f, \boldsymbol{f} \rangle\colon \begin{array}{ccc} X \times X & \xrightarrow{\langle f,f \rangle} & Y \times Y \\ {\scriptstyle\lambda}\big\downarrow & & \big\downarrow{\scriptstyle\mu} \\ \mathcal{S} & \xrightarrow[\boldsymbol{f}]{} & \mathcal{T} \end{array}$$

with both f and $\boldsymbol{f}$ one-to-one.
(3) $\mathbf{Dist_2}$, the full subcategory of $\mathbf{Dist}$ of all V-distances of type 2.

Furthermore, denote by $\Pi\colon \mathbf{Dist} \to \mathbf{DSem_m}$ the forgetful functor (see Definition 3.1).

THEOREM 9.1. *There exists a direct limits preserving functor*

$$\Phi\colon \mathbf{DSem_m} \to \mathbf{Dist_2}$$

such that the composition $\Pi \circ \Phi$ *is equivalent to the identity.*

Hence the functor Φ assigns to each distributive $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S}$ a set $X_{\mathcal{S}}$ together with a surjective $\mathcal{S}$-valued V-distance $\delta_{\mathcal{S}}\colon X_{\mathcal{S}} \times X_{\mathcal{S}} \twoheadrightarrow \mathcal{S}$ of type 2.

PROOF. The proof of Proposition 4.6 depends of the enumeration order of a certain transfinite sequence of quadruples $\langle x, y, \boldsymbol{a}, \boldsymbol{b} \rangle$, which prevents it from being functorial. We fix this by adjoining all such quadruples simultaneously, and by describing the corresponding extension. So, for a distance $\delta\colon X \times X \to \mathcal{S}$, we put $\mathcal{S}^- = \mathcal{S} \setminus \{\mathbf{0}\}$, and

$$\mathcal{H}(\delta) = \{\langle x, y, \boldsymbol{a}, \boldsymbol{b} \rangle \in X \times X \times \mathcal{S}^- \times \mathcal{S}^- \mid \delta(x, y) = \boldsymbol{a} \vee \boldsymbol{b}\}.$$

For $\xi := \langle x, y, \boldsymbol{a}, \boldsymbol{b} \rangle \in \mathcal{H}(\delta)$, we put $x_\xi^0 = x$, $x_\xi^1 = y$, $\boldsymbol{a}_\xi = \boldsymbol{a}$, and $\boldsymbol{b}_\xi = \boldsymbol{b}$. Now we put $X' := X \cup \{u_\xi^i \mid \xi \in \mathcal{H}(\delta)\}$ and $i \in \{0, 1\}$, where the elements u_ξ^i

are pairwise distinct symbols outside X. We define a map $\delta'\colon X' \times X' \to \boldsymbol{S}$ by requiring δ' to extend δ, with value zero on the diagonal, and by the rule

$$\delta'(u_\xi^i, u_\eta^j) = \begin{cases} |i-j| \cdot \boldsymbol{b}_\xi, & \text{if } \xi = \eta, \\ \boldsymbol{a}_\xi \vee \boldsymbol{a}_\eta \vee \delta(x_\xi^i, x_\eta^j), & \text{if } \xi \neq \eta, \end{cases}$$

$$\delta'(u_\xi^i, z) = \delta'(z, u_\xi^i) = \delta(z, x_\xi^i) \vee \boldsymbol{a}_\xi,$$

for all $\xi, \eta \in \mathcal{H}(\delta)$, all $i, j \in \{0, 1\}$, and all $z \in X$.

It is straightforward, though somewhat tedious, to verify that δ' is an $\boldsymbol{S}$-valued distance on X', that it extends δ, and that its kernel is the identity of X' in case the kernel of δ is the identity of X (because the semilattice elements $\boldsymbol{a}_\xi$ and $\boldsymbol{b}_\xi$ are non-zero). Furthermore, if $\boldsymbol{S}$ is distributive, then every V-condition problem for δ of the form $\delta(x, y) \leq \boldsymbol{a} \vee \boldsymbol{b}$ can be refined to a problem of the form $\delta(x, y) = \boldsymbol{a}' \vee \boldsymbol{b}'$, for some $\boldsymbol{a}' \leq \boldsymbol{a}$ and $\boldsymbol{b}' \leq \boldsymbol{b}$ (because $\boldsymbol{S}$ is distributive), and such a problem has a solution of type 2 for δ'. Namely, in case both $\boldsymbol{a}'$ and $\boldsymbol{b}'$ are non-zero (otherwise the problem can be solved in X), put $\xi := \langle x, y, \boldsymbol{a}', \boldsymbol{b}' \rangle$, and observe that $\delta'(x, u_\xi^0) = \boldsymbol{a}'$, $\delta'(u_\xi^0, u_\xi^1) = \boldsymbol{b}'$, and $\delta'(u_\xi^1, y) = \boldsymbol{a}'$.

Hence, if we put $\langle X_0, \delta_0 \rangle := \langle X, \delta \rangle$, then $\langle X_{n+1}, \delta_{n+1} \rangle = \langle X_n', \delta_n' \rangle$, for all $n < \omega$, and finally $\overline{X} := \bigcup_{n<\omega} X_n$ and $\overline{\delta} := \bigcup_{n<\omega} \delta_n$, the pair $\Psi(\langle X, \delta \rangle) = \langle \overline{X}, \overline{\delta} \rangle$ is an $\boldsymbol{S}$-valued V-distance of type 2 extending $\langle X, \delta \rangle$. Every morphism $\langle f, \boldsymbol{f} \rangle\colon \langle X, \lambda \rangle \to \langle Y, \mu \rangle$ in $\boldsymbol{S}$ extends canonically to a morphism $\langle f', \boldsymbol{f} \rangle\colon \langle X', \lambda' \rangle \to \langle Y', \mu' \rangle$ (the underlying semilattice map $\boldsymbol{f}$ is the same), by defining

$$f'(u_\xi^i) := u_{f\xi}^i, \text{ for all } \xi \in \mathcal{H}(\lambda) \text{ and all } i < 2,$$

where we put, of course,

$$f\langle x, y, \boldsymbol{a}, \boldsymbol{b} \rangle := \langle f(x), f(y), \boldsymbol{f}(\boldsymbol{a}), \boldsymbol{f}(\boldsymbol{b}) \rangle, \text{ for all } \langle x, y, \boldsymbol{a}, \boldsymbol{b} \rangle \in \mathcal{H}(\lambda).$$

Hence, by an easy induction argument, $\langle f, \boldsymbol{f} \rangle$ extends canonically to a morphism $\Psi(\langle f, \boldsymbol{f} \rangle) = \langle \overline{f}, \boldsymbol{f} \rangle\colon \langle \overline{X}, \overline{\lambda} \rangle \to \langle \overline{Y}, \overline{\mu} \rangle$, and the correspondence $\langle f, \boldsymbol{f} \rangle \mapsto \langle \overline{f}, \boldsymbol{f} \rangle$ is itself a functor. As the construction defining the correspondence $\langle X, \delta \rangle \mapsto \langle X', \delta' \rangle$ is local, the functor Ψ preserves direct limits.

It remains to find something to start with, to which we can apply Ψ. A possibility is to use the distance $\mu_{\boldsymbol{S}}$, given by (4.2), introduced in the proof of Proposition 4.6. The correspondence $\boldsymbol{S} \mapsto \mu_{\boldsymbol{S}}$ defines a functor, in particular, if $f\colon \boldsymbol{S} \to \boldsymbol{T}$ is an embedding of distributive $\langle \vee, \mathbf{0} \rangle$-semilattices, then the equality $\mu_{\boldsymbol{T}}(f(x), f(y)) = \boldsymbol{f}(\mu_{\boldsymbol{S}}(x, y))$ holds, for all $x, y \in \boldsymbol{S}$. The desired functor Φ is given by $\Phi(\boldsymbol{S}) = \Psi(\langle \boldsymbol{S}, \mu_{\boldsymbol{S}} \rangle)$, for any distributive $\langle \vee, \mathbf{0} \rangle$-semilattice $\boldsymbol{S}$. $\square$

In contrast with the result of Theorem 9.1, we shall isolate a finite, "combinatorial" reason for the forgetful functor from V-distances of type 3/2 to distributive $\langle \vee, \mathbf{0} \rangle$-semilattices not to admit any left inverse. By contrast, we recall that for V-distances of type 2, the corresponding result is positive, see Theorem 9.1. In order to establish the negative result, we shall use the example $\mathcal{D}_{ac}$ of [**74**, Section 7], and extend the corresponding result from lattices with almost permutable congruences to arbitrary V-distances of type 3/2.

We recall that $\mathcal{D}_{ac}$ is the (commutative) cube of finite Boolean semilattices represented on Figure 3, where $\mathcal{P}(X)$ denotes the powerset algebra of a set X and $\boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g}, \boldsymbol{h}_0, \boldsymbol{h}_1$, and $\boldsymbol{h}_2$ are the $\langle \vee, \mathbf{0} \rangle$-homomorphisms (and, in fact, $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-embeddings) defined by their values on atoms as follows:

$$\boldsymbol{e}(1) = \{0, 1\},$$

$$\boldsymbol{f} \colon \begin{cases} \{0\} \mapsto \{0, 1\}, \\ \{1\} \mapsto \{2, 3\}, \end{cases} \quad \boldsymbol{g} \colon \begin{cases} \{0\} \mapsto \{0, 2\}, \\ \{1\} \mapsto \{1, 3\}, \end{cases}$$

$$\boldsymbol{h}_0 \colon \begin{cases} \{0\} \mapsto \{0, 4, 7\}, \\ \{1\} \mapsto \{3, 5, 6\}, \\ \{2\} \mapsto \{2, 5, 6\}, \\ \{3\} \mapsto \{1, 4, 7\}, \end{cases} \quad \boldsymbol{h}_1 \colon \begin{cases} \{0\} \mapsto \{0, 4, 5, 7\}, \\ \{1\} \mapsto \{1, 4, 6, 7\}, \\ \{2\} \mapsto \{2, 5, 6, 7\}, \\ \{3\} \mapsto \{3, 4, 5, 6\}, \end{cases} \quad \boldsymbol{h}_2 \colon \begin{cases} \{0\} \mapsto \{0, 4, 6\}, \\ \{1\} \mapsto \{1, 5, 7\}, \\ \{2\} \mapsto \{3, 5, 7\}, \\ \{3\} \mapsto \{2, 4, 6\}, \end{cases}$$

$$\begin{tikzcd}
& \mathcal{P}(8) & \\
\mathcal{P}(4) \arrow[ur, "\boldsymbol{h}_2"] & \mathcal{P}(4) \arrow[u, "\boldsymbol{h}_1"] & \mathcal{P}(4) \arrow[ul, "\boldsymbol{h}_0"'] \\
\mathcal{P}(2) \arrow[u, "\boldsymbol{f}"] \arrow[ur, "\boldsymbol{f}"] & \mathcal{P}(2) \arrow[ul, "\boldsymbol{g}"] \arrow[ur, "\boldsymbol{f}"] & \mathcal{P}(2) \arrow[ul, "\boldsymbol{g}"] \arrow[u, "\boldsymbol{g}"'] \\
& \mathbf{2} \arrow[ul, "\boldsymbol{e}"] \arrow[u, "\boldsymbol{e}"] \arrow[ur, "\boldsymbol{e}"'] &
\end{tikzcd}$$

FIGURE 3. The cube $\mathcal{D}_{ac}$, unliftable by V-distances of type 3/2.

THEOREM 9.2. *The diagram $\mathcal{D}_{ac}$ has no lifting, with respect to the forgetful functor, by distances, surjective at level* 0 *and satisfying the V-condition of type* 3/2 *at level* 1.

PROOF. Suppose that the diagram of Figure 3 is lifted by a diagram of distances, with distances $\lambda \colon X \times X \to \mathbf{2}$, $\lambda_i \colon X_i \times X_i \to \mathcal{P}(2)$, $\mu_i \colon Y_i \times Y_i \to \mathcal{P}(4)$, and $\mu \colon Y \times Y \to \mathcal{P}(8)$, for all $i \in \{0, 1, 2\}$, see Figure 4.

We assume that λ is surjective and that λ_i is a V-distance of type 3/2, for all $i \in \{0, 1, 2\}$. Denote by $f_{U,V}$ the canonical map from U to V given by this lifting, for U below V among $X, X_0, X_1, X_2, Y_0, Y_1, Y_2, Y$. After having replaced each of those sets U by its quotient by the kernel of the corresponding distance, and then by its image in Y under $f_{U,Y}$, we may assume

$$\begin{tikzcd}
 & \mu & \\
\mu_2 \arrow[ur] & \mu_1 \arrow[u] & \mu_0 \arrow[ul] \\
\lambda_0 \arrow[u] \arrow[ur] & \lambda_1 \arrow[ul] \arrow[ur] & \lambda_2 \arrow[ul] \arrow[u] \\
 & \lambda \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

FIGURE 4. A commutative diagram of semilattice-valued distances

that $f_{U,V}$ is the inclusion map from U into V, for all U below V among $X, X_0, X_1, X_2, Y_0, Y_1, Y_2, Y$.

Since λ is surjective, there are $x, y \in X$ such that $\lambda(x, y) = 1$. For all $i \in \{0, 1, 2\}$,

$$\lambda_i(x, y) = \boldsymbol{e}(\lambda(x, y)) = \boldsymbol{e}(1) = \{0, 1\} = \{0\} \cup \{1\},$$

thus, since λ_i satisfies the V-condition of type $3/2$, there exists $z_i \in X_i$ such that

$$\begin{aligned} &\text{either } \lambda_i(x, z_i) = \{0\} \text{ and } \lambda_i(z_i, y) = \{1\} \quad (\text{say}, P(i)) \\ &\text{or } \lambda_i(x, z_i) = \{1\} \text{ and } \lambda_i(z_i, y) = \{0\} \quad (\text{say}, Q(i)). \end{aligned} \tag{9.1}$$

So we have eight cases to consider, according to which combination of P and Q occurs in (9.1) for $i \in \{0, 1, 2\}$. In each case, we shall obtain the inequality

$$\mu(z_0, z_2) \nsubseteq \mu(z_0, z_1) \cup \mu(z_1, z_2), \tag{9.2}$$

which will contradict the triangular inequality for μ.

Case 1. $P(0)$, $P(1)$, and $P(2)$ hold. Then $\mu_2(z_0, x) = \boldsymbol{f}(\lambda_0(x, z_0)) = \{0, 1\}$ and $\mu_2(x, z_1) = \boldsymbol{g}(\lambda_1(x, z_1)) = \{0, 2\}$, whence $\mu_2(z_0, z_1) \subseteq \{0, 1, 2\}$. Similarly, replacing x by y in the argument above, $\mu_2(z_0, y) = \boldsymbol{f}(\lambda_0(z_0, y)) = \{2, 3\}$ and $\mu_2(y, z_1) = \boldsymbol{g}(\lambda_1(z_1, y)) = \{1, 3\}$, whence $\mu_2(z_0, z_1) \subseteq \{1, 2, 3\}$. Therefore, $\mu_2(z_0, z_1) \subseteq \{1, 2\}$. On the other hand, from

$$\mu_2(x, z_0) \cup \mu_2(z_0, z_1) = \mu_2(x, z_1) \cup \mu_2(z_0, z_1)$$

the converse inclusion follows, whence $\mu_2(z_0, z_1) = \{1, 2\}$. Similar computations yield that $\mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{1, 2\}$. Hence, we obtain the equalities

$$\begin{aligned} \mu(z_0, z_1) &= \boldsymbol{h}_2\mu_2(z_0, z_1) = \{1, 3, 5, 7\}, \\ \mu(z_0, z_2) &= \boldsymbol{h}_1\mu_1(z_0, z_2) = \{1, 2, 4, 5, 6, 7\}, \\ \mu(z_1, z_2) &= \boldsymbol{h}_0\mu_0(z_1, z_2) = \{2, 3, 5, 6\}. \end{aligned}$$

Observe that 4 belongs to $\mu(z_0, z_2)$ but not to $\mu(z_0, z_1) \cup \mu(z_1, z_2)$.

Case 2. $P(0)$, $P(1)$, and $Q(2)$ hold. As in Case 1, we obtain

$$\mu_2(z_0, z_1) = \{1, 2\} \text{ and } \mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{0, 3\},$$

thus $\mu(z_0, z_1) = \{1, 3, 5, 7\}$, $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$, which confirms (9.2) and thus causes a contradiction.

Case 3. $P(0)$, $Q(1)$, and $P(2)$ hold. We obtain $\mu_2(z_0, z_1) = \mu_0(z_1, z_2) = \{0, 3\}$ and $\mu_1(z_0, z_2) = \{1, 2\}$, thus $\mu(z_0, z_2) = \{1, 2, 4, 5, 6, 7\}$, $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$.

Case 4. $P(0)$, $Q(1)$, and $Q(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_1(z_0, z_2) = \{0, 3\} \text{ and } \mu_0(z_1, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, and $\mu(z_1, z_2) = \{2, 3, 5, 6\}$.

Case 5. $Q(0)$, $P(1)$, and $P(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_1(z_0, z_2) = \{0, 3\} \text{ and } \mu_0(z_1, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{2, 3, 5, 6\}$.

Case 6. $Q(0)$, $P(1)$, and $Q(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_0(z_1, z_2) = \{0, 3\} \text{ and } \mu_1(z_0, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{0, 2, 4, 6\}$, $\mu(z_0, z_2) = \{1, 2, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$.

Case 7. $Q(0)$, $Q(1)$, and $P(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \{1, 2\} \text{ and } \mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{0, 3\},$$

thus $\mu(z_0, z_1) = \{1, 3, 5, 7\}$, $\mu(z_0, z_2) = \{0, 3, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{0, 1, 4, 7\}$.

Case 8. $Q(0)$, $Q(1)$, and $Q(2)$ hold. We obtain

$$\mu_2(z_0, z_1) = \mu_1(z_0, z_2) = \mu_0(z_1, z_2) = \{1, 2\},$$

thus $\mu(z_0, z_1) = \{1, 3, 5, 7\}$, $\mu(z_0, z_2) = \{1, 2, 4, 5, 6, 7\}$, and $\mu(z_1, z_2) = \{2, 3, 5, 6\}$. In all cases, we obtain a contradiction. □

A "global" version of Theorem 9.2 is presented in Theorem 5.6. The following corollary extends [**74**, Theorem 7.1] from lattices to arbitrary algebras.

Corollary 9.3. *The diagram $\mathcal{D}_{ac}$ has no lifting, with respect to the congruence lattice functor, by algebras with almost permutable congruences.*

About other commonly encountered structures, we obtain the following.

Corollary 9.4. *The diagram $\mathcal{D}_{ac}$ has no lifting by groups with respect to the* Sub *functor, and no lifting by modules with respect to the* Sub *functor.*

The following example offers a significant difference between the situations for groups and modules.

EXAMPLE 9.1. The diagonal map $\mathbf{2} \hookrightarrow \mathbf{2}^2$ has no lifting, with respect to the Sub functor, by modules over any ring. Indeed, suppose that $\boldsymbol{A} \hookrightarrow \boldsymbol{B} \times \boldsymbol{C}$ is such a lifting, with $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ simple modules. Projecting on $\boldsymbol{B}$ and on $\boldsymbol{C}$ yields that $\boldsymbol{A}$ is isomorphic to a submodule of both $\boldsymbol{B}$ and $\boldsymbol{C}$, whence, by simplicity, $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are pairwise isomorphic. But then, $\boldsymbol{B} \times \boldsymbol{C} \simeq \boldsymbol{B} \times \boldsymbol{B}$ has the diagonal as a submodule, so its submodule lattice cannot be isomorphic to $\mathbf{2}^2$.

By contrast, every square of finite Boolean $\langle \vee, \mathbf{0} \rangle$-semilattices can be lifted, with respect to the Sub functor, by groups, see Lemma 7.2.

CHAPTER 3

Free trees and the optimal bound in Wehrung's solution of the Congruence Lattice Problem

1. Introduction

Congruence lattices of universal algebras correspond to algebraic lattice. By the theorem of N. Funayama and T. Nakayama [**20**], the congruence lattice of a lattice is, in addition, distributive (see also [**27**, II. 3. Theorem 11]). On the other hand, R. P. Dilworth proved that every finite distributive lattice is isomorphic to the congruence lattice of a finite lattice (the result was first published in Grätzer's and Schmidt's [**29**]) and he conjectured that every distributive algebraic lattice is isomorphic to the congruence lattice of a lattice (see again [**29**]). This conjecture, referred to as the *Congruence Lattice Problem*, despite many attempts (see surveys [**27**, Appendix C] and [**75**]), remained open for over sixty years until, recently, F. Wehrung disproved it in [**83**].

The Wehrung's solution involves a combination of new ideas, see, in particular, Lemmas 4.4, 5.1, and 6.2 in [**83**], and methods developed in earlier papers, which originated in [**77**] and were pursued further in [**59, 60, 68, 74, 78**]. In these papers, counterexamples to various problems related to the Congruence Lattice Problem were obtained. The optimal cardinality bound for all these counterexamples is $\aleph_2$, however Wehrung's argument requires an algebraic distributive lattice with at least $\aleph_{\omega+1}$ compact elements. In the present paper, we improve Wehrung's result by proving that there is a counterexample of size $\aleph_2$. As in the related cases, $\aleph_2$ turns out to be the optimal cardinality bound for a negative solution of the Congruence Lattice Problem. Our proof closely follows Wehrung's ideas. The main difference consists in an enhancement of Kuratowski's Free Set Theorem by a new combinatorial principle which involves finite trees.

The Wehrung's construction in [**83**] uses a "*free*" distributive extension of a $\langle\vee,\mathbf{0}\rangle$-semilattice; a functor that assigns to every $\langle\vee,\mathbf{0}\rangle$-semilattice a distributive $\langle\vee,\mathbf{0}\rangle$-semilattice, constructed previously by M. Ploščica and J. Tůma in [**59**]. The main features of this construction for the refutation of the Congruence Lattice Problem are extracted in the so-called Evaporation Lemma [**83**, Lemma 4.4]. We generalize this idea by defining a *diluting functor* whose properties are sufficient to prove the Evaporation Lemma, and we prove that the free distributive extension of a $\langle\vee,\mathbf{0}\rangle$-semilattice is, indeed, a diluting functor.

Further, we modify Kuratowski's Free Set Theorem, the combinatorial essence of the above mentioned counterexamples. Given a set W and a map $\phi\colon [W]^{<\omega} \to [W]^{<\omega}$, we define a *free k-tree* (*with respect to* ϕ), for every positive integer k, which is a k-ary tree with some combinatorial properties derived from the Kuratowski's Free Set Theorem. We prove that a free k-tree exists whenever the cardinality of the set W is at least $\aleph_{k-1}$, and we apply the existence of a free 3-tree in every set of cardinality at least $\aleph_2$ to attain the optimal cardinality bound in the Wehrung's result.

2. Diluting functors

We denote by $\mathbf{Sem_0}$ the category of $\langle\vee,\mathbf{0}\rangle$-semilattices (with $\langle\vee,\mathbf{0}\rangle$-homomorphisms).

DEFINITION 2.1. An *expanding functor* on $\mathbf{Sem_0}$ is a pair $\langle\Psi,\iota\rangle$, where Ψ is an endofunctor $\mathbf{Sem_0}\to\mathbf{Sem_0}$ and ι is a natural transformation from the identity functor on $\mathbf{Sem_0}$ to Ψ such that $\iota_{\boldsymbol{\mathcal{S}}}\colon\boldsymbol{\mathcal{S}}\to\Psi(\boldsymbol{\mathcal{S}})$ is an embedding, for every $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}$. We shall denote the expanding functor above by Ψ once the natural transformation ι is understood, and we shall identify $\iota_{\boldsymbol{\mathcal{S}}}(\boldsymbol{x})$ with $\boldsymbol{x}$, for all $\boldsymbol{x}\in\boldsymbol{\mathcal{S}}$.

An expanding functor $\Psi\colon\mathbf{Sem_0}\to\mathbf{Sem_0}$ is a *diluting functor*, provided that for all $\langle\vee,\mathbf{0}\rangle$-semilattices $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{T}}$ and every $\langle\vee,\mathbf{0}\rangle$-homomorphism $\boldsymbol{f}\colon\boldsymbol{\mathcal{S}}\to\boldsymbol{\mathcal{T}}$, the following property is satisfied: for all $\boldsymbol{v}\in\Psi(\boldsymbol{\mathcal{S}})$, and $\boldsymbol{u}_0,\boldsymbol{u}_1\in\Psi(\boldsymbol{\mathcal{T}})$, the inequality $\Psi(\boldsymbol{f})(\boldsymbol{v})\le\boldsymbol{u}_0\vee\boldsymbol{u}_1$ implies that there are $\boldsymbol{x}_0,\boldsymbol{x}_1\in\Psi(\boldsymbol{\mathcal{S}})$ and $\boldsymbol{y}\in\boldsymbol{\mathcal{S}}$ such that

$$\boldsymbol{f}(\boldsymbol{y})\le\boldsymbol{u}_0\vee\boldsymbol{u}_1,\quad \Psi(\boldsymbol{f})(\boldsymbol{x}_i)\le\boldsymbol{u}_i,\text{ for all } i=1,2,\text{ and } \boldsymbol{v}\le\boldsymbol{x}_0\vee\boldsymbol{x}_1\vee\boldsymbol{y}.$$

Let U, V be subsets of a $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}$. We shall use the notation

$$U\vee V:=\{\boldsymbol{u}\vee\boldsymbol{v}\mid\boldsymbol{u}\in U\text{ and }\boldsymbol{v}\in V\}.$$

LEMMA 2.2. *Let $\boldsymbol{\mathcal{S}}$ be a $\langle\vee,\mathbf{0}\rangle$-semilattice and $\boldsymbol{\mathcal{S}}_i$, $i=0,1$, $\langle\vee,\mathbf{0}\rangle$-subsemilattices of $\boldsymbol{\mathcal{S}}$ such that $\boldsymbol{\mathcal{S}}=\boldsymbol{\mathcal{S}}_0\vee\boldsymbol{\mathcal{S}}_1$ and there are retractions $\boldsymbol{r}_i\colon\boldsymbol{\mathcal{S}}\to\boldsymbol{\mathcal{S}}_i$, for $i=0,1$. Put $\boldsymbol{s}_i=\Psi(\boldsymbol{r}_i)$, for every $i=0,1$. Let $\boldsymbol{u}_i\in\Psi(\boldsymbol{\mathcal{S}}_i)$, $i=0,1$, be such that $\boldsymbol{s}_i(\boldsymbol{u}_{1-i})=0$, for all $i=0,1$. Then for every $\boldsymbol{y}\in\boldsymbol{\mathcal{S}}$ such that $\boldsymbol{y}\le\boldsymbol{u}_0\vee\boldsymbol{u}_1$, there are $\boldsymbol{y}_i\in\boldsymbol{\mathcal{S}}_i$, $i=0,1$, such that $\boldsymbol{y}\le\boldsymbol{y}_0\vee\boldsymbol{y}_1$ and $\boldsymbol{y}_i\le\boldsymbol{u}_i$, for all $i=0,1$.*

PROOF. We put $\boldsymbol{y}_i:=\boldsymbol{r}_i(\boldsymbol{y})$, for all $i=0,1$. Since $\boldsymbol{\mathcal{S}}=\boldsymbol{\mathcal{S}}_0\vee\boldsymbol{\mathcal{S}}_1$, there are elements $\boldsymbol{y}'_i\in\boldsymbol{\mathcal{S}}_i$, $i=0,1$, such that $\boldsymbol{y}=\boldsymbol{y}'_0\vee\boldsymbol{y}'_1$. Since the maps $\boldsymbol{r}_i$, $i=0,1$, are retractions, $\boldsymbol{y}'_i\le\boldsymbol{r}_i(\boldsymbol{y})=\boldsymbol{y}_i$, for all $i=0,1$, whence $\boldsymbol{y}\le\boldsymbol{y}_0\vee\boldsymbol{y}_1$.

It remains to prove that $\boldsymbol{y}_i\le\boldsymbol{u}_i$, for all $i=0,1$. We fix $i\in\{0,1\}$. Since $\boldsymbol{s}_i\restriction\boldsymbol{\mathcal{S}}=\boldsymbol{r}_i$ and $\boldsymbol{s}_i\colon\Psi(\boldsymbol{\mathcal{S}})\to\Psi(\boldsymbol{\mathcal{S}}_i)$ is a retraction, $\boldsymbol{s}_i(\boldsymbol{u}_i)=\boldsymbol{u}_i$. Since $s_i(\boldsymbol{u}_{1-i})=0$, according to the assumption, we conclude that

$$\boldsymbol{y}_i=\boldsymbol{s}_i(\boldsymbol{y})\le\boldsymbol{s}_i(\boldsymbol{u}_0\vee\boldsymbol{u}_1)=\boldsymbol{s}_i(\boldsymbol{u}_0)\vee\boldsymbol{s}_i(\boldsymbol{u}_1)=\boldsymbol{s}_i(\boldsymbol{u}_i)=\boldsymbol{u}_i.\tag{2.1}$$

□

We define Ψ^0 to be the identity functor and, inductively, $\Psi^{n+1}=\Psi\circ\Psi^n$, for all positive integers n. By our assumption, the inclusion map defines a natural transformation from the identity functor on $\mathbf{Sem_0}$ to Ψ, therefore we can define $\Psi^\infty(\boldsymbol{\mathcal{S}})=\bigcup_{n\in\omega}\Psi^n(\boldsymbol{\mathcal{S}})$ and $\Psi^\infty(\boldsymbol{f})=\bigcup_{n\in\omega}\Psi^n(\boldsymbol{f})$, for every $\langle\vee,\mathbf{0}\rangle$-semilattice $\boldsymbol{\mathcal{S}}$, and every $\langle\vee,\mathbf{0}\rangle$-homomorphism $\boldsymbol{f}\colon\boldsymbol{\mathcal{S}}\to\boldsymbol{\mathcal{T}}$, respectively. again, the inclusion map defines a natural transformation from the identity functor on $\mathbf{Sem_0}$ to Ψ^∞. In particular, if Ψ is an expanding functor on $\mathbf{Sem_0}$, then the functor Ψ^∞ is expanding as well.

LEMMA 2.3. *Let Ψ be a diluting functor on $\mathbf{Sem_0}$. Then the functor Ψ^∞ is diluting as well.*

PROOF. Let $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{T}}$ be $\langle\vee,\mathbf{0}\rangle$-semilattices, and let $\boldsymbol{f}\colon \boldsymbol{\mathcal{S}} \to \boldsymbol{\mathcal{T}}$ be a $\langle\vee,\mathbf{0}\rangle$-homomorphism. Let $\boldsymbol{v} \in \Psi^\infty(\boldsymbol{\mathcal{S}})$ and $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \Psi^\infty(\boldsymbol{\mathcal{T}})$ be such that $\Psi^\infty(\boldsymbol{f})(\boldsymbol{v}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$. We are looking for $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Psi^\infty(\boldsymbol{\mathcal{S}})$ and $\boldsymbol{y} \in \boldsymbol{\mathcal{S}}$ such that

$$\boldsymbol{f}(\boldsymbol{y}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \Psi^\infty(\boldsymbol{f})(\boldsymbol{x}_i) \leq \boldsymbol{u}_i, \text{ for all } i = 1, 2, \text{ and } \boldsymbol{v} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}.$$

We shall argue by induction on the least natural number n such that $\boldsymbol{v} \in \Psi^n(\boldsymbol{\mathcal{S}})$. If $n = 0$, we put $\boldsymbol{x}_0 = \boldsymbol{x}_1 = 0$, $\boldsymbol{y} = \boldsymbol{v}$, and we are done. Suppose that $\boldsymbol{v} \in \Psi^{n+1}(\boldsymbol{\mathcal{S}})$, for some positive integer n, and that the property is proved at stage n. Let $k \geq n$ be a positive integer such that $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \Psi^{k+1}(\boldsymbol{\mathcal{T}})$. Denote by $\boldsymbol{g}$ the composition of the $\langle\vee,\mathbf{0}\rangle$-homomorphism $\Psi^n(\boldsymbol{f})$ and the inclusion map $\Psi^n(\boldsymbol{\mathcal{T}}) \hookrightarrow \Psi^k(\boldsymbol{\mathcal{T}})$. By applying the assumption that Ψ is a diluting functor to the $\langle\vee,\mathbf{0}\rangle$-homomorphism $\boldsymbol{g}\colon \Psi^m(\boldsymbol{\mathcal{S}}) \to \Psi^k(T)$, we obtain elements $\boldsymbol{x}'_0, \boldsymbol{x}'_1 \in \Psi^{m+1}(\boldsymbol{\mathcal{S}})$ and $\boldsymbol{y}' \in \Psi^m(\boldsymbol{\mathcal{S}})$ such that

$$\boldsymbol{g}(\boldsymbol{y}') \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \Psi(\boldsymbol{g})(\boldsymbol{x}'_i) \leq \boldsymbol{u}_i, \text{ for all } i = 0, 1, \text{ and } \ \boldsymbol{v} \leq \boldsymbol{x}'_0 \vee \boldsymbol{x}'_1 \vee \boldsymbol{y}'.$$

The inequality $\boldsymbol{g}(\boldsymbol{y}') \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$ implies that $\Psi^\infty(\boldsymbol{f})(\boldsymbol{y}') \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$. Therefore, by the induction hypothesis, there are elements $\boldsymbol{x}''_0, \boldsymbol{x}''_1 \in \Psi^\infty(\boldsymbol{\mathcal{S}})$ and $\boldsymbol{y} \in \boldsymbol{\mathcal{S}}$ such that

$$\boldsymbol{f}(\boldsymbol{y}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \Psi^\infty(\boldsymbol{f})(\boldsymbol{x}''_i) \leq \boldsymbol{u}_i, \text{ for all } i = 0, 1, \text{ and } \boldsymbol{y}' \leq \boldsymbol{x}''_0 \vee \boldsymbol{x}''_1 \vee \boldsymbol{y}.$$

Now it is easy to conclude that $\boldsymbol{x}_i = \boldsymbol{x}'_i \vee \boldsymbol{x}''_i$, for $i = 0, 1$, and $\boldsymbol{y}$ are the desired elements. □

Let **Set** denote the category of all sets. Similarly as in [**83**], we denote by $\Lambda\colon \mathbf{Set} \to \mathbf{Sem_0}$ the functor which assigns to a set W the $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice $\Lambda(W)$ defined by generators 1, and $\boldsymbol{a}_0^x$, $\boldsymbol{a}_1^x$, for $x \in W$, subjected to the relations

$$\boldsymbol{a}_0^x \vee \boldsymbol{a}_1^x = 1, \text{ for all } x \in W, \tag{2.2}$$

and which assigns to a map $f\colon X \to Y$ the unique $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-homomorphism $\Lambda(f)\colon \Lambda(X) \to \Lambda(Y)$ such that $\Lambda(f)(\boldsymbol{a}_i^x) = \boldsymbol{a}_i^{f(x)}$, for all $x \in X$ and all $i = 0, 1$.

Given a finite subset A of W and a map $\varphi\colon A \to 2$, we put $\boldsymbol{a}_\varphi^A = \bigvee_{x\in A} \boldsymbol{a}_{\varphi(x)}^x$. By the coming Corollary 3.2, the following lemma is a generalization of Wehrung's original "Evaporation Lemma" [**83**, Lemma 4.4].

LEMMA 2.4. *Let Ψ be a diluting functor on $\mathbf{Sem_0}$. We set $\Phi := \Psi \circ \Lambda$. Let W be a set, A_0, A_1 finite disjoint subsets of W, and $w \in W \setminus (A_0 \cup A_1)$. Let $\boldsymbol{v} \in \Phi(W \setminus \{w\})$, $\varphi_i\colon A_i \to 2$ be maps, and $\boldsymbol{u}_i \in \Phi(W \setminus A_{1-i})$, for all $i = 0, 1$. If*

$$\boldsymbol{v} \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1 \textit{ and } \boldsymbol{u}_i \leq \boldsymbol{a}_{\varphi_i}^{A_i}, \boldsymbol{a}_i^w, \textit{ for all } i = 0, 1,$$

then $\boldsymbol{v} = 0$.

PROOF. We denote by f the inclusion map $W \setminus \{w\} \hookrightarrow W$. Observe that $\Lambda(f)$ corresponds to the inclusion map $\Lambda(W \setminus \{w\}) \hookrightarrow \Lambda(W)$. Since the functor Ψ is diluting, there are elements $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Phi(W \setminus \{w\})$ and $\boldsymbol{y} \in \Lambda(W \setminus \{w\})$ such that

$$\boldsymbol{y} \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \quad \Phi(f)(\boldsymbol{x}_i) \leq \boldsymbol{u}_i, \text{ for all } i = 0, 1, \text{ and } \boldsymbol{v} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}.$$

Fix $i \in 0, 1$. There is a unique retraction $\boldsymbol{p}_i \colon \Lambda(W) \to \Lambda(W \setminus \{w\})$ satisfying $\boldsymbol{p}_i(\boldsymbol{a}_i^w) = 0$ and $\boldsymbol{p}_i(\boldsymbol{a}_{1-i}^w) = 1$. Set $\boldsymbol{q}_i := \Psi(p_i) \colon \Phi(W) \to \Phi(W \setminus \{w\})$, and observe that $\boldsymbol{q}_i$ is a retraction of $\Phi(f)$. Since $\boldsymbol{x}_i \in \Phi(W \setminus \{w\})$, the equality $\boldsymbol{q}_i(\Phi(f)(\boldsymbol{x}_i)) = \boldsymbol{x}_i$ holds true. Since $\boldsymbol{q}_i(\boldsymbol{a}_i^w) = 0$, by our assumptions, and $\Phi(f)(\boldsymbol{x}_i) \leq \boldsymbol{u}_i \leq \boldsymbol{a}_i^w$, we conclude that $\boldsymbol{x}_i = 0$.

Let $\boldsymbol{r}_i \colon \Lambda(W) \to \Lambda(W \setminus A_{1-i})$ be the unique retraction satisfying the equality $\boldsymbol{r}_i(\boldsymbol{a}_{\varphi_{1-i}}^{A_{1-i}}) = 0$. We put $\boldsymbol{s}_i = \Psi(\boldsymbol{r}_i)$. From $\boldsymbol{u}_{1-i} \leq \boldsymbol{a}_{\varphi_{1-i}}^{A_{1-i}}$, it follows that $\boldsymbol{s}_i(\boldsymbol{u}_{1-i}) = 0$. By Lemma 2.2, there are $\boldsymbol{y}_j \in \Lambda(W \setminus A_{1-j})$ with $\boldsymbol{y}_j \leq \boldsymbol{u}_j$, for all $j = 0, 1$, such that $\boldsymbol{y} \leq \boldsymbol{y}_0 \vee \boldsymbol{y}_1$. Since $\boldsymbol{y}_j \leq \boldsymbol{u}_j \leq \boldsymbol{a}_{\varphi_j}^{A_j}, \boldsymbol{a}_j^w$ and $w \notin A_j$, we conclude that $\boldsymbol{y}_j = 0$, for all $j = 0, 1$. □

3. Free Distributive Extension is Diluting

We summarize the main properties of the construction of the extension $\Delta(\mathcal{S})$ of a $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S}$ (see [**59**, Section 2]) referring to the outline in [**83**, Sections 3,4]. We shall prove that the functor Δ is diluting. For a $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S}$, we set $\Gamma(\mathcal{S}) = \{\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \mathcal{S}^3 \mid \boldsymbol{c} \leq \boldsymbol{a} \vee \boldsymbol{b}\}$. We say that a finite subset $\boldsymbol{v}$ of $\Gamma(\mathcal{S})$ is *reduced*, if the following properties are satisfied:

(i) the subset $\boldsymbol{v}$ contains exactly one triple of the form $\langle \boldsymbol{a}, \boldsymbol{a}, \boldsymbol{a} \rangle$; we define $\pi(\boldsymbol{v}) := \boldsymbol{a}$ and $\boldsymbol{v}^* := \boldsymbol{v} \setminus \{\langle \boldsymbol{a}, \boldsymbol{a}, \boldsymbol{a} \rangle\}$.

(ii) if both $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}$ and $\langle \boldsymbol{b}, \boldsymbol{a}, \boldsymbol{c} \rangle \in \boldsymbol{v}$, then $\boldsymbol{a} = \boldsymbol{b} = \boldsymbol{c}$, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{S}$.

(iii) if $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}^*$, then $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \not\leq \pi(\boldsymbol{v})$, for all $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathcal{S}$.

Observe that if $\boldsymbol{v}$ is a reduced subset of $\Gamma(\mathcal{S})$ and $\boldsymbol{u} \subseteq \boldsymbol{v}^*$, then the subset $\boldsymbol{u} \cup \{\langle 0, 0, 0 \rangle\}$ is reduced as well.

We denote by $\Delta(\mathcal{S})$ the set of all reduced subsets of $\Gamma(\mathcal{S})$. By [**59**, Lemma 2.1] (see also [**83**, Corollary 3.2]), $\Delta(\mathcal{S})$ is a $\langle \vee, \mathbf{0} \rangle$-semilattice with respect to the partial ordering $\leq$ defined by

$$\boldsymbol{v} \leq \boldsymbol{u} \overset{\text{def}}{\iff} (\forall \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v} \setminus \boldsymbol{u})(\text{ either } \boldsymbol{a} \leq \pi(\boldsymbol{u}) \text{ or } \boldsymbol{c} \leq \pi(\boldsymbol{u})) \tag{3.1}$$

and the assignment $\boldsymbol{v} \mapsto \{\langle \boldsymbol{v}, \boldsymbol{v}, \boldsymbol{v} \rangle\}$ is a $\langle \vee, \mathbf{0} \rangle$-embedding from $\mathcal{S}$ into $\Delta(\mathcal{S})$.

As in [**83**], we use the symbol $\bowtie_{\mathcal{S}}$ to denote the elements of $\Delta(\mathcal{S})$ defined by

$$\bowtie_{\mathcal{S}} \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle := \begin{cases} \boldsymbol{c} & \text{if either } \boldsymbol{a} = \boldsymbol{b} \text{ or } \boldsymbol{b} = 0 \text{ or } \boldsymbol{c} = 0, \\ 0 & \text{if } \boldsymbol{a} = 0, \\ \{\langle 0, 0, 0 \rangle, \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle\} & \text{otherwise ,} \end{cases}$$

for all $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \Gamma(\mathbf{S})$. Recall that

$$\boldsymbol{x} = \bigvee \langle \bowtie_{\mathbf{S}} \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \mid \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{x} \rangle, \text{ for all } \boldsymbol{x} \in \Delta(\mathbf{S}). \tag{3.2}$$

due to [**83**, formula (3.3)]. By [**83**, Proposition 3.5], every $\langle \vee, \mathbf{0} \rangle$-homomorphism $\boldsymbol{f} \colon \mathbf{S} \to \mathbf{T}$ extends to a unique $\langle \vee, \mathbf{0} \rangle$-homomorphism $\Delta(\boldsymbol{f}) \colon \Delta(\mathbf{S}) \to \Delta(\mathbf{T})$ such that

$$\Delta(\boldsymbol{f})(\bowtie_{\mathbf{S}} \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle) = \bowtie_{\mathbf{T}} \langle f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}) \rangle, \text{ for all } \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \Gamma(\mathbf{S}), \tag{3.3}$$

and the assignments $\mathbf{S} \mapsto \Delta(\mathbf{S})$ and $\boldsymbol{f} \mapsto \Delta(\boldsymbol{f})$ define a functor $\mathbf{Sem_0} \to \mathbf{Sem_0}$. It follows that if $\boldsymbol{f} \colon \mathbf{S} \to \mathbf{T}$ is a $\langle \vee, \mathbf{0} \rangle$-homomorphism, $\boldsymbol{v} \in \mathbf{S}$, and $\boldsymbol{u} \in \mathbf{T}$, then

$$\Delta(\boldsymbol{f})(\boldsymbol{v}) \leq \boldsymbol{u} \text{ if and only if } \bowtie_{\mathbf{T}} \langle \boldsymbol{f}(\boldsymbol{a}), \boldsymbol{f}(\boldsymbol{b}), \boldsymbol{f}(\boldsymbol{c}) \rangle \leq \boldsymbol{u}, \tag{3.4}$$

for all $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}$.

