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Classes of rings determined by a categorical property

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Contents

Preface

The core of the presented habilitation thesis consists of the following articles:

A. Simion Breaz, Jan Žemlička, *When every self-small module is finitely generated*, J. Algebra 315/2 (2007), 885–893.

B. Jan Žemlička, *When products of self-small modules are self-small*, Commun. Algebra 36/7 (2008), 2570–2576.

C. Jan Žemlička, *Steadiness is tested by a single module*, in: Kelarev, A. V. (ed.) et al., Abelian groups, rings and modules. Proceedings of the AGRAM 2000 conference, Perth, Australia, July 9-15, 2000. American Mathematical Society (AMS), Providence, RI. (2001), Contemp. Math. 273, 301–308.

D. Jan Žemlička, *Steadiness of regular semiartinian rings with primitive factors artinian*, J. Algebra 304/1 (2006), 500–509.

E. Simion Breaz, Jan Žemlička, *The defect functor of homomorphisms and direct unions*, Algebr. Represent. Theor. 19/1 (2016), 181–208.

F. Jan Žemlička, *Socle chains of abelian regular semiartinian rings*, J. Pure Appl. Algebra 217/6 (2013), 1018–1025.

G. Tomáš Penk, Jan Žemlička, Commutative tall rings, J. Algebra Appl., 13/4 (2014).

H. M. Tamer Kosan, Jan Žemlička, Mod-retractable rings, Commun. Algebra 42/3 (2014) 998-1010.

I. M. Tamer Kosan, Jan Žemlička, On modules and rings with restricted minimum condition, Colloq. Math., 140,1 (2015), 75–86.

Chapter 1

Introduction

This chapter contains a survey summarizing several particular concepts and tools useful in research of connections between a ring structure and a structure of categories of modules.

CLASSES OF RINGS DETERMINED BY A CATEGORICAL PROPERTY

There are many properties of the category of all modules over a ring which can be easily recognized from the structure of rings. This phenomenon can be illustrated by the classical theorem characterizing left perfect rings:

Theorem 0.1. [17, Theorem P] *The following statements are equivalent for a ring R:*

(1) *Every left module has a projective cover.*
(2) *$R/J(R)$ is semisimple and $J(R)$ is left T-nilpotent .*
(3) *$R/J(R)$ is semisimple and every non-zero left R module contains a maximal submodule.*
(4) *R satisfies the descending chain condition on principal right ideals.*
(5) *Every flat left module is projective.*

Note that the conditions (1) and (5) deals with the structure of the category of all modules over the ring, however the conditions (2), (3), and (4) are expressed in the language of ring structure.

Characterization by both ring-theoretical and categorical properties are known for various classical classes of rings such semisimple, hereditary, semihereditary or abelian regular ones. Recall for example a characterization of von Neumann regular rings [49, Theorem 1.1 and Corollary 1.13] which appears to be useful as a test class for ring theoretical characterization of some categorical properties.

Theorem 0.2. *The following statements are equivalent for a ring R:*

(1) *For every $x \in R$ there exists an element $y \in R$ such that $x = xyx$.*
(2) *Every principal left (right) ideal is generated by an idempotent.*
(3) *Every finitely generated left (right) ideal is generated by an idempotent.*
(4) *Every finitely generated submodule of