Lemma 3.1. *The functor Δ is diluting.*

Proof. Let $\mathbf{S}$ and $\mathbf{T}$ be $\langle \vee, \mathbf{0} \rangle$-semilattices and $\boldsymbol{f} \colon \mathbf{S} \to \mathbf{T}$ a $\langle \vee, \mathbf{0} \rangle$-homomorphism. We have to verify that for every $\boldsymbol{v} \in \Delta(\mathbf{S})$ and every $\boldsymbol{u}_0, \boldsymbol{u}_1 \in \Delta(\mathbf{T})$ such that $\Delta(\boldsymbol{f})(\boldsymbol{v}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$, there are elements $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Delta(\mathbf{S})$ and $\boldsymbol{y} \in \mathbf{S}$ satisfying

$$\boldsymbol{f}(\boldsymbol{y}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1, \ \Delta(\boldsymbol{f})(\boldsymbol{x}_i) \leq \boldsymbol{u}_i, \text{ for all } i = 0, 1, \text{ and } \boldsymbol{v} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}.$$

For all $i = 0, 1$ we set

$$\boldsymbol{x}_i := \{\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v} \mid \langle f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}) \rangle \in \boldsymbol{u}_i^* \} \cup \{\langle 0, 0, 0 \rangle\}.$$

Observe that $\boldsymbol{x}_0$, $\boldsymbol{x}_1$, as subsets of $\boldsymbol{v}^* \cup \{\langle 0, 0, 0 \rangle\}$, are reduced. Therefore we have that $\boldsymbol{x}_0, \boldsymbol{x}_1 \in \Delta(\mathbf{S})$. It follows from (3.3) that $\Delta(\boldsymbol{f})(\boldsymbol{x}_i) \leq \boldsymbol{u}_i$, for all $i = 0, 1$. An easy application of [**83**, Lemma 3.1] yields that $(\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^* \subseteq \boldsymbol{u}_0^* \cup \boldsymbol{u}_1^*$, and so $\bowtie_{\mathbf{S}} \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1$, for every $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}$ such that $\langle f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}) \rangle \in (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^*$.

We define

$$\varrho \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle := \begin{cases} \boldsymbol{a} & \text{if } f(\boldsymbol{a}) \leq \pi(\boldsymbol{u}_0 \vee \boldsymbol{u}_1), \\ \boldsymbol{c} & \text{otherwise ,} \end{cases}$$

and we put

$$\boldsymbol{y} := \bigvee \langle \varrho \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \mid \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v} \text{ and } \langle f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}) \rangle \notin (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^* \rangle,$$

for all $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}$. Clearly, $\boldsymbol{y} \in \mathbf{S}$, and, by (3.1), $\bowtie_{\mathbf{S}} \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \leq \boldsymbol{y}$, for all $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}$ such that $\langle f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}) \rangle \notin (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^*$. Thus we have proved that $\bowtie_{\mathbf{S}} \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}$, for all $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}$, and so $\boldsymbol{v} \leq \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \boldsymbol{y}$, due to (3.2).

Since $\Delta(f)(\boldsymbol{v}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$, it follows from (3.1) that $f(\varrho \langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle) \leq \pi(\boldsymbol{u}_0 \vee \boldsymbol{u}_1)$, for every $\langle \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \rangle \in \boldsymbol{v}$ such that $\langle f(\boldsymbol{a}), f(\boldsymbol{b}), f(\boldsymbol{c}) \rangle \notin (\boldsymbol{u}_0 \vee \boldsymbol{u}_1)^*$. We conclude that $f(\boldsymbol{y}) \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$. $\square$

Observe that $\Delta(\mathcal{S})$ is distributive "relatively to" the $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{S}$, that is, for every $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\mathcal{S}$ with $\boldsymbol{c}\leq\boldsymbol{a}\vee\boldsymbol{b}$, there are $\boldsymbol{a}'\leq\boldsymbol{a}$, $\boldsymbol{b}'\leq\boldsymbol{b}$ in $\Delta(\mathcal{S})$ such that $\boldsymbol{c}=\boldsymbol{a}'\vee\boldsymbol{b}'$. It follows that the $\langle\vee,\mathbf{0}\rangle$-semilattice $\Delta^\infty(\mathcal{S})$ is distributive. Applying Lemma 2.3 we conclude that

COROLLARY 3.2. *The functor Δ^∞ is diluting. Moreover, $\Delta^\infty(\mathcal{S})$ is a distributive $\langle\vee,\mathbf{0}\rangle$-semilattice, for every $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{S}$.*

Note that the functor Δ^∞ corresponds to the functor $\mathcal{D}$ from [**83**].

4. Free Trees

DEFINITION 4.1. Let k be a positive integer and W a set. Given a map $\phi\colon [W]^{k-1}\to[W]^{<\omega}$, we say that a k-element subset A of W is *free* (*with respect to the map* ϕ) provided that $a\notin\phi(A\setminus\{a\})$, for all $a\in A$.

THE KURATOWSKI'S FREE SET THEOREM [**46**]. Let k be a positive integer, W a set, and $\phi\colon[W]^{k-1}\to[W]^{<\omega}$ a map. If $|W|\geq\aleph_{k-1}$, then there is a k-element free subset of W.

Let $0\leq n$ and $0<k$ be integers. Given an integer $0\leq m\leq n$ and a map $g\colon\{m,\dots,n-1\}\to k$, we shall put

$$\mathrm{Tree}_{n,k}(g):=\{f\colon n\to k\mid f\text{ extends }g\}.$$

In particular, we will use the notation

$$\mathrm{Tree}_{n,k}:=\mathrm{Tree}_{n,k}(\emptyset)=\{f\mid f\colon n\to k\}.$$

Given integers $0\leq m<n$, $0\leq i<k$, and a map $g\colon\{m+1,\dots,n-1\}\to k$, we shall use the notation

$$\mathrm{Tree}_{n,k}(g,i):=\{f\in\mathrm{Tree}_{n,k}(g)\mid f(m)=i\},$$
$$\mathrm{Tree}_{n,k}(g,\neg i):=\{f\in\mathrm{Tree}_{n,k}(g)\mid f(m)\neq i\}.$$

DEFINITION 4.2. Let W be a set and $\phi\colon[W]^{<\omega}\to[W]^{<\omega}$ a map. Let $0<k$ and $0\leq n$ be integers. We say that a one-to-one map $\boldsymbol{\tau}\colon\mathrm{Tree}_{n,k}\to W$ is a *free k-tree of height n* (*with respect to the map* ϕ) provided that

$$\boldsymbol{\tau}(\mathrm{Tree}_{n,k}(g,i))\cap\phi(\boldsymbol{\tau}(\mathrm{Tree}_{n,k}(g,\neg i)))=\emptyset,\tag{4.1}$$

for all $0\leq m<n$, all $0\leq i<k$, and all maps $g\colon\{m+1,\dots,n-1\}\to k$. We call the set $\operatorname{rng}\boldsymbol{\tau}:=\boldsymbol{\tau}(\mathrm{Tree}_{n,k})$ the *range* of $\boldsymbol{\tau}$.

LEMMA 4.3. *Let W be a set and $\phi\colon[W]^{<\omega}\to[W]^{<\omega}$ a map. Let k and n be positive integers. Then each $A\subseteq W$ with $|A|\geq\aleph_{k-1}$ contains the range of a free k-tree of height n.*

PROOF. We fix k and argue by induction on n. If $n=0$, we pick $\boldsymbol{\tau}(\emptyset)$ to be an arbitrary element of the set A. Suppose that the statement holds up to an integer $n\geq 0$. We shall prove that the set A contains the range of a free k-tree, say $\boldsymbol{\tau}$, of height $n+1$. We cut up the set A as a union of pairwise disjoint subsets A_w, for $w<\omega_{k-1}$, each of cardinality at least

$\aleph_{k-1}$. By the induction hypothesis, each of the sets A_w contains the range of a free tree $\boldsymbol{\tau}_w$ of height n. We define a map $\psi\colon [\omega_{k-1}]^{k-1} \to [\omega_{k-1}]^{<\omega}$ by

$$\psi(X) := \langle \nu < \omega_{k-1} \mid \operatorname{rng} \boldsymbol{\tau}_\nu \cap \phi(\bigcup_{w \in X} \operatorname{rng} \boldsymbol{\tau}_w) \neq \emptyset \rangle, \tag{4.2}$$

for all $X \in [\omega_{k-1}]^{k-1}$. Since the sets $\operatorname{rng} \mathcal{T}_w$ are pairwise disjoint and finite, $\psi(F)$ is finite, for all $F \in [\omega_{k-1}]^{k-1}$. By Kuratowski's Free Set Theorem there is a k-element free subset, $F := \{w_0, \dots, w_{k-1}\} \subseteq A$, with respect to the map ψ. We set $\boldsymbol{\tau}(f) = \boldsymbol{\tau}_{w_{f(n)}}(f \restriction n)$, for all $f\colon (n+1) \to k$. We claim that $\boldsymbol{\tau}\colon \mathrm{Tree}_{n+1,k} \to A$ is a free k-tree with respect to ϕ. In order to prove the claim, we fix $0 \geq m < n+1$ and a map $g\colon \{m+1, \dots, n\} \to k$.

If $m = n$, the only possibility is $g = \emptyset$. Then

$$\boldsymbol{\tau}(\mathrm{Tree}_{n+1,k}(g, i)) = \operatorname{rng} \boldsymbol{\tau}_{w_i}, \text{ and}$$
$$\boldsymbol{\tau}(\mathrm{Tree}_{n+1,k}(g, \neg i)) = \bigcup_{j<k,\ j \neq i} \operatorname{rng} \boldsymbol{\tau}_{w_j},$$

for all $i < k$. Since F is a free set with respect to ψ, we have that

$$\operatorname{rng} \boldsymbol{\tau}_{w_i} \cap \phi(\bigcup_{j<k,\ j \neq i} \operatorname{rng} \boldsymbol{\tau}_{w_j}) = \emptyset,$$

by (4.2).

Suppose that $m < n$ and set $g' := g \restriction \{m+1, \dots, n-1\}$. Let $i < k$. Then the equalities

$$\boldsymbol{\tau}(\mathrm{Tree}_{n+1,k}(g, i)) = \boldsymbol{\tau}_{w_{g(n)}}(\mathrm{Tree}_{n,k}(g', i)),$$
$$\boldsymbol{\tau}(\mathrm{Tree}_{n+1,k}(g, \neg i)) = \boldsymbol{\tau}_{w_{g(n)}}(\mathrm{Tree}_{n,k}(g', \neg i)).$$

hold true. Since $\boldsymbol{\tau}_{w_{g(n)}}$ is a free k-tree with respect to ϕ, we conclude that

$$\boldsymbol{\tau}_{w_{g(n)}}(\mathrm{Tree}_{n,k}(g', i)) \cap \phi(\boldsymbol{\tau}_{w_{g(n)}}(\mathrm{Tree}_{n,k}(g', \neg i))) = \emptyset,$$

due to (4.1). $\square$

5. The optimal bound in Wehrung's Theorem

Let Ψ be an expanding functor on $\mathbf{Sem_0}$ satisfying the following properties:

- For all $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S}$ and all families $\langle \mathcal{S}_i \mid i \in I \rangle$ of $\langle \vee, \mathbf{0} \rangle$-subsemilattices of $\mathcal{S}$, the equality

$$\bigcap_{i \in I} \Psi(\mathcal{S}_i) = \Psi\Big(\bigcap_{i \in I} \mathcal{S}_i\Big) \tag{5.1}$$

  holds true.

- For all nonempty upwards directed posets P and all families $\langle \mathcal{S}_p \mid p \in P \rangle$ of $\langle \vee, \mathbf{0} \rangle$-semilattices such that $\mathcal{S}_p$ is a $\langle \vee, \mathbf{0} \rangle$-subsemilattice of $\mathcal{S}_q$, whenever $p \leq q$ in P, the equality

$$\bigcup_{p \in P} \Psi(\mathcal{S}_p) = \Psi\Big(\bigcup_{p \in P} \mathcal{S}_p \Big) \tag{5.2}$$

holds true.

We put $\Phi := \Psi \circ \Lambda$. Then for every set W and every family $\langle U_i \mid i \in I \rangle$ of subsets of W, we have that

$$\bigcap_{i \in I} \Phi(U_i) = \Phi\Big(\bigcap_{i \in I} U_i \Big),$$

and for every nonempty upwards directed poset P and every family $\langle U_p \mid p \in P \rangle$ of subsets of W such that $U_p \subseteq U_q$, whenever $p \leq q$ in P, the equality

$$\bigcup_{p \in P} \Phi(U_p) = \Phi\Big(\bigcup_{p \in P} U_p \Big)$$

holds true.

It follows that given a set W and an element $\boldsymbol{a} \in \Phi(W)$, there is a smallest finite $F \subseteq W$ such that $\boldsymbol{a} \in \Phi(F)$. We shall call F the *support* of $\boldsymbol{a}$and denote it by $\operatorname{supp}(\boldsymbol{a})$ (see [**83**]). Now we are ready to rephrase [**83**, Theorem 6.1]:

THEOREM 5.1. *Let W be a set of cardinality at least $\aleph_2$, Ψ a diluting functor satisfying properties* (5.1) *and* (5.2)*, and let $\boldsymbol{A}$ be an algebra having either a congruence-compatible structure of a $\langle \vee, \mathbf{1} \rangle$-semilattice or a congruence compatible structure of a lattice. Then there does not exist a weakly distributive $\langle \vee, \mathbf{0} \rangle$-homomorphism $\operatorname{Con_c} A \to (\Psi \circ \Lambda)(W)$ containing 1 in its range.*

PROOF. As above, we put $\Phi := \Psi \circ \Lambda$. We assume for a contradiction that there is a weakly distributive $\langle \vee, \mathbf{0} \rangle$-homomorphism $\boldsymbol{h} \colon \operatorname{Con_c} \boldsymbol{A} \to \Phi(W)$ having 1 in its range. Since 1 is in the range of $\boldsymbol{h}$, there is a finite subset F of $\boldsymbol{A}$ such that $\boldsymbol{h}(\Theta_{\boldsymbol{A}}(F)) = 1$. We set $\mathbf{a} := 1$, respectively, $\mathbf{a} := \bigvee F$, if $\boldsymbol{A}$ has a congruence-compatible structure of a $\langle \vee, \mathbf{1} \rangle$-semilattice, respectively, a lattice. We can without loss of generality assume that $\mathbf{a} \in F$. Then we have that

$$\bigvee_{\mathbf{x} \in F} \boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{x})) = \boldsymbol{h}(\bigvee_{\mathbf{x} \in F} \Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{x})) = \boldsymbol{h}(\Theta_{\boldsymbol{A}}(F)) = 1.$$

We pick an arbitrary element $w \in W$. Since the homomorphism $\boldsymbol{h}$ is weakly distributive, there are congruences $\Theta_0^w, \Theta_1^w \in \operatorname{Con_c} \boldsymbol{A}$ such that

$$\bigvee_{\mathbf{x} \in F} \Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{x}) \leq \Theta_0^w \vee \Theta_1^w \text{ and } \boldsymbol{h}(\Theta_i^w) \leq \boldsymbol{a}_i^w, \text{ for all } i = 0, 1.$$

In particular, we have that $\Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{x}) \leq \Theta_0^w \vee \Theta_1^w$, for all $\mathbf{x} \in F$.

We fix an element $\mathbf{x} \in F$. Since $\Theta_{\boldsymbol{A}}(\mathbf{a},\mathbf{x}) \leq \Theta_0^w \vee \Theta_1^w$, there are a positive integer n_w and elements $\mathbf{x} = \mathbf{z}_0^w, \mathbf{z}_1^w, \dots, \mathbf{z}_{n_w}^w = \mathbf{a}$ in $\boldsymbol{A}$, such that

$$\boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{z}_i^w, \mathbf{z}_{i+1}^w)) \leq \boldsymbol{a}_{\varepsilon(i)}^w, \text{ for all } i < n_w. \tag{5.3}$$

(Recall that $\varepsilon(i) = i \mod 2$.)

If $\boldsymbol{A}$ has a congruence-compatible structure of a $\langle \vee, \mathbf{1} \rangle$-semilattice, we replace each $\mathbf{z}_i^w$ with $\mathbf{z}_0^w \vee \cdots \vee \mathbf{z}_i^w$ and if $\boldsymbol{A}$ has a congruence-compatible structure of a lattice, we replace each $\mathbf{z}_i^w$ with $\mathbf{a} \wedge (\mathbf{z}_0^w \vee \cdots \vee \mathbf{z}_i^w)$. In both the cases we obtain an increasing chain $\mathbf{x} = \mathbf{z}_0^w \leq \mathbf{z}_1^w \leq \cdots \leq \mathbf{z}_{n_w}^w = \mathbf{a}$ in $\boldsymbol{A}$ such that (5.3) remains satisfied.

Let X be a subset of the algebra $\boldsymbol{A}$. As in [**83**, Section 6], we denote by $\operatorname{Con_c}^X \boldsymbol{A}$ the $\langle \vee, \mathbf{0} \rangle$-subsemilattice of $\operatorname{Con_c} \boldsymbol{A}$ generated by all principal congruences $\Theta_{\boldsymbol{A}}(\mathbf{x},\mathbf{y})$, where $\mathbf{x},\mathbf{y} \in X$. We denote by $\mathbf{S}_U$ the join-subsemilattice of $\boldsymbol{A}$ generated by the set $\{\mathbf{z}_i^u \mid u \in U \text{ and } 0 \leq i \leq n_w\}$, and we put

$$\phi(U) = \bigcup \{\operatorname{supp}(\boldsymbol{h}(\Theta)) \mid \Theta \in \operatorname{Con_c}^{\mathbf{S}_U} \boldsymbol{A}\}, \tag{5.4}$$

for all $U \subseteq W$. Observe that if the subset U is finite, then both $\mathbf{S}_U$ and $\phi(U)$ are finite.

Since the size of the set W is at least $\aleph_2$, there are a positive integer n and a subset U of W of cardinality at least $\aleph_2$ such that $n_u = n$, for all $u \in U$. The following crucial claim is analogous to [**83**, Lemma 6.2], giving another illustration of the "erosion method".

Claim 1. *Let $\boldsymbol{\tau} \colon \mathrm{Tree}_{n,3} \to U$ be a free 3-tree with respect to the map ϕ defined by* (5.4). *Then*

$$\Theta_{\boldsymbol{A}}(\mathbf{a}, \bigvee \{\mathbf{z}_{n-m}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g)\}) = 0, \tag{5.5}$$

for all integers $0 \leq m \leq n$ and all maps $g \colon \{m, \dots, n-1\} \to 2$.

Proof of Claim 1. We shall argue by induction on m. If $m = 0$, then the equality (5.5) is trivially satisfied for all maps $g \colon \{m, \dots, n-1\} \to 2$. Let $0 \leq m < n$, $g \colon \{m+1, \dots, n-1\} \to 2$, and suppose that (5.5) is satisfied at stage n. We put

$$\mathbf{y}_i := \bigvee \{\mathbf{z}_{n-m-1}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g,i)\},$$

for all $i = 0, 1$. We fix $i \in \{0,1\}$. It is straightforward that

$$\begin{aligned} \boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a},\mathbf{y}_i)) = \boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, &\bigvee \{\mathbf{z}_{n-m-1}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g,i)\})) \leq \\ &\Big(\bigvee \{\boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{z}_{n-m-1}^{\boldsymbol{\tau}(f)}, \mathbf{z}_{n-m}^{\boldsymbol{\tau}(f)})) \mid f \in \mathrm{Tree}_{n,2}(g,i)\}\Big) \vee \\ &\boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, \bigvee \{\mathbf{z}_{n-m}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g,i)\})). \end{aligned}$$

We set

$$\boldsymbol{v} := \boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, \bigvee \{\mathbf{z}_{n-m-1}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g)\})) = \boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{y}_0 \vee \mathbf{y}_1)),$$

$U_i := \{\boldsymbol{\tau}(f) \mid f \in \mathrm{Tree}_{n,2}(g,i)\}$, and we let $\varphi_i\colon U_i \to 2$ be the constant map with the value $\varepsilon(n-m-1)$. By the induction hypothesis we have that

$$\boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, \bigvee\{\mathbf{z}_{n-m}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g,i)\})) = 0,$$

and from (5.3) we conclude that

$$\boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{z}_{n-m-1}^{\boldsymbol{\tau}(f)}, \mathbf{z}_{n-m}^{\boldsymbol{\tau}(f)})) \leq \boldsymbol{a}_{\varepsilon(n-m-1)}^{\boldsymbol{\tau}(f)},$$

for all $f \in \mathrm{Tree}_{n,2}(g,i)$. It follows that

$$\boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{y}_i)) \leq \bigvee\{\boldsymbol{a}_{\varepsilon(n-m-1)}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g,i)\} = \boldsymbol{a}_{\varphi_i}^{U_i}.$$

Let $u \in U$ be arbitrary. Applying the Erosion Lemma [**83**, Lemma 5.1], we infer that there are $\boldsymbol{u}_j \in \mathrm{Con}_c^{\mathbf{S}_{U_j \cup \{u\}}} \boldsymbol{A}$ such that $\boldsymbol{v} \leq \boldsymbol{u}_0 \vee \boldsymbol{u}_1$ and both $\boldsymbol{u}_j \leq \boldsymbol{a}_{\varepsilon(j)}^u$ and $\boldsymbol{u}_j \leq \boldsymbol{h}(\Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{y}_j))$, for all $j = 0, 1$. It follows that $\boldsymbol{u}_j \leq \boldsymbol{a}_{\varepsilon(j)}^u$ and $\boldsymbol{u}_j \leq \boldsymbol{a}_{\varphi_j}^{U_j}$, for all $j = 0, 1$.

Now suppose that $u = \boldsymbol{\tau}(f)$, for some $f \in \mathrm{Tree}_{n,3}(g,2)$. It follows from (5.4) that $\mathrm{supp}(\boldsymbol{v}) \subseteq \phi(\{\boldsymbol{\tau}(f) \mid f \in \mathrm{Tree}_{n,2}(g)\}) = \phi(U_0 \cup U_1)$ and $\mathrm{supp}(\boldsymbol{u}_j) \subseteq \phi(U_j \cup \{u\})$, for all $j = 0, 1$. Since $\boldsymbol{\tau}$ is a free 3-tree with respect to ϕ, we have that $u \notin \phi(U_0 \cup U_1)$ and $U_{1-j} \cap \phi(U_j \cup \{u\}) = \emptyset$, for all $j = 0, 1$. It follows that $\boldsymbol{v} \in \Phi(W \setminus \{u\})$ and $\boldsymbol{u}_j \in \Phi(W \setminus U_{1-j})$, for all $j = 0, 1$. By our assumptions the functor Ψ is diluting. Therefore applying Lemma 2.4, we conclude that $\boldsymbol{v} = 0$ as desired . □ Claim 1.

According to Lemma 4.3 there is a free 3-tree $\boldsymbol{\tau}\colon \mathrm{Tree}_{n,3} \to U$. Applying Claim 1, we get that

$$\Theta_{\boldsymbol{A}}(\mathbf{a}, \bigvee\{\mathbf{z}_{n-m}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g)\}) = 0,$$

for all $0 \leq m \leq n$ and all maps $g\colon \{m, \dots, n-1\} \to 2$. If $m = n$ and $g = \emptyset$ we have that

$$\bigvee\{\mathbf{z}_{n-m}^{\boldsymbol{\tau}(f)} \mid f \in \mathrm{Tree}_{n,2}(g)\} = \bigvee\{\mathbf{z}_0^{\boldsymbol{\tau}(f)} \mid f\colon n \to 2\} = \mathbf{x},$$

hence $\Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{x}) = 0$ for all $\mathbf{x} \in F$. Therefore $\bigvee_{\mathbf{x} \in F} \Theta_{\boldsymbol{A}}(\mathbf{a}, \mathbf{x}) = 0$, which leads to a contradiction. □

The functor Δ^∞ is diluting due to Corollary 3.2 and it satisfies both (5.1) and (5.2) due to [**83**, Lemma 3.6]. We put $\Phi := \Delta^\infty \circ \Lambda$.[1] Since the $\langle\vee, \mathbf{0}\rangle$-semilattice $\Delta^\infty(\mathbf{S})$ is distributive for all $\langle\vee, \mathbf{0}\rangle$-semilattices $\mathbf{S}$, we get that

Corollary 5.2. *Let W be a set of cardinality at most $\aleph_2$. Then there is no lattice $\boldsymbol{\mathcal{L}}$ such that the $\langle\vee, \mathbf{0}\rangle$-semilattice $\Phi(W)$ is isomorphic to* $\mathrm{Con_c}\,\boldsymbol{\mathcal{L}}$.

[1] Note that this is the same Φ as in [**83**].

A. P. Huhn [**41, 42**] (see also [**27**, Theorem 13 in Appendix C]) proved that every distributive $\langle\vee,\mathbf{0}\rangle$-semilattice of size at most $\aleph_1$ is isomorphic to $\mathrm{Con_c}\,\boldsymbol{\mathcal{L}}$, for some lattice $\boldsymbol{\mathcal{L}}$. Moreover, the lattice $\boldsymbol{\mathcal{L}}$ can be taken sectionally complemented and modular [**79**, Corollary 5.3] or relatively complemented, locally finite, and with zero [**28**]. In particular, in any of these cases, the lattice $\boldsymbol{\mathcal{L}}$ has permutable congruences (cf. [**15**]).

CHAPTER 4

Countable chains of distributive lattices and dimension groups

1. Introduction

Given a ring $\boldsymbol{R}$, we denote by $\mathrm{FP}(\boldsymbol{R})$ the class of all finitely generated projective right $\boldsymbol{R}$ modules. We denote by $[\,\boldsymbol{A}\,]$ the isomorphism class of a module $\boldsymbol{A} \in \mathrm{FP}(\boldsymbol{R})$ and by $\boldsymbol{V}(\boldsymbol{R})$ the monoid of all isomorphism classes of modules from $\mathrm{FP}(\boldsymbol{R})$, with the operation of addition defined by $[\,\boldsymbol{A}\,]+[\,\boldsymbol{B}\,] = [\,\boldsymbol{A} \oplus \boldsymbol{B}\,]$. If the ring $\boldsymbol{R}$ is von Neumann regular, then the monoid $\boldsymbol{V}(\boldsymbol{R})$ satisfies the Riesz refinement property and the semilattice $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ of finitely generated two-sided ideals of $\boldsymbol{R}$ is isomorphic to the maximal semilattice quotient of $\boldsymbol{V}(\boldsymbol{R})$ [**78**, Proposition 4.6].

Modules $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{FP}(\boldsymbol{R})$ are said to be *stably equivalent* provided that there exists $\boldsymbol{C} \in \mathrm{FP}(\boldsymbol{R})$ such that $\boldsymbol{A} \oplus \boldsymbol{C} \simeq \boldsymbol{B} \oplus \boldsymbol{C}$. We denote by $[\,\boldsymbol{A}\,]_s$ the stable equivalence class of $\boldsymbol{A} \in \mathrm{FP}(\boldsymbol{R})$, and by $\boldsymbol{V}_s(\boldsymbol{R})$ the quotient monoid $\{[\,\boldsymbol{A}\,]_s \mid \boldsymbol{A} \in \mathrm{FP}(\boldsymbol{R})\}$ of $\boldsymbol{V}(\boldsymbol{R})$ modulo the stable equivalence. We set

$$K_0(\boldsymbol{R}) := \{[\,\boldsymbol{A}\,]_s - [\,\boldsymbol{B}\,]_s \mid \boldsymbol{A}, \boldsymbol{B} \in \mathrm{FP}(\boldsymbol{R})\}$$

and we define a binary operation on $K_0(\boldsymbol{R})$ by

$$([\,\boldsymbol{A}\,]_s - [\,\boldsymbol{B}\,]_s) + ([\,\boldsymbol{C}\,]_s - [\,\boldsymbol{D}\,]_s) = [\,\boldsymbol{A} \oplus \boldsymbol{C}\,]\,s - [\,\boldsymbol{B} \oplus \boldsymbol{D}\,]_s.$$

This makes $K_0(\boldsymbol{R})$ be an abelian group equipped with the preorder $\leq$ determined by the positive cone $\boldsymbol{V}_s(\boldsymbol{R})$, in particular, $([\,\boldsymbol{A}\,]_s - [\,\boldsymbol{B}\,]_s) \leq ([\,\boldsymbol{C}\,]_s - [\,\boldsymbol{D}\,]_s)$ if and only if there is $\boldsymbol{E} \in \mathrm{FP}(\boldsymbol{R})$ such that $\boldsymbol{A} \oplus \boldsymbol{D} \oplus \boldsymbol{E} \simeq \boldsymbol{C} \oplus \boldsymbol{B}$.

If the ring $\boldsymbol{R}$ is unit-regular, then the equivalence and the stable equivalence of modules from $\mathrm{FP}(\boldsymbol{R})$ coincide, $\boldsymbol{V}(\boldsymbol{R}) = \boldsymbol{V}_s(\boldsymbol{R})$, and $K_0(\boldsymbol{R})$ is a partially ordered abelian group. Moreover $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ is isomorphic to the maximal semilattice quotient of $\boldsymbol{V}(\boldsymbol{R})$ (denoted by $\nabla(\boldsymbol{V}(\boldsymbol{R}))$). The monoid $\boldsymbol{V}(\boldsymbol{R})$ satisfies the Riesz refinement property and it generates $K_0(\boldsymbol{R})$. If the ring $\boldsymbol{R}$ is a direct limit of von Neumann regular rings whose primitive factors are artinian, in particular, if $\boldsymbol{R}$ is a locally matricial algebra (over a field), then $K_0(\boldsymbol{R})$ is in addition unperforated [**22**, Theorem 15.12], that is, $K_0(\boldsymbol{R})$ is a *dimension group* (see [**21, 16**]).

Our study of representations of distributive $\langle\vee, \mathbf{0}\rangle$-semilattices in maximal semilattice quotients of dimension groups is motivated by the study of representations of distributive $\langle\vee, \mathbf{0}\rangle$-semilattices as semilattices of two-sided ideals of locally matricial algebras. G.M. Bergman [**9**] proved that every countable distributive $\langle\vee, \mathbf{0}\rangle$-semilattice is isomorphic to the join-semilattice of finitely generated ideals of some locally matricial algebra. By [**24**, Theorem 1.1], a dimension group of size at most $\aleph_1$ is isomorphic to $K_0(\boldsymbol{R})$ of some locally matricial algebra. It follows that a distributive $\langle\vee, \mathbf{0}\rangle$-semilattice of size $\aleph_1$ is isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra if and only if it is isomorphic to the maximal semilattice quotient of the positive cone of some dimension group (such a group, if it exists, can be always taken of size at most $\aleph_1$).

It follows from a direct construction in [**79**] that a distributive $\langle\vee, \mathbf{0}\rangle$-semilattice of size $\leq \aleph_1$ is isomorphic to the $\langle\vee, \mathbf{0}\rangle$-semilattice $\mathrm{Id}_c(\boldsymbol{R})$ of all finitely generated two sided ideals of a von Neumann regular ring $\boldsymbol{R}$. On the other hand, the construction of F. Wehrung [**81**] gives an example of a distributive $\langle \vee, \mathbf{0}\rangle$-semilattice of size $\aleph_1$ not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group, and therefore not isomorphic to the semilattice of finitely generated two-sided ideals of any locally matricial algebra. The key idea of his construction consists of the formulation of a semilattice property, denoted by $\mathrm{URP_{sr}}$ [**81**, Definition 4.2], that is satisfied by the maximal semilattice quotient of the positive cone of any dimension group, and the construction of a distributive $\langle \vee, \mathbf{0}\rangle$-semilattice S_{ω_1} of size $\aleph_1$ that does not satisfy this property. In the same paper [**81**] F. Wehrung proved that a direct limit of a countable chain of distributive lattices and join-homomorphisms satisfies $\mathrm{URP_{sr}}$ [**81**, Section 7] and asked whether

Problem 1 [**81**]. Let $\mathcal{S} = \varinjlim_{n<\omega} \mathcal{D}_n$ with all $\mathcal{D}_n$-s being distributive lattices with zero and all transition maps being $\langle \vee, \mathbf{0}\rangle$-homomorphisms. Does there exists a dimension group $\boldsymbol{G}$ such that $\mathcal{S} \simeq \nabla(\boldsymbol{G}^+)$?

We solve this problem by constructing a union of a countable chain of Boolean semilattices, resp.strongly distributive $\langle \vee, \mathbf{0}, \mathbf{1}\rangle$-semilattices (such that all inclusions are $\langle \vee, \mathbf{0}, \mathbf{1}\rangle$-homomorphisms), not isomorphic to the maximal semilattice quotient of any Riesz monoid in which every nonzero element is anti-idempotent, and therefore not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group.

2. Notation and terminology

We say that an element e of a monoid $\boldsymbol{M}$ is *anti-idempotent* provided that $2ne \not\leq ne$ (equivalently, $(n+1)e \not\leq ne$), for every positive integer n.

We say that a subset D of a $\langle \vee, \mathbf{0}\rangle$-semilattice $\mathcal{S}$ is *dense in* $\mathcal{S}$, if D is a dense subset of the poset $\mathcal{S} \setminus \{\mathbf{0}\}$.

Let $\boldsymbol{G}$ be a partially ordered abelian group. We will use the notation $\boldsymbol{G}^+ := \{x \in \boldsymbol{G} \mid x \geq 0\}$ for the *positive cone* of $\boldsymbol{G}$. The partially ordered abelian group $\boldsymbol{G}$ is *unperforated* if $nx \geq 0$ implies $x \geq 0$ for every positive integer n and all $x \in \boldsymbol{G}$, and it is *directed* provided that each of its element is the difference of a pair of elements from $\boldsymbol{G}^+$. It is easy to see that a partially ordered abelian group is directed if and only if it is directed as a partially ordered set. Furhermore, $\boldsymbol{G}$ is an *interpolation group* provided that for all x_0, x_1, y_0, and y_1 in $\boldsymbol{G}$ with $x_i \leq y_j$, for all $i, j \in \{0, 1\}$, there exists $z \in \boldsymbol{G}$ such that $x_i \leq z \leq y_j$, for all $i, j \in \{0, 1\}$. A partially ordered abelian group is an interpolation group if and only if its positive cone is a refinement monoid [**21**, Proposition 2.1]. A *dimension group* is an unperforated directed interpolation group.

By an *ordered vector space* we mean a partially ordered vector space over the field of rational numbers such that the multiplication by positive scalars is order-preserving. A *dimension vector space* is an ordered vector space which, as a partially ordered abelian group, is a dimension group.

Given a Boolean algebra $\boldsymbol{B}$ and an element $\boldsymbol{b} \in \boldsymbol{B}$, we denote by $\boldsymbol{B} \restriction \boldsymbol{b}$ the Boolean algebra with the universe $\{\boldsymbol{x} \in \boldsymbol{B} \mid \boldsymbol{x} \leq \boldsymbol{b}\}$ and join and meet operations inherited from $\boldsymbol{B}$.

3. The construction

Let $\boldsymbol{B}$ be a Boolean algebra, $\boldsymbol{F}$ a filter of $\boldsymbol{B}$, and $\boldsymbol{I}$ the dual ideal of the filter $\boldsymbol{F}$. Given a distributive $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S}$, we set

$$\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B} := \big((\mathcal{S} \smallsetminus \{\mathbf{0}\}) \times \boldsymbol{F}\big) \cup \big(\{\mathbf{0}\} \times \boldsymbol{I}\big) \subseteq \mathcal{S} \times \mathcal{B}$$

(see [**63**] and [**81**]). One readily sees that $\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B}$ is a join-subsemilattice of the cartesian product $\mathcal{S} \times \mathcal{B}$. It could be proved similarly as [**63**, Lemma 3.3] that if $\mathcal{S}$ is a distributive $\langle \vee, \mathbf{0} \rangle$-semilattice, then $\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B}$ is distributive as well. We are going to reprove this fact by presenting the $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B}$ as the union of a directed system of distributive $\langle \vee, \mathbf{0} \rangle$-semilattices.

Lemma 3.1. *Let $\mathcal{B}$ be a Boolean algebra, $\boldsymbol{F}$ a filter of $\mathcal{B}$ and $\boldsymbol{I}$ the ideal dual to $\boldsymbol{F}$. If $\mathcal{S}$ is a distributive $\langle \vee, \mathbf{0} \rangle$-semilattice, then the $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B}$ is distributive.*

Proof. Let $X \subseteq \mathcal{B}$ be such that $\boldsymbol{I} = \downarrow_X (\mathcal{B})$. We set

$$\mathcal{S}_{\boldsymbol{a}} := \{\langle \mathbf{0}, \boldsymbol{b} \rangle \mid \boldsymbol{b} \in \mathcal{B} \restriction \boldsymbol{a}\} \cup \{\langle \boldsymbol{x}, \boldsymbol{b} \vee (-\boldsymbol{a}) \rangle \mid \boldsymbol{x} \in \mathcal{S} \smallsetminus \{\mathbf{0}\} \text{ and } \boldsymbol{b} \in \mathcal{B} \restriction \boldsymbol{a}\},$$

for all $\boldsymbol{a} \in X$. It is easy to see that $\mathcal{S}_{\boldsymbol{a}}$ is a $\langle \vee, \mathbf{0} \rangle$-subsemilattice of $\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B}$ isomorphic to $\mathcal{S} \times (\mathcal{B} \restriction \boldsymbol{a})$.

We prove that $\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B}$ is a directed union of the distributive $\langle \vee, \mathbf{0} \rangle$-semilattices $\mathcal{S}_{\boldsymbol{a}}$-s. Trivially we have that $\{\mathbf{0}\} \times \boldsymbol{I} \subseteq \bigcup_{\boldsymbol{a} \in X} \mathcal{S}_{\boldsymbol{a}}$. Let $\boldsymbol{x}$ be a nonzero element of $\mathcal{S}$ and $\boldsymbol{b} \in \boldsymbol{F}$. Then $-\boldsymbol{a} \leq \boldsymbol{b}$, for some $\boldsymbol{a} \in X$, hence $(\boldsymbol{a} \wedge \boldsymbol{b}) \vee (-\boldsymbol{a}) = \boldsymbol{b}$, whence $\langle \boldsymbol{x}, \boldsymbol{b} \rangle \in \mathcal{S}_{\boldsymbol{a}}$. Therefore $(\mathcal{S} \smallsetminus \{\mathbf{0}\}) \times \boldsymbol{F} \subseteq \bigcup_{\boldsymbol{a} \in X} \mathcal{S}_{\boldsymbol{a}}$, and so we have proved that $\mathcal{S} \times_{\boldsymbol{F}} \mathcal{B} = \bigcup_{\boldsymbol{a} \in X} \mathcal{S}_{\boldsymbol{a}}$. We get readily from the definition that $\boldsymbol{a} \leq \boldsymbol{c}$ in X implies that $\mathcal{S}_{\boldsymbol{a}} \subseteq \mathcal{S}_{\boldsymbol{c}}$, which implies that the union is directed. This completes the proof. $\square$

Remark 3.2. Let

$$\mathfrak{F} := \{X \subseteq \omega \mid \omega \setminus X \text{ is finite}\}.$$

denote the *Fréchet filter* on $\mathcal{P}(\omega)$. Then

$$\mathcal{S} \times_{\mathfrak{F}} \mathcal{P}(\omega) = \varinjlim \langle \mathcal{S} \times \mathcal{P}(n+1) \mid n \in \omega \rangle,$$

with the transition maps $\boldsymbol{f}_{n,m}$ being $\langle \vee, \mathbf{0} \rangle$-embeddings

$$\boldsymbol{f}_{n,m}(\langle \boldsymbol{a}, F \rangle) := \begin{cases} \langle \boldsymbol{a}, F \cup \{n+1, \dots, m\} \rangle & \text{if } \mathbf{0} < \boldsymbol{a}, \\ \langle \boldsymbol{a}, F \rangle & \text{if } \mathbf{0} = \boldsymbol{a}, \end{cases}$$

where $n < m$ in ω, $\boldsymbol{a} \in \mathcal{S}$, and $F \subseteq \{0, \dots, n\}$. In particular, if the $\langle \vee, \mathbf{0} \rangle$-semilattice $\mathcal{S}$ is Boolean or strongly distributive respectively then $\mathcal{S} \times_{\mathfrak{F}} \mathcal{P}(\omega)$ is a directed union of a countable chain of Boolean $\langle \vee, \mathbf{0} \rangle$-semilattices or strongly distributive $\langle \vee, \mathbf{0} \rangle$-semilattices. Furthermore if $\mathcal{S}$ has a greatest element, then the transition maps are $\langle \vee, \mathbf{0}, \mathbf{1} \rangle$-homomorphisms.

We modify some notation from [**63**]. Let x, y be elements of a monoid $\boldsymbol{M}$. We set

$$\mathrm{Q}(x/y) := \{n/m \mid n, m \in \mathbb{N} \text{ and } kny \leq kmx \text{ for some } k \in \mathbb{N}\}.$$

Observe that the set $\mathrm{Q}(x/y)$ forms a lower interval in $\mathbb{Q}^+$. Indeed, if $n'/m' \leq n/m$ and $n/m \in \mathrm{Q}(x/y)$, then $kny \leq kmx$ for some $k \in \mathbb{N}$, whence $knn'y \leq kmn'x \leq knm'x$. We define $(x/y) := \sup \mathrm{Q}(x/y)$.

LEMMA 3.3. *Let x, y, and z be elements of a monoid $\boldsymbol{M}$. Then the following hold true:*

(i) $(na/y) = n(x/y)$ *for all* $n \in \mathbb{N}$.
(ii) $(x + y/z) \geq (x/z) + (y/z)$.
(iii) *Suppose that $\boldsymbol{M}$ is a Riesz monoid and that $y \wedge z = 0$. Then $z \leq x + y$ implies that $z \leq x$. In particular, we have that $(x + y/z) = (x/z)$ (cf.* [**63**, Corollary 2.5]).

PROOF. Property (i) follows from $n'/nm \in \mathrm{Q}(x/y)$ if and only if $n'/m \in \mathrm{Q}(nx/y)$, for all $n', m \in \mathbb{N}$.

(ii) It is obvious that if $k/n \in \mathrm{Q}(x/z)$ and $l/n \in \mathrm{Q}(y/z)$, then $k/n + l/n \in \mathrm{Q}(x + y/z)$.

(iii) Suppose that $z \leq x + y$ in $\boldsymbol{M}$. Since $\boldsymbol{M}$ is a Riesz monoid, there are $x' \leq x$ and $y' \leq y$ satisfying $z = x' + y'$. From $y \wedge z = 0$, it follows that $y' = 0$, whence $z \leq x$. For the equality $(x + y/z) = (x/z)$, it suffices to check that $(x + y/z) \leq (x/z)$. If $m, n, k \in \mathbb{N}$ are such that $kmz \leq kn(x + y) = knx + kny$, then $kmz \leq knx$. This proves that $(x + y/z) \leq (x/z)$. □

We denote by $\mathbb{R}_+^\omega$ and $\mathbb{R}_+^{(\omega)}$ respectively the monoid of all maps from $\omega \to \mathbb{R}_+$ and the monoid of all maps from $\omega \to \mathbb{R}_+$ with a finite support, and we let $\boldsymbol{R}$ denote the quotient $\widehat{\boldsymbol{R}} := \mathbb{R}_+^\omega / \mathbb{R}_+^{(\omega)}$. Furthermor, for every $f \in \mathbb{R}_+^\omega$ we denote by $\widehat{f}$ the image of f in $\widehat{\boldsymbol{R}}$ and by $\widehat{\boldsymbol{f}}$ the corresponding element of $\nabla(\widehat{\boldsymbol{R}})$.

Let $\mathbf{\mathcal{S}}$ be a $\langle \vee, \mathbf{0} \rangle$-semilattice, $\boldsymbol{M}$ a monoid, and $\boldsymbol{h} \colon \mathbf{\mathcal{S}} \times_{\mathfrak{F}} \mathcal{P}(\omega) \to \nabla(\boldsymbol{M})$ an isomorphism. We fix $\mathcal{Q} := \{q_i \mid i \in \omega\} \subseteq \boldsymbol{M}$ such that $\boldsymbol{q}_i = \boldsymbol{h}(\langle \mathbf{0}, \{i\} \rangle)$, for all $i \in \omega$, and a map $f_x \colon \omega \to \mathbb{Q}^+$ given by the correspondence $i \mapsto (x/q_i)$, for all $x \in \boldsymbol{M}$.

LEMMA 3.4. *Let $\boldsymbol{M}$ be a Riesz monoid. Suppose that the set $\mathcal{Q} = \{q_i \mid i \in \omega\}$ consists of anti-idempotent elements from $\boldsymbol{M}$. Then $(x/q_i) < \infty$, for all $i \in \omega$ and all $x \in \boldsymbol{M}$. Therefore f_x is a map $\omega \to \mathbb{R}_+$, for all $x \in \boldsymbol{M}$.*

PROOF. We fix $i \in \omega$ and $x \in \boldsymbol{M}$. Let $\langle \boldsymbol{a}, A \rangle \in \mathbf{\mathcal{S}} \times_{\mathfrak{F}} \mathcal{P}(\omega)$ be such that $\boldsymbol{x} = \boldsymbol{h}(\langle \boldsymbol{a}, A \rangle)$. Pick $y \in \boldsymbol{M}$ such that $\boldsymbol{y} = \boldsymbol{h}(\langle \boldsymbol{a}, A \smallsetminus \{i\} \rangle)$. Then $\boldsymbol{x} \leq \boldsymbol{y} \vee \boldsymbol{q}_i$, hence $x \leq ny + nq_i$, for some positive integer n. We prove that $(x/q_i) \leq 2n$. Suppose otherwise. Then $2nkq_i \leq kx$, for some $k \in \mathbb{N}$. It follows that $2nkq_i \leq kny + knq_i$. Since $\boldsymbol{y} \wedge \boldsymbol{q}_i = \mathbf{0}$, we get from Lemma 3.3(iii) that $2nkq_i \leq knq_i$, which contradicts the assumption that q_i is anti-idempotent. Therefore $(x/q_i) \leq 2n$. □

LEMMA 3.5. *If $\boldsymbol{x} = \boldsymbol{h}(\langle \boldsymbol{a}, A\rangle)$ and $\boldsymbol{y} = \boldsymbol{h}(\langle \boldsymbol{a}, B\rangle)$, then $\widehat{\boldsymbol{f}}_x = \widehat{\boldsymbol{f}}_y$, for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{M}$.*

PROOF. There exists a finite $F \subseteq \omega$ such that $A \cup F = B \cup F$. We pick $z \in \boldsymbol{M}$ satisfying $\boldsymbol{z} = \boldsymbol{h}(\langle \boldsymbol{0}, F\rangle)$. As follows, we get that $\langle \boldsymbol{a}, A\rangle \leq \langle \boldsymbol{a}, B\rangle \vee \langle \boldsymbol{0}, F\rangle$ in $\mathcal{S} \times_{\mathfrak{F}} \mathcal{P}(\omega)$, hence $\boldsymbol{x} \leq \boldsymbol{y} \vee \boldsymbol{z}$, whence $x \leq n(y+z)$ for some $n \in \mathbb{N}$. Since $\boldsymbol{z} \wedge \boldsymbol{q}_i = \boldsymbol{0}$, for all $i \in \omega \smallsetminus F$, we have that

$$f_x(i) \leq f_{n(y+z)}(i) = (ny + nz/q_i) = n(y/q_i)) = nf_y(i),$$

due to Lemma 3.3. It follows that $\widehat{\boldsymbol{f}}_x \leq \widehat{\boldsymbol{f}}_y$. Similarly we prove the opposite inequality $\widehat{\boldsymbol{f}}_y \leq \widehat{\boldsymbol{f}}_x$. $\square$

Lemmas 3.4 and 3.5 entitle us to define a monotone map $\varphi_{\boldsymbol{h},\mathcal{Q}} \colon \mathcal{S} \to \nabla(\widehat{\boldsymbol{R}})$ as follows: Given $x \in \mathcal{S}$, we pick $A \subseteq \omega$ such that $\langle \boldsymbol{a}, A\rangle \in \mathcal{S} \times_{\mathfrak{F}} \mathcal{P}(\omega)$, we put $\boldsymbol{x} := \boldsymbol{h}(\langle \boldsymbol{a}, A\rangle)$, and we define $\varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a}) := \widehat{\boldsymbol{f}}_x$.

LEMMA 3.6. *Let $\boldsymbol{M}$ be a Riesz monoid, $\mathcal{S}$ a distributive $\langle \vee, \boldsymbol{0}\rangle$-semilattice, and $\boldsymbol{h} \colon \mathcal{S} \times_{\mathfrak{F}} \mathcal{P}(\omega) \to \nabla(\boldsymbol{M})$ an isomorphism. Let $\mathcal{Q} := \{q_i \mid i \in \omega\}$ be a set of anti-idempotent elements of $\boldsymbol{M}$ satisfying $\boldsymbol{q}_i = \boldsymbol{h}(\langle \boldsymbol{0}, \{i\}\rangle)$, for all $i \in \omega$. Finally, let $\boldsymbol{a} \in \mathcal{S} \smallsetminus \{\boldsymbol{0}\}$ and $\{\boldsymbol{b}_w \mid w \in W\}$ be an uncountable set of elements of $\mathcal{S} \smallsetminus \{\boldsymbol{0}\}$ such that $\boldsymbol{b}_w \leq \boldsymbol{a}$, for all $w \in W$, and $\boldsymbol{b}_w \wedge \boldsymbol{b}_v = \boldsymbol{0}$, for all v in $W \setminus \{w\}$ (we will call $\{\boldsymbol{b}_w \mid w \in W\}$ a* decomposition under $\boldsymbol{a}$*). Then there exists $w \in W$ with $\varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{b}_w) < \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a})$.*

PROOF. Let x, and y_w, $w \in W$, be elements of $\boldsymbol{M}$ such that $\boldsymbol{x} = \boldsymbol{h}(\langle \boldsymbol{a}, \omega\rangle)$ and $\boldsymbol{y}_w = \boldsymbol{h}(\langle \boldsymbol{b}_w, \omega\rangle)$, for all $w \in W$. Since $\boldsymbol{b}_w \leq \boldsymbol{a}$, we have that $\boldsymbol{y}_w \leq \boldsymbol{x}$, for all $w \in W$, and so there are positive integers m_w, $w \in W$, such that $y_w \leq m_w x$, for all $w \in W$. Since the set W is uncountable, there are a positive integer m and an uncountable $V \subseteq W$ such that $m_v = m$, for all $v \in V$. We can replace the element $\boldsymbol{a}$ with its multiple $m\boldsymbol{a}$. Therefore we can without loss of generality assume that $m = 1$.

Since the map $\varphi_{\boldsymbol{h},\mathcal{Q}}$ is monotone, we have that $\varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{b}_v) \leq \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a})$, for all $v \in V$. Suppose for a contradiction that the equality $\varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{b}_v) = \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a})$ holds for all $v \in V$. Then there are positive integers n_v and finite subsets $F_v \subseteq \omega$, $v \in V$, such that $f_x(j) \leq n_v f_{y_v}(j)$, for all $j \in \omega \smallsetminus F_v$. Since the set V is uncountable, there are $n \in \mathbb{N}$ and an infinite[1] $U \subseteq V$ such that $n_u = n$, for all $u \in U$. Let $u_0, \dots, u_n$ be distinct elements from U. According to [**81**, Lemma 2.3] there are a finite $F \subseteq \omega$ and an element $q_F \in \boldsymbol{M}$ with $\boldsymbol{q}_F = \boldsymbol{h}(\langle \boldsymbol{0}, F\rangle)$ satisfying

$$\sum_{i=0}^{n} y_{u_i} \leq x + q_F.$$

[1] Observe that we can pick U uncountable.

According to Lemma 3.3(ii), we have that

$$\sum_{i=0}^{n}(y_{u_i}/q_j) \leq (\sum_{i=0}^{n} y_{u_i}/q_j),$$

hence

$$\sum_{i=0}^{n} f_{y_{u_i}}(j) \leq f_{x+q_F}(j), \tag{3.1}$$

for all $j \in \omega$. For all $j \in \omega \setminus F$, the equality

$$(x + q_F/q_j) = (x/q_j)$$

holds due to Lemma 3.3(iii), hence we infer from (3.1) that

$$\sum_{i=0}^{n} f_{y_{u_i}}(j) \leq f_x(j). \tag{3.2}$$

Since $(F \cup \bigcup_{i=0}^{n} F_{u_i})$ is finite, we can pick $j \in \omega \setminus (F \cup \bigcup_{i=0}^{n} F_{u_i})$. We conclude from (3.2) that

$$nf_x(j) \geq n\sum_{i=0}^{n} f_{y_{u_i}}(j) = \sum_{i=0}^{n} nf_{y_{u_i}}(j) \geq (n+1)f_x(j),$$

hence $f_x(j) = 0$, whence $(x/v_j) = 0$. This contradicts $\langle \mathbf{0}, \{j\}\rangle \leq \langle \boldsymbol{a}, \omega\rangle$. □

Definition 3.7. Let $\varkappa$ be an infinite cardinal. We define the following properties of a partially ordered set P:

($\mathrm{A}_\varkappa$) Every decreasing sequence of elements of P of length at most $\varkappa$ has a nonzero lower bound.

(B) For every $p \in P$, there exists an uncountable $\{p_w \mid w \in W\} \subseteq P$ such that $p_w < p$ for all $w \in W$ and $p_v \perp p_w$, for all $v \neq w$ in W.

Lemma 3.8. *For every infinite cardinal $\varkappa$, there exists a Boolean algebra $\boldsymbol{B}_\varkappa$ of size $2^\varkappa$ such that $\boldsymbol{B}_\varkappa \smallsetminus \{\mathbf{0}\}$ satisfies both* ($\mathrm{A}_\varkappa$) *and* (B).

Proof. Given an ordinal number α, we denote by ${}^{\alpha}W$ the set of all maps $\alpha \to W$. We set

$$P_\varkappa := \bigcup_{\varkappa \leq \alpha < \varkappa^+} {}^{\alpha}W,$$

and we define an order on the $P_\varkappa$ by reverse inclusion, that is, $f \leq g$, if f is an extension of g, for all $f, g \in P_\varkappa$. It is easy to see that $P_\varkappa$ is a tree of cardinality $2^\varkappa$ satisfying both ($\mathrm{A}_\varkappa$) and (B). We denote by $\boldsymbol{\mathcal{L}}_\varkappa$ the sublattice of $\mathrm{Her}(P_\varkappa)$ generated by $P_\varkappa$. Denote by $\boldsymbol{B}_\varkappa$ the Boolean algebra R-generated by $\boldsymbol{\mathcal{L}}_\varkappa$ [**27**, II.4. Definition 2].[2] Observe that for every $\boldsymbol{a} \ngeq \boldsymbol{b}$ in $\boldsymbol{\mathcal{L}}_\varkappa$, there is $f \in P_\varkappa$ such that $f \leq \boldsymbol{b}$ and $f \wedge \boldsymbol{a} = \mathbf{0}$. By [**27**, II.4. Lemma 3] there are $\boldsymbol{a} < \boldsymbol{b}$ in $L_\varkappa$ such that $\boldsymbol{b} - \boldsymbol{a} \leq \boldsymbol{c}$, for all $\boldsymbol{c} \in \boldsymbol{\mathcal{L}}_\varkappa$. We pick $f \in P_\varkappa$ with $f \leq \boldsymbol{b}$ and $f \wedge \boldsymbol{a} = \mathbf{0}$. It follows that $f \leq \boldsymbol{c}$, and so $P_\varkappa$ is a dense subset of

[2] $\boldsymbol{B}_\varkappa$ is generated by $\boldsymbol{\mathcal{L}}_\varkappa$ as a ring and both $\boldsymbol{B}_\varkappa$ and $\boldsymbol{\mathcal{L}}_\varkappa$ have the same bounds.

$\boldsymbol{B}_{\varkappa}$. We conclude that $\boldsymbol{B}_{\varkappa} \setminus \{\mathbf{0}\}$ satisfies both properties ($\mathrm{A}_{\varkappa}$) and (B). It is straightforward that the cardinality of $\boldsymbol{B}_{\varkappa}$ is $2^{\varkappa}$. $\square$

PROPOSITION 3.9. *Let $\varkappa$ be an infinite cardinal and $\boldsymbol{\mathcal{S}}$ a distributive $\langle\vee, \mathbf{0}\rangle$-semilattice such that the partially ordered set $\boldsymbol{\mathcal{S}} \setminus \{\mathbf{0}\}$ satisfies both* ($\mathrm{A}_{\varkappa}$) *and* (B). *Suppose that there is an isomorphism $\boldsymbol{h}\colon \boldsymbol{\mathcal{S}} \times_{\mathfrak{F}} \mathcal{P}(\omega) \to \nabla(\boldsymbol{M})$, where $\boldsymbol{M}$ a Riesz monoid and there are anti-idempotent elements q_i, $i \in \omega$, in $\boldsymbol{M}$ such that $\boldsymbol{q}_i = \boldsymbol{h}(\langle 0, \{i\}\rangle)$, for all $i \in \omega$. Then the $\langle\vee, \mathbf{0}, \mathbf{1}\rangle$-semilattice $\nabla(\widehat{\boldsymbol{R}})$ contains a strictly decreasing sequence of length $\varkappa^+$.*

PROOF. By transfinite induction up to $\varkappa^+$, we define a sequence $\langle \boldsymbol{a}_\alpha \mid \alpha < \varkappa^+\rangle$ of elements of $\boldsymbol{\mathcal{S}} \setminus \{\mathbf{0}\}$ inducing a strictly decreasing sequence $\langle \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a}_\alpha) \mid \alpha < \varkappa^+\rangle$ of elements of $\nabla(\widehat{\boldsymbol{R}})$. We pick $\boldsymbol{a}_0$ to be any nonzero element of $\boldsymbol{\mathcal{S}}$. Suppose now that the sequence $\langle \boldsymbol{a}_\alpha \mid \alpha \leq \beta\rangle$ is defined for an ordinal $\beta < \varkappa^+$. Since the poset $\boldsymbol{\mathcal{S}} \setminus \{\mathbf{0}\}$ satisfies property (B), there is a decomposition $\{\boldsymbol{b}_w \mid w \in W\}$ under $\boldsymbol{a}_\beta$. It foolows from Lemma 3.6 that $\varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a}_\beta) > \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{b}_w)$ for some $w \in W$, and therefore we can define $\boldsymbol{a}_{\beta+1} := \boldsymbol{b}_w$. Let $\beta < \varkappa^+$ be a limit ordinal and suppose that we have already defined a sequence $\langle \boldsymbol{a}_\alpha \mid \alpha < \beta\rangle$ such that $\langle \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a}_\alpha) \mid \alpha < \beta\rangle$ is strictly decreasing. According to ($\mathrm{A}_{\varkappa}$) there is a lower bound $\boldsymbol{a}_\beta$ of $\{\boldsymbol{a}_\alpha \mid \alpha < \beta\}$ in $\boldsymbol{\mathcal{S}} \setminus \{\mathbf{0}\}$. Since the map $\varphi_{\boldsymbol{h},\mathcal{Q}}$ is monotone, we conclude that $\varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a}_\alpha) > \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a}_{\alpha+1}) \geq \varphi_{\boldsymbol{h},\mathcal{Q}}(\boldsymbol{a}_\beta)$, for all $\alpha < \beta < \varkappa^+$. $\square$

We denote by λ the supremum of the lengths of all strictly decreasing sequences in $\nabla(\widehat{\boldsymbol{R}})$.

THEOREM 3.10. *There is a directed union $\boldsymbol{\mathcal{D}}$ of a countable chain of Boolean join-semilattices* (*with* $(\vee, 0, 1)$*-preserving inclusion maps*), *of cardinality 2^λ, which is not isomorphic to $\nabla(\boldsymbol{M})$ for any Riesz monoid $\boldsymbol{M}$ in which every nonzero element is anti-idempotent.*

PROOF. We set $\boldsymbol{\mathcal{D}} := \boldsymbol{B}_\lambda \times_{\mathfrak{F}} \mathcal{P}(\omega)$. As we have discussed in Remark 3.2, the $\langle\vee, \mathbf{0}, \mathbf{1}\rangle$-semilattice $\boldsymbol{\mathcal{D}}$ is the direct limit of a countable chain of Boolean lattices and one-to-one $(\vee, 0, 1)$-preserving transition maps. It follows from Lemma 3.8 that the poset $\boldsymbol{B}_\lambda \setminus \{\mathbf{0}\}$ satisfies both (A_λ) and (B). Since $\boldsymbol{M}$ is a Riesz monoid in which every nonzero element is anti-idempotent, the assertion follows from Proposition 3.9. It is straightforward that $|\boldsymbol{\mathcal{D}}| = |\boldsymbol{B}_\lambda \times_{\mathfrak{F}} \mathcal{P}(\omega)| = 2^\lambda$. $\square$

REMARK 3.11. This result contrasts with the answer to the analogue of [**81**, Problem 1] for $\langle\vee, \mathbf{0}\rangle$-semilattice of compact congruences of lattices: Every direct limit of a countable sequence of distributive lattices with zero and $\langle\vee, \mathbf{0}\rangle$-homomorphisms is isomorphic to the semilattice $\mathrm{Con_c}\,\boldsymbol{\mathcal{L}}$ of all compact congruences of some relatively complemented lattice $\boldsymbol{\mathcal{L}}$ with zero [**80**, Corollary 21.3].

THEOREM 3.12. *There is a union $\boldsymbol{\mathcal{H}}$ of a countable chain of strongly distributive $\langle\vee, \mathbf{0}, \mathbf{1}\rangle$-semilattices* (*with $\langle\vee, \mathbf{0}, \mathbf{1}\rangle$-preserving inclusion maps*) *which is not isomorphic to the maximal semilattice quotient of any Riesz monoid in which every nonzero element is anti-idempotent.*

PROOF. Let $\mathcal{H}'$ denote the $\langle\vee,\mathbf{0}\rangle$-semilattice of all compact elements of $\operatorname{Her}(P_\lambda)$. Similarly as in the proof of Theorem 3.10, we set $\mathcal{H} = \mathcal{H}' \times_{\mathfrak{F}} \mathcal{P}(\omega)$. The $\langle\vee,\mathbf{0}\rangle$-semilattice $\mathcal{H}$ is a direct limit of a countable chain of strongly distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattices and one-to-one $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-preserving transition maps due to Remark 3.2. We argue as in the proof of Theorem 3.10. $\square$

Recall that a monoid $\boldsymbol{M}$ is said to be *conical* provided that $x+y=0$ if and only if $x=y=0$, for all $x,y\in\boldsymbol{M}$. Since $2ne+x=ne$ implies that $2(ne+x)=ne+x$, for all $x\in\boldsymbol{M}$, the conical monoids without nonzero idempotent elements are exactly conical monoids with all elements anti-idempotent. Notice that the positive cone of a dimension group forms a conical monoid without nonzero idempotent elements satisfying the Riesz refinement property.

COROLLARY 3.13. *There is the union of a countable chain of Boolean join-semilattices, respectively, the union of a countable chain of strongly distributive $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattices, with $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-preserving inclusion maps, not isomorphic to $\nabla(\boldsymbol{M})$ for any conical Riesz monoid $\boldsymbol{M}$ without nonzero-idempotent elements. In particular, it is not isomorphic to the maximal semilattice quotient of the positive cone, $\nabla(\boldsymbol{G}^+)$, for any dimension group $\boldsymbol{G}$.*

Recall (e.g. from [**81**]) that a monoid $\boldsymbol{M}$ is *strongly separative* provided that $x+y=2y$ implies that $x=y$ for all $x,y\in\boldsymbol{M}$. An element e of a monoid $\boldsymbol{M}$ has *finite stable rank* if there is a positive integer k such that $ke+x\le e+y$ implies that $x\le y$, for all $x,y\in\boldsymbol{M}$. Observe that every element of a strongly separative monoid has finite stable rank. In a conical monoid, every nonzero idempotent element has infinite stable rank. Therefore, we can replace the assumption that the monoid $\boldsymbol{M}$ has no nonzero idempotent elements by any of the following statements:

- every element of $\boldsymbol{M}$ has finite stable rank,
- the monoid $\boldsymbol{M}$ is strongly separative

(compare to [**81**, Corollary 5.3]). We derive from Corollary 3.13 similar consequences to the ones obtained from [**81**, Corollary 5.3] in [**81**, Section 6]. In particular, neither the $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice $\mathcal{D}$ nor the $\langle\vee,\mathbf{0},\mathbf{1}\rangle$-semilattice $\mathcal{H}$ (defined in Theorem 3.10 and Theorem 3.12, respectively) is isomorphic to the join-semilattice of all finitely generated ideals of a strongly separative von Neumann regular ring or the join-semilattice $\operatorname{Con_c}\boldsymbol{\mathcal{L}}$ of all compact congruences of a modular lattice $\boldsymbol{\mathcal{L}}$ of locally finite length.

REMARK 3.14. Observe that every element $\widehat{\boldsymbol{f}}\in\nabla(\widehat{\boldsymbol{R}})$ is represented by a map with rational values. It follows that the cardinality of $\nabla(\widehat{\boldsymbol{R}})$ is $2^{\aleph_0}$, and so we have the estimate $\aleph_1\le\lambda\le 2^{\aleph_0}$. Of course, if $2^{\aleph_0}=\aleph_1$ and $2^{\aleph_1}=\aleph_2$, then $2^\lambda=\aleph_2$. On the other hand, $\aleph_2<2^{\aleph_1}$ implies that $\aleph_2<2^\lambda$.

CHAPTER 5

Construction and realization of some wild refinement monoids

1. Introduction

The commutative monoid $\boldsymbol{V}(\boldsymbol{R})$, assigned to a unital associative ring $\boldsymbol{R}$, consists of all isomorphism classes of finitely generated projective right $\boldsymbol{R}$-modules, with the operation induced from direct sums. Alternatively, the monoid $\boldsymbol{V}(\boldsymbol{R})$ is defined as *Murray-von Neumann equivalence* classes of idempotent $\omega \times \omega$-matrices with finitely many nonzero entries over $\boldsymbol{R}$.

For a von Neumann regular ring $\boldsymbol{R}$, the monoid $\boldsymbol{V}(\boldsymbol{R})$ faithfully reflects the structure of the ring. Not surprisingly, many of direct sum decomposition problems of von Neumann regular rings have reformulation in terms of the corresponding refinement monoids. Let us mention the separativity problem whether there are non-isomorphic finitely generated projective right $\boldsymbol{R}$-modules $\boldsymbol{M}$, $\boldsymbol{N}$ such that $\boldsymbol{M} \oplus \boldsymbol{M} \simeq \boldsymbol{M} \oplus \boldsymbol{N} \simeq \boldsymbol{N} \oplus \boldsymbol{N}$ as a prominent example (cf. [**22**, Problem 1]).

If $\boldsymbol{R}$ is a von Neumann regular ring or a C^*-algebra with real rank zero, then the monoid $\boldsymbol{V}(\boldsymbol{R})$ satisfies the *Riesz refinement property. The realization problem* [**23**] asks which refienement monoids are realized as $\boldsymbol{V}(\boldsymbol{R})$ of von Neumann regular rings. As shows an example of F. Wehrung [**77**, Corollary 2.12], not all of them. But the size of the Wehrung's counter-example is $\geq \aleph_2$, which leaves the realization problem open for refinement monoids of smaller cardinalities. The countable case is particularly important for the direct sum decomposition problems of von Neumann rings are usually reduced to realization problems of certain countable refinement monoids.

There are classes of refinement monoids for which the realization problem has a positive solution. The monoids $\boldsymbol{M}(E)$ associated to row-finite directed graphs (cf. [**7**]) are realized functorially in [**4**]. The method used in [**4**] is extended in [**3**], where finitely generated primitive monoids are realized. We refer to [**2**] for a survey on this result.

The refinement monoids obtained by these canonical constructions have a common feature; they are direct limits of finitely generated refinement monoids. Such refinement monoids are called *tame* in [**5**]. The remaining ones are *wild*. Two examples of wild monoids $\boldsymbol{M}$ and $\overline{\boldsymbol{M}}$ are studied in detail in [**5**] and realized in [**6**].

The refinement monoid $\boldsymbol{M}$ is non-cancellative but admits faithful state, consequently, it cannot be realized as $\boldsymbol{V}(\boldsymbol{R})$ for any von Neumann regular algebra over an uncountable field [**3**, Proposition 4.1]. Surprisingly, $\boldsymbol{M}$ is realized by an exchange algebra over any field with involution [**6**, Theorem 4.10] as well as a regular algebra over a countable field [**6**, Theorem 5.5]. Note that the first such example goes back to [**14**].

The monoid $\overline{\boldsymbol{M}}$ is a factor of $\boldsymbol{M}$ by an o-ideal and it is isomorphic to $\boldsymbol{V}(\boldsymbol{S})$ for a regular algebra $\boldsymbol{S}$ invented by Bergman and Goodearl [**22**, Example 5.10]. It is in some sense a canonical example of a wild monoid. the modification of this construction was used by Moncasi who constructed a directly finite regular Hermite ring such that $K_0(R)$ is not a Riesz group

[**51**], in particular, the monoid $\boldsymbol{V}(\boldsymbol{R})$ does not satisfy the Riesz interpolation property. Modifications of the Bergman-Goodearl construction play a crucial role also in this paper.

The chapter consists of three parts. Firstly, we develop quite elementary but useful methods of computing the monoid $\boldsymbol{V}(\boldsymbol{R})$ for a regular ring $\boldsymbol{R}$. We define a *partial H-map* from a hereditary subset H of a monoid and we understand when the partial H-map map uniquely extends to a monoid isomorphism. This idea leads to Lemma 3.5 that allows us to compute the monoid $\boldsymbol{V}(\boldsymbol{R})$ of a regular ring $\boldsymbol{R}$ knowing the structure of the partial monoid of its finitely generated right ideals. We refine Lemma 3.5 in Corollary 3.9, which is designed to compute $\boldsymbol{V}(\boldsymbol{R})$ of directly finite regular rings $\boldsymbol{R}$; in this case it suffices to describe the ordered set of traces of idempotents of the ring $\boldsymbol{R}$.

In the second part of the chapter, consisting of Sections 4 and 5, we construct a class of directly finite non-cancellative refinement monoids. In Sections 4 we aim to construct a class of refinement monoids rich enough to provide interesting examples with potential of further applications. In Section 5 we restrict ourselves to particular refinement monoids $\boldsymbol{B}_{2n}$, $n \in \mathbb{N}$, obtained by the previous construction. We prove that the monoids $\boldsymbol{B}_{2n}$ for $n \geq 2$, do not satisfy the Riesz interpolation property.

The remaining Sections 6-8 are devoted to construction of regular rings $\boldsymbol{R}_{2n}$ and the proof that $\boldsymbol{V}(\boldsymbol{R}_{2n}) \simeq \boldsymbol{B}_{2n}$, for all positive integers n. The auxiliary Section 6 is elementary linear algebra. In Section 7 we recall the Goodearl's modification [**22**, Example 5.10] of the Bergman's example, denoted by $\boldsymbol{R}_2$, and we prove that $\boldsymbol{B}_2 \simeq \boldsymbol{V}(\boldsymbol{R}_2)$. In the final Section 8, we generalize the constructions of Bergman and Goodearl. This results in rings $\boldsymbol{R}_{2n}$ such that $\boldsymbol{V}(\boldsymbol{R}_{2n}) \simeq \boldsymbol{B}_{2n}$.

2. Preliminaries

A subset H of a monoid $\boldsymbol{M}$ is called *hereditary* provided that $y \in H$ and $x \leq_{\boldsymbol{M}} y$ implies that $x \in H$, for all $x, y \in \boldsymbol{M}$. Given a subset X of the monoid $\boldsymbol{M}$, we set

$$\downarrow_{\boldsymbol{M}}(X) := \{x \in \boldsymbol{M} \mid \exists y \in X \colon x \leq_{\boldsymbol{M}} y\}.$$

Thus $\downarrow_{\boldsymbol{M}}(X)$ is the least hereditary subset of $\boldsymbol{M}$ containing X. A hereditary submonoid of the monoid $\boldsymbol{M}$ will be called an *o-ideal* of $\boldsymbol{M}$. We will denote by $\mathcal{O}(X)_{\boldsymbol{M}}$ the least o-ideal of $\boldsymbol{M}$ containing the set X, i.e.,

$$\mathcal{O}(X)_{\boldsymbol{M}} := \{x \in \boldsymbol{M} \mid \exists y_1, \dots, y_n \in X \colon x \leq_{\boldsymbol{M}} y_1 + \cdots + y_n\}.$$

When $X = \{x\}$ is a singleton set, we will write shortly $\downarrow_{\boldsymbol{M}}(x)$ and $\mathcal{O}(x)_{\boldsymbol{M}}$. An element $\boldsymbol{u} \in \boldsymbol{M}$ is an *order unit* of $\boldsymbol{M}$ provided that $\mathcal{O}(\boldsymbol{u})_{\boldsymbol{M}} = \boldsymbol{M}$; equivalently, there is a positive integer λ such that $x \leq_{\boldsymbol{M}} \lambda\boldsymbol{u}$, for each $x \in \boldsymbol{M}$.

3. Partial H-maps and their applications

Let $\boldsymbol{M}$, $\boldsymbol{N}$ be monoids and H a hereditary subset of $\boldsymbol{M}$. A *partial H-map* is a one-to-one map $\alpha\colon H \to \boldsymbol{N}$ such that for all $z \in H$ and all $u, v \in \boldsymbol{N}$, the equality $\alpha(z) = u + v$ holds true if and only if there are (necessarily unique) $x, y \in H$ such that $\alpha(x) = u$, $\alpha(y) = v$ and $x + y = z$.

By induction we readily prove that if $\alpha\colon H \to \boldsymbol{N}$ is a partial H-map, then for all $x \in \boldsymbol{M}$, all $n \in \mathbb{N}$ and all $u_1, \dots, u_n \in \boldsymbol{N}$: $\alpha(x) = u_1 + \cdots + u_n$ if and only if $x = x_1 + \cdots + x_n$ for (necessarily unique) $x_i \in H$, $i \in \{1, 2, \dots, n\}$, such that $u_i = \alpha(x_i)$ for all $i = \{1, 2, \dots, n\}$.

Lemma 3.1. *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be monoids and let H be a hereditary subset of $\boldsymbol{M}$. If $\alpha\colon H \to \boldsymbol{N}$ is a partial H-map then for all $x, y, z \in H$:*

$$z = x + y \iff \alpha(z) = \alpha(x) + \alpha(y).$$

Proof. If $z = x+y$, then $\alpha(z) = \alpha(x)+\alpha(y)$ readily by the definition of a partial H-map. Conversely, the equality $\alpha(z) = \alpha(x) + \alpha(y)$ implies that there are $x', y' \in H$ such that $z = x' + y'$, $\alpha(x) = \alpha(x')$ and $\alpha(y) = \alpha(y')$. Since a partial H-map is by definition one-to-one, we conclude that $x = x'$ and $y = y'$. $\square$

Keeping the setting of Lemma 3.1, we get by induction that for every $n \in \mathbb{N}$ and all $x, y_1, \dots, y_n \in H$:

$$x = \sum_{i=1}^{n} y_i \iff \alpha(x) = \sum_{i=1}^{n} \alpha(y_i). \tag{3.1}$$

Lemma 3.2. *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be refinement monoids and H a hereditary subset of $\boldsymbol{M}$. Then every partial H-map $\alpha\colon H \to \boldsymbol{N}$ extends to a unique isomorphism $\beta\colon \mathcal{O}(H)_{\boldsymbol{M}} \to \mathcal{O}(\alpha(H))_{\boldsymbol{N}}$.*

Proof. By the definition, for every $x \in \mathcal{O}(H)_{\boldsymbol{M}}$ there are $n \in \mathbb{N}$ and $y_1, \dots, y_n \in H$ with $x \leq_{\boldsymbol{M}} y_1 + \cdots + y_n$. Since $\boldsymbol{M}$ is a refinement monoid, there are $x_i \leq_{\boldsymbol{M}} y_i$, $i = 1, \dots, n$, such that $x = x_1 + \cdots + x_n$. We define a map $\beta\colon \mathcal{O}(H)_{\boldsymbol{M}} \to \boldsymbol{N}$ by $x \mapsto \alpha(x_1) + \cdots + \alpha(x_n)$.

Claim 2. *The map β is a well-defined monoid homomorphism.*

Proof of Claim 2. Let $x_1 + \cdots + x_m = y_1 + \cdots + y_n$ for some $m, n \in \mathbb{N}$ and $x_1, \dots, x_m, y_1, \dots, y_n \in H$. Since $\boldsymbol{M}$ is a refinement monoid, there are $z_{ij} \in H$ such that $x_i = \sum_{j=1}^{n} z_{ij}$ for all $i \leq m$ and $y_j = \sum_{i=1}^{m} z_{ij}$ for all $j \leq n$. By (3.1) we have that $\alpha(x_i) = \sum_{j=1}^{n} \alpha(z_{ij})$ for all $i \leq m$ and $\alpha(y_j) = \sum_{i=1}^{m} \alpha(z_{ij})$ for all $j \leq n$. It follows that

$$\sum_{i=1}^{m} \alpha(x_i) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha(z_{ij}) = \sum_{j=1}^{n} \sum_{i=1}^{m} \alpha(z_{ij}) = \sum_{j=1}^{n} \alpha(y_j).$$

Thus the map $\beta\colon \mathcal{O}(H)_{\boldsymbol{M}} \to N$ is well-defined. It is straightforward that it is a monoid homomorphism. $\square$ Claim 2.

CLAIM 3. *The homomorphism β is one-to-one.*

PROOF OF CLAIM 3. Suppose that $\beta(x) = \beta(y)$ for some $x, y \in \mathcal{O}(H)_{\boldsymbol{M}}$. By the definition, there are $m, n \in \mathbb{N}$ and $x'_1, \dots, x'_m, y'_1, \dots, y'_n \in H$ such that $x \leq_{\boldsymbol{M}} x'_1 + \cdots + x'_m$ and $y \leq_{\boldsymbol{M}} y'_1 + \cdots + y'_n$. Since $\boldsymbol{M}$ is a refinement monoid, there are $x_i \leq_{\boldsymbol{M}} x'_i$, $i = 1, \dots, m$, and $y_j \leq_{\boldsymbol{M}} y'_j$, $j = 1, \dots, n$, in H such that $x = x_1 + \cdots + x_m$ and $y = y_1 + \cdots + y_n$. Since $\beta(x) = \beta(y)$, we get that $\sum_{i=1}^m \alpha(x_i) = \sum_{j=1}^n \alpha(y_j)$. Since $\boldsymbol{N}$ is a refinement monoid, there are $w_{i,j} \in \boldsymbol{N}$ such that $\alpha(x_i) = \sum_{j=1}^n w_{i,j}$, for all $i = 1, \dots, m$, and $\alpha(y_j) = \sum_{i=1}^m w_{i,j}$, for all $j = 1, \dots, n$. Since α is a partial H-map, there are elements $z_{i,j} \in H$ such that

$$w_{i,j} = \alpha(z_{i,j}), \text{ for all } i \in \{1, 2, \dots, m\},\ j \in \{1, 2, \dots, n\}, \tag{3.2}$$

and

$$x_i = \sum_{j=1}^n z_{i,j}, \text{ for all } i \in \{1, 2, \dots, m\}.$$

Applying that α is a partial H-map again, we infer that there are elements $z'_{i,j} \in H$ such that

$$w_{i,j} = \alpha(z'_{i,j}), \text{ for all } i \in \{1, 2, \dots, m\},\ j \in \{1, 2, \dots, n\}, \tag{3.3}$$

and

$$y_j = \sum_{i=1}^m z'_{i,j}, \text{ for all } j \in \{1, 2, \dots, n\}.$$

Since the map $\alpha \colon H \to \boldsymbol{N}$ is by definition one-to-one, we get from (3.2) and (3.3) that $z_{i,j} = z'_{i,j}$ for all $i \in \{1, 2, \dots, m\}$ and $j \in \{1, 2, \dots, n\}$. It follows that

$$x = \sum_{i=1}^m x_i = \sum_{i=1}^m \sum_{j=1}^n z_{i,j} = \sum_{j=1}^n \sum_{i=1}^m z'_{i,j} = \sum_{j=1}^n y_j = y.$$

This proves that β is one-to-one. □ Claim 3.

CLAIM 4. *The equality $\beta(\mathcal{O}(H)_{\boldsymbol{M}}) = \mathcal{O}(\alpha(H))_{\boldsymbol{N}}$ holds true.*

PROOF OF CLAIM 4. As we have shown above, each $x \in \mathcal{O}(H)_{\boldsymbol{M}}$ is a sum of elements from H. It follows that $\beta(\mathcal{O}(H)_{\boldsymbol{M}}) \subseteq \mathcal{O}(\alpha(H))_{\boldsymbol{N}}$. It is straightforward to see from the definition of a partial H-map, that the image $\alpha(H)$ is a hereditary subset of $\boldsymbol{N}$. Since $\boldsymbol{N}$ is a refinement monoid, each element of $\mathcal{O}(\alpha(H))_{\boldsymbol{N}}$ is a sum of elements of $\downarrow_{\boldsymbol{N}} (\alpha(H))$. Therefore $\mathcal{O}(\alpha(H))_{\boldsymbol{N}}$ is a submonoid of $\boldsymbol{N}$ generated by $\alpha(H)$. From $H \subseteq \mathcal{O}(H)_{\boldsymbol{M}}$ we infer that $\alpha(H) \subseteq \beta(\mathcal{O}(H)_{\boldsymbol{M}})$. Since $\beta(\mathcal{O}(H)_{\boldsymbol{M}})$ is a submonid of $\boldsymbol{N}$, we conclude that $\mathcal{O}(\alpha(H))_{\boldsymbol{N}} \subseteq \beta(\mathcal{O}(H)_{\boldsymbol{M}})$. □ Claim 4.

The three claims prove the lemma. □

COROLLARY 3.3. *Let $\boldsymbol{M}$, $\boldsymbol{N}$ be refinement monoids, $\boldsymbol{u} \in \boldsymbol{M}$, and $\alpha \colon \downarrow_{\boldsymbol{M}} (\boldsymbol{u}) \to \boldsymbol{N}$ a partial $\downarrow_{\boldsymbol{M}} (\boldsymbol{u})$-map. If $\boldsymbol{u}$ is an order unit in $\boldsymbol{M}$ and $\alpha(\boldsymbol{u})$ is an order unit in $\boldsymbol{N}$, then α extends to a unique isomorphism $\beta \colon \boldsymbol{M} \to \boldsymbol{N}$.*

Let $\boldsymbol{R}$ be a ring. Given a finitely generated right $\boldsymbol{R}$-module $\boldsymbol{A}$, we denote by $[\boldsymbol{A}]$ the isomorphism class of the module $\boldsymbol{A}$, and by $\boldsymbol{V}(\boldsymbol{R})$ the monoid of all isomorphism classes of finitely generated projective right $\boldsymbol{R}$-modules with the operation of addition defined by

$$[\boldsymbol{A}] + [\boldsymbol{B}] = [\boldsymbol{A} \oplus \boldsymbol{B}],$$

for all $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{FP}(\boldsymbol{R})$. As above, we will use $\leq_{\boldsymbol{V}(\boldsymbol{R})}$ to denote the algebraic preorder on $\boldsymbol{V}(\boldsymbol{R})$ and $\equiv_{\boldsymbol{V}(\boldsymbol{R})}$ to denote the corresponding equivalence relation. If the ring $\boldsymbol{R}$ is regular, then $\boldsymbol{V}(\boldsymbol{R})$ is a refinement monoid due to [**22**, Theorem 2.8].

Lemma 3.4. *Let $\boldsymbol{R}$ be a ring and $\boldsymbol{A}$, $\boldsymbol{B}$ finitely generated right $\boldsymbol{R}$-modules. Then $[\boldsymbol{A}] + [\boldsymbol{B}] \leq_{\boldsymbol{V}(\boldsymbol{R})} [\boldsymbol{R}]$ if and only if there are orthogonal idempotents $e, f \in \boldsymbol{R}$ such that $[e\boldsymbol{R}] = [\boldsymbol{A}]$ and $[f\boldsymbol{R}] = [\boldsymbol{B}]$.*

Proof. ($\Leftarrow$) Let e, f be orthogonal idempotents such that $e\boldsymbol{R} \simeq \boldsymbol{A}$ and $f\boldsymbol{R} \simeq \boldsymbol{B}$. Since the idempotents e and f are orthogonal, $\boldsymbol{R} = e\boldsymbol{R} \oplus f\boldsymbol{R} \oplus (1-e-f)\boldsymbol{R}$. Therefore $\boldsymbol{A} \oplus \boldsymbol{B} \lesssim^{\oplus} \boldsymbol{R}$, hence $[\boldsymbol{A}] + [\boldsymbol{B}] \leq_{\boldsymbol{V}(\boldsymbol{R})} [\boldsymbol{R}]$. ($\Rightarrow$) By the assumption $[\boldsymbol{A}] + [\boldsymbol{B}] \leq_{\boldsymbol{V}(\boldsymbol{R})} [\boldsymbol{R}]$, hence $\boldsymbol{A} \oplus \boldsymbol{B} \lesssim^{\oplus} \boldsymbol{R}$. It follows that $\boldsymbol{R} = \boldsymbol{A}' \oplus \boldsymbol{B}' \oplus \boldsymbol{C}$ for some $\boldsymbol{A}' \simeq \boldsymbol{A}$ and $\boldsymbol{B}' \simeq \boldsymbol{B}$. The projection $\boldsymbol{R} \to \boldsymbol{A}'$ with the kernel $\boldsymbol{B}' \oplus \boldsymbol{C}$ corresponds to a left multiplication by an idempotent, say e. Similarly, the projection $\boldsymbol{R} \to \boldsymbol{B}'$ with the kernel $\boldsymbol{A}' \oplus \boldsymbol{C}$ corresponds to a left multiplication by an idempotent, say f. As the composition of these projections, in whatever order, is the zero endomorphism, the idempotents e and f are orthogonal. Clearly $e\boldsymbol{R} = \boldsymbol{A}' \simeq \boldsymbol{A}$ and $f\boldsymbol{R} = \boldsymbol{B}' \simeq \boldsymbol{B}$, hence $[e\boldsymbol{R}] = [\boldsymbol{A}]$ and $[f\boldsymbol{R}] = [\boldsymbol{B}]$. ☐

Lemma 3.5. *Let $\boldsymbol{R}$ be a regular ring, $\boldsymbol{N}$ a refinement monoid, and $\gamma\colon \mathrm{Idem}(\boldsymbol{R}) \to \boldsymbol{N}$ a map satisfying:*

(i) $\gamma(e) = \gamma(f) \iff [e\boldsymbol{R}] = [f\boldsymbol{R}]$*, for all* $e, f \in \mathrm{Idem}(\boldsymbol{R})$.

(ii) *The equality $x + y = \gamma(g)$ holds true for some $x, y \in \boldsymbol{N}$ and $g \in \mathrm{Idem}(\boldsymbol{R})$ if and only if there are orthogonal idempotents $e, f \in \boldsymbol{R}$ such that $\gamma(e) = x$, $\gamma(f) = y$, and $e + f = g$.*

(iii) *$\gamma(1)$ is an order unit in $\boldsymbol{N}$.*

Then the map $\alpha\colon \{[e\boldsymbol{R}] \mid e \in \mathrm{Idem}(\boldsymbol{R})\} \to \boldsymbol{N}$ given by the correspondence $[e\boldsymbol{R}] \mapsto \gamma(e)$ extends to a (unique) isomorphism $\beta\colon \boldsymbol{V}(\boldsymbol{R}) \to \boldsymbol{N}$.

Proof. Put $\boldsymbol{M} := \{[e\boldsymbol{R}] \mid e \in \mathrm{Idem}(\boldsymbol{R})\}$. It follows from (1) that the $\alpha\colon \boldsymbol{M} \to \boldsymbol{N}$ given by $[e\boldsymbol{R}] \mapsto \gamma(e)$ is a well-defined one-to-one map. In view of Lemma 3.4 property (2) implies that α is a partial $\downarrow_{\boldsymbol{V}([\boldsymbol{R}])}([\boldsymbol{R}])$-map. Property (3) says that $\alpha([R])$ is an order unit in $\boldsymbol{N}$ and since $[R]$ is clearly an order unit in $\boldsymbol{V}([R])$, the map α extends to a (unique) isomorphism $\beta\colon \boldsymbol{V}(\boldsymbol{R}) \to \boldsymbol{N}$ due to Corollary 3.3. ☐

We set

$$\mathrm{tr}_{\boldsymbol{R}}(b) := \{abc \mid b, c \in \boldsymbol{R}\} = \bigcup_{a \in \boldsymbol{R}} ab\boldsymbol{R} = \bigcup_{c \in \boldsymbol{R}} \boldsymbol{R}ac,$$

for every $b \in \boldsymbol{R}$.

LEMMA 3.6. *Let e and f be idempotents of a ring $\boldsymbol{R}$. Then*

$$[e\boldsymbol{R}] \leq [f\boldsymbol{R}] \iff \operatorname{tr}_{\boldsymbol{R}}(e) \subseteq \operatorname{tr}_{\boldsymbol{R}}(f). \tag{3.4}$$

PROOF. ($\Rightarrow$) Suppose that $[e\boldsymbol{R}] \leq [f\boldsymbol{R}]$. Then $e\boldsymbol{R} \lesssim^{\oplus} f\boldsymbol{R}$, by the definition. It follows that there is a surjective homomorphism $\varphi\colon f\boldsymbol{R} \to e\boldsymbol{R}$. Since f is an idempotent, φ extends to a homomorphism $\Phi\colon \boldsymbol{R} \to e\boldsymbol{R}$. The homomorphism Φ corresponds to a left multiplication by an element $a = \Phi(1) \in \boldsymbol{R}$. It follows that $e\boldsymbol{R} = af\boldsymbol{R}$, and consequently $\operatorname{tr}_{\boldsymbol{R}}(e) \subseteq \operatorname{tr}_{\boldsymbol{R}}(f)$.

($\Leftarrow$) If $\operatorname{tr}_{\boldsymbol{R}}(e) \subseteq \operatorname{tr}_{\boldsymbol{R}}(f)$, then $e \in af\boldsymbol{R}$ for some $a \in \boldsymbol{R}$. Since e is an idempotent, the left multiplication by ea determines a surjective map $f\boldsymbol{R} \to e\boldsymbol{R}$. Since $e\boldsymbol{R}$ is a projective right $\boldsymbol{R}$-module, we infer that $e\boldsymbol{R} \lesssim^{\oplus} f\boldsymbol{R}$. Therefore $[e\boldsymbol{R}] \leq [f\boldsymbol{R}]$. □

A right $\boldsymbol{R}$-module $\boldsymbol{A}$ is *directly finite* provided that $\boldsymbol{A} \simeq \boldsymbol{A} \oplus \boldsymbol{B}$ implies that $\boldsymbol{B} = \boldsymbol{0}$ for all right $\boldsymbol{R}$-modules $\boldsymbol{B}$, i.e, the module $\boldsymbol{A}$ it is not isomorphic to any of its proper direct summands [**22**, page 49]. A ring $\boldsymbol{R}$ is *directly finite* if it is directly finite as a right $\boldsymbol{R}$-module. Note that this notion is left-right symmetric as a ring $\boldsymbol{R}$ is directly finite if and only if $ab = 1$ implies $ba = 1$ for all $a, b \in \boldsymbol{R}$ (cf. [**22**, Lemma 5.1]).

LEMMA 3.7. *If a ring $\boldsymbol{R}$ is directly finite then*

$$[e\boldsymbol{R}] \equiv_{\boldsymbol{V}(\boldsymbol{R})} [f\boldsymbol{R}] \implies [e\boldsymbol{R}] = [f\boldsymbol{R}],$$

for all $e, f \in \operatorname{Idem}(\boldsymbol{R})$.

PROOF. Let $e, f \in \operatorname{Idem}(\boldsymbol{R})$ and suppose that $[e\boldsymbol{R}] \equiv_{\boldsymbol{V}(\boldsymbol{R})} [f\boldsymbol{R}]$. Then there are $\boldsymbol{A}, \boldsymbol{B} \in \operatorname{FP}(\boldsymbol{R})$ such that $[f\boldsymbol{R}] = [e\boldsymbol{R}] + [\boldsymbol{A}]$ and $[e\boldsymbol{R}] = [f\boldsymbol{R}] + [\boldsymbol{B}]$, i.e., $f\boldsymbol{R} \simeq e\boldsymbol{R} \oplus \boldsymbol{A}$ and $e\boldsymbol{R} \simeq f\boldsymbol{R} \oplus \boldsymbol{B}$. It follows that

$$e\boldsymbol{R} = f\boldsymbol{R} \oplus \boldsymbol{B} \simeq e\boldsymbol{R} \oplus \boldsymbol{A} \oplus \boldsymbol{B},$$

hence

$$\boldsymbol{R} = (1-e)\boldsymbol{R} \oplus e\boldsymbol{R} \simeq (1-e)\boldsymbol{R} \oplus e\boldsymbol{R} \oplus \boldsymbol{A} \oplus \boldsymbol{B} = \boldsymbol{R} \oplus \boldsymbol{A} \oplus \boldsymbol{B}.$$

Since the ring $\boldsymbol{R}$ is directly finite, we conclude that $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{0}$, hence $e\boldsymbol{R} \simeq f\boldsymbol{R}$, whence $[e\boldsymbol{R}] = [f\boldsymbol{R}]$ □

Applying Lemma 3.6 we get that

COROLLARY 3.8. *Let $\boldsymbol{R}$ be a directly finite ring. Then*

$$[e\boldsymbol{R}] = [f\boldsymbol{R}] \iff \operatorname{tr}_{\boldsymbol{R}}(e) = \operatorname{tr}_{\boldsymbol{R}}(f),$$

for all $e, f \in \operatorname{Idem}(\boldsymbol{R})$.

Combining Lemma 3.5 and Corollary 3.8 we conclude with

COROLLARY 3.9. *Let $\boldsymbol{R}$ be a directly finite regular ring, let $\boldsymbol{N}$ be a refinement monoid, and let $\gamma\colon \operatorname{Idem}(\boldsymbol{R}) \to \boldsymbol{N}$ be a map satisfying:*

(i) $\gamma(e) = \gamma(f) \iff \operatorname{tr}_{\boldsymbol{R}}(e) = \operatorname{tr}_{\boldsymbol{R}}(f)$, *for all $e, f \in \operatorname{Idem}(\boldsymbol{R})$.*

(ii) *The equality* $x + y = \gamma(g)$ *holds true for some* $x, y \in \boldsymbol{N}$ *and* $g \in \operatorname{Idem}(\boldsymbol{R})$ *if and only if there are orthogonal idempotents* $e, f \in \boldsymbol{R}$ *such that* $\gamma(e) = x$, $\gamma(f) = y$, *and* $e + f = g$.

(iii) $\gamma(1)$ *is an order unit in* $\boldsymbol{N}$.

Then the map $\alpha\colon \{[e\boldsymbol{R}] \mid e \in \operatorname{Idem}(\boldsymbol{R})\} \to \boldsymbol{N}$ *given by the correspondence* $[e\boldsymbol{R}] \mapsto \gamma(e)$ *extends to a (unique) isomorphism* $\beta\colon \boldsymbol{V}(\boldsymbol{R}) \to \boldsymbol{N}$.

4. Non-cancellative refinement monoids

In this section we recall a construction of refinement monoids that are, under some simple conditions, non cancellative directly finite. It leads to examples that will be realized as $\boldsymbol{V}(\boldsymbol{R})$ of regular rings, $\boldsymbol{R}$, in the rest of the paper. We seek both simplicity and generality hoping for further applications of the construction.

Definition 4.1. Let $\boldsymbol{M}$, $\boldsymbol{G}$ be monoids and $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ a monoid homomorphism. Given $H \subseteq \boldsymbol{M}$ a hereditary subset (w.r.t. the algebraic preorder on $\boldsymbol{M}$) and a submonoid $\boldsymbol{F}$ of $\boldsymbol{G}$, we define a relation $\Theta_H^{\boldsymbol{F}}$ on the monoid $\boldsymbol{M}$ by

(4.1)
$$x \equiv y\,(\Theta_H^{\boldsymbol{F}}) \quad \overset{\text{df}}{\Longleftrightarrow} \quad \begin{cases} \iota(x) + p = \iota(y) + q \text{ for some } p, q \in \boldsymbol{F} & : x, y \notin H, \\ x = y & : \text{otherwise,} \end{cases}$$

for all $x, y \in \boldsymbol{M}$.

Lemma 4.2. *Let* $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ *be a monoid homomorphism,* H *a hereditary subset of* $\boldsymbol{M}$, *and* $\boldsymbol{F}$ *a submonoid of* $\boldsymbol{G}$. *Then the relation* $\Theta_H^{\boldsymbol{F}}$ *defined by* (4.1) *is a congruence of* $\boldsymbol{M}$.

Proof. We shall prove separately that $\Theta_H^{\boldsymbol{F}}$ is an equivalence relation on $\boldsymbol{M}$ and that $\Theta_H^{\boldsymbol{F}}$ is compatible with the operation of addition.

Claim 5. $\Theta_H^{\boldsymbol{F}}$ *is an equivalence relation.*

Proof of Claim 5. The relation $\Theta_H^{\boldsymbol{F}}$ is clearly symmetric and reflexive. Suppose that

$$x \equiv y\,(\Theta_H^{\boldsymbol{F}}) \text{ and } y \equiv z\,(\Theta_H^{\boldsymbol{F}}) \tag{4.2}$$

for some $x, y, z \in \boldsymbol{M}$. Observe from definition (4.1) that $x \equiv y\,(\Theta_H^{\boldsymbol{F}})$ implies that either both x and y belong to H, in which case they are equal, or none of them belong to H. Therefore, in order to verify transitivity of $\Theta_H^{\boldsymbol{F}}$, there are two cases to discuss:

Case 1: None of the elements x, y, z belong to H. In this case there are $p, q, r, s \in \boldsymbol{F}$ such that

$$\iota(x) + p = \iota(y) + q \text{ and } \iota(y) + r = \iota(z) + s.$$

It follows that $\iota(x) + (p + r) = \iota(z) + (q + s)$, and since $\boldsymbol{F}$ is a submonoid of $\boldsymbol{G}$, we conclude that $x \equiv z\,(\Theta_H^{\boldsymbol{F}})$.

Case 2: All the elements x, y, z belong to H. In this case it follows readily from (4.2) that $x = y = z$, and thus trivially $x \equiv z\,(\Theta_H^{\boldsymbol{F}})$.

We conclude that $\Theta_H^{\boldsymbol{F}}$ is an equivalence relation on $\boldsymbol{M}$. $\square$ Claim 5.

CLAIM 6. *$\Theta_H^{\boldsymbol{F}}$ is compatible with addition.*

PROOF OF CLAIM 6. Let $x_i \equiv y_i\,(\Theta_H^{\boldsymbol{F}})$ for some $x_i, y_i \in \boldsymbol{M}$, $i = 1, 2$. If all the elements x_i, y_i, $i = 1, 2$, belong to H, definition (4.1) gives that $x_i = y_i$, for all $i = 1, 2$. It follows that $x_1 + x_2 = y_1 + y_2$,

Suppose that not all the elements x_i, y_i, $i = 1, 2$, belong to H. By symmetry we can without loss of generality assume that $x_1 \notin H$. From $x_1 \equiv y_1\,(\Theta_H^{\boldsymbol{F}})$ we infer that $y_1 \notin H$ as well. Since H is a hereditary subset of $\boldsymbol{M}$, we get that $x_1 + x_2, y_1 + y_2 \notin H$. By definition (4.1), there are $p_i, q_i \in \boldsymbol{F}$, $i = 1, 2$ (p_2, q_2 possibly zero when $x_2, y_2 \in H$) such that

$$\iota(x_i) + p_i = \iota(y_i) + q_i,$$

for all $i = 1, 2$. It follows that

$$\iota(x_1 + x_2) + (p_1 + p_2) = \iota(y_1 + y_2) + (q_1 + q_2).$$

Since $\boldsymbol{F}$ is closed under addition and none of the elements $x_1 + x_2$, $y_1 + y_2$ belongs to H, we conclude from (4.1) that $x_1 + x_2 \equiv y_1 + y_2\,(\Theta_H^{\boldsymbol{F}})$. $\square$ Claim 6.

This concludes the proof. $\square$

Let $\boldsymbol{M}$ be a monoid and Θ a congruence of $\boldsymbol{M}$. Given an element $x \in \boldsymbol{M}$, we denote by $[\,x\,]_\Theta$ the Θ-block of x, i.e., $[\,x\,]_\Theta := \{y \in \boldsymbol{M} \mid x \equiv y\,(\Theta)\}$. We denote by $\boldsymbol{M}/\Theta$ the quotient monoid of $\boldsymbol{M}$ by the congruence Θ.

LEMMA 4.3. *Let $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ be a monoid homomorphism, H a proper hereditary subset of $\boldsymbol{M}$, and $\boldsymbol{F}$ a submonoid of $\boldsymbol{G}$. Suppose that there are $x \neq y$ in H and $p, q \in \boldsymbol{F}$ such that*

$$\iota(x) + p = \iota(y) + q. \tag{4.3}$$

Then the quotient monoid $\boldsymbol{M}/\Theta_H^{\boldsymbol{F}}$ is not cancellative.

PROOF. Since H is a proper subset of $\boldsymbol{M}$, there is $z \in \boldsymbol{M} \setminus H$. From (4.3) we get that

$$\iota(z + x) + p = \iota(z) + \iota(x) + q = \iota(z) + \iota(y) + q = \iota(z + y) + q \tag{4.4}$$

From (4.4) we infer that

$$z + x \equiv z + y\,(\Theta_H^{\boldsymbol{F}}),$$

hence

$$[\,z\,]_{\Theta_H^{\boldsymbol{F}}} + [\,x\,]_{\Theta_H^{\boldsymbol{F}}} = [\,z + x\,]_{\Theta_H^{\boldsymbol{F}}} = [\,z + y\,]_{\Theta_H^{\boldsymbol{F}}} = [\,z\,]_{\Theta_H^{\boldsymbol{F}}} + [\,y\,]_{\Theta_H^{\boldsymbol{F}}}.$$

On the other hand since $x \neq y$ in H, we get from Definition 4.1 that

$$[\,x\,]_{\Theta_H^{\boldsymbol{F}}} = \{x\} \neq \{y\} = [\,y\,]_{\Theta_H^{\boldsymbol{F}}}.$$

Therefore $\boldsymbol{M}/\Theta_H^{\boldsymbol{F}}$ is not cancellative. $\square$

In the next lemma we show that under the assumptions that $H = \boldsymbol{O}$ is an o-ideal and both $\boldsymbol{O}$ and $\boldsymbol{G}$ are cancellative, we can cancel elements from the given hereditary subset.

LEMMA 4.4. *Let $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ be a monoid homomorphism, $\boldsymbol{O}$ an order ideal of $\boldsymbol{M}$, and $\boldsymbol{F}$ a submonoid of $\boldsymbol{G}$. Suppose that both $\boldsymbol{O}$ and $\boldsymbol{G}$ are cancellative. Let $x, y \in \boldsymbol{M}$ and $o \in \boldsymbol{O}$ satisfy*

$$[\,x\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} + [\,o\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} = [\,y\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} + [\,o\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}}. \tag{4.5}$$

Then $[\,x\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} = [\,y\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}}$.

PROOF. Equation (4.5) is equivalent to

$$x + o \equiv y + o\,(\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}).$$

First suppose that $x + o \in \boldsymbol{O}$. Then also $y + o \in \boldsymbol{O}$, and consequently $x, y \in \boldsymbol{O}$ for $\boldsymbol{O}$ is an o-ideal. By Definition 4.1 we have that $x + o = y + o \in \boldsymbol{O}$. Since $\boldsymbol{O}$ is cancellative, we get that $x = y$.

Assume that $x + o \notin \boldsymbol{O}$. Since $o \in \boldsymbol{O}$ and $\boldsymbol{O}$ is an o-ideal, we infer that $x \notin \boldsymbol{O}$. Similarly we get that $y \notin \boldsymbol{O}$. According to Definition 4.1 there are elements $p, q \in \boldsymbol{F}$ such that

$$\iota(x) + \iota(o) + p = \iota(x + o) + p = \iota(y + o) + q = \iota(y) + \iota(o) + q.$$

Since $\boldsymbol{F}$ is cancellative, we get that

$$\iota(x) + p = \iota(y) + q,$$

hence $[\,x\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} = [\,y\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}}$, due to Definition 4.1. □

Let $\boldsymbol{G}$ be a group and $\boldsymbol{F}$ a submonoid of $\boldsymbol{G}$. We set

$$\boldsymbol{F}^{\natural} := \{p - q \mid p, q \in \boldsymbol{F}\}.$$

Clearly, $\boldsymbol{F}^{\natural}$ is the subgroup of $\boldsymbol{G}$ generated by the monoid $\boldsymbol{F}$.

LEMMA 4.5. *Let $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ be a monoid homomorphism, H a hereditary subset of $\boldsymbol{M}$. Suppose that $\boldsymbol{G}$ is a group and let $\boldsymbol{F}$ be a submonoid of $\boldsymbol{G}$. Then $\Theta_H^{\boldsymbol{F}} = \Theta_H^{\boldsymbol{F}^{\natural}}$.*

PROOF. It is clear that $\Theta_H^{\boldsymbol{F}} \subseteq \Theta_H^{\boldsymbol{F}^{\natural}}$. We prove the opposite inclusion. Let x and y be elements of $\boldsymbol{M}$ such that $x \equiv y\,(\Theta_H^{\boldsymbol{F}^{\natural}})$. By Definition 4.1, we have that $x = y$ unless both x, y belong to $\boldsymbol{M} \setminus H$. In this case there are $p, q \in \boldsymbol{F}^{\natural}$ such that

$$\iota(x) + p = \iota(y) + q. \tag{4.6}$$

Then there are $p_i, q_i \in \boldsymbol{F}$, $i = 1, 2$, such that $p = p_1 - p_2$ and $q = q_2 - q_1$. Substituting to (4.6) we get that

$$\iota(x) + p_1 + q_1 = \iota(y) + q_2 + p_2.$$

Therefore $x \equiv y\,(\Theta_H^{\boldsymbol{F}})$. □

Under the assumptions of Lemma 4.5 we may restrict ourselves to the case when $\boldsymbol{F}$ is a subgroup of the group $\boldsymbol{G}$. Notice also that when $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ is the inclusion map and $\boldsymbol{F}$ is a group, we have that

$$x \equiv y\,(\Theta_H^{\boldsymbol{F}}) \iff \begin{cases} x = y + q \text{ for some } q \in \boldsymbol{F} & : x, y \notin H, \\ x = y & : \text{otherwise}, \end{cases} \tag{4.7}$$

for all $x, y \in \boldsymbol{M}$.

A monoid $\boldsymbol{M}$ is said to be *directly finite* provided that $x + y = x$ implies that $y = 0$ for all $x, y \in \boldsymbol{M}$. We can see readily from the definitions, that the monoid $\boldsymbol{V}(\boldsymbol{R})$ is directly finite if and only if all finitely generated projective right $\boldsymbol{R}$-modules are directly finite. Following [**22**, p. 50], this is equivalent to all matrix rings $\mathbb{M}_n(\boldsymbol{R})$ being directly finite. As far as we know it is still an open question whether the monoid $\boldsymbol{V}(\boldsymbol{R})$ of a directly finite regular ring must be directly finite (cf. [**22**, Problem 1 on p. 344]). A sufficient conditions for direct finiteness of the quotient monoids $\boldsymbol{M}/\Theta_H^{\boldsymbol{F}}$ is given by the following lemma:

LEMMA 4.6. *Let $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ be a monoid homomorphism, H a hereditary subset of $\boldsymbol{M}$. Suppose that $\boldsymbol{G}$ is a group and let $\boldsymbol{F}$ be a subgroup of $\boldsymbol{G}$ such that $\iota^{-1}(\boldsymbol{F}) = \boldsymbol{0}$. Then the quotient $\boldsymbol{M}/\Theta_H^{\boldsymbol{F}}$ is directly finite whenever the monoid $\boldsymbol{M}$ is directly finite.*

PROOF. Suppose that elements $x, y \in \boldsymbol{M}$ satisfy

$$[\,x\,]_{\Theta_H^{\boldsymbol{F}}} + [\,y\,]_{\Theta_H^{\boldsymbol{F}}} = [\,x\,]_{\Theta_H^{\boldsymbol{F}}}.$$

If $x \in H$, then $x + y = x$ by the definition of $\Theta_H^{\boldsymbol{F}}$ and since $\boldsymbol{M}$ is directly finite, we conclude that $y = 0$. Suppose that $x \notin H$. According to (4.7) there is $q \in \boldsymbol{F}$ such that

$$\iota(x) + \iota(y) = \iota(x) + q. \tag{4.8}$$

Since $\boldsymbol{G}$ is a group, we get from (4.8) that $\iota(y) = q$, and so $y \in \iota^{-1}(\boldsymbol{F}) = \boldsymbol{0}$. Therefore $y = 0$. □

In the proof of forthcoming Lemma 4.8 we will repeatedly make use of the following:

LEMMA 4.7. *Let $\boldsymbol{M}$ be a monoid and Θ a congruence of $\boldsymbol{M}$. Let $x_i, y_i \in \boldsymbol{M}$, $i = 1, 2$, be such that*

$$[\,x_1\,]_\Theta + [\,x_2\,]_\Theta = [\,y_1\,]_\Theta + [\,y_2\,]_\Theta \tag{4.9}$$

and suppose that here are x_i', y_i', $i = 1, 2$, in $\boldsymbol{M}$ with $x_i \equiv x_i'\,(\Theta)$ and $y_i \equiv y_i'\,(\Theta)$ for all $i = 1, 2$ and

$$x_1' + x_2' = y_1' + y_2'. \tag{4.10}$$

If z_{ij}, $i, j = 1, 2$, is a refinement of (4.10), *then $[\,z_{ij}\,]_\Theta$, $i, j = 1, 2$, is a refinement of* (4.9).

Proof. Since Θ is a congruence of $\boldsymbol{M}$, the equality $x_i' = z_{i1} + z_{i2}$ implies that $[\,x_i\,]_\Theta = [\,x_i'\,]_\Theta = [\,z_{i1}\,]_\Theta + [\,z_{i2}\,]_\Theta$ and $y_j' = z_{1j} + z_{2j}$ implies that $[\,y_j\,]_\Theta = [\,y_j'\,]_\Theta = [\,z_{1j}\,]_\Theta + [\,z_{2j}\,]_\Theta$, for all $i, j \in 1, 2$. Therefore, if z_{ij}, $i, j = 1, 2$, is a refinement of (4.10), then $[\,z_{ij}\,]_\Theta$, $i, j = 1, 2$, is a refinement of (4.9). $\square$

Lemma 4.8. *Let $\boldsymbol{M}$, $\boldsymbol{G}$ be monoids, $\boldsymbol{O}$ and order ideal of $\boldsymbol{M}$, and $\boldsymbol{F}$ a submonoid of $\boldsymbol{G}$. Let $\iota\colon \boldsymbol{M} \to \boldsymbol{G}$ be a one-to-one monoid homomorphism such that for every $x, y \in \boldsymbol{M} \setminus \boldsymbol{O}$ and every $p, q \in \boldsymbol{F}$ there is $r \in \boldsymbol{F}$ satisfying both $\iota(x)+p+r \in \iota(\boldsymbol{M}\setminus\boldsymbol{O})$ and $\iota(y)+q+r \in \iota(\boldsymbol{M}\setminus\boldsymbol{O})$. If $\boldsymbol{M}$ is a refinement monoid, then the quotient $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}$ is a refinement monoid as well.*

Proof. We are to verify that the quotient monoid $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}$ is conical and that it satisfies the Riesz refinement property.

Claim 7. *The quotient $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}$ is conical.*

Proof of Claim 7. Let

$$[\,x\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} + [\,y\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} = [\,0\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}},$$

for some $x, y \in \boldsymbol{M}$. This is equivalent to $x + y \equiv 0\,(\Theta_{\boldsymbol{O}}^{\boldsymbol{F}})$. Since $0 \in \boldsymbol{O}$, we get from (4.1) that $x + y = 0$. Since the monois $\boldsymbol{M}$ is conical, we conclude that $x = y = 0$. $\square$ Claim 7.

Claim 8. *The quotient $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}$ satisfies the Riesz refinement property.*

Proof of Claim 8. Suppose that $x_i, y_i \in \boldsymbol{M}$, $i = 1, 2$, satisfy

$$[\,x_1\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} + [\,x_2\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} = [\,y_1\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}} + [\,y_2\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}}, \tag{4.11}$$

and so equivalently

$$x_1 + x_2 \equiv y_1 + y_2\,(\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}). \tag{4.12}$$

We are going to discuss two complementary cases:

Case 1: Suppose that $x_1 + x_2 \in \boldsymbol{O}$. With regard to definition (4.1), we get from (4.12) that $y_1 + y_2 \in \boldsymbol{O}$ as well and that

$$x_1 + x_2 = y_1 + y_2. \tag{4.13}$$

Since equality (4.13) has a refinement, (4.11) has a refinement as well due to Lemma 4.7

Case 2: If $x_1 + x_2 \notin \boldsymbol{O}$, then $y_1 + y_2 \notin \boldsymbol{O}$ as well, due to (4.12) and (4.1). Since $\boldsymbol{O}$ is an o-ideal of $\boldsymbol{M}$, in particular, it is closed under addition, at least one of the elements x_1, x_2, as well as at least one of the elements y_1, y_2 does not belong to $\boldsymbol{O}$. By symmetry, we can assume without loss of generality that both x_2 and y_2 are not in $\boldsymbol{O}$. Since (4.12) holds true, there are $p, q \in \boldsymbol{F}$ such that

$$\iota(x_1 + x_2) + p = \iota(y_1 + y_2) + q, \tag{4.14}$$

due to definition (4.1). According to the assumptions, there is an element $r \in \boldsymbol{F}$ such that $\iota(x_2) + p + r \in \iota(\boldsymbol{M} \setminus \boldsymbol{O})$ and $\iota(y_2) + q + r \in \iota(\boldsymbol{M} \setminus \boldsymbol{O})$.

Let $x_2', y_2' \in \boldsymbol{M} \setminus \boldsymbol{O}$ be the elements satisfying $\iota(x_2') = \iota(x_2) + p + r$ tand $\iota(y_2') = \iota(y_2) + q + r$. It follows from (4.14) that
(4.15)
$$\begin{aligned}\iota(x_1 + x_2') &= \iota(x_1) + \iota(x_2) + p + r = \iota(x_1 + x_2) + p + r\\ &= \iota(y_1 + y_2) + q + r = \iota(y_1) + \iota(y_2) + q + r = \iota(y_1 + y_2').\end{aligned}$$

From (4.15) and the injectivity of ι we conclude that

$$x_1 + x_2' = y_1 + y_2'. \tag{4.16}$$

Since $\boldsymbol{M}$ is a refinement monoid, equation (4.16) has a refinement that induces a refinemnt of (4.11) due to Lemma 4.7. □ Claim 8.

The properties verified by Claims 7 and 8 mean that $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}$ is a refinement monoid. □

We state a corollary of Lemma 4.8 describing some cases when the formulation of the assumptions can be reasonably simplified. It is going to be applied in the next section.

COROLLARY 4.9. *Let $\boldsymbol{M}$, $\boldsymbol{G}$ be monoids, $\boldsymbol{O}$ and order ideal of $\boldsymbol{M}$, and $\boldsymbol{F}$ a submonoid of $\boldsymbol{G}$. Let $\iota \colon \boldsymbol{M} \to \boldsymbol{G}$ be a one-to-one monoid homomorphism such that*

$$\iota(\boldsymbol{M} \setminus \boldsymbol{O}) + \boldsymbol{F} \subseteq \iota(\boldsymbol{M} \setminus \boldsymbol{O}). \tag{4.17}$$

If $\boldsymbol{M}$ is a refinement monoid, then the quotient $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}$ is a refinement monoid. If $\boldsymbol{G}$ is a group then $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}^\natural}$ is s a refinement monoid as well.

PROOF. The fact that $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}}$ is a refinement monoid follows readily from Lemma 4.8 as the assumptions of the lemma follow from (4.17). The quotient $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{F}^\natural}$ is a refinement monoid due to Lemma 4.5. □

5. The monoid $\boldsymbol{A}_{2n}$, $\boldsymbol{B}_{2n}$, and $\boldsymbol{C}_{2n}$

Let $\boldsymbol{O}$ be an o-ideal in a monoid $\boldsymbol{M}$. We denote by $\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$ the relation on $\boldsymbol{M}$ defined by $x \equiv y\,(\Theta_{\boldsymbol{O}}^{\boldsymbol{M}})$ provided that there are $o, p \in \boldsymbol{O}$ such that $x + o = y + p$. Note that this definition is consistent with the notation of the previous section assuming that we are given the identity map $\iota \colon \boldsymbol{M} \to \boldsymbol{M}$.

LEMMA 5.1. *Let $\boldsymbol{M}$ be a conical cancellative monoid. Let $\boldsymbol{O}$ be an o-ideal of $\boldsymbol{M}$ such that*

$$o \leq x \text{ for all } o \in \boldsymbol{O} \text{ and all } x \in \boldsymbol{M} \setminus \boldsymbol{O}. \tag{5.1}$$

Then $\boldsymbol{M}$ is a refinement monoid if and only if both $\boldsymbol{O}$ and $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$ are refinement monoids.

PROOF. ($\Rightarrow$) Suppose that $\boldsymbol{M}$ is a refinement monoid. An o-ideal of a refinement monoid is clearly a refinement monoid, in particular $\boldsymbol{O}$ is a refinement monoid.

Suppose that

$$[x]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} + [y]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = [x+y]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = [0]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}},$$

for some $x, y \in \boldsymbol{M}$. Note that it follows readily from the definition of the congruence $\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$ that $[0]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = \boldsymbol{O}$. Therefore, $x + y \in \boldsymbol{O}$, hence both x, y belong to $\boldsymbol{O}$, for $\boldsymbol{O}$ is an o-ideal. We conclude that $[x]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = [y]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = [0]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}}$, and so the quotient monoid $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$ is conical.

We are going to prove that $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$ satisfies the Riesz refinement property. Let

$$[x_1]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} + [x_2]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = [y_1]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} + [y_2]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} \tag{5.2}$$

in $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$. Then, by the definition, there are $o, p \in \boldsymbol{O}$ such that $x_1 + x_2 + o = y_1 + y_2 + p$. We set $x_2' := x_2 + o$ and $y_2' := y_2 + p$. Then

$$x_1 + x_2' = x_2 + y_2', \tag{5.3}$$

and since $\boldsymbol{M}$ satisfies the Riesz refinement property, the equation (5.3) has a refinement. Clearly $x_2' \equiv x_2 + o\,(\Theta_{\boldsymbol{O}}^{\boldsymbol{M}})$ and $y_2' \equiv y_2 + p\,(\Theta_{\boldsymbol{O}}^{\boldsymbol{M}})$, and so this refinement leads to a refinement of (5.2) in the quotient monoid $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$.

($\Leftarrow$). Suppose that both $\boldsymbol{O}$ and $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$ are refinement monoids. Note that a monoid having a conical o-ideal is conical, in particular the monoid $\boldsymbol{M}$ is conical. It remains to prove that $\boldsymbol{M}$ satisfies the Riesz refinement property. Given elements $o \in \boldsymbol{O}$ and $x \in \boldsymbol{M} \setminus \boldsymbol{O}$, we denote by $x - o$ the unique element of $\boldsymbol{M}$ satisfying $x = o + (x - o)$. Such an element exists due to (5.1) and it is unique since $\boldsymbol{M}$ is cancellative.

Suppose that

$$x_1 + x_2 = y_1 + y_2 \tag{5.4}$$

for some $x_i, y_j \in \boldsymbol{M}$, $i, j = 1, 2$. We aim to prove that the equation (5.4) has a refinement. Up to symmetry, there are three cases to discuss.

Case 1: All x_i, y_j, $i, j \in 1, 2$, are from $\boldsymbol{O}$. Since $\boldsymbol{O}$ satisfies the Riesz refinement property, we find a refinement of (5.4) within $\boldsymbol{O}$.

Case 2: Some but not all the elements appearing in (5.4) are in $\boldsymbol{O}$. Observe that in this case at most one of x_i, $i = 1, 2$, as well as at most one of y_j, $j = 1, 2$, are from $\boldsymbol{M} \setminus \boldsymbol{O}$. Therefore, we can without loss of generality assume that $x_1, y_1 \in \boldsymbol{M} \setminus \boldsymbol{O}$ while $y_2 \in \boldsymbol{O}$. We put

$$z_{11} := x_1 - y_2, \quad z_{12} := y_2, \quad z_{21} := x_2, \text{ and } z_{22} := 0.$$

Clearly

$$\begin{aligned} x_1 &= z_{11} + z_{12} = (x_1 - y_2) + y_2, \\ x_2 &= z_{21} + z_{22} = x_2 + 0, \text{ and} \\ y_2 &= z_{12} + z_{22} = y_2 + 0. \end{aligned}$$

Thus we only need to verify that $y_1 = z_{11} + z_{21}$. This follows from

$$z_{11} + z_{21} + y_2 = (x_1 - y_2) + x_2 + y_2 = x_1 + x_2 = y_1 + y_2$$

and the cancellativity of $\boldsymbol{M}$.

Case 3: All the elements x_i, y_j, $i, j = 1, 2$, are in $\boldsymbol{M} \setminus \boldsymbol{O}$. Since $\boldsymbol{M}/\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}$ is a refinement monoid, there are z_{ij}, $i, j = 1, 2$, such that

$$[\,x_i\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = [\,z_{i1}\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} + [\,z_{i2}\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}}, \text{ for all } i = 1, 2, \text{ and}$$
$$[\,y_j\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} = [\,z_{1j}\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} + [\,z_{2j}\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}}, \text{ for all } j = 1, 2.$$

This particularly means that there are $o_i, p_i \in \boldsymbol{O}$, $i = 1, 2$, satisfying

$$x_i + o_i = z_{i1} + z_{i2} + p_i, \text{ for both } i = 1, 2.$$

Observe that since $x_i, y_j \in \boldsymbol{M} \setminus \boldsymbol{O}$, for all $i, j = 1, 2$, either $z_{11}, z_{22} \in \boldsymbol{M} \setminus \boldsymbol{O}$ or $z_{12}, z_{21} \in \boldsymbol{M} \setminus \boldsymbol{O}$. We can without loss of generality assume that the first one holds true. Set

$$u_{ii} := z_{ii} + o_i - p_i, \text{ for all } i = 1, 2 \text{ and } u_{ij} := z_{ij} \text{ for all } i \neq j \text{ in } \{1, 2\},$$

and observe that

$$\begin{aligned} x_i &= u_{i1} + u_{i2}, \text{ for all } i = 1, 2, \\ [\,y_j\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} &= [\,u_{1j}\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}} + [\,u_{2j}\,]_{\Theta_{\boldsymbol{O}}^{\boldsymbol{M}}}, \text{ for all } j = 1, 2, \end{aligned} \tag{5.5}$$

and both u_{11}, u_{22} belong to $\boldsymbol{M} \setminus \boldsymbol{O}$. It follows from (5.5) that

$$y_j + q_j = u_{1j} + u_{2j} + r_j, \; j = 1, 2, \tag{5.6}$$

for some $q_j, r_j \in \boldsymbol{O}$, $j = 1, 2$. Therefore

$$y_1 + y_2 + q_1 + q_2 = \left(\sum_{i=1}^{2} \sum_{j=1}^{2} u_{ij} \right) + r_1 + r_2 = x_1 + x_2 + r_2 + r_2. \tag{5.7}$$

Since $\boldsymbol{M}$ is cancellative, we conclude from (5.4) and (5.7) that

$$q_1 + q_2 = r_1 + r_2.$$

Since $\boldsymbol{O}$ satisfies the Riesz refinement property, there are elements $s_{ij} \in \boldsymbol{O}$, $i, j = 1, 2$, such that

$$q_j = s_{j1} + s_{j2} \text{ and } r_j = s_{1j} + s_{2j} \text{ for all } j = 1, 2. \tag{5.8}$$

Substituting from (5.8) to (5.6), we get that

$$y_j + s_{j1} + s_{j2} = u_{1j} + u_{2j} + s_{1j} + s_{2j}, \text{ for all } j = 1, 2. \tag{5.9}$$

Since the monoid $\boldsymbol{M}$ is cancellative, we conclude from (5.9) that

$$\begin{aligned} y_1 + s_{12} &= u_{11} + u_{21} + s_{21} \text{ and} \\ y_2 + s_{21} &= u_{12} + u_{22} + s_{12}. \end{aligned} \tag{5.10}$$

It follows from (5.5) and (5.10) that setting

$$\begin{aligned} v_{11} &:= u_{11} - s_{12}, \quad & v_{12} &:= u_{12} + s_{12}, \\ v_{21} &:= u_{21} + s_{21}, \quad & v_{22} &:= u_{22} - s_{21}, \end{aligned}$$

we get a refinement of (5.4) in $\boldsymbol{M}$. ☐

Let n be a non-negative integer. Let

$$\boldsymbol{A}_n := (\mathbf{0} \times \mathbb{N}_0^n) \cup (\mathbb{N} \times \mathbb{Z}^n) \tag{5.11}$$

be a submonoid of the Cartesian power $\mathbb{Z}^{n+1}$. Note that being a submonoid of a group, the monoid $\boldsymbol{A}_n$ is cancellatice. We denote by $\boldsymbol{O}_n$ the o-ideal of $\boldsymbol{A}_n$ defined by $\boldsymbol{O}_n := \mathbf{0} \times \mathbb{N}_0^n$, and we set $\boldsymbol{U}_n := \boldsymbol{A}_n \setminus \boldsymbol{O}_n = \mathbb{N} \times \mathbb{Z}^n$.

Corollary 5.2. *The monoid $\boldsymbol{A}_n$ is a refinement monoid, for every non-negative integer n.*

Proof. It is straightforward to see that $o \leq_{\boldsymbol{A}_n} x$ for every $o \in \boldsymbol{O}_n$ and every $x \in \boldsymbol{U}_n$. Therefore property (5.1) of Lemma 5.1 is satisfied. Clearly $\boldsymbol{O}_n$, being a Cartesian product of refinement monoids, is a refinement monoid. Observing that

$$\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{A}_n} \simeq \mathbb{N}_0,$$

which is a refinement monoid as well, we conclude from Lemma 5.1 that $\boldsymbol{A}_n$ is a refinement monoid. $\square$

Lemma 5.3. *Let n be a non-negative integer and $\iota\colon \boldsymbol{A}_n \to \mathbb{Z}^{n+1}$ the inclusion map. Then $\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}$ is a refinement monoid for every submonoid $\boldsymbol{F}$ of $\mathbb{Z}^{n+1}$. Moreover*

(a) *if $\boldsymbol{A}_n \cap \boldsymbol{F}^\natural = \mathbf{0}$ holds true, then $\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}$ is directly finite;*

(b) *if $\boldsymbol{O}_n^\natural \cap \boldsymbol{F}^\natural \neq \mathbf{0}$, then $\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}$ is not cancellative.*

Proof. Firstly note that according to Lemma 4.5 we can without loss of generality assume that $\boldsymbol{F}$ is a subgroup of $\mathbb{Z}^{n+1}$, i.e, that $\boldsymbol{F} = \boldsymbol{F}^\natural$. Put $\boldsymbol{F}_+ := \boldsymbol{F} \cap (\mathbb{N}_0 \times \mathbb{Z}^n)$ and observe that $\iota(\boldsymbol{U}_n) + \boldsymbol{F}_+ \subseteq \iota(\boldsymbol{U}_n)$. Applying Corollary 4.9 we conclude that $\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}$ is a refinement monoid.

Being a submonoid of $\mathbb{Z}^{n+1}$, the monoid $\boldsymbol{A}_n$ is cancellative and, *a fortiori*, directly finite. Then (a) follows readily from Lemma 4.6.

The assumption $\boldsymbol{O}_n^\natural \cap \boldsymbol{F}^\natural \neq \mathbf{0}$ implies that there are $x \neq y$ in $\boldsymbol{O}_n$ and $p, q \in \boldsymbol{F}$ such that $x - y = q - p$, and so, equivalently, $x + p = y + q$. Since ι is an inclusion map, the monoid $\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}$ is not cancellative due to Lemma 4.3. $\square$

Although the monoid $\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}$ might not be cancellative we can cancel the elements from $\boldsymbol{O}_n$ due to Lemma 4.4.

Lemma 5.4. *Let $\boldsymbol{F}$ be a non-trivial submonoid of $\mathbb{Z}^{n+1}$. If $x, y \in \boldsymbol{A}_n$ and $o \in \boldsymbol{O}_n$ satisfy*

$$[\,x\,]_{\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}} + [\,o\,]_{\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}} = [\,y\,]_{\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}} + [\,o\,]_{\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}} \tag{5.12}$$

Then $[\,x\,]_{\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}} = [\,y\,]_{\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}}$.

Fix a positive integer n. For an element $x = \langle x_0, x_1, \dots, x_n \rangle \in \mathbb{Z}^{n+1}$ we set

$$\sigma x := x_0 + x_1 + \cdots + x_n.$$

We put $\Sigma_n^0 = \{x \in \mathbb{Z}^{n+1} \mid x_0 = 0 \text{ and } \sigma x = 0\}$ and observe that Σ_n^0 is a subgroup of $\mathbb{Z}^{n+1}$.

COROLLARY 5.5. *Let $\boldsymbol{F}$ be a non-trivial subgroup of Σ_n^0. Then $\boldsymbol{A}_n/\Theta_{\boldsymbol{O}_n}^{\boldsymbol{F}}$ is a non-cancellative directly finite refinement monoid.*

PROOF. Observe that $\boldsymbol{A}_n \cap \Sigma_n^0 = \boldsymbol{0}$ and $\boldsymbol{O}_n{}^\natural \cap \Sigma_n^0 = \Sigma_n^0$. Now apply Lemma 5.3. □

Given a positive integer n, let $\boldsymbol{F}_{2n}$ denote a subgroup of $\mathbb{Z}^{2n+1}$ generated by $\langle 0, 1, -1, \ldots, 1, -1\rangle$. We set

$$\boldsymbol{B}_{2n} := \boldsymbol{A}_{2n}/\Theta_{\boldsymbol{O}_{2n}}^{\boldsymbol{F}_{2n}}.$$

As $\boldsymbol{F}_{2n}$ is clearly a non-trivial subgroup of Σ_{2n}^0, $\boldsymbol{B}_{2n}$ is a non-cancellative directly finite refinement monoid. We are going to realize the monoids $\boldsymbol{B}_{2n}$ as $\boldsymbol{V}(\boldsymbol{R}_{2n})$ of regular rings $\boldsymbol{R}_{2n}$.

Before that, we prove that the monoid $\boldsymbol{B}_4$ (and consequently the monoids $\boldsymbol{B}_{2n}$ for all $n \geq 2$) does not satisfy the Riesz interpolation property.

PROPOSITION 5.6. *The monoid $\boldsymbol{B}_4$ does not satisfy the Riesz interpolation property.*

PROOF. Let $x = \langle x_0, x_1, \ldots, x_4\rangle$ and $y = \langle y_0, y_1, \ldots, y_4\rangle$ be elements of $\boldsymbol{A}_4$. We observe readily from the definitions that if $x_0 = y_0$, then

(5.13)
$$[\,x\,]_{\Theta_{\boldsymbol{O}_2}^{\boldsymbol{F}_2}} <_{\boldsymbol{B}_4} [\,y\,]_{\Theta_{\boldsymbol{O}_2}^{\boldsymbol{F}_2}} \implies \sigma x < \sigma y \text{ and } [\,x\,]_{\Theta_{\boldsymbol{O}_2}^{\boldsymbol{F}_2}} = [\,y\,]_{\Theta_{\boldsymbol{O}_2}^{\boldsymbol{F}_2}} \implies \sigma x = \sigma y.$$

We set

$$x^1 := (1,1,1,0,0), \quad x^2 := (1,1,0,1,0),$$
$$y^1 := (1,1,1,1,0), \quad y^2 := (1,1,1,0,1).$$

We see that $x^1, x^2 \leq_{\boldsymbol{A}_4} y^1$ and $x^1 \leq_{\boldsymbol{A}_4} y^2$. Since $\sigma x^1 = \sigma x^2 = 3 < 4 = \sigma y^1 = \sigma y^2$, we get that $[\,x^1\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}}, [\,x^2\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}} <_{\boldsymbol{B}_4} [\,y^1\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}}$ and $[\,x^1\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}} <_{\boldsymbol{B}_4} [\,y^2\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}}$. Since

$$y^2 \equiv \langle 1,2,0,1,0\rangle\,(\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}),$$

we have that also $[\,x^2\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}} <_{\boldsymbol{B}_4} [\,y^2\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}}$. Suppose that there is a tuple $z = \langle z_0, z_1, \ldots, z_4\rangle$ with

$$[\,x^1\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}}, [\,x^2\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}} <_{\boldsymbol{B}_4} [\,z\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}} <_{\boldsymbol{B}_4} [\,y^1\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}}, [\,y^2\,]_{\Theta_{\boldsymbol{O}_4}^{\boldsymbol{F}_4}}. \tag{5.14}$$

Since $x_0^i = y_0^i = 1$, for all $i = 1, 2$, we get that $z_0 = 1$. From (5.14) and (5.13) we get that $3 = \sigma x^i < \sigma z < \sigma y^j = 4$, $i, j = 1, 2$. This is absurd. □

Let n be a positive integer. We set $x_{\{2i-1,2j\}} := x_{2i-1} + x_{2j}$, for each $i, j \in \{1, 2, \ldots, n\}$, and we define

$$\boldsymbol{V}_{2n} := \{\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j \leq n} \mid x_0 \in \mathbb{N} \text{ and } x_{\{2i-1,2j\}} \in \mathbb{Z} \text{ for all } i, j \leq n\}$$

and we set $\boldsymbol{C}_{2n} := \boldsymbol{O}_{2n} \cup \boldsymbol{V}_{2n}$. Observe that $\boldsymbol{V}_{2n}$ is a commutative semigroup isomorphic to $\mathbb{N} \times \mathbb{Z}^{n^2}$ and that $\boldsymbol{C}_{2n}$ is a monoid with the operation of addition defined coordinate-wise on the two components $\boldsymbol{O}_{2n}$ and $\boldsymbol{V}_{2n}$ and by

$$p + x := \langle x_0, p_{2i-1} + p_{2j} + x_{\{2i-1,2j\}} \rangle_{i,j \in \{1,2,\ldots,n\}}$$

for all $p = \langle 0, p_1, \ldots, p_{2n} \rangle \in \boldsymbol{O}_{2n}$ and $x = \langle x_0, x_{\{2i-1,2j\}} \rangle_{i,j \leq n} \in \boldsymbol{V}_{2n}$.

Let $\varphi_{2n} \colon \boldsymbol{A}_{2n} \to \boldsymbol{C}_{2n}$ be a map corresponding to the identity on $\boldsymbol{O}_{2n}$ and sending

$$\langle x_0, x_1, x_2 \ldots, x_{2n} \rangle \mapsto \langle x_0, x_{\{2i-1,2j\}} \rangle_{i,j \leq n} \in \boldsymbol{V}_{2n},$$

whenever $x_0 > 0$. It is straightforward to see that φ_{2n} is a monoid homomorphism.

Let $x = \langle x_0, \ldots, x_{2n} \rangle$, $y = \langle y_0, \ldots, y_{2n} \rangle$ be elements from $\boldsymbol{A}_{2n}$ satisfying $\varphi_{2n}(x) = \varphi_{2n}(y)$. Readily from the definition we see that $x_0 = y_0$. If $x_0 = y_0 = 0$, then necessarily $x = y$. Suppose that $x_0 = y_0 > 0$. In this case the equality $\varphi_{2n}(x) = \varphi_{2n}(y)$ is equivalent to

$$x_{2i-1} + x_{2j} = y_{2i-1} + y_{2j} \tag{5.15}$$

for all $i, j \in \{1, 2, \ldots, n\}$. This is equivalent to

$$x_1 - y_1 = y_2 - x_2 = \cdots = x_{2n-1} - y_{2n-1} = y_{2n} - x_{2n},$$

which happens if and only if

$$x = y + \lambda \langle 1, -1, \ldots, 1, -1 \rangle,$$

for some $\lambda \in \mathbb{Z}$. Therefore the kernel of the homomorphism φ_{2n} coincides with the congruence $\Theta_{\boldsymbol{O}_{2n}}^{\boldsymbol{F}_{2n}}$, and so φ_{2n} factors through an embedding $\psi_{2n} \colon \boldsymbol{B}_{2n} \to \boldsymbol{C}_{2n}$. This one is given by

$$\psi_{2n}([\, x \,]_{\Theta_{\boldsymbol{O}_{2n}}^{\boldsymbol{F}_{2n}}}) = \begin{cases} x = \langle 0, x_1, x_2, \ldots, x_{2n} \rangle & \text{if } x \in \boldsymbol{O}_{2n}, \\ \langle x_0, x_{\{2i-1,2j\}} \rangle_{i,j \leq n} & \text{if } x \in \boldsymbol{U}_{2n}, \end{cases} \tag{5.16}$$

for every $x = \langle x_0, x_1, \ldots, x_{2n} \rangle \in \boldsymbol{A}_{2n}$.

We say that a tuple $\langle x_0, x_{\{2i-1,2j\}} \rangle_{i,j \leq n} \in \boldsymbol{V}_{2n}$ is *balanced* provided that

$$x_{\{2i-1,2j\}} + x_{\{2k-1,2l\}} = x_{\{2k-1,2j\}} + x_{\{2i-1,2l\}} \tag{5.17}$$

holds true for all $i, j, k, l \in \{1, 2, \ldots, n\}$. We denote by $\boldsymbol{W}_{2n}$ the set of all balanced tuples from $\boldsymbol{V}_{2n}$ and we set

$$\boldsymbol{D}_{2n} := \boldsymbol{O}_{2n} \cup \boldsymbol{W}_{2n}.$$

It is straightforward to show that $\boldsymbol{D}_{2n}$ is a submonoid of $\boldsymbol{C}_{2n}$. Observe also that $\boldsymbol{D}_2 = \boldsymbol{C}_2$.

Lemma 5.7. *The monoid $\boldsymbol{D}_{2n}$ corresponds to $\varphi_{2n}(\boldsymbol{A}_{2n})$, the image of $\boldsymbol{A}_{2n}$ under the monoid homomorphism $\varphi_{2n} \colon \boldsymbol{A}_{2n} \to \boldsymbol{C}_{2n}$.*

Proof. As $\varphi_{2n} \restriction \boldsymbol{O}_{2n}$ is the identity map, we have that $\varphi_{2n}(\boldsymbol{O}_{2n}) = \boldsymbol{O}_{2n}$. We are going to prove that $\varphi_{2n}(\boldsymbol{U}_{2n}) = \boldsymbol{V}_{2n}$.

Let $x = \langle x_0, x_1, \ldots, x_{2n} \rangle \in \boldsymbol{U}_{2n}$. By the definition,

$$\varphi_{2n}(x) = \langle x_0, x_{\{2i-1,2j\}} \rangle_{i,j \leq n},$$

where

$$x_{\{2i-1,2j\}} = x_{2i-1} + x_{2j} \text{ for all } i,j \in \{1,2,\ldots,n\}.$$

Given $i,j,k,l \in \{1,2,\ldots,n\}$, we get straightaway that

$$x_{\{2i-1,2j\}} + x_{\{2k-1,2l\}} = x_{2i-1} + x_{2j} + x_{2k-1} + x_{2l} = x_{\{2i-1,2l\}} + x_{\{2k-1,2j\}},$$

and so $\varphi_{2n}(x)$ is a balanced tuple. Therefore $\varphi_{2n}(\boldsymbol{U}_{2n}) \subseteq \boldsymbol{V}_{2n}$.

Let $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} \in \boldsymbol{V}_{2n}$ be a balanced tuple. We set

$$x_{2i-1} := x_{\{2i-1,2n\}} \text{ and } x_{2j} := x_{\{2j-1,2j\}} - x_{\{2j-1,2n\}} \tag{5.18}$$

for all $i,j = \{1,2,\ldots,n\}$ and we put $x := \langle x_0, x_1, \ldots, x_{2n}\rangle$. Since the tuple $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n}$ is balanced, we have the equality

$$x_{\{2i-1,2j\}} + x_{\{2j-1,2n\}} = x_{\{2i-1,2n\}} + x_{\{2j-1,2j\}},$$

hence

$$x_{2i-1,2j} = x_{\{2i-1,2n\}} + x_{\{2j-1,2j\}} - x_{\{2j-1,2n\}} = x_{2i-1} + x_{2j},$$

for all $i,j \in \{1,2,\ldots,n\}$. It follows that $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} = \varphi_{2n}(x)$. Since $\langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} \in \boldsymbol{V}_{2n}$, we have that $x_0 > 0$, and so $x \in \boldsymbol{U}_{2n}$. Therefore $\boldsymbol{V}_{2n} \subseteq \varphi_{2n}(\boldsymbol{U}_{2n})$. $\square$

Corollary 5.8. *The map defined by correspondence* (5.16) *is an isomorphism*

$$\psi_{2n}\colon \boldsymbol{B}_{2n} \to \boldsymbol{D}_{2n}.$$

It is easy to gain insight into the algebraic preorder on $\boldsymbol{A}_{2n}$. Indeed,

$$x = \langle x_0, x_1, \ldots, x_{2n}\rangle \leq_{\boldsymbol{A}_{2n}} y = \langle y_0, y_1, \ldots, y_{2n}\rangle$$

if and only if either $x_0 < y_0$ or $x_0 = y_0$ and $x_i \leq y_i$ for all $i \in \{1,2,\ldots,n\}$. We are going to show that the algebraic preorder on the monoid $\boldsymbol{D}_{2n}$ behaves analogously.

Let $x = \langle x_0, \ldots\rangle$ and $y = \langle y_0, \ldots\rangle$ be elements of $\boldsymbol{D}_{2n}$. We set

$$x \ll y \iff \begin{cases} x_0 < y_0 \\ x_0 = y_0 = 0 \text{ and } x_i \leq y_i, \text{ for all } i \in \{1,2,\ldots,2n\}, \\ x_0 = y_0 > 0 \text{ and } x_{\{2i-1,2j\}} \leq y_{\{2i-1,2j\}}, \text{ for all } i,j \leq n. \end{cases}$$

It is easy to see that $\ll$ is a partial order on the set $\boldsymbol{D}_{2n}$.

Lemma 5.9. *Let* $x = \langle x_0, x_{\{2i-1,2j\}}\rangle_{i,j\leq n} \in \boldsymbol{V}_{2n}$, $z = \langle z_0, z_1, \ldots, z_{2n}\rangle \in \boldsymbol{U}_{2n}$ *be such that* $x \ll \varphi_{2n}(z)$. *There is* $w \in \boldsymbol{U}_{2n}$ *such that* $w \leq_{\boldsymbol{A}_{2n}} z$ *and* $x = \varphi_{2n}(w)$.

Proof. We set

$$\mu := \min\{z_{2j} - x_{\{1,2j\}} + z_1 \mid j = 1,2,\ldots,n\}$$

and

$$w_0 := x_0,\ w_{2j} := x_{\{1,2j\}} - z_1 + \mu, \text{ and } w_{2i-1} := x_{\{2i-1,2\}} + z_1 - x_{\{1,2\}} - \mu,$$

for every $i, j \in \{1, 2, \dots, n\}$. Since the tuple x is balanced, we have that

$$x_{\{1,2j\}} + x_{\{2i-1,2\}} = x_{\{1,2\}} + x_{\{2i-1,2j\}},$$

hence

$$x_{\{2i-1,2\}} - x_{\{1,2\}} = x_{\{2i-1,2j\}} - x_{\{1,2j\}},$$

whence

$$w_{2i-1} = x_{\{2i-1,2j\}} + z_1 - x_{\{1,2j\}} - \mu, \tag{5.19}$$

for all $i, j \in \{1, 2, \dots, n\}$. It follows that

$$w_{2i-1} + w_{2j} = x_{\{2i-1,2j\}} + z_1 - x_{\{1,2j\}} - \mu + x_{\{1,2j\}} - z_1 + \mu = x_{\{2i-1,2j\}},$$

for all $i, j \in \{1, 2, \dots, n\}$. Since $x_0 = w_0$ by definition, we conclude that $x = \varphi_{2n}(w)$. Let $j \in \{1, 2, \dots, n\}$. From $\mu \le z_{2j} - x_{\{1,2j\}} + z_1$ we get that

$$w_{2j} = x_{\{1,2j\}} - z_1 + \mu \le x_{\{1,2j\}} - z_1 + z_{2j} - x_{\{1,2j\}} + z_1 = z_{2j}. \tag{5.20}$$

Let $k \in \{1, 2, \dots, n\}$ be such that $\mu = z_{2k} - x_{\{1,2k\}} + z_1$. Then, with regard to (5.19), we compute that

$$\begin{aligned} w_{2i-1} &= x_{\{2i-1,2k\}} + z_1 - x_{\{1,2k\}} - \mu \\ &= x_{\{2i-1,2k\}} + z_1 - x_{\{1,2k\}} - z_{2k} - x_{\{1,2k\}} + z_1 \\ &= x_{\{2i-1,2k\}} - z_{2k} \le x_{\{2i-1,2k\}} - z_{2k}. \end{aligned} \tag{5.21}$$

Since $x \ll \varphi_{2n}(z)$, we have that $x_{\{2i-1,2k\}} \le z_{2i-1} + z_{2k}$. Substituting to (5.21), we conclude that

$$w_{2i-1} \le x_{\{2i-1,2k\}} - z_{2k} \le z_{2i-1} + z_{2k} - z_{2k} = z_{2i-1}, \tag{5.22}$$

for all $i = \{1, 2, \dots, n\}$. Since $x \ll \varphi_{2n}(z)$, we have $w_0 = x_0 \le z_0$. This together with (5.20) and (5.22) implies that $w \le_{\boldsymbol{A}_{2n}} z$, which was to prove. □

PROPOSITION 5.10. *Let $x = \langle x_0, \dots \rangle$ and $y = \langle y_0, \dots \rangle$ be elements of $\boldsymbol{D}_{2n}$. Then $x \ll y$ if and only if $x \le_{\boldsymbol{V}_{2n}} y$.*

PROOF. If $x \le_{\boldsymbol{D}_{2n}} y$, there are $x' \le_{\boldsymbol{A}_{2n}} y'$ in $\boldsymbol{A}_{2n}$ sutisfying $x = \varphi_{2n}(x')$ and $y = \varphi_{2n}(y')$. Using the description of the algebraic preoreder in $\boldsymbol{A}_{2n}$, it is easy to see that the relation $x \ll y$ holds true. On the other hand, suppose that $x \ll y$. If $x_0 \le y_0$ or $x_0 = y_0 = 0$, then $x \ll y$ clearly implies that $x \le_{\boldsymbol{D}_{2n}} y$. In the remaining case when $0 < x_0 = y_0$, the implication $x \ll y \implies x \le_{\boldsymbol{D}_{2n}} y$ follows from Lemma 5.9. □

6. Some linear algebra

We fix an arbitrary field $\mathbb{F}$. All vector spaces are supposed to be over $\mathbb{F}$. Let $\boldsymbol{U}$, $\boldsymbol{V}$ be vector spaces and $f\colon \boldsymbol{U} \to \boldsymbol{V}$ a linear map. We define a *dimension* and a *codimension* of the map f by

(i) $\dim f := \operatorname{codim} \ker f + \dim \operatorname{img} f$,
(ii) $\operatorname{codim} f := \dim \ker f + \operatorname{codim} \operatorname{img} f$.

Observe that $\dim f = 2\dim\operatorname{img} f$ and $\dim f + \operatorname{codim} f = \dim U + \dim V$. In particular, if $\dim f$ is finite, it is even.

LEMMA 6.1. *Let $\boldsymbol{U}$ be a vector space. Let $f, g\colon \boldsymbol{U} \to \boldsymbol{V}$ be linear maps such that $\dim f$ and $\operatorname{codim} g$ are finite, and let $h := f + g$ be the sum of the linear maps. Then $\operatorname{codim} h$ is finite and*

$$\dim\ker h - \operatorname{codim}\operatorname{img} h = \dim\ker g - \operatorname{codim}\operatorname{img} g. \tag{6.1}$$

PROOF. We decompose $\boldsymbol{U} = \ker g \oplus \boldsymbol{X}$ and we put $\boldsymbol{Y} := \boldsymbol{X} \cap \ker f$. Now we set $\boldsymbol{Z} := h(\boldsymbol{Y}) = g(\boldsymbol{Y})$ and we use $g', h'\colon \boldsymbol{U}/\boldsymbol{Y} \to \boldsymbol{V}/\boldsymbol{Z}$ to denote the quotients of the maps g, h, respectively.

Observe that $\ker h' = \ker h + \boldsymbol{Y}$ and $\ker g' = \ker g + \boldsymbol{Y}$. Since $\boldsymbol{Y} \subseteq \boldsymbol{X}$, we have that $\boldsymbol{Y} \cap \ker g = \boldsymbol{0}$. Since $\boldsymbol{Y} \subseteq \ker f$, we have that $h \restriction \boldsymbol{Y} = g \restriction \boldsymbol{Y}$, and so $\boldsymbol{Y} \cap \ker h = \boldsymbol{0}$. It follows that

$$\dim\ker h' = \dim\ker h \text{ and } \dim\ker g' = \dim\ker g. \tag{6.2}$$

Clearly $\operatorname{img} h' = \operatorname{img} h + \boldsymbol{Z}$ and $\operatorname{img} g' = \operatorname{img} g + \boldsymbol{Z}$. Since $\boldsymbol{Z} \subseteq \operatorname{img} h \cap \operatorname{img} g$, we conclude that

$$\operatorname{codim}\operatorname{img} h' = \operatorname{codim}\operatorname{img} h \text{ and } \operatorname{codim}\operatorname{img} g' = \operatorname{codim}\operatorname{img} g. \tag{6.3}$$

Since both $\operatorname{codim}\ker f$ and $\operatorname{codim} \boldsymbol{X} = \dim\ker g$ are finite, we have that $\operatorname{codim} \boldsymbol{Y}$ is finite. As $\operatorname{codim}\operatorname{img} g = \operatorname{codim} g(\boldsymbol{X})$ is finite, and the codimension of $\boldsymbol{Y}$ in $\boldsymbol{X}$ is finite (cf. $\operatorname{codim} \boldsymbol{Y}$ is finite), $\operatorname{codim} \boldsymbol{Z} = \operatorname{codim} g(\boldsymbol{Y})$ is finite.

Clearly $\dim\ker h' + \operatorname{codim}\ker h' = \operatorname{codim} \boldsymbol{Y}$ and $\dim\operatorname{img} h' + \operatorname{codim}\operatorname{img} h' = \operatorname{codim} \boldsymbol{Z}$. Since $\operatorname{codim}\ker h' \leq \operatorname{codim} \boldsymbol{Y}$ is finite, we have that $\operatorname{codim}\ker h' = \dim\operatorname{img} h'$. We conclude that

$$\dim\ker h' - \operatorname{codim}\operatorname{img} h' = \operatorname{codim} \boldsymbol{Y} - \operatorname{codim} \boldsymbol{Z}$$

Similarly we prove that

$$\dim\ker g' - \operatorname{codim}\operatorname{img} g' = \operatorname{codim} \boldsymbol{Y} - \operatorname{codim} \boldsymbol{Z},$$

and so

$$\dim\ker h' - \operatorname{codim}\operatorname{img} h' = \dim\ker g' - \operatorname{codim}\operatorname{img} g'. \tag{6.4}$$

Equation (6.4) together with equalities (6.2) and (6.3) give (6.1). ☐

LEMMA 6.2. *Let $f\colon \boldsymbol{U} \to \boldsymbol{V}$ and $g\colon \boldsymbol{V} \to \boldsymbol{W}$ be homomorphisms of vector spaces and $h := g \circ f$ their composition. Let $\boldsymbol{X}$ be a subspace of $\ker g$ such that $\ker g$ decomposes as $\ker g = \boldsymbol{X} \oplus (\operatorname{img} f \cap \ker g)$. Then*

$$\operatorname{codim} f + \operatorname{codim} g = \operatorname{codim} h + 2(\dim \boldsymbol{X}). \tag{6.5}$$

PROOF. The lemma follows from these straightforward equalities:

$$\begin{aligned}
\dim\ker h &= \dim\ker f + \dim(\operatorname{img} f \cap \ker g),\\
\operatorname{codim}\operatorname{img} h &= \operatorname{codim}\operatorname{img} g + \operatorname{codim}(\operatorname{img} f + \ker g),\\
\operatorname{codim}\operatorname{img} f &= \dim \boldsymbol{X} + \operatorname{codim}(\operatorname{img} f + \ker g),\\
\dim\ker g &= \dim \boldsymbol{X} + \dim(\operatorname{img} f \cap \ker g).
\end{aligned}$$

□

The next lemma is "the reason why it works". It is a crucial part of Lemma 7.10.

LEMMA 6.3. *Let* $\boldsymbol{U}$ *be a vector space. Let* x, u, y_i, v_i, $i = 1, 2$, *be endomorphisms of the vector space* $\boldsymbol{U}$ *such that both* $\operatorname{codim} x$ *and* $\operatorname{codim} u$ *are finite as well as all* $\dim y_i$ *and* $\dim v_i$, $i = 1, 2$, *are finite. Put* $f_i := x + y_i$ *and* $g_i := u + v_i$, $i = 1, 2$, *and set*

$$h_1 := g_1 \circ f_1 = (u + v_1) \circ (x + u_1),$$
$$h_2 := f_2 \circ g_2 = (x + u_2) \circ (u + v_2).$$

Then

$$\operatorname{codim} h_1 + \operatorname{codim} h_2 \geq \max\{\operatorname{codim} f_1 + \operatorname{codim} f_2, \operatorname{codim} g_1 + \operatorname{codim} g_2\}. \tag{6.6}$$

PROOF. We are going to prove that

$$\operatorname{codim} h_1 + \operatorname{codim} h_2 \geq \operatorname{codim} g_1 + \operatorname{codim} g_2. \tag{6.7}$$

The other inequality, namely $\operatorname{codim} h_1 + \operatorname{codim} h_2 \geq \operatorname{codim} f_1 + \operatorname{codim} f_2$, is symmetric. We choose decompositions

$$\begin{aligned} \ker g_2 &= \boldsymbol{X} \oplus (\operatorname{img} f_1 \cap \ker g_2) \text{ and} \\ \ker f_2 &= \boldsymbol{Y} \oplus (\operatorname{img} g_1 \cap \ker f_2). \end{aligned} \tag{6.8}$$

Applying Lemma 6.2, we get that

$$\operatorname{codim} h_1 + 2 \dim \boldsymbol{X} = \operatorname{codim} f_1 + \operatorname{codim} g_1 \text{ and}$$
$$\operatorname{codim} h_2 + 2 \dim \boldsymbol{Y} = \operatorname{codim} f_2 + \operatorname{codim} g_2.$$

Since, by the initial assumptions, $\operatorname{codim} u$ is finite and both $\dim v_i$, $i = 1, 2$, are finite, the co-dimensions $\operatorname{codim} g_i$, $i = 1, 2$, are finite due to Lemma 6.1. Thus it suffices to prove that

$$2(\dim \boldsymbol{X} + \dim \boldsymbol{Y}) \leq \operatorname{codim} g_1 + \operatorname{codim} g_2. \tag{6.9}$$

Applying Lemma 6.1 again we get that

$$\dim \ker g_1 - \operatorname{codim} \operatorname{img} g_1 = \dim \ker u - \operatorname{codim} \operatorname{img} u = \dim \ker g_2 - \operatorname{codim} \operatorname{img} g_2,$$

hence

$$\dim \ker g_1 + \operatorname{codim} \operatorname{img} g_2 = \dim \ker g_2 + \operatorname{codim} \operatorname{img} g_1,$$

whence

$$\operatorname{codim} g_1 + \operatorname{codim} g_2 = 2(\dim \ker g_2 + \operatorname{codim} \operatorname{img} g_1). \tag{6.10}$$

It follows from (6.8) that $\dim \boldsymbol{X} \leq \dim \ker g_2$ and $\dim \boldsymbol{Y} \leq \operatorname{codim} \operatorname{img} g_1$. This together with previous equality (6.10) implies inequality (6.9), and, consequently, inequality (6.7). This concludes the proof. □

LEMMA 6.4. *Let* $f\colon \boldsymbol{U} \to \boldsymbol{U}$ *be an endomorphism of a vector space* $\boldsymbol{U}$ *of a finite dimension. We denote by* 1 *the identity endomorphism of* $\boldsymbol{U}$. *Then* $\operatorname{codim}(1 + f)$ *is finite and even.*

PROOF. We apply Lemma 6.1 putting $g := 1$ and $h := g + f = 1 + f$. Note that $\dim\ker 1 = \operatorname{codim}\operatorname{img} 1 = 0$. Thus it follows from (6.1) that $\dim\ker(1 + f) = \operatorname{codim}\operatorname{img}(1 + f)$, hence $\operatorname{codim}(1 + f) = \dim\ker(1 + f) + \operatorname{codim}\operatorname{img}(1 + f)$ is even. □

7. The example of Bergman and Goodearl

In this section we recall Goodearl's modification [**22**, Example 5.10] of the Bergman's example [**22**, Example 4.26] of a regular ring $\boldsymbol{R}_2$ which is not unit-regular but the matrix rings $\mathbb{M}_n(\boldsymbol{R}_2)$ are directly finite for all positive integers. The ring $\boldsymbol{R}_2$ is constructed as follows: Let $\boldsymbol{T}$ denote the ring $\mathbb{F}[[\,t\,]]$ of all formal power series over a field $\mathbb{F}$ in an indeterminate t, and let $\Bbbk$ denote the quotient field of $\boldsymbol{T}$. Denote by $\boldsymbol{S}$ the ring of all $a \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ such that there is a positive integer n and $b \in \Bbbk$ with $(a - b)t^n\boldsymbol{T} = 0$ (i.e., $b\,t^n\boldsymbol{T} \subseteq \boldsymbol{T}$ and the restriction $a \restriction t^n\boldsymbol{T}$ coincides with the multiplication by b). It turns out that the element $b \in \Bbbk$ is unique and the correspondence $a \mapsto b := \varphi(a)$ determines an $\mathbb{F}$-algebra homomorphism $\varphi\colon \boldsymbol{S} \to \Bbbk$ (cf. [**22**, Example 4.26]). Finally let us denote by $\boldsymbol{S}^{\mathrm{op}}$ the opposite ring of the ring $\boldsymbol{S}$ and set

$$\boldsymbol{R}_2 := \{\langle a_1, a_2\rangle \in \boldsymbol{S} \times \boldsymbol{S}^{\mathrm{op}} \mid \varphi(a_1) = \varphi(a_2)\}\}.$$

Observe that every nonzero element a of $\boldsymbol{T} = \mathbb{F}[[t]]$ is a product $a = t^n a'$ for some $n \in \mathbb{N}_0$ and some invertible $a' \in \boldsymbol{T}$. Moreover, every nonzero $b \in \Bbbk$ is of the form $b = t^z b'$ for a unique (possibly negative) integer z and $b' \in \boldsymbol{T}$ invertible in $\boldsymbol{T}$. Denote the unique exponent z by $\nu(b)$ and set $\nu(0) := 0$. Let $b \neq 0$ be an element of $\Bbbk$. Observe that whenever $n + \nu(b) \geq 0$ for a positive integer n, the left multiplication by b determines a bijection $t^n\boldsymbol{T} \to t^{n+\nu(b)}\boldsymbol{T}$.

Given an element $\boldsymbol{a} = \langle a_1, a_2\rangle \in \boldsymbol{R}_2$, we define $\varphi(\boldsymbol{a}) := \varphi(a_1) = \varphi(a_2)$. For elements $a \in \boldsymbol{S}$ and $\boldsymbol{a} \in \boldsymbol{R}_2$ we set $\nu(a) := \nu(\varphi(a))$ and $\nu(\boldsymbol{a}) = \nu(\varphi(\boldsymbol{a}))$, respectively. Finally given an element $\boldsymbol{a} = \langle a_1, a_2\rangle \in \boldsymbol{R}_2$, we define $\dim\boldsymbol{a} := \dim a_1 + \dim a_2$ and $\operatorname{codim}\boldsymbol{a} := \operatorname{codim} a_1 + \operatorname{codim} a_2$.[1]

Let $\boldsymbol{a} \in \boldsymbol{R}_2$. Observe that $\varphi(\boldsymbol{a}) = 0$ implies that $\dim\boldsymbol{a}$ is finite while $\varphi(\boldsymbol{a}) \neq 0$ implies that $\operatorname{codim}\boldsymbol{a}$ is finite; the latter follows from the first statement of Lemma 6.1.

LEMMA 7.1. *Let a be an element of the ring $\boldsymbol{S}$. Then*

$$\begin{aligned} \varphi(a) = 0 &\implies \dim a \textit{ is even}, \\ \varphi(a) = 1 &\implies \operatorname{codim} a \textit{ is even}. \end{aligned} \tag{7.1}$$

PROOF. It follows from the finiteness of $\dim a$, $\operatorname{codim} a$, respectively, and Lemma 6.4. □

COROLLARY 7.2. *Let $\boldsymbol{a} = (a_1, a_2)$ be an element of the ring $\boldsymbol{R}_2$. Then*

(i) *$\varphi(\boldsymbol{a}) = 0$ implies that both the dimensions $\dim a_1$ and $\dim a_2$ are even;*

(ii) *$\varphi(\boldsymbol{a}) \neq 0$ implies that $\operatorname{codim}\boldsymbol{a}$ is even.*

---

[1]Note that this is consistent with the notation introduced in Section 6

Proof. If $\varphi(\boldsymbol{a}) = 0$, then both the dimensions $\dim a_1$ and $\dim a_2$ are finite and (1) follows readily from Lemma L011. Suppose that $\boldsymbol{a} \in \boldsymbol{R}_2$ satisfy $\varphi(\boldsymbol{a}) \neq 0$. Since $\boldsymbol{R}_2$ is a regular ring, there is an idempotent $\boldsymbol{e} \in \boldsymbol{R}_2$ such that $\boldsymbol{e}\boldsymbol{R}_2 = \boldsymbol{a}\boldsymbol{R}_2$. Then clearly $\operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a})$. From $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e})$ and $\boldsymbol{e} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a})$ we get that $\operatorname{codim} \boldsymbol{a} \leq \operatorname{codim} \boldsymbol{e}$ and $\operatorname{codim} \boldsymbol{e} \leq \operatorname{codim} \boldsymbol{a}$, due to Lemma 6.3. Therefore $\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{e}$. Since $\boldsymbol{e}$ is an idempotent of a finite codimension, $\varphi(\boldsymbol{e}) = 1$, and so the codimension of $\boldsymbol{e}$ is even due to Lemma 7.1. □

Lemma 7.3. *Let $\boldsymbol{U}_i$, $\boldsymbol{V}_i$, $i = 1, 2$, be finite-dimensional vector spaces over a common field $\mathbb{F}$, let $a \colon \boldsymbol{U}_1 \to \boldsymbol{U}_2$ and $b \colon \boldsymbol{V}_1 \to \boldsymbol{V}_2$ be linear maps. Then $\dim a \leq \dim b$ if and only if there are linear maps $r \colon \boldsymbol{U}_1 \to \boldsymbol{V}_1$ and $s \colon \boldsymbol{V}_2 \to \boldsymbol{U}_2$ such that $a = sbr$.*

Proof. Folklore. □

Lemma 7.4. *Let $\boldsymbol{U}$ be a vector space,*

$$\boldsymbol{U} = \boldsymbol{U}_0 \supseteq \boldsymbol{U}_1 \supseteq \boldsymbol{U}_2 \supseteq \dots$$

be a decreasing sequence of subspaces of $\boldsymbol{U}$, and $\boldsymbol{V}$ a finite-dimensional subspace of $\boldsymbol{U}$. Suppose that

$$\boldsymbol{V} \cap \left(\bigcap_{i \in \mathbb{N}} \boldsymbol{U}_i \right) = \boldsymbol{0},$$

then there is a positive integer n such that $\boldsymbol{V} \cap \boldsymbol{U}_n = \boldsymbol{0}$.

Proof. For every positive integer n we set $\boldsymbol{V}_n := \boldsymbol{V} \cap \boldsymbol{U}_n$. Note that

$$\boldsymbol{V} = \boldsymbol{V}_0 \supseteq \boldsymbol{V}_1 \supseteq \boldsymbol{V}_2 \supseteq \dots \tag{7.2}$$

is a decreasing sequence of subspaces of $\boldsymbol{V}$ such that $\bigcap_{i \in \mathbb{N}} \boldsymbol{V}_i = \boldsymbol{0}$. Since $\boldsymbol{V}$ is finite-dimensional, the sequence (7.2) is eventually stationary. Therefore there is $n \in \mathbb{N}$ such that $\boldsymbol{0} = \boldsymbol{V}_n = \boldsymbol{V} \cap \boldsymbol{U}_n$. □

We set

$$\boldsymbol{I} := \{a \in \boldsymbol{S} \mid \varphi(a) = 0\}.$$

It is straightforward to see that $\boldsymbol{I}$ is an ideal of the ring $\boldsymbol{S}$.

Lemma 7.5. *For all $a, b \in \boldsymbol{I}$, the following properties are equivalent:*

(i) $\dim a \leq \dim b$.
(ii) $a \in \operatorname{tr}_{\boldsymbol{I}}(b)$.
(iii) $a \in \operatorname{tr}_{\boldsymbol{S}}(b)$.

Proof. $(1 \Rightarrow 2)$ Let $\boldsymbol{U}_1$ and $\boldsymbol{V}_1$ denote complements of $\ker a$ and $\ker b$, respectively, in $\boldsymbol{T}$ (viewed as a vector space over the field $\mathbb{F}$). We set $\boldsymbol{U}_2 := \operatorname{img} a$ and $\boldsymbol{V}_2 = \operatorname{img} b$ and we denote by $a' \colon \boldsymbol{U}_1 \to \boldsymbol{U}_2$, resp. $b' \colon \boldsymbol{V}_1 \to \boldsymbol{V}_2$ the restrictions $a' := a \restriction \boldsymbol{U}_1$, resp. $b' := b \restriction \boldsymbol{V}_1$. Observe that $\dim a' \leq \dim b'$. Applying Lemma 7.3, we find homomorphisms $r' \colon \boldsymbol{U}_1 \to \boldsymbol{V}_1$ and $s' \colon \boldsymbol{V}_2 \to \boldsymbol{U}_2$ such that $a' = s'b'r'$. There are positive integers m and n

such that $\boldsymbol{U}_1 \cap t^m\boldsymbol{T} = \boldsymbol{0} = \boldsymbol{V}_1 \cap t^n\boldsymbol{T}$ due to Lemma 7.4. It follows that there are r and s in $\operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ extending r' and s', satisfying $t^m\boldsymbol{T} \subseteq \ker r$ and $t^nT \subseteq \ker s$, respectively. It follows that $r, s \in \boldsymbol{I}$ and that $a = sbr$, hence $a \in \operatorname{tr}_{\boldsymbol{I}}(b)$. The implication (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) follows from Lemma 7.3. □

We set

$$\boldsymbol{J}_2 := \{\boldsymbol{a} \in \boldsymbol{R}_2 \mid \varphi(\boldsymbol{a}) = 0\},$$

and observe that $\boldsymbol{J}_2$ is an ideal of the ring $\boldsymbol{R}_2$. The next corollary follows readily from Lemma 7.5.

COROLLARY 7.6. *Let $\boldsymbol{a} = \langle a_1, a_2\rangle$ and $\boldsymbol{b} = \langle b_1, b_2\rangle$ be elements of the ideal $\boldsymbol{J}_2$. The following properties are equivalent:*

(i) $\dim a_i \leq \dim b_i$, *for all* $i = 1, 2$.
(ii) $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{J}_2}(\boldsymbol{b})$.
(iii) $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$.

For each ordered pair $m \leq n$ of non-negative integers we denote by $e_{m,n} \colon \boldsymbol{T} \to \boldsymbol{T}$ the projection onto $\bigoplus_{i=m}^{n} t^i\mathbb{F}$ given by

$$\sum_{i=0}^{\infty} a_i t^i \mapsto \sum_{i=m}^{n} a_i t^i.$$

LEMMA 7.7. *Let λ be a positive integer and $\boldsymbol{e} = \langle e_1, e_2\rangle$ an idempotent of the ring $\boldsymbol{R}_2$ such that $\varphi(\boldsymbol{e}) \neq 0$. Then the following hold true:*

(i) *If* $\operatorname{codim} e_1 \geq 2\lambda$, *there is* $\boldsymbol{f} = \langle f_1, f_2\rangle \in \operatorname{Idem}(\boldsymbol{R}_2)$ *with*

$$\operatorname{codim} f_1 = \operatorname{codim} e_1 - 2\lambda$$
$$\operatorname{codim} f_2 = \operatorname{codim} e_2 + 2\lambda$$

and elements $\boldsymbol{r}, \boldsymbol{s} \in \boldsymbol{R}_2$ *such that* $\varphi(\boldsymbol{r}) = t^{\lambda}$, $\varphi(\boldsymbol{s}) = t^{-\lambda}$, *and* $\boldsymbol{f} = \boldsymbol{ser}$.

(ii) *If* $\operatorname{codim} e_2 \geq 2\lambda$, *there is* $\boldsymbol{f} = \langle f_1, f_2\rangle \in \operatorname{Idem}(\boldsymbol{R}_2)$ *with*

$$\operatorname{codim} f_1 = \operatorname{codim} e_1 + 2\lambda$$
$$\operatorname{codim} f_2 = \operatorname{codim} e_2 - 2\lambda$$

and elements $\boldsymbol{r}, \boldsymbol{s} \in \boldsymbol{R}_2$ *such that* $\varphi(\boldsymbol{r}) = t^{-\lambda}$, $\varphi(\boldsymbol{s}) = t^{\lambda}$, *and* $\boldsymbol{f} = \boldsymbol{ser}$.

PROOF. We prove property (1). Property (2) is symmetric. Since $\varphi(\boldsymbol{e}) \neq 0$ and $\boldsymbol{e}$ is an idempotent, we have that $\varphi(\boldsymbol{e}) = 1$. By the definition (of the ring $\boldsymbol{R}_2$) there is a natural number n such that $(e_i - 1)t^n\boldsymbol{T} = 0$, in particular $\ker e_i \cap t^n\boldsymbol{T} = \boldsymbol{0}$, for all $i = 1, 2$. For each $i = 1, 2$ we pick a complement $\boldsymbol{U}_i$ of $t^n\boldsymbol{T} \oplus \ker e_i$ in $\boldsymbol{T}$.

Observe that the restrictions $e_i \restriction (t^n\boldsymbol{T} \oplus \boldsymbol{U}_i)$ are one-to-one. Since $e_i \restriction t^n\boldsymbol{T}$ coincides with identity, we conclude that $e_i\boldsymbol{U}_i \cap t^n\boldsymbol{T} = \boldsymbol{0}$, for all $i = 1, 2$. Since $\operatorname{codim} t^n\boldsymbol{T} = n$ is finite, we get that

$$\operatorname{codim} \operatorname{img} e_i = \operatorname{codim}(e_i\boldsymbol{U}_i \oplus t^n\boldsymbol{T}) = \operatorname{codim}(\boldsymbol{U}_i \oplus t^n\boldsymbol{T}) = \dim \ker e_i,$$

hence $\operatorname{codim} e_i = 2 \dim \ker e_i$, for both $i = 1, 2$. Since $\operatorname{codim} e_1 \geq 2\lambda$, we get that $\dim \ker e_1 \geq \lambda$, hence $\dim \boldsymbol{U}_1 = \operatorname{codim} t^n \boldsymbol{T} - \dim \ker e_1 \leq n - \lambda$. It follows that there are $\mathbb{F}$-linear maps

$$r_1' \colon \bigoplus_{i=0}^{n-\lambda-1} t^i \mathbb{F} \to \boldsymbol{U}_1 \text{ and } s_1' \colon e_1 \boldsymbol{U}_1 \to \bigoplus_{i=0}^{n-\lambda-1} t^i \mathbb{F}$$

such that the composition $s_1' e_1 r_1'$ is an idempotent linear map with $\dim s_1' e_1 r_1' = 2 \dim \boldsymbol{U}_1$. Clearly $\dim \boldsymbol{U}_2 \leq \operatorname{codim} t^n \boldsymbol{T} = n$. Therefore there are

$$r_2' \colon \bigoplus_{i=0}^{n+\lambda-1} t^i \mathbb{F} \to \boldsymbol{U}_2 \text{ and } s_2' \colon e_2 \boldsymbol{U}_2 \to \bigoplus_{i=0}^{n+\lambda-1} t^i \mathbb{F}$$

such that $s_2' e_2 r_2'$ is idempotent and $\dim s_2' e_2 r_2' = 2 \dim \boldsymbol{U}_2$. We extend the linear maps r_i', s_i', $i = 1, 2$, to $\mathbb{F}$-endomorphisms of $\boldsymbol{T}$ by setting

$$r_1'(t^{n-\lambda} \boldsymbol{T}) = \boldsymbol{0}, \; r_2'(t^{n+\lambda} \boldsymbol{T}) = \boldsymbol{0}, \text{ and } s_i'(t^n \boldsymbol{T} \oplus \boldsymbol{W}_i) = \boldsymbol{0},$$

where $\boldsymbol{W}_i$ are complements of $t^n \boldsymbol{T} \oplus e_i \boldsymbol{U}_i$, for both $i = 1, 2$. Observe that r_i', s_i' belong to $\boldsymbol{I}$.

Let us define $r \colon t^{n-\lambda} \boldsymbol{T} \to \boldsymbol{T}^n$, resp. $s \colon t^n \boldsymbol{T} \to \boldsymbol{T}^{n-\lambda}$, to be the $\mathbb{F}$-liner maps corresponding to multiplications by t^λ, resp. $t^{-\lambda}$, and we extend these maps to $\mathbb{F}$-endomorphisms of $\boldsymbol{T}$ by setting $r(\bigoplus_{i=0}^{n-\lambda-1} t^i \mathbb{F}) = s(\bigoplus_{i=0}^{n} t^i \mathbb{F}) = \boldsymbol{0}$. Observe that both r and s belong to $\boldsymbol{S}$.

We set $r_i := r + r_i'$ and $s_i = s + s_i'$, for both $i = 1, 2$. Then it is straightforward from the constructions of the endomorphisms r, r_i, s, and s_i that $\varphi(r_i) = \varphi(r) = t^\lambda$, $\varphi(s_i) = \varphi(s) = t^{-\lambda}$, and that $f_i = s_i' e_i r_i' = s_i e_i r_i$ are idempotents, for all $i = 1, 2$. Furthermore we have that

$$\operatorname{codim} f_1 = \operatorname{codim} e_1 - 2\lambda \text{ and } \operatorname{codim} f_2 = \operatorname{codim} e_2 + 2\lambda.$$

Finally setting $\boldsymbol{f} := \langle f_1, f_2 \rangle$, $\boldsymbol{r} = \langle r_1, r_2 \rangle$, and $\boldsymbol{s} = \langle s_1, s_2 \rangle$, we get the desired idempotent and elements of $\boldsymbol{R}_2$ such that $\boldsymbol{f} = \boldsymbol{ser}$. $\square$

Observe that $\boldsymbol{f} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e})$ and since $\operatorname{codim} \boldsymbol{f}$ is finite, and it is an idempotent, we have that $\varphi(\boldsymbol{f}) = 1$.

Lemma 7.8. *Let $e, f \in \boldsymbol{S} \setminus \boldsymbol{I}$ be idempotents. Then* $\operatorname{codim} e \geq \operatorname{codim} f$ *if and only if there are elements $r, s \in \boldsymbol{S}$ such that $\varphi(r) = \varphi(s) = 1$ and $e = sfr$. In particular, if any of the equivalent properties is satisfied, then $e \in \operatorname{tr}_{\boldsymbol{S}}(f)$.*

Proof. ($\Rightarrow$) First suppose that $e = sfr$ for some $r, s \in \boldsymbol{T}$ with $\varphi(r) = \varphi(s) = 1$. Since $e, f \in \operatorname{Idem}(\boldsymbol{S}) \setminus \boldsymbol{I}$, there is a positive integer n such that $(e-1)t^n \boldsymbol{T} = (s-1)t^n \boldsymbol{T} = (f-1)t^n \boldsymbol{T} = (r-1)t^n \boldsymbol{T} = \boldsymbol{0}$. It follows that $et^n \boldsymbol{T} = ft^n \boldsymbol{T} = rt^n \boldsymbol{T} = st^n \boldsymbol{T} = t^n \boldsymbol{T}$, hence $e, f, r, s \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ induce endomorphisms e', f', r', and s' of the finite-dimensional $\mathbb{F}$-vector space $\boldsymbol{T}/t^n \boldsymbol{T}$. From $\operatorname{codim} e = \operatorname{codim} e'$, $\operatorname{codim} f = \operatorname{codim} f'$, and $\dim e' = \dim s' f' r' \leq \dim f'$, we deduce that

$$\operatorname{codim} e = \operatorname{codim} e' = 2n - \dim e' \geq 2n - \dim f' = \operatorname{codim} f' = \operatorname{codim} f.$$

($\Leftarrow$) Suppose now that $\operatorname{codim} e \geq \operatorname{codim} f$. Since e and f are idempotents not in $\boldsymbol{I}$, we have that $\varphi(e) = \varphi(f) = 1$. It follows that there is a positive integer n such that $(e-1)t^n\boldsymbol{T} = (f-1)t^n\boldsymbol{T} = \boldsymbol{0}$. Therefore $\ker e \cap t^n\boldsymbol{T} = \ker f \cap t^n\boldsymbol{T} = \boldsymbol{0}$. We pick subspaces $\boldsymbol{U}$ and $\boldsymbol{V}$ of the $\mathbb{F}$-vector space $\boldsymbol{T}$ such that

$$\boldsymbol{T} = \boldsymbol{U} \oplus \ker e \oplus t^n\boldsymbol{T} = \boldsymbol{V} \oplus \ker f \oplus t^n\boldsymbol{T}$$

and we set $e' := e \restriction \boldsymbol{U}$, $f' := f \restriction \boldsymbol{V}$. Since $\ker e \cap (\boldsymbol{U} \oplus t^n\boldsymbol{T}) = \boldsymbol{0}$ and the restriction $e \restriction t^n\boldsymbol{T}$ coincides with the identity map, we have that $e\boldsymbol{T} = e\boldsymbol{U} \oplus t^n\boldsymbol{T}$. Similarly we prove that $f\boldsymbol{T} = f\boldsymbol{V} \oplus t^n\boldsymbol{T}$. It follows that

$$\dim \boldsymbol{U} = \dim e\boldsymbol{U} = n - \frac{\operatorname{codim} e}{2} \leq n - \frac{\operatorname{codim} f}{2} = \dim f\boldsymbol{V} = \dim \boldsymbol{V}$$

and there are linear maps $r' \colon \boldsymbol{U} \to \boldsymbol{V}$ and $s' \colon f\boldsymbol{V} \to e\boldsymbol{U}$ such that $e' = s'f'r'$. There are $r, s \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{W})$ such that

$$r \restriction \boldsymbol{U} = r', \ \ker r \geq \ker e, \ \text{and } (r-1)t^n\boldsymbol{T} = \boldsymbol{0},$$
$$s \restriction \boldsymbol{V} = s', \ \ker s \geq \ker f, \ \text{and } (s-1)t^n\boldsymbol{T} = \boldsymbol{0}.$$

We conclude that r and s are elements of $\boldsymbol{S}$ satisfying $\varphi(r) = \varphi(s) = 1$ and $e = sfr$. As an immediate consequence we have that $e \in \operatorname{tr}_{\boldsymbol{S}}(f)$. $\square$

The next corollary will be applied in the forthcoming section.

COROLLARY 7.9. *Let λ be a positive integer and $\boldsymbol{e} = \langle e_1, e_2\rangle$ an idempotent in $\boldsymbol{R}_2 \setminus \boldsymbol{J}_2$. Then the following hold true:*

(i) *Suppose that* $\operatorname{codim} e_1 \geq 2\lambda$ *and let $\boldsymbol{f} = \langle f_1, f_2\rangle$ be the idempotent constructed in Lemma 7.7. Then there are elements $\boldsymbol{r}^*, \boldsymbol{s}^* \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}^*) = t^{-\lambda}$ and $\varphi(\boldsymbol{s}^*) \in t^{\lambda}$ such that $\boldsymbol{e} = \boldsymbol{s}^*\boldsymbol{f}\boldsymbol{r}^*$.*

(ii) *Suppose that* $\operatorname{codim} e_2 \geq 2\lambda$ *and let $\boldsymbol{f} = \langle f_1, f_2\rangle$ be the idempotent constructed in Lemma 7.7. Then there are elements $\boldsymbol{r}^*, \boldsymbol{s}^* \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}^*) = t^{\lambda}$ and $\varphi(\boldsymbol{s}^*) \in t^{-\lambda}$ such that $\boldsymbol{e} = \boldsymbol{s}^*\boldsymbol{f}\boldsymbol{r}^*$.*

PROOF. Both the cases are symmetric, we only prove (1). Suppose that $\operatorname{codim} e_1 \geq 2\lambda$. Then $\operatorname{codim} f_2 = \operatorname{codim} e_2 + 2\lambda \geq 2\lambda$, and so there is an idempotent $\boldsymbol{g} = \langle g_1, g_2\rangle \in \boldsymbol{R}_2$ with

$$\begin{aligned} &\operatorname{codim} g_1 = \operatorname{codim} f_1 + 2\lambda = \operatorname{codim} e_1 \text{ and} \\ &\operatorname{codim} g_2 = \operatorname{codim} f_2 - 2\lambda = \operatorname{codim} e_2, \end{aligned} \tag{7.3}$$

and element $\boldsymbol{r}', \boldsymbol{s}' \in \boldsymbol{R}_2$ such that $\varphi(\boldsymbol{r}') = t^{-\lambda}$, $\varphi(\boldsymbol{s}') = t^{\lambda}$, and $\boldsymbol{g} = \boldsymbol{s}'\boldsymbol{f}\boldsymbol{r}'$ due to Lemma 7.7. Applying Lemma 7.8, we get elements $\boldsymbol{r}'', \boldsymbol{s}'' \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}'') = \varphi(\boldsymbol{s}'') = 1$ and

$$\boldsymbol{e} = \boldsymbol{s}''\boldsymbol{g}\boldsymbol{r}'' = \boldsymbol{s}''\boldsymbol{s}'\boldsymbol{f}\boldsymbol{r}'\boldsymbol{r}''.$$

We put $\boldsymbol{r}^* = \boldsymbol{r}'\boldsymbol{r}''$ and $\boldsymbol{s}^* = \boldsymbol{s}''\boldsymbol{s}'$. It is straightforward to compute that

$$\varphi(\boldsymbol{r}^*) = \varphi(\boldsymbol{r}'\boldsymbol{r}'') = \varphi(\boldsymbol{r}')\varphi(\boldsymbol{r}'') = t^{-\lambda} \text{ and } \varphi(\boldsymbol{s}^*) = \varphi(\boldsymbol{s}''\boldsymbol{r}') = \varphi(\boldsymbol{s}'')\varphi(\boldsymbol{s}') = t^{\lambda}.$$

$\square$

LEMMA 7.10. *Let $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$. Then $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$ if and only if* $\operatorname{codim} \boldsymbol{a} \geq \operatorname{codim} \boldsymbol{b}$.

PROOF. ($\Rightarrow$) It follows from Lemma 6.3 that

$$\operatorname{codim} \boldsymbol{cd} \geq \max\{\operatorname{codim} \boldsymbol{c}, \operatorname{codim} \boldsymbol{d}\}, \tag{7.4}$$

for all $\boldsymbol{c}, \boldsymbol{d} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$. If $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$, then $\boldsymbol{a} = \boldsymbol{sbr}$ for some $\boldsymbol{s}, \boldsymbol{r} \in \boldsymbol{R}_2$. Observe that $\boldsymbol{s}, \boldsymbol{r} \notin \boldsymbol{J}_2$, for otherwise $\boldsymbol{a} \in \boldsymbol{J}_2$. Applying (7.4) twice, we get that

$$\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{sbr} \geq \operatorname{codim} \boldsymbol{br} \geq \operatorname{codim} \boldsymbol{b}.$$

($\Leftarrow$) Suppose that $\operatorname{codim} \boldsymbol{a} \geq \operatorname{codim} \boldsymbol{b}$. Since $\boldsymbol{R}_2$ is regular there are idempotents $\boldsymbol{e} = \langle e_1, e_2 \rangle$ and $\boldsymbol{f} = \langle f_1, f_2 \rangle$ such that $\boldsymbol{eR}_2 = \boldsymbol{aR}_2$ and $\boldsymbol{fR}_2 = \boldsymbol{bR}_2$, respectively. As a consequence we get that

$$\operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a}) \text{ and } \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b}). \tag{7.5}$$

By the already proved implication we have that

$$\operatorname{codim} \boldsymbol{e} = \operatorname{codim} \boldsymbol{a} \geq \operatorname{codim} \boldsymbol{b} = \operatorname{codim} \boldsymbol{f}.$$

By Lemma 7.7, there is an idempotent $\boldsymbol{g} = \langle g_1, g_2 \rangle \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f})$ such that $\operatorname{codim} e_1 \geq \operatorname{codim} g_1$ and $\operatorname{codim} e_2 \geq \operatorname{codim} g_2$. By Lemma 7.8, there are elements $r_i, s_i \in \boldsymbol{S}$, $i = 1, 2$, such that $\varphi(r_i) = \varphi(s_i) = 1$ and $e_i = s_i g_i r_i$. It follows that $\boldsymbol{e} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{g}) \subseteq \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f})$, and so $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{e}) \subseteq \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{f}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$ due to (7.5). □

LEMMA 7.11. *Let $\boldsymbol{a} = \langle a_1, a_2 \rangle$ and $\boldsymbol{b} = \langle b_1, b_2 \rangle$ be elements of the ring $\boldsymbol{R}_2$. Then $\operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{a}) = \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b})$ if and only if either both $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to $\boldsymbol{J}_2$ and $\dim a_i = \dim b_i$ for both $i = 1, 2$, or none of the elements $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to $\boldsymbol{J}_2$ and then $\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{b}$.*

PROOF. ($\Rightarrow$) Assume that $\operatorname{tr}_{\boldsymbol{R}_2}(a) = \operatorname{tr}_{\boldsymbol{R}_2}(b)$. Since $\boldsymbol{J}_2$ is a two-sided ideal of $\boldsymbol{R}_2$, either both the elements $\boldsymbol{a}$ and $\boldsymbol{b}$ or none of them belong to $\boldsymbol{J}_2$. If $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{J}_2$, then both $\dim a_i = \dim b_i$, $i = 1, 2$, due to Corollary 7.6. In the other case when $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$, the equality $\operatorname{codim} \boldsymbol{a} = \operatorname{codim} \boldsymbol{b}$ holds true due to Lemma 7.10. ($\Leftarrow$) This implication follows readily from Corollary 7.6 and Lemma 7.10. □

LEMMA 7.12. *Let $g \in \boldsymbol{I}$ be an idempotent, λ and μ non-negative integers such that $\dim g = 2\lambda + 2\mu$. Then there is a pair $e, f \in \boldsymbol{I}$ of orthogonal idempotents such that $\dim e = 2\lambda$, $\dim f = 2\mu$, and $g = e + f$.*

PROOF. Since $g \in \boldsymbol{I}$, it is of a finite dimension, and so $\dim \operatorname{img} g = (\dim g)/2 = \lambda + \mu$. We pick a decomposition $\operatorname{img} g = \boldsymbol{U} \oplus \boldsymbol{V}$ with $\dim \boldsymbol{U} = \lambda$ and $\dim \boldsymbol{V} = \mu$. Let e be an endomorphism of $\boldsymbol{T}$ such that $\ker e = \ker g \oplus \boldsymbol{U}$ and $e \restriction \boldsymbol{V} = g \restriction \boldsymbol{V}$. Putting $f = g - e$, we get a pair e, f of orthogonal idempotents with the desired properties. □

LEMMA 7.13. *Let $g \in \boldsymbol{S} \setminus \boldsymbol{I}$ be an idempotent, λ and μ non-negative integers such that $2\lambda = \operatorname{codim} g + 2\mu$. Then there is a pair $e \in \boldsymbol{S} \setminus \boldsymbol{I}$ and*

$f \in \boldsymbol{I}$ *of orthogonal idempotents such that* $\operatorname{codim} e = 2\lambda$, $\dim f = 2\mu$, *and* $g = e + f$.

Proof. From $g \in \boldsymbol{S} \setminus \boldsymbol{I}$ we infer that $\dim \operatorname{img} g$ is infinite. We find a decomposition $\operatorname{img} g = \boldsymbol{U} \oplus \boldsymbol{V}$ such that $\dim \boldsymbol{U} = \mu$. Let $f \in \operatorname{End}_{\mathbb{F}}(\boldsymbol{T})$ be such that $\ker f = \ker g \oplus \boldsymbol{V}$ and $f \restriction \boldsymbol{U} = g \restriction \boldsymbol{U}$. Putting $e = g - f$, we get a pair of orthogonal idempotents $e \in \boldsymbol{S} \setminus \boldsymbol{I}$ and $f \in \boldsymbol{I}$ satisfying the desired properties. $\square$

Applying Lemma 7.13 we get that

Corollary 7.14. *Let* $\boldsymbol{g} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$ *be an idempotent. Let* λ, μ_1, μ_2 *be non-negative integers such that* $2\lambda = \operatorname{codim} \boldsymbol{g} + 2\mu_1 + 2\mu_2$. *Then there are orthogonal idempotents* $\boldsymbol{e} \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$ *and* $\boldsymbol{f} = \langle f_1, f_2 \rangle \in \boldsymbol{J}_2$ *such that* $\operatorname{codim} \boldsymbol{e} = 2\lambda$, $\dim f_i = 2\mu_i$, *for all* $i = 1, 2$, *and* $\boldsymbol{g} = \boldsymbol{e} + \boldsymbol{f}$.

Theorem 7.15. *The monoid* $\boldsymbol{V}(\boldsymbol{R}_2)$ *is isomorphic to* $\boldsymbol{C}_2 = \boldsymbol{D}_2$ *and, via the isomorphism* $\psi_2^{-1} \colon \boldsymbol{D}_2 \to \boldsymbol{B}_2$, *also to* $\boldsymbol{B}_2$.

Proof. We define a map $\gamma \colon \operatorname{Idem}(\boldsymbol{R}_2) \to \boldsymbol{C}_2$ by

$$\boldsymbol{e} = \langle e_1, e_2 \rangle \mapsto \begin{cases} \langle 0, \frac{\dim e_1}{2}, \frac{\dim e_2}{2} \rangle & \text{if } \varphi(\boldsymbol{e}) = 0, \\ \langle 1, -\frac{\operatorname{codim} \boldsymbol{e}}{2}, \rangle & \text{if } \varphi(\boldsymbol{e}) = 1, \end{cases}$$

and we verify that the properties (1-3) of Corollary 3.9 are satisfied. Property (1) follows from Lemma 7.11.

Property (2) is a consequence of Lemma 7.12 in the case that $\varphi(\boldsymbol{e}) = 0$ and Corollary 7.14 if $\varphi(\boldsymbol{e}) = 1$. Observe that in the latter case, when $\varphi(\boldsymbol{e}) = 1$, if $\gamma(\boldsymbol{e}) = u + v$ for some $u, v \in \boldsymbol{C}_2$, one of them belongs to $\boldsymbol{O}_2$. This is because

$$\gamma(\boldsymbol{e}) = \langle 1, -\frac{\operatorname{codim} \boldsymbol{e}}{2} \rangle,$$

and so $u_0 + v_0 = 1$.

By the definition, $\gamma(1) = \langle 1, 0 \rangle$ which is an order-unit in $\boldsymbol{C}_2$, thus property (3) holds true as well.

Since the ring $\boldsymbol{R}_2$ is directly finite due to [**22**, Example 5.10], the map γ extends to a unique isomorphism $\beta \colon \boldsymbol{V}(\boldsymbol{R}_2) \to \boldsymbol{C}_2$, due to Corollary 3.9. $\square$

8. Representing the monoids $\boldsymbol{B}_{2n}$

Let $\boldsymbol{R}_2$ and $\boldsymbol{S}$ be the rings defined in the previous section. Given a positive integer n, we set

$$\boldsymbol{R}_{2n} := \{\langle a_1, a_2, \dots, a_{2n} \rangle \mid a_{2i-1} \in \boldsymbol{S}, a_{2i} \in \boldsymbol{S}^{\mathrm{op}},\ \varphi(a_1) = \dots = \varphi(a_{2n})\}.$$

Observe that $\boldsymbol{R}_{2n}$ is a sub-direct product of copies of the ring $\boldsymbol{R}_2$. Therefore it is regular and directly finite (cf. [**22**, Proposition 1.4] and [**22**, Lemma 5.1], respectively). Further, we set

$$\boldsymbol{J}_{2n} := \{\langle a_1, a_2, \dots, a_{2n} \rangle \in \boldsymbol{R}_{2n} \mid \varphi(a_1) = \dots = \varphi(a_{2n}) = 0\}.$$

Clearly, the set $\boldsymbol{J}_{2n}$ forms a two-sided ideal of the ring $\boldsymbol{R}_{2n}$. Applying Lemma 7.5 we get, similarly as in the previous section, that

LEMMA 8.1. *For a pair of elements $\boldsymbol{a} = \langle a_1, \dots, a_{2n}\rangle$ and $\boldsymbol{b} = \langle b_1, \dots, b_{2n}\rangle$ from $\boldsymbol{J}_{2n}$, the following properties are equivalent:*

(i) $\dim a_i \leq \dim b_i$ *for all* $i = 1, \dots, 2n$.
(ii) $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{J}_{2n}}(\boldsymbol{b})$.
(iii) $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$.

Let $\boldsymbol{a} = \langle a_1, \dots, a_{2n}\rangle$ be an element of the ring $\boldsymbol{R}_{2n}$. For each $i, j \in \{1, 2, \dots, n\}$ we set $\boldsymbol{a}_{\{2i-1,2j\}} := \langle a_{2i-1}, a_{2j}\rangle$. Observe that $\boldsymbol{a}_{\{2i-1,2j\}}$ is an element of the ring $\boldsymbol{R}_2$.

LEMMA 8.2. *Let $n \in \mathbb{N}$ and a_i, b_i, $i \in \{1, 2, \dots, 2n\}$, integers such that*

$$a_{2i-1} + a_{2j} \geq b_{2i-1} + b_{2j}. \tag{8.1}$$

for all $i, j \in \{1, 2, \dots, n\}$. Then there is an integer λ such that

$$a_{2i-1} + \lambda \geq b_{2i-1} \text{ and } a_{2j} - \lambda \geq b_{2j}, \tag{8.2}$$

for all $i, j = \{1, 2, \dots, n\}$.

PROOF. The equations (8.1) are equivalent to

$$a_{2j} - b_{2j} \geq b_{2i-1} - a_{2i-1},$$

for all $i, j \in \{1, 2, \dots, n\}$, hence

$$\min\{a_{2j} - b_{2j} \mid j = 1, \dots, n\} \geq \max\{b_{2i-1} - a_{2i-1} \mid i = 1, \dots, n\}.$$

We pick any integer λ with

$$\min\{a_{2j} - b_{2j} \mid j = 1, \dots, n\} \geq \lambda \geq \max\{b_{2i-1} - a_{2i-1} \mid i = 1, \dots, n\}$$

and observe that (8.2) holds true. □

LEMMA 8.3. *Let $\boldsymbol{a} = \langle a_1, \dots, a_{2n}\rangle$ and $\boldsymbol{b} = \langle b_1, \dots, b_{2n}\rangle$ be elements of $\boldsymbol{R}_{2n} \backslash \boldsymbol{J}_{2n}$. Then $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$ if and only if $\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} \geq \operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}}$ for all $i, j \in \{1, 2, \dots, n\}$.*

PROOF. ($\Rightarrow$) Suppose that $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$. Then $\boldsymbol{a}_{\{2i-1,2j\}} \in \operatorname{tr}_{\boldsymbol{R}_2}(\boldsymbol{b}_{\{2i-1,2j\}})$, which implies that $\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} \geq \operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}}$, for all $i, j \in \{1, 2, \dots, n\}$, due to Lemma 7.10.

($\Leftarrow$) Since the ring $\boldsymbol{R}_{2n}$ is regular, it contains idempotents $\boldsymbol{e} = \langle e_1, e_2, \dots, e_{2n}\rangle$ and $\boldsymbol{f} = \langle f_1, f_2, \dots, f_{2n}\rangle$ such that $\operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{a}) = \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{e})$ and $\operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b}) = \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{f})$. As we have just proved, this implies that $\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} = \operatorname{codim} \boldsymbol{e}_{\{2i-1,2j\}}$ and $\operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}} = \operatorname{codim} \boldsymbol{f}_{\{2i-1,2j\}}$, for all $i, j \in \{1, 2, \dots, n\}$. According to the assumption we have that

$$\operatorname{codim} e_{2i-1} + \operatorname{codim} e_{2j} \geq \operatorname{codim} f_{2i-1} + \operatorname{codim} f_{2j},$$

for all $i, j \in \{1, 2, \dots, n\}$. By Lemma 7.1 all $\operatorname{codim} e_i$ and $\operatorname{codim} f_i$, $i = 1, \dots, 2n$, are even. Applying Lemma 8.2, there is an integer 2λ such that

$$\operatorname{codim} e_{2i-1} + 2\lambda \geq \operatorname{codim} f_{2i-1} \text{ and } \operatorname{codim} e_{2j} - 2\lambda \geq \operatorname{codim} f_{2j},$$

for all $i, j \in \{1, 2, \ldots, n\}$. Applying Corollary 7.9 we find idempotents $\boldsymbol{g}_{\{2i-1,2i\}} = (g_{2i-1}, g_{2i}) \in \boldsymbol{R}_2 \setminus \boldsymbol{J}_2$, and elements $\boldsymbol{r}^*_{\{2i-1,2i\}} = \langle r^*_{2i-1}, r^*_{2i}\rangle$, $\boldsymbol{s}^*_{\{2i-1,2i\}} = \langle s^*_{2i-1}, s^*_{2i}\rangle \in \boldsymbol{R}_2$ with $\varphi(\boldsymbol{r}_{\{2i-1,2i\}}) = t^{\lambda}$, $\varphi(\boldsymbol{s}^*_{\{2i-1,2i\}}) = t^{-\lambda}$, for all $i \in \{1, 2, \ldots, n\}$, satisfying

$$\operatorname{codim} g_{2i-1} = \operatorname{codim} e_{2i-1} + 2\lambda,$$
$$\operatorname{codim} g_{2i} = \operatorname{codim} e_{2i} - 2\lambda,$$

and

$$\boldsymbol{e}_{\{2i-1,2i\}} = \boldsymbol{s}^*_{\{2i-1,2i\}}\boldsymbol{g}_{\{2i-1,2i\}}\boldsymbol{r}^*_{\{2i-1,2i\}},$$

for all $i \in \{1, 2, \ldots, n\}$. Putting $\boldsymbol{g} := \langle g_1, g_2, \ldots, g_{2n}\rangle$, $\boldsymbol{r}^* := \langle r^*_1, r^*_2, \ldots, r^*_{2n}\rangle$, and $\boldsymbol{s}^* := \langle s^*_1, s^*_2, \ldots, s^*_{2n}\rangle$, we get elements of $\boldsymbol{R}_{2n}$ with $\varphi(\boldsymbol{g}) = 1$, $\varphi(\boldsymbol{r}^*) = t^{\lambda}$, and $\varphi(\boldsymbol{s}^*) = t^{-\lambda}$, satisfying $\boldsymbol{e} = \boldsymbol{s}^*\boldsymbol{g}\boldsymbol{r}^*$. Since $\operatorname{codim} g_i \geq \operatorname{codim} f_i$ for all $i = 1, \ldots, 2n$, we have $\boldsymbol{r}, \boldsymbol{s} \in \boldsymbol{R}_{2n}$ with $\varphi(\boldsymbol{r}) = \varphi(\boldsymbol{s}) = 1$ satisfying $\boldsymbol{g} = \boldsymbol{s}\boldsymbol{f}\boldsymbol{r}$ due to Lemma 7.8. It follows that

$$\boldsymbol{e} = \boldsymbol{s}^*\boldsymbol{g}\boldsymbol{r}^* = \boldsymbol{s}^*\boldsymbol{s}\boldsymbol{f}\boldsymbol{r}\boldsymbol{r}^*,$$

hence $\boldsymbol{e} \in \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{f})$. Therefore $\boldsymbol{a} \in \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$. □

The next lemma is an analogy of Lemma 7.11. It follows readily as a combination of Lemmas 8.1 and 8.3.

LEMMA 8.4. *Let $\boldsymbol{a} = \langle a_1, a_2, \ldots, a_{2n}\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, \ldots, b_{2n}\rangle$ be elements of the ring $\boldsymbol{R}_{2n}$. Then $\operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{a}) = \operatorname{tr}_{\boldsymbol{R}_{2n}}(\boldsymbol{b})$ if and only if either both $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{J}_{2n}$ and*

$$\dim a_i = \dim b_i$$

for all $i \in \{1, 2, \ldots, 2n\}$, or both $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{R}_{2n} \setminus \boldsymbol{J}_{2n}$ and

$$\operatorname{codim} \boldsymbol{a}_{\{2i-1,2j\}} = \operatorname{codim} \boldsymbol{b}_{\{2i-1,2j\}}$$

for all $i, j \in \{1, 2, \ldots, n\}$.

THEOREM 8.5. *Let n be a positive integer. The monoid $\boldsymbol{V}(\boldsymbol{R}_{2n})$ is isomorphic to $\boldsymbol{D}_{2n}$ and, via the isomorphism $\psi_{2n}^{-1}\colon \boldsymbol{D}_{2n} \to \boldsymbol{B}_{2n}$, also to $\boldsymbol{B}_{2n}$.*

PROOF. We define a map $\gamma\colon \operatorname{Idem}(\boldsymbol{R}_{2n}) \to \boldsymbol{D}_{2n}$ by

$$\boldsymbol{e} = \langle e_1, e_2, \ldots, e_{2n}\rangle \mapsto \begin{cases} \langle 0, \frac{\dim e_1}{2}, \frac{\dim e_2}{2}, \ldots, \frac{\dim e_{2n}}{2}\rangle \in \boldsymbol{O}_{2n} & \text{if } \varphi(\boldsymbol{e}) = 0, \\ \langle 1, -\frac{\operatorname{codim} \boldsymbol{e}_{\{i,j\}}}{2}\rangle_{\{i,j\}} \in \boldsymbol{V}_{2n} & \text{if } \varphi(\boldsymbol{e}) = 1, \end{cases}$$

and we verify that the properties (1-3) of Corollary 3.9 are satisfied. Property (1) follows from Lemma 8.4.

We are going to prove that (2) holds true. Let $x = \langle x_0, \ldots\rangle, y = \langle y_0, \ldots\rangle \in \boldsymbol{D}_{2n}$ and $\boldsymbol{g} = \langle g_1, g_2, \ldots, g_{2n}\rangle \in \operatorname{Idem}(\boldsymbol{R}_{2n})$. The implication ($\Leftarrow$) is trivial. In order to prove the opposite one, ($\Rightarrow$), assume that $\gamma(g) = x+y$. We are going to discuss two cases.

The first case is when $\boldsymbol{g} \in \boldsymbol{J}_{2n}$. Then $0 = \varphi(\boldsymbol{g}) = x_0 + y_0$, hence $x_0 = y_0 = 0$ and both x_0 and y_0 belong to $\boldsymbol{O}_{2n}$. Applying Lemma 7.12, we find, for each $i \in \{1, 2, \ldots, 2n\}$, a pair of orthogonal idempotents $e_i, f_i \in \boldsymbol{I}$ such that $\dim e_i = x_i$, $\dim f_i = y_i$, and $g_i = e_i + f_i$. Putting $\boldsymbol{e} = \langle e_1, e_2, \ldots, e_{2n}\rangle$ and $\boldsymbol{f} = \langle f_1, f_2, \ldots, f_{2n}\rangle$, we get a pair of orthogonal idempotents $\boldsymbol{e}, \boldsymbol{f}$ such that $\gamma(\boldsymbol{e}) = x$, $\gamma(\boldsymbol{f}) = y$, and $\boldsymbol{g} = \boldsymbol{e} + \boldsymbol{f}$.

The latter case is when $\boldsymbol{g}$ is an idempotent from $\boldsymbol{R}_{2n} \setminus \boldsymbol{J}_{2n}$. We can without loss of generality assume that $x_0 \geq y_0$. Since $x_0 + y_0 = z_0 = 1$, we get that $x_0 = 1$, hence $x \in \boldsymbol{V}_{2n}$, and $y_0 = 0$, hence $y \in \boldsymbol{O}_{2n}$. Applying Lemma 7.13 we find for each $i \in \{1, 2, \ldots, 2n\}$ a pair of orthogonal idempotents $e_i \in \boldsymbol{S} \setminus \boldsymbol{I}$, and $f_i \in \boldsymbol{I}$ such that

$$\frac{\operatorname{codim} e_i}{2} = \frac{\operatorname{codim} g_i}{2} + y_i, \quad \frac{\dim f_i}{2} = y_i, \text{ and } g_i = e_i + f_i.$$

Set $\boldsymbol{e} := \langle e_1, e_2, \ldots, e_{2n}\rangle$ and $\boldsymbol{f} := \langle f_1, f_2, \ldots, f_{2n}\rangle$. Then $\boldsymbol{e} \in \boldsymbol{R}_{2n} \setminus \boldsymbol{J}_{2n}$ and $\boldsymbol{f} \in \boldsymbol{R}_{2n}$ are orthogonal idempotents such that $\boldsymbol{g} = \boldsymbol{e} + \boldsymbol{f}$ and $\gamma(\boldsymbol{f}) = y$. It follows that

$$\gamma(\boldsymbol{e}) + y = \gamma(\boldsymbol{g}) = x + y.$$

Applying Lemma 5.4, we infer from $y \in \boldsymbol{O}_{2n}$ that $\gamma(\boldsymbol{e}) = x$. Therefore property (2) is satisfied.

By the definition

$$\gamma(1) = \langle 1, \underbrace{0, \ldots, 0}_{n^2 \times}\rangle,$$

which is an order-unit in $\boldsymbol{D}_{2n}$, thus property (3) holds true as well.

Since the ring $\boldsymbol{R}_{2n}$ is directly finite, the map γ extends to a unique isomorphism $\beta \colon \boldsymbol{V}(\boldsymbol{R}_{2n}) \to \boldsymbol{D}_{2n}$, due to Corollary 3.9. $\square$

CHAPTER 6

Boolean ranges of Banaschewski functions

1. Introduction

In [**84**] Friedrich Wehrung defined a *Banaschewski function* on a bounded complemented lattice $\boldsymbol{\mathcal{L}}$ as an antitone (i.e., order-reversing) map sending each element of $\boldsymbol{\mathcal{L}}$ to one of its complements, being motivated by the earlier result of Bernhard Banaschewski that such a function exists on the lattice of all subspaces of a vector space [**8**]. Wehrung extended Banaschewski's result by proving that every countable complemented modular lattice has a Banaschewski function with Boolean range and that all the possible ranges of Banaschewski functions with Boolean range on $\boldsymbol{\mathcal{L}}$ are isomorphic [**84**, Corollary 4.8].

Still in [**84**] Wehrung defined a ring-theoretical analogue of the Banaschewski function that, for a von Neuman regular ring $\boldsymbol{R}$, is closely connected to the lattice-theoretical Banaschewski function on the lattice $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ of all finitely generated right ideals of $\boldsymbol{R}$. He made use of these ideas to construct a unit-regular ring $\boldsymbol{S}$ (in fact of bounded index 3) of size $\aleph_1$ with no Banaschewski function [**85**].

Furthermore in [**84**] Wehrung defined notions of a Banaschewski measure and a Banaschewski trace on sectionally complemented modular lattices and he proved that a sectionally complemented lattice which is either modular with a large 4-frame or Arguesian with a large 3-frame is coordinatizable (i.e. isomorphic to $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ for a possibly non-unital von Neumann regular ring $\boldsymbol{R}$) if and only if it has a Banaschewski trace. Applying those results, he constructed a non-coordinatizable sectionally complemented modular lattice, of size $\aleph_1$, with a large 4-frame [**84**, Theorem 7.5].

The aim of this chapter is to solve the second problem from [**84**]:

Problem (Problem 2 from [**84**]). *Is every maximal Boolean sublattice of an at most countable complemented modular lattice $\boldsymbol{\mathcal{L}}$ the range of some Banaschewski function on $\boldsymbol{\mathcal{L}}$? Are any two such Boolean sublattices isomorphic?*

We construct a countable complemented modular lattice $\mathbf{\mathcal{S}}$ with two non-isomorphic maximal Boolean sublattices $\mathbf{\mathcal{G}}$ and $\mathbf{\mathcal{H}}$. We represent $\mathbf{\mathcal{G}}$ as the range of a Banaschewski function on $\mathbf{\mathcal{S}}$ and we prove that $\mathbf{\mathcal{H}}$ is not the range of any Banaschewski function. We represent the lattice $\mathbf{\mathcal{S}}$ as a bounded sublattice of the subspace lattice of a vector space over an arbitrary given field. The lattice $\mathbf{\mathcal{S}}$ is constructed as a bounded sublattice of $\mathbf{\mathcal{M}}_3[\mathbf{\mathcal{F}}(\kappa)]$. We prove that there is no 3-frame in the lattice $\mathbf{\mathcal{M}}_3[\mathbf{\mathcal{D}}]$ for any distributive lattice $\mathbf{\mathcal{D}}$. As a consequence we get that there is no 3-frame in the lattice $\mathbf{\mathcal{S}}$. On the other hand we show that lattices $\mathbf{\mathcal{M}}_3[\mathbf{\mathcal{B}}]$ are cordinatizated by Boolean powers of the ring of 2×2 matrices over a two-element field $\mathbb{F}_2$ by a Boolean lattice $\mathbf{\mathcal{B}}$. We find a regular $\mathbb{F}_2$-algebra $\boldsymbol{S}$ such that $\mathbf{\mathcal{S}} \simeq \boldsymbol{\mathcal{L}}(\boldsymbol{S})$ and we show that the maximal Boolean sublattices $\mathbf{\mathcal{G}}$ and $\mathbf{\mathcal{H}}$ correspond to maximal Abelian regular subalgebras of the algebra $\boldsymbol{S}$.

2. Preliminaries

We start with recalling same basic notions as well as the precise definition of a Banaschewski function adopted from [**84**]. Next we outline Schmidt's $\mathcal{M}_3[\mathcal{L}]$ construction, which we then apply to define the bounded modular lattice $\mathcal{S}$ containing a pair of non-isomorphic maximal Boolean sublattices.

2.1. Complemented lattices and the Banaschewski function. A lattice $\mathcal{L}$ is *bounded* if it has both the least element and the greatest element, denoted by $0_{\mathcal{L}}$ and $1_{\mathcal{L}}$, respectively. A *bounded sublattice* of a bounded lattice is a sublattice containing the bounds. Given elements a, b, c of a lattice $\mathcal{L}$ with zero, we will use the notation $c = a \oplus b$ when $a \wedge b = 0_{\mathcal{L}}$ and $a \vee b = c$. A *complement* of an element a of a bounded lattice $\mathcal{L}$ is an element a' of $\mathcal{L}$ such that $a \oplus a' = 1_{\mathcal{L}}$. A lattice $\mathcal{L}$ is said to be *complemented* provided that it is bounded and each element of $\mathcal{L}$ has a (not necessarily unique) complement. A lattice $\mathcal{L}$ is *relatively complemented* if each of its closed intervals is complemented. Note that a relatively complemented lattice is not necessarily bounded.

We say that a lattice $\mathcal{L}$ is *uniquely complemented* if it is bounded and each element of $\mathcal{L}$ has a unique complement. By a *Boolean lattice* we mean a lattice reduct of a Boolean algebra, that is, a complemented distributive lattice. For the clarity, let us recall the formal definition of a Banaschewski function [**84**, Definition 3.1]:

DEFINITION 2.1. Let $\mathcal{L}$ be a bounded lattice. A *Banaschewski function* on $\mathcal{L}$ is a map $\beta\colon \mathcal{L} \to \mathcal{L}$ such that both

(i) $x \leq y$ implies $\beta(x) \geq \beta(y)$, for all $x, y \in \mathcal{L}$, and
(ii) $\beta(x) \oplus x = 1_{\mathcal{L}}$ for all $x \in \mathcal{L}$,

hold true.

2.2. The $\mathcal{M}_3[\mathcal{L}]$-construction. Let $\mathcal{L}$ be a lattice. We will call a triple $\langle a, b, c\rangle \in \mathcal{L}^3$ *balanced*, if it satisfies

$$a \wedge b = a \wedge c = b \wedge c$$

and we denote by $\mathcal{M}_3[\mathcal{L}]$ the set of all balanced triples. It is readily seen that $\mathcal{M}_3[\mathcal{L}]$ is a meet-subsemilattice of the cartesian product $\mathcal{L}^3$. However, it is not necessarily a join-subsemilattice, for one easily observes that the componentwise join of balanced triples may not be balanced. The $\mathcal{M}_3[\mathcal{L}]$-construction was introduced by E. T. Schmidt [**69, 71**] for a bounded distributive lattice $\mathcal{L}$. He proved [**71**, Lemma 1] that in this case $\mathcal{M}_3[\mathcal{L}]$ is a bounded modular lattice and that it is a congruence-preserving extension of the distributive lattice $\mathcal{L}$. This result was later extended by Grätzer and Schmidt in various directions [**30, 31**]. In particular, in [**30**] they proved that every lattice with a non-trivial distributive interval has a proper congruence-preserving extension. This was further improved by Grätzer and Wehrung in [**35**], where they introduced a modification of the $\mathcal{M}_3[\mathcal{L}]$-construction, called $\mathbf{M}_3\langle\boldsymbol{\mathcal{L}}\rangle$-construction. Using this new idea they proved that every non-trivial lattice admits a proper congruence-preserving extension.

The lattice constructions $\mathbf{M}_3[\boldsymbol{\mathcal{L}}]$ and $\mathbf{M}_3\langle\boldsymbol{\mathcal{L}}\rangle$ appeared in the series of papers by Grätzer and Wehrung [**32, 33, 34, 35, 36, 37, 38**] dealing with semilattice tensor product and its related structures, namely the box product and the lattice tensor product [**34**, Definition 2.1 and Definition 3.3]. Indeed, $\mathbf{M}_3 \boxtimes \boldsymbol{\mathcal{L}} \simeq \mathbf{M}_3\langle\boldsymbol{\mathcal{L}}\rangle$ for every lattice $\boldsymbol{\mathcal{L}}$ and $\mathbf{M}_3 \otimes \boldsymbol{\mathcal{L}} \simeq \mathbf{M}_3[\boldsymbol{\mathcal{L}}]$ whenever $\boldsymbol{\mathcal{L}}$ has a zero and $\mathbf{M}_3 \otimes \boldsymbol{\mathcal{L}}$ is a lattice (see [**38**, Theorem 6.5] and [**33**, Corollary 6.3]). In particular, the latter is satisfied when the lattice $\boldsymbol{\mathcal{L}}$ is modular with zero. Note also, that if $\boldsymbol{\mathcal{L}}$ is a bounded distributive lattice both the constructions $\mathbf{M}_3[\boldsymbol{\mathcal{L}}]$ and $\mathbf{M}_3\langle\boldsymbol{\mathcal{L}}\rangle$ coincide. In our paper we get by with this simple case.

Let $\boldsymbol{\mathcal{L}}$ be a distributive lattice. Given a triple $\langle a, b, c\rangle \in \boldsymbol{\mathcal{L}}^3$, we define

$$\mu\langle a, b, c\rangle = (a \wedge b) \vee (a \wedge c) \vee (b \wedge c) \tag{2.1}$$

and we set

$$\overline{\langle a, b, c\rangle} = \langle a \vee \mu\langle a, b, c\rangle, b \vee \mu\langle a, b, c\rangle, c \vee \mu\langle a, b, c\rangle\rangle. \tag{2.2}$$

Using the distributivity of $\boldsymbol{\mathcal{L}}$ one easily sees that $\overline{\langle a, b, c\rangle}$ is the least balanced triple $\geq \langle a, b, c\rangle$ in $\boldsymbol{\mathcal{L}}^3$ and that the map $\overline{\langle - \rangle}\colon \boldsymbol{\mathcal{L}}^3 \to \boldsymbol{\mathcal{L}}^3$ determines a closure operator on the lattice $\boldsymbol{\mathcal{L}}^3$ (see [**33**, Lemma 2.3] for a refinement of this observation). It is also clear that

$$\begin{aligned} a \vee \mu\langle a, b, c\rangle &= a \vee (b \wedge c), \\ b \vee \mu\langle a, b, c\rangle &= b \vee (a \wedge c), \\ c \vee \mu\langle a, b, c\rangle &= c \vee (a \wedge b). \end{aligned}$$

A triple $\langle a, b, c\rangle \in \boldsymbol{\mathcal{L}}^3$ is closed with respect to the closure operator if and only if it is balanced. Therefore the set of all balanced triples, denoted by $\mathbf{M}_3[\boldsymbol{\mathcal{L}}]$, forms a lattice [**33**, Lemma 2.1], where

$$\langle a, b, c\rangle \vee \langle a', b', c'\rangle = \overline{\langle a \vee a', b \vee b', c \vee c'\rangle} \tag{2.3}$$

and

$$\langle a, b, c\rangle \wedge \langle a', b', c'\rangle = \langle a \wedge a', b \wedge b', c \wedge c'\rangle. \tag{2.4}$$

By [**33**, Lemma 2.9] the lattice $\mathbf{M}_3[\boldsymbol{\mathcal{L}}]$ is modular if and only if the lattice $\boldsymbol{\mathcal{L}}$ is distributive. The "if" part of the equivalence is included in the above mentioned [**71**, Lemma 1].

2.3. Coordinatizability. Finitely generated right ideals of a regular ring $\boldsymbol{R}$ form a sectionally complemented modular lattice [**22**, Theorem 2.3]. We will denote this lattice by $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$. Note that for a regular ring the correspondence $e\boldsymbol{R} \mapsto \boldsymbol{R}(1-e)$ determines an anti-isomorphisms from the lattice $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$, of all finitely generated right ideals of the ring $\boldsymbol{R}$, to the lattice of all finitely generated left ideals of the ring $\boldsymbol{R}$ (cf. [**22**, Theorem 2.5]).

A lattice, necessarily sectionally complemented modular, is *coordinatizable* if it is isomorphic to the lattice $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ for a regular ring $\boldsymbol{R}$. For a lucid introduction into the problem of coordinatizability of sectionally complemented modular lattice we refer to [**27**, Appendix D] and [**85**]. Here we will limit ourselves to Jónsson's coordinatization theorem [**44**], to our knowledge the most complete description of coordinatizable lattices.

We say a set X of non-zero elements of a lattice $\mathcal{L}$ with zero is *independent* provided that for every finite $F, G \subseteq X$, the equality

$$\bigvee F \wedge \bigvee G = \bigvee (F \cap G)$$

holds true. If the lattice $\mathcal{L}$ is modular then an n-element set $\{a_1, \dots, a_n\}$ of distinct non-zero elements of $\mathcal{L}$ is independent if and only if $a_{j+1} \wedge \bigvee_{i=1}^{j} a_i = 0$ for all $j = 1, \dots, n-1$ (see [**27**, Theorem IV.1.11]). If the lattice $\mathcal{L}$ is distributive, a subset $X \subseteq \mathcal{L} \setminus \{0\}$ is independent if and only if $a \wedge b = 0$ for all distinct $a, b \in X$.

Elements a, b of a bounded lattice $\mathcal{L}$ are *perspective* provided that there is $c \in \mathcal{L}$ such that

$$1 = a \oplus c = b \oplus c. \tag{2.5}$$

The notation $a \sim_c b$ means that equalities (2.5) hold true. The notation $a \sim b$ means that $a \sim_c b$ for some $c \in \mathcal{L}$, i.e. that a and b are perspective.

An element a of a lattice $\mathcal{L}$ is *neutral* provided that the sublattice of $\mathcal{L}$ generated by a triple $\{a, b, c\}$ is distributive for all $b, c \in \mathcal{L}$ [**27**, Section III.2]. An ideal I of a lattice $\mathcal{L}$ is *neutral* if it is a neutral element in the ideal lattice of $\mathcal{L}$. An *n-frame* in a lattice $\mathcal{L}$ is a pair $\langle \langle a_i \mid i = 0, \dots, n-1 \rangle, \langle c_i \mid i = 1, \dots, n-1 \rangle \rangle$ of families of elements of $\mathcal{L}$ such that the set $\{a_0, \dots, a_{n-1}\}$ is independent and $a_0 \sim_{c_i} a_i$ for all $i = 1, \dots, n-1$. An n-frame is *large* if the neutral ideal generated by a_0 is the entire $\mathcal{L}$. In particular, an n-frame such that $\bigvee_{i=0}^{n-1} a_i = 1$ is large.

Theorem 2.2 (Jónsson's coordinatization theorem [**44**]). *A modular complemented lattice $\mathcal{L}$ that has a large n-frame for some $n \geq 4$ or that is Arguesian and has a large n-frame with $n \geq 3$ is coordinatizable.*

2.4. Stone duality and Boolean powers. In this subsection we follow [**13**, Chapter IV,§§4-5]. For topological notions we refer to [**18**]. A *Boolean space* is a compact Hausdorff topological space with a basis consisting of clopen (i.e. closed and open) subsets. Let $\mathcal{B}$ be a Boolean lattice. We denote by $\mathcal{B}^*$ the collection of all ultrafilters on $\mathcal{B}$. For each $a \in \mathcal{B}$ we set

$$N_a := \{\mathfrak{u} \in \mathcal{B}^* \mid a \in \mathfrak{u}\}. \tag{2.6}$$

The collection of all N_a, $a \in \mathcal{B}$, is a basis of a topology on $\mathcal{B}^*$, and $\mathcal{B}^*$ equipped with this topology is a Boolean space called the *Stone space* of $\mathcal{B}$.

All clopen subsets of a topological space $\mathcal{T}$ form a sulattice, denoted by $\mathcal{T}^*$, of the Boolean lattice of all subsets of $\mathcal{T}$. Every Boolean lattice $\mathcal{B}$ is isomorphic to $\mathcal{B}^{**}$ via the map $a \mapsto N_a$ and every Boolean space $\mathcal{T}$ is homeomorphic to $\mathcal{T}^{**}$ via $x \mapsto \{N \in \mathcal{T}^* \mid x \in N\}$.

Let $\boldsymbol{A}$ be an algebra and $\mathfrak{B}$ a Boolean lattice. We equip the set A with the discrete topology and we denote by $A[\mathfrak{B}]^*$ the set of all continuous functions from the Boolean space $\mathfrak{B}^*$ to A. By [**13**, Lemma IV.5.2], $A[\mathfrak{B}]^*$ is a subuniverse of the Cartesian power $\boldsymbol{A}^{\mathfrak{B}^*}$. We denote by $\boldsymbol{A}[\mathfrak{B}]^*$ the subalgebra of $\boldsymbol{A}^{\mathfrak{B}^*}$ with the universe $A[\mathfrak{B}]^*$ and we will call the subalgebra the *Boolean power* of $\boldsymbol{A}$ by $\mathfrak{B}$.

3. The lattice

Fix an infinite cardinal κ. As it is customary, we identify κ with the set of all ordinals of cardinality less than κ. Let us denote by $\mathfrak{P}(\kappa)$ the Boolean lattice of all subsets of κ and set

$$\mathfrak{F}(\kappa) := \{X \subseteq \kappa \mid X \text{ is finite or } \kappa \setminus X \text{ is finite}\}.$$

It is well-known that $\mathfrak{F}(\kappa)$ is a bounded Boolean sublattice of $\mathfrak{P}(\kappa)$.

Given sets X, Y, the notation $X \leq_{\text{fin}} Y$ means that $X \setminus Y$ is finite. Clearly $\leq_{\text{fin}}$ is a quasiorder on the class of all sets. We define

$$\mathfrak{E} = \{\langle A, B, C\rangle \in \mathfrak{F}(\kappa)^3 \mid C \leq_{\text{fin}} A \cup B\}.$$

Since for all A, A', B, B', C, C' we have that

$$(C \cup C') \setminus ((A \cup A') \cup (B \cup B')) \subseteq (C \setminus (A \cup B)) \cup (C' \setminus (A' \cup B')), \tag{3.1}$$

the set $\mathfrak{E}$ is closed under finite joins. Both $0_{\mathfrak{F}(\kappa)^3} = \langle \emptyset, \emptyset, \emptyset\rangle$ and $1_{\mathfrak{F}(\kappa)^3} = \langle \kappa, \kappa, \kappa\rangle$ clearly belong to $\mathfrak{E}$, thus we can conclude that $\mathfrak{E}$ forms a bounded join-subsemilattice of $\mathfrak{F}(\kappa)^3$.

Let $\mathfrak{S} := \mathfrak{E} \cap \mathfrak{M}_3[\mathfrak{F}(\kappa)]$ denote the set of all balanced triples from $\mathfrak{E}$. Since $A \cap C = B \cap C$ for every balanced triple $\langle A, B, C\rangle$, we have that

$$\begin{aligned} \mathfrak{S} &= \{\langle A, B, C\rangle \in \mathfrak{M}_3[\mathfrak{F}(\kappa)] \mid C \leq_{\text{fin}} A\} \\ &= \{\langle A, B, C\rangle \in \mathfrak{M}_3[\mathfrak{F}(\kappa)] \mid C \leq_{\text{fin}} B\}. \end{aligned} \tag{3.2}$$

Note that since for a balanced triple $\langle A, B, C\rangle$ the equality $A \cap C = \mu\langle A, B, C\rangle$ holds true, we get from (3.2) that

$$\mathfrak{S} = \{\langle A, B, C\rangle \in \mathfrak{M}_3[\mathfrak{F}(\kappa)] \mid C \leq_{\text{fin}} \mu\langle A, B, C\rangle\}. \tag{3.3}$$

LEMMA 3.1. *The set $\mathfrak{S}$ forms a bounded sublattice of the lattice $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$.*

PROOF. Observe that

$$C \setminus (A \cup B) = (C \cup \mu\langle A, B, C\rangle) \setminus (A \cup B \cup \mu\langle A, B, C\rangle),$$

for all $\langle A, B, C\rangle \in \mathfrak{F}(\kappa)^3$. Therefore the join-semilattice $\mathfrak{E}$ is closed under the operation μ. It follows that $\mathfrak{S}$ forms a bounded join-subsemilattice of $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$. It remains to prove that $\mathfrak{S}$ is closed under finite meets. However, this is a consequence of the inequality

$$(C \cap C') \setminus (A \cap A') \subseteq (C \setminus A) \cup (C' \setminus A'),$$

that holds for all sets A, A', C, C'. $\square$

As discussed in Section 2, since the lattice $\mathcal{F}(\kappa)$ is distributive, the lattice $\mathcal{M}_3[\mathcal{F}(\kappa)]$ is modular. Observe that the mapping $A \mapsto \langle A, A, A\rangle$ embeds $\mathcal{F}(\kappa)$ into $\mathcal{S}$, from which we deduce that

$$|\mathcal{F}(\kappa)| \leq |\mathcal{S}| \leq |\mathcal{F}(\kappa)^3|.$$

Since the size of both $\mathcal{F}(\kappa)$ and $\mathcal{F}(\kappa)^3$ is κ, we get that $|\mathcal{S}| = \kappa$. Let us sum up these observations in the following corollary to Lemma 3.1.

COROLLARY 3.2. *For $\kappa = \omega_0$, the lattice $\mathcal{S}$ is countable infinite.*

REMARK 3.3. Note that unlike $\mathcal{S}$, the lattice $\mathcal{E}$ is not a meet-subsemilattice of $\mathcal{F}(\kappa)^3$. Indeed, both $\langle \kappa, \emptyset, \kappa\rangle, \langle \emptyset, \kappa, \kappa\rangle \in \mathcal{E}$ while $\langle \kappa, \emptyset, \kappa\rangle \wedge \langle \emptyset, \kappa, \kappa\rangle = \langle \emptyset, \emptyset, \kappa\rangle \notin \mathcal{E}$.

4. A Banaschewski function on $\mathcal{S}$

In this section we define a Banaschewski function $\beta\colon \mathcal{S} \to \mathcal{S}$ and describe, element-wise, its range $\mathcal{G}$.

LEMMA 4.1. *The map $\beta\colon \mathcal{S} \to \mathcal{S}$ defined by*

$$\beta\langle A, B, C\rangle := \langle \kappa \setminus A, \kappa \setminus (B \cup C), \kappa \setminus (A \cup B \cup C)\rangle, \quad \textit{for all } \langle A, B, C\rangle \in \mathcal{S}, \tag{4.1}$$

is a Banaschewski function on $\mathcal{S}$. Consequently, $\mathcal{S}$ is a complemented modular lattice.

PROOF. First we prove that $\mathcal{S}$ contains the range of the map β. Observe that if we put $A' := \kappa \setminus A$ and $B' := \kappa \setminus (B \cup C)$, then $\beta\langle A, B, C\rangle = \langle A', B', A' \cap B'\rangle$. Since $\mathcal{F}(\kappa)$ is a Boolean lattice, the sets A', B' and $A' \cap B'$ all belong to $\mathcal{F}(\kappa)$. Furthermore, we have that

$$A' \cap B' = \mu\langle A', B', A' \cap B'\rangle = \mu\beta\langle A, B, C\rangle.$$

In particular, $A' \cap B' \setminus \mu\beta\langle A, B, C\rangle = \emptyset$, whence $\beta\langle A, B, C\rangle \in \mathcal{S}$.

It is clear from (4.1) that the map β is antitone. Finally, we check that

$$1_{\mathcal{S}} = \langle \kappa, \kappa, \kappa\rangle = \langle A, B, C\rangle \oplus \beta\langle A, B, C\rangle, \quad \text{for all } \langle A, B, C\rangle \in \mathcal{S}.$$

It follows immediately from the definition of β that

$$\langle A, B, C\rangle \wedge \beta\langle A, B, C\rangle = \langle \emptyset, \emptyset, \emptyset\rangle = 0_{\mathcal{S}}.$$

To prove that $\langle A, B, C\rangle \vee \beta\langle A, B, C\rangle = 1_{\mathcal{S}}$, let us verify that

$$\kappa = \mu\langle A \cup (\kappa \setminus A), B \cup (\kappa \setminus (B \cup C)), C \cup (\kappa \setminus (A \cup B \cup C))\rangle. \tag{4.2}$$

Note that each element of κ that is *not* contained in C belongs to $B \cup (\kappa \setminus (B \cup C))$. Together with $A \cup (\kappa \setminus A) = \kappa$, we get that (4.2) holds, which concludes the proof. $\square$

LEMMA 4.2. *Let $\mathcal{G}$ denote the range of the Banaschewski function $\beta\colon \mathcal{S} \to \mathcal{S}$. Then*

$$\mathcal{G} = \{\langle A, B, A \cap B\rangle \mid A, B \in \mathcal{F}(\kappa)\}$$

and the mapping

$$\langle A, B, A \cap B\rangle \mapsto \langle A, B\rangle \tag{4.3}$$

determines an isomorphism from $\mathfrak{G}$ *onto the Boolean lattice* $\mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa)$.

PROOF. While proving Lemma 4.1, we have observed that

$$\begin{aligned}\mathfrak{G} \subseteq &\{\langle A, B, C\rangle \in \mathfrak{S} \mid C = A \cap B\} = \\ &\{\langle A', B', A' \cap B'\rangle \mid A', B' \in \mathfrak{F}(\kappa)\}.\end{aligned} \tag{4.4}$$

A straightforward computation gives that

$$\beta(\beta\langle A', B', A' \cap B'\rangle) = \langle A', B', A' \cap B'\rangle,$$

and so the lattice $\mathfrak{G}$ is equal to the right-hand side of (4.4). Finally, it is readily seen that the correspondence (4.3) determines an isomorphism $\mathfrak{G} \to \mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa)$. $\square$

It was noted in [**84**] that if the range of a Banaschewski function on a lattice $\mathcal{L}$ is Boolean, then it is a *maximal* Boolean sublattice of $\mathcal{L}$. Thus we derive from Theorem 4.2 that $\mathfrak{G}$ is a maximal Boolean sublattice of $\mathfrak{S}$.

5. The counter-example

In the present section, we construct another maximal Boolean sublattice $\mathfrak{H}$ of the lattice $\mathfrak{S}$. We show that the lattices $\mathfrak{H}$ and $\mathfrak{G}$ are not isomorphic and we prove directly that the lattice $\mathfrak{H}$ is not the range of any Banaschewski function on $\mathfrak{S}$.

LEMMA 5.1. *The assignment* $\langle A, C\rangle \mapsto g\langle A, C\rangle := \langle A, A \cap C, C\rangle$ *defines a bounded lattice embedding* $g\colon \mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa) \to \mathfrak{M}_3[\mathfrak{F}(\kappa)]$. *In particular, the range of* g *is a bounded Boolean sublattice of* $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$ *isomorphic to* $\mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa)$.

PROOF. It is clear from the definition of the map g that it is injective and that its range is included in $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$. Further, for any $A, A', C, C' \subseteq \kappa$, the equality

$$g\langle A, C\rangle \wedge g\langle A', C'\rangle = g\langle A \cap A', C \cap C'\rangle$$

holds by (2.4), while

$$g\langle A, C\rangle \vee g\langle A', C'\rangle = g\langle A \cup A', C \cup C'\rangle \tag{5.1}$$

can be easily deduced from (2.2) and (2.3). Finally, observe that $g\langle \kappa, \kappa\rangle = \langle \kappa, \kappa, \kappa\rangle$ and $g\langle \emptyset, \emptyset\rangle = \langle \emptyset, \emptyset, \emptyset\rangle$, which concludes the proof. $\square$

For any $A, C \in \mathfrak{F}(\kappa)$, we say that $\langle A, C\rangle$ is *finite* if both A and C are finite, and we say that $\langle A, C\rangle$ is *co-finite* if both $\kappa \setminus A$ and $\kappa \setminus C$ are finite. Let us write $A \approx C$ if $\langle A, C\rangle$ is either finite or co-finite. Note that there are pairs $A, C \in \mathfrak{F}(\kappa)$ such that $\langle A, C\rangle$ is neither finite nor co-finite; namely, $A \approx C$ if and only if the symmetric difference $(A \setminus C) \cup (C \setminus A)$ is finite.

LEMMA 5.2. *The set*

$$\boldsymbol{\mathcal{A}} = \{\langle A, C\rangle \in \mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa) \mid A \approx C\}$$

forms a bounded Boolean sublattice of $\mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa)$.

PROOF. Let $\langle A, C\rangle$, $\langle A', C'\rangle$ be a pair of elements from $\boldsymbol{\mathcal{A}}$. If at least one of them is finite, then $\langle A \cap A', C \cap C'\rangle$ is clearly finite as well. If both $\langle A, C\rangle$ and $\langle A', C'\rangle$ are co-finite, then so is $\langle A \cap A', C \cap C'\rangle$. In either case, $\langle A \cap A', C \cap C'\rangle \in \boldsymbol{\mathcal{A}}$.

If at least one of the pairs $\langle A, C\rangle, \langle A', C'\rangle$ is co-finite, then $\langle A \cup A', C \cup C'\rangle$ is co-finite as well, while if both $\langle A, C\rangle$ and $\langle A', C'\rangle$ are finite, then so is $\langle A \cup A', C \cup C'\rangle$. In particular, $\langle A \cup A', C \cup C'\rangle \in \boldsymbol{\mathcal{A}}$ whenever $\langle A, C\rangle$, $\langle A', C'\rangle \in \boldsymbol{\mathcal{A}}$.

We have shown that $\boldsymbol{\mathcal{A}}$ is a sublattice of $\mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa)$. To complete the proof, observe that $\langle \emptyset, \emptyset\rangle$ is finite and $\langle \kappa, \kappa\rangle$ is co-finite and that the unique complement in $\mathfrak{F}(\kappa) \times \mathfrak{F}(\kappa)$ of each $\langle A, C\rangle \in \boldsymbol{\mathcal{A}}$, namely $\langle \kappa \setminus A, \kappa \setminus C\rangle$ belongs to $\boldsymbol{\mathcal{A}}$. □

LEMMA 5.3. *The g-image $\mathfrak{H} = g(\boldsymbol{\mathcal{A}})$ of $\boldsymbol{\mathcal{A}}$ is a bounded Boolean sublattice of $\mathfrak{S}$.*

PROOF. Due to Lemma 5.1 and Lemma 5.2, $\mathfrak{H}$ is a bounded Boolean sublattice of $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$. Thus in view of Lemma 3.1, it suffices to verify that $\mathfrak{H} \subseteq \mathfrak{S}$, that is, that $C \setminus (A \cap C)$ is finite for every $\langle A, C\rangle \in \boldsymbol{\mathcal{A}}$. This is clear when $\langle A, C\rangle$ is finite. If $\langle A, C\rangle$ is co-finite, then $C \setminus (A \cap C) = C \setminus A \subseteq \kappa \setminus A$ is finite and we are done. □

Observe that if $\langle A, B, C\rangle$ is a balanced triple then $B \subseteq A$ if and only if $B = A \cap B = A \cap C$. It follows that

$$\mathfrak{H} = \{\langle A, B, C\rangle \in \mathfrak{S} \mid A \approx C \text{ and } B \subseteq A\}. \tag{5.2}$$

LEMMA 5.4. *Let $\langle A, B, C\rangle \in \mathfrak{S} \setminus \mathfrak{H}$ and let $\langle A', B', C'\rangle$ be a complement of $\langle A, B, C\rangle$ in $\mathfrak{S}$. If $B \subseteq A$, then $B' \nsubseteq A'$.*

PROOF. Since $\langle A, B, C\rangle \notin \mathfrak{H}$ and $B \subseteq A$, it follows from (5.2) that $A \not\approx C$. Hence exactly one of the two sets A, C is finite. From $B \subseteq A$ and $C \setminus B$ being finite we conclude that C and $\kappa \setminus A$ are both finite. Furthermore from $B \subseteq A$ and $A \cap B = B \cap C$, we infer that $B = B \cap C$. It follows that the set B is finite as well.

Suppose now that $B' \subseteq A'$. Since $\langle A, B, C\rangle \wedge \langle A', B', C'\rangle = 0_{\mathfrak{S}}$, we have that $A \cap A' = \emptyset$, whence the set $A' \subseteq \kappa \setminus A$ is finite. A fortiori, the set B' is also finite due to the assumption that $B' \subseteq A'$. As $C' \setminus B' = C' \setminus (B' \cap A') = C' \setminus \mu\langle A', B', C'\rangle$ is also finite, we conclude that so is C'. But then

$$\mu\langle A \cup A', B \cup B', C \cup C'\rangle \subseteq B \cup B' \cup C \cup C'$$

is a finite set, which contradicts the assumption that $\langle A, B, C\rangle \vee \langle A', B', C'\rangle = \langle \kappa, \kappa, \kappa\rangle = 1_{\mathfrak{S}}$. □

Corollary 5.5. *Every complemented bounded sublattice $\mathbf{C}$ of $\mathbf{S}$ such that $\mathbf{H} \subsetneq \mathbf{C}$ contains an element $\langle A, B, C\rangle$ with $B \nsubseteq A$.*

Proof. Let $\langle A, B, C\rangle \in \mathbf{C} \setminus \mathbf{H}$ and let $\langle A', B', C'\rangle$ be one of its complements in $\mathbf{C}$. Applying Lemma 5.4, we get that either $B \nsubseteq A$ or $B' \nsubseteq A'$. □

Proposition 5.6. *The lattice $\mathbf{H}$ is a maximal Boolean sublattice of $\mathbf{S}$.*

Proof. Let $\mathbf{C}$ be a complemented bounded sublattice of $\mathbf{S}$ satisfying $\mathbf{H} \subsetneq \mathbf{C}$. There is $\langle A, B, C\rangle \in \mathbf{C}$ with $B \nsubseteq A$ by Corollary 5.5. We can pick a finite nonempty $F \subseteq B \setminus A$. Since the triple $\langle A, B, C\rangle$ is balanced,

$$\emptyset = F \cap A = F \cap B \cap A = F \cap B \cap C = F \cap C. \tag{5.3}$$

Now observe that both $g\langle F, \emptyset\rangle$ and $g\langle \emptyset, F\rangle$ are in $\mathbf{H}$. Applying (5.1) and (5.3), we get that

$$\langle A, B, C\rangle \wedge \big(g\langle F, \emptyset\rangle \vee g\langle \emptyset, F\rangle\big) = \langle A, B, C\rangle \wedge g\langle F, F\rangle = \langle \emptyset, F, \emptyset\rangle, \tag{5.4}$$

while

$$\big(\langle A, B, C\rangle \wedge g\langle F, \emptyset\rangle\big) \vee \big(\langle A, B, C\rangle \wedge g\langle \emptyset, F\rangle\big) = \langle \emptyset, \emptyset, \emptyset\rangle. \tag{5.5}$$

It follows from (5.4) and (5.5) that the lattice $\mathbf{C}$ is not distributive, *a fortiori* it is not Boolean. □

Proposition 5.7. *The sublattice $\mathbf{H}$ of $\mathbf{S}$ is not the range of any Banaschewski function on $\mathbf{S}$.*

Proof. The range of a Banaschewski function on $\mathbf{S}$ must contain a complement of each element of $\mathbf{S}$. We show that no complement of $\langle \kappa, \emptyset, \emptyset\rangle$ in $\mathbf{S}$ belongs to $\mathbf{H}$.

Suppose the contrary, that is, that there is $\langle A, B, C\rangle = g\langle A, C\rangle \in \mathbf{H}$ satisfying $\langle \kappa, \emptyset, \emptyset\rangle \oplus \langle A, B, C\rangle = 1_{\mathbf{S}}$. Then $A = A \cap \kappa = \emptyset$, and by (5.2) also $B = \emptyset$. Then from $B = \emptyset$ and $\langle \kappa, \emptyset, \emptyset\rangle \vee \langle A, B, C\rangle = 1_{\mathbf{S}}$, one infers that $C = \kappa$. It follows that $\langle A, B, C\rangle \notin \mathbf{S}$; indeed, $C \setminus \mu\langle A, B, C\rangle = C \setminus \emptyset = \kappa$ is not finite. Thus $\langle A, B, C\rangle \notin \mathbf{H}$, which is a contradiction. □

Remark 5.8. Note that for the particular case of $\kappa = \aleph_0$, the assertion of Proposition 5.7 follows from Proposition 5.9 together with [**84**, Corollary 4.8], which states that the ranges of two Boolean Banaschewski functions on a *countable* complemented modular lattice are isomorphic.

Proposition 5.9. *The lattices $\mathbf{H}$ and $\mathbf{G}$ are not isomorphic.*

Proof. In $\mathbf{H}$, every *finite* element $g\langle A, C\rangle$ is a join of a finite set of atoms, namely

$$g\langle A, C\rangle = \left(\bigvee_{\alpha \in A} g\langle \{\alpha\}, \emptyset\rangle\right) \vee \left(\bigvee_{\gamma \in C} g\langle \emptyset, \{\gamma\}\rangle\right),$$

and, dually, every co-finite element is a meet of a finite set of co-atoms. On the other hand, there are elements in $\mathbf{F}(\kappa) \times \mathbf{F}(\kappa)$ that are neither finite joins of atoms nor finite meets of co-atoms. Recall that in Lemma 4.2, we have observed that the lattice $\mathcal{G}$ is isomorphic to $\mathcal{F}(\kappa) \times \mathcal{F}(\kappa)$. Therefore the lattices $\mathcal{H}$ and $\mathcal{G}$ are not isomorphic. □

6. Representing $\mathcal{S}$ in a subspace lattice

Although the construction in the three previous sections was performed for an *infinite* cardinal κ, the results of the present section on embedding the lattice $\mathcal{M}_3[\mathcal{P}(\kappa)]$ into $\mathrm{Sub}(\mathcal{V})$ (namely Theorem 6.4) work just as well for κ finite. In particular, Proposition 6.5 (an enhancement of [**33**, Lemma 2.9]) holds for lattices of any cardinality.

Let $\mathbb{F}$ be an arbitrary field and let $\boldsymbol{V}$ denote the vector space over the field $\mathbb{F}$ presented by generators $x_\alpha, y_\alpha, z_\alpha$, $\alpha \in \kappa$, and relations $x_\alpha + y_\alpha + z_\alpha = 0$. For a subset X of the vector space $\boldsymbol{V}$ we denote by $\mathrm{Span}(X)$ the subspace of $\boldsymbol{V}$ generated by X. Given subspaces of $\boldsymbol{V}$, say $\boldsymbol{X}$ and $\boldsymbol{Y}$, we will use the notation $\boldsymbol{X} + \boldsymbol{Y} = \mathrm{Span}(\boldsymbol{X} \cup \boldsymbol{Y})$. Let $\mathrm{Sub}(\boldsymbol{V})$ denote the lattice of all subspaces of the vector space $\boldsymbol{V}$.

For all $A, B, C \subseteq \kappa$ we put $\boldsymbol{X}_A = \mathrm{Span}(\{x_\alpha \mid \alpha \in A\})$, $\boldsymbol{Y}_B = \mathrm{Span}(\{y_\beta \mid \beta \in B\})$, and $\boldsymbol{Z}_C = \mathrm{Span}(\{z_\gamma \mid \gamma \in C\})$.

We define the map $F \colon \mathcal{P}(\kappa)^3 \to \mathrm{Sub}(\boldsymbol{V})$ by the correspondence

$$\langle A, B, C\rangle \mapsto \boldsymbol{X}_A + \boldsymbol{Y}_B + \boldsymbol{Z}_C. \tag{6.1}$$

Each of the sets $\{x_\alpha \mid \alpha \in \kappa\}$, $\{y_\beta \mid \beta \in \kappa\}$, and $\{z_\gamma \mid \gamma \in \kappa\}$ is clearly linearly independent. It follows that $\boldsymbol{X}_{A \cup A'} = \boldsymbol{X}_A + \boldsymbol{X}_{A'}$ for all $A, A' \subseteq \kappa$ and, similarly, $\boldsymbol{Y}_{B \cup B'} = \boldsymbol{Y}_B + \boldsymbol{Y}_{B'}$ and $\boldsymbol{Z}_{C \cup C'} = \boldsymbol{Z}_C + \boldsymbol{Z}_{C'}$ for all $B, B', C, C' \subseteq \kappa$. A straightforward computation gives the following lemma:

LEMMA 6.1. *The map $F \colon \mathcal{P}(\kappa)^3 \to \mathrm{Sub}(\boldsymbol{V})$ is a bounded join-homomorphism.*

PROOF. Clearly $F\langle\emptyset,\emptyset,\emptyset\rangle = \mathbf{0}$ and $F\langle\kappa,\kappa,\kappa\rangle = \boldsymbol{V}$. Following the definitions, we compute $F(\langle A, B, C\rangle) + F(\langle A', B', C'\rangle) = \boldsymbol{X}_A + \boldsymbol{Y}_B + \boldsymbol{Z}_C + \boldsymbol{X}_{A'} + \boldsymbol{Y}_{B'} + \boldsymbol{Z}_{C'} = \boldsymbol{X}_{A \cup A'} + \boldsymbol{Y}_{B \cup B'} + \boldsymbol{Z}_{C \cup C'} = F(\langle A \cup A', B \cup B', C \cup C'\rangle)$. □

Let $G \colon \mathrm{Sub}(\boldsymbol{V}) \to \mathcal{P}(\kappa)^3$ be a map defined by

$$\boldsymbol{W} \mapsto \langle\{\alpha \mid x_\alpha \in \boldsymbol{W}\}, \{\beta \mid y_\beta \in \boldsymbol{W}\}, \{\gamma \mid z_\gamma \in \boldsymbol{W}\}\rangle,$$

for all $\boldsymbol{W} \in \mathrm{Sub}(\boldsymbol{V})$.

It is straightforward that G is a bounded meet-homomorphism and that it is the right adjoint of F (i.e., replacing the lattice $\mathrm{Sub}(\boldsymbol{V})$ with its dual, the maps F and G form a Galois correspondence [**54**]). Indeed, one readily sees that

$$F\langle A, B, C\rangle \subseteq \boldsymbol{W} \text{ iff } \langle A, B, C\rangle \leq G(\boldsymbol{W}).$$

The maps F and G induce a closure operator GF on $\mathcal{P}(\kappa)^3$.

LEMMA 6.2. *The composition $GF \colon \mathcal{P}(\kappa)^3 \to \mathcal{P}(\kappa)^3$ is precisely the closure operator $\overline{\langle - \rangle}$ on $\mathcal{P}(\kappa)^3$ defined by* (2.2).

Proof. We shall prove that $GF\langle A, B, C\rangle = \overline{\langle A, B, C\rangle}$, for all $\langle A, B, C\rangle$ from $\mathcal{P}(\kappa)^3$. By symmetry, it suffices to prove that

$$\{\alpha \in \kappa \mid x_\alpha \in F\langle A, B, C\rangle\} = A \cup (B \cap C).$$

Let $\alpha \in A \cup (B \cap C)$. If $\alpha \in A$, then $x_\alpha \in F\langle A, B, C\rangle$ by the definition (6.1), while if $\alpha \in B \cap C$, then $x_\alpha = -y_\alpha - z_\alpha \in F\langle A, B, C\rangle$ by (6.1) and the defining relations of $\boldsymbol{V}$. It follows that $A \cup (B \cap C) \subseteq \{\alpha \in \kappa \mid x_\alpha \in F\langle A, B, C\rangle\}$.

In order to prove the opposite inclusion, take any $x \in \kappa \setminus A$ satisfying $x_x \in F\langle A, B, C\rangle$; if there is none, there is nothing to prove. We need to show that then $x \in B \cap C$. Certainly

$$x_x = \sum_{\alpha \in A} a_\alpha x_\alpha + \sum_{\beta \in B} b_\beta y_\beta + \sum_{\gamma \in C} c_\gamma z_\gamma \tag{6.2}$$

for suitable a_α, b_β, and $c_\gamma \in \mathbb{F}$ such that all but finitely many of them are zero. We set $a_\alpha = 0$ for $\alpha \notin A$, $b_\beta = 0$ for $\beta \notin B$, and $c_\gamma = 0$ for $\gamma \notin C$. Since $z_\gamma + x_\gamma + y_\gamma = 0$ for every $\gamma \in \kappa$, it follows from (6.2) that

$$x_x = \left(\sum_{\alpha \in A} a_\alpha x_\alpha - \sum_{\gamma \in C} c_\gamma x_\gamma\right) + \left(\sum_{\beta \in B} b_\beta y_\beta - \sum_{\gamma \in C} c_\gamma y_\gamma\right). \tag{6.3}$$

It easily follows from the defining relations of $\boldsymbol{V}$ that $\{x_\alpha, y_\alpha \mid \alpha \in \kappa\}$ forms a basis of $\boldsymbol{V}$. Thus, applying (6.3) we get that

$$a_x - c_x = 1 \text{ and } b_x - c_x = 0. \tag{6.4}$$

Since by our assumption $x \notin A$, we get from (6.2) that $a_x = 0$. Substituting to (6.4) we get that $b_x = c_x = -1$, hence $x \in B \cap C$. This concludes the proof that $A \cup (B \cap C) \supseteq \{\alpha \in \kappa \mid x_\alpha \in F\langle A, B, C\rangle\}$. □

The next lemma shows that $F \restriction \mathcal{M}_3[\mathcal{P}(\kappa)]$ preserves meets. Note that with Lemma 6.1, this means that $F \restriction \mathcal{M}_3[\mathcal{P}(\kappa)]$ is a lattice embedding of $\mathcal{M}_3[\mathcal{P}(\kappa)]$ into the lattice $\mathrm{Sub}(\boldsymbol{V})$.

Lemma 6.3. *Let $\langle A, B, C\rangle, \langle A', B', C'\rangle \in \mathcal{M}_3[\mathcal{P}(\kappa)]$ be balanced triples. Then*

$$F\langle A, B, C\rangle \cap F\langle A', B', C'\rangle = F\langle A \cap A', B \cap B', C \cap C'\rangle.$$

Proof. Since, by Lemma 6.1, F is a join-homomorphism, it is monotone, whence $F\langle A \cap A', B \cap B', C \cap C'\rangle \subseteq F\langle A, B, C\rangle \cap F\langle A', B', C'\rangle$. Thus it remains to prove the opposite inclusion.

Let $v \in F\langle A, B, C\rangle \cap F\langle A', B', C'\rangle$ be a non-zero vector. Then v can be expressed as

$$v = \sum_{\alpha \in A} a_\alpha x_\alpha + \sum_{\beta \in B} b_\beta y_\beta + \sum_{\gamma \in C} c_\gamma z_\gamma = \sum_{\alpha \in A'} a'_\alpha x_\alpha + \sum_{\beta \in B'} b'_\beta y_\beta + \sum_{\gamma \in C'} c'_\gamma z_\gamma. \tag{6.5}$$

Consider such an expression of v with

$$|\{\alpha \mid a_\alpha \neq 0\}| + |\{\beta \mid b_\beta \neq 0\}| + |\{\gamma \mid c_\gamma \neq 0\}| \tag{6.6}$$

minimal possible. Put $a_\alpha = 0$ for $\alpha \notin A$, $b_\beta = 0$ for $\beta \notin B$, and $c_\gamma = 0$ for $\gamma \notin C$. By symmetry, we can assume that $a_\alpha \neq 0$ for some $\alpha \in A$. Suppose for a contradiction that $\alpha \notin A'$. Since the triple $\langle A', B', C' \rangle$ is balanced, $B' \cap C' \subseteq A'$, whence $\alpha \notin B' \cap C'$. Without loss of generality we can assume that $\alpha \notin B'$. If all a_α, b_α, and c_α were non-zero, we could replace $c_\alpha z_\alpha$ with $-c_\alpha x_\alpha - c_\alpha y_\alpha$ and reduce the value of the expression in (6.6) which is assumed minimal possible. Thus either $b_\alpha = 0$ or $c_\alpha = 0$ (recall that we assume that $a_\alpha \neq 0$). We will deal with these two cases separately. If $b_\alpha = 0$, then the equality

$$a_\alpha x_\alpha + c_\alpha z_\alpha = c'_\alpha z_\alpha \tag{6.7}$$

must hold true. Since x_α and z_α are linearly independent, it follows from (6.7) that $a_\alpha = 0$ which contradicts our choice of α. The remaining case is when $c_\alpha = 0$. Under this assumption we have that

$$a_\alpha x_\alpha + b_\alpha y_\alpha = c'_\alpha z_\alpha.$$

It follows that

$$a_\alpha x_\alpha = c'_\alpha z_\alpha - b_\alpha y_\alpha = -c'_\alpha x_\alpha - (c'_\alpha + b_\alpha) y_\alpha. \tag{6.8}$$

Since x_α and y_α are linearly independent, we infer from (6.8) that $a_\alpha = -c'_\alpha = b_\alpha$. Then we could reduce the value of (6.6) by replacing $a_\alpha x_\alpha + b_\alpha y_\alpha$ with $c'_\alpha z_\alpha$ in (6.5). This contradicts the minimality of (6.6). $\square$

Combining Lemma 6.1, Lemma 6.2, and Lemma 6.3, we conclude:

THEOREM 6.4. *The restrictions $F \restriction \mathbf{M}_3[\mathbf{P}(\kappa)] \colon \mathbf{M}_3[\mathbf{P}(\kappa)] \to \mathrm{Sub}(\boldsymbol{V})$ and,* a fortiori*, $F \restriction \mathbf{S} \colon \mathbf{S} \to \mathrm{Sub}(\boldsymbol{V})$ are bounded lattice embeddings. In particular, the lattice $\mathbf{S}$ is isomorphic to a bounded sublattice of the subspace lattice of a vector space.*

It is well-known that a distributive lattice $\mathcal{L}$ embeds (via a bounds-preserving lattice embedding) into the lattice $\mathbf{P}(\kappa)$, where κ is the cardinality of the set of all maximal ideals of $\mathcal{L}$. Such embedding induces an embedding $\mathbf{M}_3[\mathcal{L}] \hookrightarrow \mathbf{M}_3[\mathbf{P}(\kappa)]$ (cf. Lemma 3.1). By Theorem 6.4, the lattice $\mathbf{M}_3[\mathbf{P}(\kappa)]$ embeds into the lattice $\mathrm{Sub}(\boldsymbol{V})$ for a suitable vector space $\boldsymbol{V}$ (note again that we now also admit finite κ). Since the lattice $\mathrm{Sub}(\mathcal{V})$ is Arguesian, so are $\mathbf{M}_3[\mathbf{P}(\kappa)]$ and $\mathbf{M}_3[\mathcal{L}]$.

On the other hand, [**33**, Lemma 2.9] states that a lattice L is distributive if and only if $\mathbf{M}_3[\mathcal{L}]$ is modular. Hence, if $\mathbf{M}_3[\mathcal{L}]$ is modular, it follows that $\mathcal{L}$ is distributive, and, by the above argument, $\mathbf{M}_3[\mathcal{L}]$ is even Arguesian. We have thus proven the following strengthening of [**33**, Lemma 2.9]:

PROPOSITION 6.5. *Let L be a lattice. Then L is distributive iff the lattice $\mathbf{M}_3[\mathcal{L}]$ is modular iff $\mathbf{M}_3[\mathcal{L}]$ is Arguesian. If this is the case, then $\mathbf{M}_3[\mathcal{L}]$ can be embedded into the lattice of all subspaces of a suitable vector space over any given field.*

7. Non existence of 3-frames

In this section we prove that there is no 3-frame in the lattice $\mathfrak{M}_3[\mathfrak{D}]$ for any distributive lattice $\mathfrak{D}$. As a consequence, we cannot apply the Jónsson's coordinatization theorem in order to prove coordinatizability of any of these lattices, in particular, of the lattices $\mathfrak{M}_3[\mathfrak{F}(\kappa)]$ and $\mathfrak{S}$.

Lemma 7.1. *Let $\mathfrak{D}$ be a distributive lattice. Then for each $\langle a_1, a_2, a_3\rangle \in \mathfrak{D}^3$, the equality*

$$\mu\overline{\langle a_1, a_2, a_3\rangle} = \mu\langle a_1, a_2, a_3\rangle.$$

holds true.

Proof. First observe that for all $1 \leq k < l \leq 3$ we have that

$$a_k \wedge a_l \leq \bigvee_{1\leq i<j\leq 3} (a_i \wedge a_j) = \mu\langle a_1, a_2, a_3\rangle. \tag{7.1}$$

By (2.2) we have the equalities

$$\begin{aligned}\mu\overline{\langle a_1, a_2, a_3\rangle} &= \mu\langle a_1 \vee \mu\langle a_1, a_2, a_3\rangle, a_2 \vee \mu\langle a_1, a_2, a_3\rangle, a_3 \vee \mu\langle a_1, a_2, a_3\rangle\rangle \\ &= \bigvee_{1\leq i<j\leq 3} ((a_i \vee \mu\langle a_1, a_2, a_3\rangle) \wedge (a_j \vee \mu\langle a_1, a_2, a_3\rangle)).\end{aligned}$$

Since the lattice $\mathfrak{D}$ is distributive,

$$(a_i \vee \mu\langle a_1, a_2, a_3\rangle) \wedge (a_j \vee \mu\langle a_1, a_2, a_3\rangle) = (a_i \wedge a_j) \vee \mu\langle a_1, a_2, a_3\rangle,$$

for all $1 \leq i < j \leq 3$. Applying (7.1), we conclude that

$$\mu\overline{\langle a_1, a_2, a_3\rangle} = \bigvee_{1\leq i<j\leq 3} ((a_i \wedge a_j) \vee \mu\langle a_1, a_2, a_3\rangle) = \mu\langle a_1, a_2, a_3\rangle.$$

□

With regard to (2.3), we conclude from Lemma 7.1 that

Corollary 7.2. *If $\mathfrak{D}$ is a distributive lattice, then*

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\langle a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3\rangle,$$

for all $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle, \boldsymbol{b} = \langle b_1, b_2, b_3\rangle \in \mathfrak{M}_3[\mathfrak{D}]$.

Lemma 7.3. *Let $\mathfrak{D}$ be a distributive lattice and $\boldsymbol{a} = \langle a_1, a_2, a_3\rangle$ and $\boldsymbol{b} = \langle b_1, b_2, b_3\rangle$ elements of $\mathfrak{M}_3[\mathfrak{D}]$. If $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{0}$, then*

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i\right) \wedge \left(\bigvee_{j=1}^{3} b_j\right)\right).$$

Proof. Applying Corollary 7.2 and using the distributivity of $\mathfrak{D}$, we compute that

$$\begin{aligned}\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\langle a_1 \vee b_1, a_2 \vee b_2, a_3 \vee b_3\rangle &= \bigvee_{1\leq i<j\leq 3} ((a_i \vee b_i) \wedge (a_j \vee b_j)) \\ &= \bigvee_{1\leq i<j\leq 3} ((a_i \wedge a_j) \vee (b_i \wedge b_j) \vee (a_i \wedge b_j) \vee (a_j \wedge b_i)).\end{aligned}$$

Since $\boldsymbol{a}$ and $\boldsymbol{b}$ are balanced triples, $\mu\boldsymbol{a} = a_i \wedge a_j$ and $\mu\boldsymbol{b} = b_i \wedge b_j$ for all $1 \leq i < j \leq 3$. Thus

$$
\begin{aligned}
\mu(\boldsymbol{a} \vee \boldsymbol{b}) &= \bigvee_{1\leq i<j\leq 3} (\mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee (a_i \wedge b_j) \vee (a_j \wedge b_i)) \\
&= \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \bigvee_{1\leq i<j\leq 3} ((a_i \wedge b_j) \vee (a_j \wedge b_i)).
\end{aligned} \tag{7.2}
$$

¿From $\boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{0}$ we get that $a_i \wedge b_i = 0$, for all $i = 1, 2, 3$. Substituting to (7.2) we get that

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \bigvee_{1\leq i\leq j\leq 3} ((a_i \wedge b_j) \vee (a_j \wedge b_i)) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} (a_i \wedge b_j).$$

Applying the distributivity of $\mathfrak{D}$ again we conclude that

$$\mu(\boldsymbol{a} \vee \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i \right) \wedge \left(\bigvee_{j=1}^{3} b_j \right) \right).$$

$\square$

Lemma 7.4. *Let $\mathfrak{D}$ be a bounded distributive lattice and $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle, \boldsymbol{b} = \langle b_1, b_2, b_3 \rangle \in \mathfrak{M}_3[\mathfrak{D}]$. If $\boldsymbol{a} \oplus \boldsymbol{b} = \boldsymbol{t}1$, then*

$$\mu\boldsymbol{a} \oplus \bigvee_{j=1}^{3} b_j = 1.$$

Proof. Since trivially

$$\mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i \right) \wedge \left(\bigvee_{j=1}^{3} b_j \right) \right) \leq \bigvee_{j=1}^{3} b_j,$$

we infer from Lemma 7.3 that

$$1 = \mu(\boldsymbol{a} \oplus \boldsymbol{b}) = \mu\boldsymbol{a} \vee \mu\boldsymbol{b} \vee \left(\left(\bigvee_{i=1}^{3} a_i \right) \wedge \left(\bigvee_{j=1}^{3} b_j \right) \right) \leq \mu\boldsymbol{a} \vee \bigvee_{j=1}^{3} b_j \leq 1. \tag{7.3}$$

Since $\boldsymbol{a} \wedge \boldsymbol{b} = 0$, we have that $\mu\boldsymbol{a} \leq a_i \leq b_i$, for all $i = 1, 2, 3$. Since the lattice $\mathfrak{D}$ is distributive, we conclude that

$$0 = \bigvee_{j=1}^{3} (\mu\boldsymbol{a} \wedge b_j) = \mu\boldsymbol{a} \wedge \bigvee_{j=1}^{3} b_j. \tag{7.4}$$

Combining (7.3) and (7.4) we get the desired equality $\mu\boldsymbol{a} \oplus \bigvee_{j=1}^{3} b_j = 1$. $\square$

Lemma 7.5. *Let $\mathfrak{D}$ be a bounded distributive lattice and $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{a}' = \langle a'_1, a'_2, a'_3 \rangle$ perspective elements of $\mathfrak{M}_3[\mathfrak{D}]$. If $\boldsymbol{a} \wedge \boldsymbol{a}' = \boldsymbol{t}0$, then*

$$\mu\boldsymbol{a} = \mu\boldsymbol{a}' \quad \textit{and} \quad \mu(\boldsymbol{a} \vee \boldsymbol{a}') = \bigvee_{i=1}^{3} a_i = \bigvee_{i=1}^{3} a'_i.$$

PROOF. Let $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle$ be a common complement of $\boldsymbol{a}$ and $\boldsymbol{a}'$. It follows from Lemma 7.4 that both $\mu\boldsymbol{a}$ and $\mu\boldsymbol{a}'$ are complements of $\bigvee_{j=1}^{3} b_j$. Since complements in a distributive lattice are unique, we get that $\mu\boldsymbol{a} = \mu\boldsymbol{a}'$. Similarly we get that both $\bigvee_{i=1}^{3} a_i$ and $\bigvee_{i=1}^{3} a'_i$ are complements of $\mu\boldsymbol{b}$, hence they are equal. From these equalities we infer that

$$\mu\boldsymbol{a} = \mu\boldsymbol{a}' \leq \bigvee_{i=1}^{3} a'_i = \bigvee_{i=1}^{3} a_i.$$

Applying Lemma 7.3 we conclude that

$$\mu(\boldsymbol{a} \vee \boldsymbol{a}') = \bigvee_{i=1}^{3} a_i = \bigvee_{i=1}^{3} a'_i.$$

□

PROPOSITION 7.6. *There is no* 3*-frame in the lattice* $\mathcal{M}_3[\mathcal{D}]$*, for any bounded distributive lattice* $\mathcal{D}$*.*

PROOF. Suppose that there are elements $\boldsymbol{a} = \langle a_1, a_2, a_3 \rangle$, $\boldsymbol{a}' = \langle a'_1, a'_2, a'_3 \rangle$, and $\boldsymbol{a}'' = \langle a''_1, a''_2, a''_3 \rangle$ of $\mathcal{M}_3[\mathcal{D}]$ such that $\boldsymbol{a} \sim \boldsymbol{a}'$, $\boldsymbol{a} \sim \boldsymbol{a}''$ and the family $\langle \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{a}'' \rangle$ is independent. Then $\mu(\boldsymbol{a} \vee \boldsymbol{a}') = \bigvee_{i=1}^{3} a_i = \bigvee_{i=1}^{3} a''_i$ due to Lemma 7.5. It follows that $\boldsymbol{a} \vee \boldsymbol{a}' \geq \boldsymbol{a}''$ which contradicts the independence of the family $\langle \boldsymbol{a}, \boldsymbol{a}', \boldsymbol{a}'' \rangle$. □

COROLLARY 7.7. *There is no* 3*-frame in the lattice* $\mathcal{M}_3[\mathcal{B}]$*, for any Boolean lattice* $\mathcal{B}$*. In particular, neither the lattices* $\mathcal{M}_3[\mathcal{F}(\kappa)]$ *nor the lattice* $\mathcal{S}$ *has a* 3*-frame.*

REMARK 7.8. This remark is due to the anonymous referee. He pointed out that the main results of Sections 6 and 7 can be obtained by a simpler argument using the representation of a distributive lattice as a subdirect product of the two-element lattice $\mathbf{2}$. Namely, it is well-known that a distributive lattice $\mathcal{D}$ is a subdirect power of $\mathbf{2}$. In particular, there is an index set I and an embedding $\varphi\colon \mathcal{D} \hookrightarrow \mathbf{2}^I$ such that the composition $\pi_i \circ \varphi\colon \mathcal{D} \to \mathbf{2}$ with the canonical projection $\pi_i\colon \mathbf{2}^I \to \mathbf{2}$ is a surjective homomorphism for all $i \in I$. The map φ induces the embedding $\mathcal{M}_3[\mathcal{D}] \to \mathcal{M}_3[\mathbf{2}^I]$ given by $\langle a, b, c \rangle \mapsto \langle \varphi(a), \varphi(b), \varphi(c) \rangle$. Observing that $\mathcal{M}_3[\mathbf{2}] \simeq \mathcal{M}_3$ we get isomorphisms $\mathcal{M}_3[\mathbf{2}^I] \simeq \mathcal{M}_3[\mathbf{2}]^I \simeq {\mathcal{M}_3}^I$. Thus we have an embedding $\Phi\colon \mathcal{M}_3[\mathcal{D}] \hookrightarrow {\mathcal{M}_3}^I$. It is straightforward to see that the composition of Φ with the i^{th} canonical projection ${\mathcal{M}_3}^I \to \mathcal{M}_3$ is a surjective homomorphism $\mathcal{M}_3[\mathcal{D}] \to \mathcal{M}_3$. Therefore $\mathcal{M}_3[\mathcal{D}]$ is a subdirect power of $\mathcal{M}_3$. The lattice $\mathcal{M}_3$ embeds into the subspace lattice of the 2-dimensional vectors space $\boldsymbol{V}$ over an arbitrary field. Let $\psi\colon \mathcal{M}_3 \hookrightarrow \boldsymbol{V}$ be such an embedding. Then $\mathcal{M}_3^I$ embeds into $\operatorname{Sub} \boldsymbol{V}^{(I)}$ (here $\boldsymbol{V}^{(I)}$ denotes the direct sum of copies of $\boldsymbol{V}$) via the mapping $(a_i)_{i\in I} \mapsto \bigoplus_{i\in I} \psi(a_i)$. The restriction of the map to $\mathcal{M}_3[\mathcal{D}]$ is an embedding of $\mathcal{M}_3[\mathcal{D}]$ into $\operatorname{Sub} \boldsymbol{V}^{(I)}$. Clearly, if $\mathcal{D}$ is bounded, the embedding can be chosen bounds-preserving. This gives the main results of Section 6.

Let $\mathfrak{D}$ be a bounded lattice. Observe that the embedding $\Phi\colon \mathfrak{M}_3[\mathfrak{D}] \hookrightarrow {\mathfrak{M}_3}^I$ preserves the bounds. It follows that the Φ-image of a 3-frame would be a 3-frame in ${\mathfrak{M}_3}^I$. Let $i \in I$ and $\pi_i\colon {\mathfrak{M}_3}^I \to \mathfrak{M}_3$ be the corresponding canonical projection. The π_i image of the 3-frame in ${\mathfrak{M}_3}^I$ would be a 3-frame in $\mathfrak{M}_3$. However, it is easy to see that there is no 3-frame in $\mathfrak{M}_3$. Consequently, there is no 3-frame in $\mathfrak{M}_3[\mathfrak{D}]$. Thus we get Proposition 7.6.

8. Coordinatizability

We prove that despite of non-existence of 3-frames, the lattice $\mathfrak{M}_3[\mathfrak{B}]$ is coordinatizated for any Boolean lattice $\mathfrak{B}$. It is isomorphic to $\mathcal{L}(\boldsymbol{M}[\mathfrak{B}]^*)$, the lattice of all finitely generated right ideals of the Boolean power of the ring $\boldsymbol{M}$, the ring of 2×2 matrices over the two-element field, by the Boolean lattice $\mathfrak{B}$. Modifying this construction we show that the lattice $\mathfrak{S}$ introduced in Section 3 is coordinatizable as well.

Let the notation $\boldsymbol{M}$ stand for the ring of all 2×2-matrices over the two-element field $\mathbb{F}_2$. It is well known that the matrix ring over a regular ring is regular, in particular, the ring $\boldsymbol{M}$ is regular (cf. [**22**, Theorem 1.7]). We put

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad e_3 := \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

There are exactly eight idempotents in the ring $\boldsymbol{M}$, namely $0, 1, e_1, e_2, e_3, 1-e_1, 1-e_2$, and $1-e_3$, and there are exactly three proper non-zero right ideals of $\boldsymbol{M}$, namely $e_1\boldsymbol{M} = (1-e_3)\boldsymbol{M}$, $e_2\boldsymbol{M} = (1-e_1)\boldsymbol{M}$, and $e_3\boldsymbol{M} = (1-e_2)\boldsymbol{M}$. Thus the lattice $\mathcal{L}(\boldsymbol{M})$ is isomorphic to the five-element modular non-distributive lattice $\mathfrak{M}_3$ (see Figure 1).

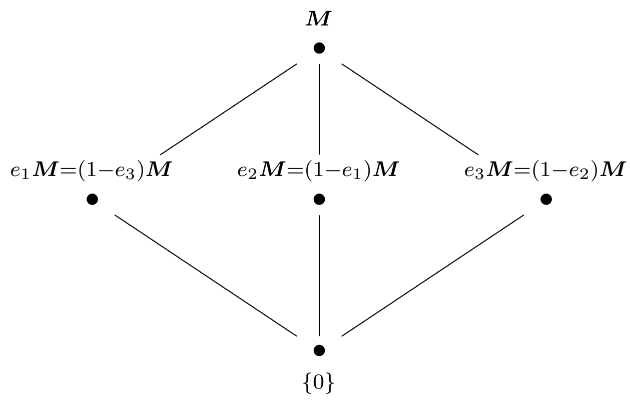

FIGURE 1. The lattice $\mathcal{L}(\boldsymbol{M})$

We denote by $\mathrm{Idem}(\boldsymbol{R})$ the set of all idempotents of a ring $\boldsymbol{R}$. We are going to make use of the next elementary lemma.

LEMMA 8.1. *Let $\boldsymbol{R}$ be a ring and $e, f \in \mathrm{Idem}(\boldsymbol{R})$. Then*

$$ef = f \iff f\boldsymbol{R} \subseteq e\boldsymbol{R}.$$

PROOF. ($\Leftarrow$) If $f\boldsymbol{R} \subseteq e\boldsymbol{R}$, then $f \in e\boldsymbol{R}$ and so $f = er$ for some $r \in \boldsymbol{R}$. It follows that $ef = eer = er = f$. ($\Rightarrow$) Conversely, $ef = f$ implies that $f \in e\boldsymbol{R}$. We get readily that $f\boldsymbol{R} \subseteq e\boldsymbol{R}$. $\square$

We equip the set $\mathrm{Idem}(\boldsymbol{R})$ with a quasi-order $\leq_{\mathrm{e}}$ defined as follows: $f \leq_{\mathrm{e}} e$ provided that $ef = f$, for all $e, f \in \mathrm{Idem}(\boldsymbol{R})$. Further, we denote by $\equiv_{\mathrm{e}}$ the corresponding equivalence relation on $\mathrm{Idem}(\boldsymbol{R})$, i.e., $e \equiv_{\mathrm{e}} f$ if and only if both $e \leq_{\mathrm{e}} f$ and $f \leq_{\mathrm{e}} e$, for all $e, f \in \mathrm{Idem}(\boldsymbol{R})$.

Suppose that $\boldsymbol{R}$ is a regular ring. Let $\iota_{\boldsymbol{R}} \colon \mathrm{Idem}(\boldsymbol{R}) \to \boldsymbol{\mathcal{L}}(\boldsymbol{R})$ be the map given by the correspondence $e \mapsto e\boldsymbol{R}$. It follows from Lemma 8.1 that the kernel of the map $\iota_{\boldsymbol{R}}$ coincides with the the equivalence relation $\equiv_{\mathrm{e}}$ and the quotient $\mathrm{Idem}(\boldsymbol{R})/\equiv_{\mathrm{e}}$ is order-isomorphic to the set $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ ordered by inclusion. Since $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ is a lattice, $\mathrm{Idem}(\boldsymbol{R})/\equiv_{\mathrm{e}}$ has finite suprema and infima, and the lattices $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ and $\mathrm{Idem}(\boldsymbol{R})/\equiv_{\mathrm{e}}$ are isomorphic.

The following lemma is a trivial consequence of the preceding two paragraphs. We leave the details of the proof to the reader.

LEMMA 8.2. *Let $\boldsymbol{\mathcal{L}}$ be a lattice and $\boldsymbol{R}$ a regular ring. Suppose that there is a surjective map $\varepsilon \colon \mathrm{Idem}(\boldsymbol{R}) \to \boldsymbol{\mathcal{L}}$ such that*

$$e \leq_{\mathrm{e}} f \iff \varepsilon(e) \leq \varepsilon(f), \quad \textit{for all } e, f \in \mathrm{Idem}(\boldsymbol{R}). \tag{8.1}$$

Then $\ker \varepsilon = \ker \iota_R$ is equal to $\equiv_{\mathrm{e}}$ and the lattice $\boldsymbol{\mathcal{L}}$ is isomorphic to $\boldsymbol{\mathcal{L}}(\boldsymbol{R})$ via the composition[1] *$\iota_R \circ \varepsilon^{-1} \colon \boldsymbol{\mathcal{L}} \to \boldsymbol{\mathcal{L}}(\boldsymbol{R})$.*

Note that in the ring $\boldsymbol{M}$ introduced above, we have $e_1 \equiv_{\mathrm{e}} 1 - e_3$, $e_2 \equiv_{\mathrm{e}} 1 - e_1$, and $e_3 \equiv_{\mathrm{e}} 1 - e_2$, and the idempotents e_1, e_2, and e_3 are pairwise incomparable. Recall from Subsection 2.4 that the Boolean power $\boldsymbol{M}[\boldsymbol{\mathcal{B}}]^*$ of the ring $\boldsymbol{M}$ by a Boolean lattice $\boldsymbol{\mathcal{B}}$ is the set of all continuous functions from the Stone space of $\boldsymbol{\mathcal{B}}$ to $\boldsymbol{M}$ equipped with the discrete topology.

LEMMA 8.3. *Let $\boldsymbol{\mathcal{B}}$ be a Boolean lattice. If a ring $\boldsymbol{R}$ is regular, then the Boolean power $\boldsymbol{R}[\boldsymbol{\mathcal{B}}]^*$ is regular as well.*

PROOF. For each $a \in \boldsymbol{R}$ we pick an element $a^* \in \boldsymbol{R}$ such that $a = aa^*a$. Given $\boldsymbol{x} \in \boldsymbol{R}[\boldsymbol{\mathcal{B}}]^*$, we define a map $\boldsymbol{x}^* \colon \boldsymbol{\mathcal{B}}^* \to \boldsymbol{R}$ by the correspondence $\mathfrak{u} \mapsto \boldsymbol{x}(\mathfrak{u})^*$, $\mathfrak{u} \in \boldsymbol{\mathcal{B}}^*$. The $\boldsymbol{x}^*$-preimage of an element $b \in \boldsymbol{R}$ is $\bigcup\{\boldsymbol{x}^{-1}(a) \mid a^* = b\}$, which is a union of open sets. It follows that the map $\boldsymbol{x}^*$ is continuous and clearly $\boldsymbol{x} = \boldsymbol{x}\boldsymbol{x}^*\boldsymbol{x}$. Therefore $\boldsymbol{R}[\boldsymbol{\mathcal{B}}]^*$ is a regular ring. $\square$

---

[1] Purists would object that the composition $\iota_R \circ \varepsilon^{-1}$ sends an element $a \in \boldsymbol{\mathcal{L}}$ to a singleton set $\{e\boldsymbol{R}\}$, where e is any idempotent from the $\equiv_{\mathrm{e}}$-block $\varepsilon^{-1}(a)$. For the sake of simplicity we identify the singleton set $\{e\boldsymbol{R}\}$ with its element $e\boldsymbol{R}$.

Given elements a, b of a Boolean lattice $\mathcal{B}$, we set $a - b := a \wedge b'$, where b' is a unique complement of b. Note that an element $\boldsymbol{x} \in \boldsymbol{M}[\mathcal{B}]^*$ is an idempotent if and only if $\boldsymbol{x}(\mathfrak{u}) \in \mathrm{Idem}(\boldsymbol{M})$ for every $\mathfrak{u} \in \mathcal{B}^*$. For each $\boldsymbol{e} \in \mathrm{Idem}(\boldsymbol{M}[\mathcal{B}]^*)$ we set $\varepsilon(\boldsymbol{e}) := \langle a_1, a_2, a_3 \rangle$, where[2]

$$
\begin{aligned}
N_{a_1} &= \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) \in \{1, e_1, 1 - e_3\}\},\\
N_{a_2} &= \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) \in \{1, e_2, 1 - e_1\}\},\\
N_{a_3} &= \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) \in \{1, e_3, 1 - e_2\}\}.
\end{aligned}
\tag{8.2}
$$

It is clear that $\varepsilon(\boldsymbol{e})$ is a balanced triple with $N_{\mu\varepsilon(\boldsymbol{e})} = \{\mathfrak{u} \mid \boldsymbol{e}(\mathfrak{u}) = 1\}$. Therefore (8.2) defines a map $\varepsilon\colon \mathrm{Idem}(\boldsymbol{M}[\mathcal{B}]^*) \to \mathcal{M}_3[\mathcal{B}]$.

Lemma 8.4. *Let $\mathcal{B}$ be a Boolean lattice. Then the map $\varepsilon\colon \mathrm{Idem}(\boldsymbol{M}[\mathcal{B}]^*) \to \mathcal{M}_3[\mathcal{B}]$ defined by* (8.2) *satisfies property* (8.1).

Proof. The implications $\boldsymbol{e} \leq_{\mathrm{e}} \boldsymbol{f} \implies \varepsilon(\boldsymbol{e}) \leq \varepsilon(\boldsymbol{f})$, $\boldsymbol{e}, \boldsymbol{f} \in \mathrm{Idem}(\boldsymbol{M}[\mathcal{B}]^*)$, are seen readily from (8.2). Let $\boldsymbol{e}, \boldsymbol{f} \in \mathrm{Idem}(\boldsymbol{M}[\mathcal{B}]^*)$ with $\varepsilon(\boldsymbol{e}) = \langle a_1, a_2, a_3 \rangle$ and $\varepsilon(\boldsymbol{f}) = \langle b_1, b_2, b_3 \rangle$. Suppose that $\varepsilon(\boldsymbol{e}) \leq \varepsilon(\boldsymbol{f})$ and let $\mathfrak{u} \in \mathcal{B}^*$. The inequality implies that $\mu\varepsilon(\boldsymbol{e}) \leq \mu\varepsilon(\boldsymbol{f})$, hence $\boldsymbol{e}(\mathfrak{u}) = 1 \implies \boldsymbol{f}(\mathfrak{u}) = 1$. From $a_1 \leq b_1$ we infer that $\boldsymbol{e}(\mathfrak{u}) \in \{e_1, 1 - e_3\} \implies \boldsymbol{f}(\mathfrak{u}) \in \{1, e_1, 1 - e_3\}$. Similarly, from $a_2 \leq b_2$ we get that $\boldsymbol{e}(\mathfrak{u}) \in \{e_2, 1 - e_1\} \implies \boldsymbol{f}(\mathfrak{u}) \in \{1, e_2, 1 - e_1\}$ and from $a_3 \leq b_3$ we conclude that $\boldsymbol{e}(\mathfrak{u}) \in \{e_3, 1 - e_2\} \implies \boldsymbol{f}(\mathfrak{u}) \in \{1, e_3, 1 - e_2\}$. Therefore $\boldsymbol{e} \leq_{\mathrm{e}} \boldsymbol{f}$. □

Theorem 8.5. *Let $\mathcal{B}$ be a Boolean lattice. The ring $\boldsymbol{M}[\mathcal{B}]^*$ is regular and*

$$\mathcal{L}(\boldsymbol{M}[\mathcal{B}]^*) \simeq \mathcal{M}_3[\mathcal{B}].$$

Proof. The ring $\boldsymbol{M}[\mathcal{B}]^*$ is regular due to Lemma 8.3.

Let $\boldsymbol{b} = \langle b_1, b_2, b_3 \rangle \in \mathcal{M}_3[\mathcal{B}]$. Note that since $\boldsymbol{b}$ is a balanced triple, each ultrafilter on $\mathcal{B}$ contains at most one element from $\{b_i - \mu\boldsymbol{b} \mid i = 1, 2, 3\} \cup \{\mu\boldsymbol{b}\}$. Thus we can define $\boldsymbol{e} \in \mathrm{Idem}(\boldsymbol{M}[\mathcal{B}]^*)$ by

$$
\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1 : \text{if } \mu\boldsymbol{b} \in \mathfrak{u},\\ e_i : \text{if } b_i - \mu\boldsymbol{b} \in \mathfrak{u},\\ 0 : \text{otherwise}, \end{cases}
$$

for all $\mathfrak{u} \in \mathcal{B}^*$. It follows from (8.2) that $\varepsilon(\boldsymbol{e}) = \boldsymbol{b}$, and so ε is a projection.

By Lemma 8.4, the map $\varepsilon\colon \mathcal{L}(\boldsymbol{M}[\mathcal{B}]^*) \to \mathcal{M}_3[\mathcal{B}]$ satisfies (8.1), and so it is an isomorphism due to Lemma 8.2. □

Corollary 8.6. *Let $\mathcal{L}$ be a bounded lattice. The lattice $\mathcal{M}_3[\mathcal{L}]$ is coordinatizable if and anly if the lattice $\mathcal{L}$ is Boolean.*

Proof. If $\mathcal{L}$ is Boolean, then the lattice $\mathcal{M}_3[\mathcal{L}]$ is coordinatizable by Theorem 8.5. In order to prove the opposite implication, suppose that the lattice $\mathcal{M}_3[\mathcal{L}]$ is modular and complemented. We will prove that $\mathcal{L}$ is Boolean. By [**33**, Lemma 2.9] the lattice $\mathcal{M}_3[\mathcal{L}]$ is modular if and only

[2] Recall definition (2.6).

if the lattice $\boldsymbol{\mathcal{L}}$ is distributive. Thus the lattice $\boldsymbol{\mathcal{L}}$ must be distributive. It follows from Lemma 7.4 that $\boldsymbol{\mathcal{L}}$ is complemented. Therefore it is a Boolean lattice. □

Let us now turn our attention to the lattice $\boldsymbol{\mathcal{S}}$ introduced in Section 3. Let κ be an infinite cardinal. There are exactly κ principal ultrafilters on $\boldsymbol{\mathcal{F}}(\kappa)$, each corresponding to an ordinal $\alpha \in \kappa$, namely $\mathfrak{u}_\alpha = \{X \in \boldsymbol{\mathcal{F}}(\kappa) \mid \alpha \in X\}$. Besides there is the only non-principal ultrafilter, $\mathfrak{f}$, consisting of all cofinite subsets of κ. The topological space $\boldsymbol{\mathcal{F}}(\kappa)^*$ is the one-point compactification of the discrete space $\{\mathfrak{u}_\alpha \mid \alpha \in \kappa\}$. In particular, the singleton sets $\{\mathfrak{u}_\alpha\}$, $\alpha \in \kappa$, are open, and neighborhoods of $\mathfrak{f}$ are of the form $\boldsymbol{\mathcal{F}}(\kappa) \setminus \{\mathfrak{u}_\alpha \mid \alpha \in F\}$, where F runs through all finite subsets of κ.

We put

$$\boldsymbol{S} := \{\boldsymbol{x} \in \boldsymbol{M}[\boldsymbol{\mathcal{F}}(\kappa)]^* \mid \boldsymbol{x}(\mathfrak{f}) \in \{0, 1, e_1, 1 - e_1\}\}.$$

Theorem 8.7. *The ring $\boldsymbol{S}$ is regular and $\boldsymbol{\mathcal{L}}(\boldsymbol{S}) \simeq \boldsymbol{\mathcal{S}}$.*

Proof. Observe that the $\boldsymbol{I} := \{\boldsymbol{x} \in \boldsymbol{S} \mid \boldsymbol{x}(\mathfrak{f}) = 0\}$ is an ideal of the ring $\boldsymbol{M}[\boldsymbol{\mathcal{F}}(\kappa)]^*$. Since the ring $\boldsymbol{M}[\boldsymbol{\mathcal{F}}(\kappa)]^*$ is regular due to Lemma 8.3, we get from [**22**, Lemma 1.3] that $\boldsymbol{I}$ is a regular ideal. Thus $\boldsymbol{I}$ is a regular ideal of the ring $\boldsymbol{S}$ and it is easy to see that $\boldsymbol{S}/\boldsymbol{I} \simeq \mathbb{F}_2 \times \mathbb{F}_2$. Applying [**22**, Lemma 1.3] again, we conclude that the ring $\boldsymbol{S}$ is regular.

Let $\varepsilon\colon \boldsymbol{M}[\boldsymbol{\mathcal{F}}(\kappa)]^* \to \boldsymbol{\mathcal{M}}_3[\boldsymbol{\mathcal{F}}(\kappa)]$ be the map defined by (8.2). The map ε satisfies (8.1) due to Lemma 8.4. To conclude that it is an isomorphism, it remains to prove that $\varepsilon(\operatorname{Idem}(\boldsymbol{S})) = \boldsymbol{\mathcal{S}}$ (cf. Lemma 8.2).

Let $\boldsymbol{e} \in \operatorname{Idem}(\boldsymbol{S})$. Then $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1, e_1, 1-e_1\}$. Since the map $\boldsymbol{e}\colon \boldsymbol{\mathcal{F}}(\kappa)^* \to \boldsymbol{M}$ is by definition continuous, it is constant on some neighborhood of $\mathfrak{f}$. It follows that the set $\{\alpha \mid \boldsymbol{e}(\mathfrak{u}_\alpha) \in \{e_3, 1 - e_2\}\}$ is finite. We infer from (8.2) that this set is in fact $C \setminus \mu\langle A, B, C\rangle$, hence the set $C \setminus \mu\langle A, B, C\rangle$ is finite. Thus $\varepsilon(\operatorname{Idem}(\boldsymbol{S})) \subseteq \boldsymbol{\mathcal{S}}$.

It now remains to prove the opposite inclusion. Given $\langle A, B, C\rangle \in \boldsymbol{\mathcal{S}}$, we define an idempotent $\boldsymbol{e} \in \boldsymbol{M}[\boldsymbol{\mathcal{F}}(\kappa)]^*$ by

$$\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1 & \text{if } \mu\langle A, B, C\rangle \in \mathfrak{u}, \\ e_1 & \text{if } A \setminus \mu\langle A, B, C\rangle \in \mathfrak{u}, \\ 1 - e_1 & \text{if } B \setminus \mu\langle A, B, C\rangle \in \mathfrak{u}, \\ e_3 & \text{if } C \setminus \mu\langle A, B, C\rangle \in \mathfrak{u}, \\ 0 & \text{otherwise}, \end{cases}$$

for all $\mathfrak{u} \in \boldsymbol{\mathcal{F}}(\kappa)^*$. Since $\langle A, B, C\rangle \in \boldsymbol{\mathcal{S}}$, the set $C \setminus \mu\langle A, B, C\rangle$ is finite by (3.3), hence $C \setminus \mu\langle A, B, C\rangle \notin \mathfrak{f}$. It follows that $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1, e_1, 1 - e_1\}$, and so $\boldsymbol{e} \in \boldsymbol{S}$. We infer that $\boldsymbol{\mathcal{S}} \subseteq \varepsilon(\operatorname{Idem}(\boldsymbol{S}))$. This concludes the proof. □

9. Maximal Abelian regular subalgebras

We prove that the maximal Boolean sublattices $\mathcal{G}$ and $\mathcal{H}$ of the lattice $\mathcal{S}$ from Sections 4 and 5, respectively, correspond to maximal Abelian regular subalgebras (over the field $\mathbb{F}_2$) of $\boldsymbol{S}$.

Observe that the diagonal matrices, namely 0, 1, e_1, and $1 - e_1$, form a subalgebra of $\boldsymbol{M}$, which we denote by $\boldsymbol{G}$. It is easy to compute by hand that the elements from $\boldsymbol{M}$ commuting with e_1 are exactly the diagonal matrices. It follows that $\boldsymbol{G}$ is a maximal Abelian regular subalgebra of the $\mathbb{F}_2$-algebra $\boldsymbol{M}$(cf. [**52**, Section 4.4]).

Proposition 9.1. *Let $\mathcal{B}$ be a Boolean lattice and $\varepsilon\colon \mathrm{Idem}(\boldsymbol{M}[\mathcal{B}]^*) \to \mathcal{M}_3[\mathcal{B}]$ the map defined by* (8.2)*. Then $\boldsymbol{G}[\mathcal{B}]^*$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}[\mathcal{B}]^*$, it is commutative, and*

$$\varepsilon(\mathrm{Idem}(\boldsymbol{G}[\mathcal{B}]^*)) = \{\langle a, b, a \wedge b\rangle \mid a, b \in \mathcal{B}\}. \tag{9.1}$$

Proof. The ring $\boldsymbol{G}[\mathcal{B}]^*$ is regular due to Lemma 8.3. (Observe that $\mathrm{Idem}(\boldsymbol{G}[\mathcal{B}]^*) = \boldsymbol{G}[\mathcal{B}]^*$.)

Since $\boldsymbol{G}$ is commutative, the Boolean power $\boldsymbol{G}[\mathcal{B}]^*$ is commutative as well. As observed above, $\boldsymbol{G} = \{a \in \boldsymbol{M} \mid ae_1 = e_1a\}$. Thus the range of each $\boldsymbol{x} \in \boldsymbol{M}[\mathcal{B}]^*$ commuting with the constant map $\mathcal{B}^* \to \{e_1\}$ must be included in $\boldsymbol{G}$. It follows that $\boldsymbol{G}[\mathcal{B}]^*$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}[\mathcal{B}]^*$.

It follows from (8.2) that $\varepsilon(\boldsymbol{e}) \in \{\langle a, b, a \wedge b\rangle \mid a, b \in \mathcal{B}\}$ for every $\boldsymbol{e} \in \mathrm{Idem}(\boldsymbol{G}[\mathcal{B}]^*)$. Conversely, given $a, b \in \mathcal{B}$ and an ultrafilter $\mathfrak{u}$ on $\mathcal{B}$, we set

$$\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1 & \text{if } a \wedge b \in \mathfrak{u},\\ e_1 & \text{if } a - b \in \mathfrak{u},\\ 1 - e_1 & \text{if } b - a \in \mathfrak{u},\\ 0 & \text{otherwise.}\end{cases}$$

Then $\boldsymbol{e} \in \mathrm{Idem}(\boldsymbol{G}[\mathcal{B}]^*)$ and $\varepsilon(\boldsymbol{e}) = \langle a, b, a \wedge b\rangle$. This proves (9.1). $\square$

In the case that $\mathcal{B} = \mathcal{F}(\kappa)$, we have $\boldsymbol{G}[\mathcal{F}(\kappa)]^* \subseteq \mathcal{S}$. Thus it follows from Proposition 9.1 that

Corollary 9.2. *The ring $\boldsymbol{G}[\mathcal{F}(\kappa)]^*$ is commutative and it forms a maximal Abelian regular subalgebra of $\mathcal{S}$. Moreover $\varepsilon(\mathrm{Idem}(\boldsymbol{G}[\mathcal{F}(\kappa)]^*)) = \mathcal{G}$, where $\mathcal{G}$ is the Boolean lattice introduced in Section 4.*

Put

$$m := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \boldsymbol{M}$$

and observe $e_3 = me_1m^{-1}$. It follows that the subalgebra $\boldsymbol{H} = \{0, 1, e_3, 1 - e_3\}$ of $\boldsymbol{M}$ is the image of $\boldsymbol{G}$ under the inner automorphism of $\boldsymbol{M}$ given by $x \mapsto mxm^{-1}$, $x \in \boldsymbol{M}$. Consequently, $\boldsymbol{H}$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}$ and also $\boldsymbol{H}[\mathcal{B}]^*$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}[\mathcal{B}]^*$ for every Boolean lattice $\mathcal{B}$.

Proposition 9.3. *The intersection* $\boldsymbol{H'} := \boldsymbol{H}[\boldsymbol{\mathcal{F}(\kappa)}]^* \cap \boldsymbol{S}$ *is a maximal Abelian regular subalgebra of* $\boldsymbol{S}$*, it is commutative, and* $\varepsilon(\mathrm{Idem}(\boldsymbol{H'})) = \mathcal{H}$*, where* $\mathcal{H}$ *is the Boolean lattice introduced in Section 5.*

Proof. Clearly $\boldsymbol{H}$, and so also $\boldsymbol{H'}$, are commutative. Put

$$\boldsymbol{J} = \{\boldsymbol{x} \in \boldsymbol{H'} \mid \boldsymbol{x}(\mathfrak{f}) = 0\}$$

and observe that $\boldsymbol{J}$ is isomorphic to a direct sum of copies of $\mathbb{F}_2$. In particular, $\boldsymbol{J}$ is a regular ideal of $\boldsymbol{H'}$. Since $\boldsymbol{H'}/\boldsymbol{J} \simeq \mathbb{F}_2$, the algebra $\boldsymbol{H'}$ is regular due to [**22**, Lemma 1.3].

Given a principal ultrafilter $\mathfrak{u} \in \boldsymbol{\mathcal{F}(\kappa)}^*$, set

$$\boldsymbol{e}_{\mathfrak{u}}(\mathfrak{v}) := \begin{cases} e_3 & \text{if } \mathfrak{v} = \mathfrak{u}, \\ 0 & \text{whenever } \mathfrak{v} \neq \mathfrak{u}, \end{cases}$$

for all $\mathfrak{v} \in \boldsymbol{\mathcal{F}(\kappa)}^*$. Observe that since $\boldsymbol{e}_{\mathfrak{u}}(\mathfrak{f}) = 0$, we have $\boldsymbol{e}_{\mathfrak{u}} \in \boldsymbol{H'}$. Let $\boldsymbol{x} \in \boldsymbol{S}$ be commuting with every element of $\boldsymbol{H'}$. Since $\boldsymbol{x}$ commutes with all $\boldsymbol{e}_{\mathfrak{u}}$ and $\boldsymbol{H}$ is a maximal Abelian regular subalgebra of $\boldsymbol{M}$, we have that $\boldsymbol{x}(\mathfrak{u}) \in \boldsymbol{H}$ for all principal ultrafilters $\mathfrak{u}$. Since the map $\boldsymbol{x}$ is continuous, it is constant on some neighborhood of $\mathfrak{f}$, and so $\boldsymbol{x}(\mathfrak{f}) \notin \{e_1, 1-e_1\}$. We conclude that $\boldsymbol{x} \in \boldsymbol{H'}$. Therefore $\boldsymbol{H'}$ is a maximal Abelian regular subalgebra of $\boldsymbol{S}$.

Let $\boldsymbol{e} \in \mathrm{Idem}(\boldsymbol{H'})$ (note that $\mathrm{Idem}(\boldsymbol{H'}) = \boldsymbol{H'}$) and put $\langle A, B, C\rangle := \varepsilon(\boldsymbol{e})$. We get readily from (8.2) that $B \subseteq A$. From $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1\}$ and $\boldsymbol{e}$ being constant on some neighborhood of $\mathfrak{f}$, we conclude that $A \approx C$. Therefore $\langle A, B, C\rangle \in \mathcal{H}$ due to (5.2). Thus we have proved that $\varepsilon(\mathrm{Idem}(\boldsymbol{H'})) \subseteq \mathcal{H}$.

Given $\langle A, B, C\rangle \in \mathcal{H}$, we define an idempotent $\boldsymbol{e} \in \boldsymbol{H}[\boldsymbol{\mathcal{F}(\kappa)}]^*$ by

$$\boldsymbol{e}(\mathfrak{u}) := \begin{cases} 1 & \text{if } B \in \mathfrak{u}, \\ 1 - e_3 & \text{if } A \setminus B \in \mathfrak{u}, \\ e_3 & \text{if } C \setminus B \in \mathfrak{u}, \\ 0 & \text{otherwise}, \end{cases}$$

for every ultrafilter $\mathfrak{u}$ on $\mathcal{F}(\kappa)$. Since $\langle A, B, C\rangle \in \mathcal{H}$, both $A \setminus B$ and $C \setminus B$ are finite, and so $\boldsymbol{e}(\mathfrak{f}) \in \{0, 1\}$. It follows that $\boldsymbol{e} \in \boldsymbol{S}$, and so $\boldsymbol{e} \in \boldsymbol{H'}$. Therefore $\mathcal{H} \subseteq \varepsilon(\mathrm{Idem}(\boldsymbol{H'}))$. $\square$

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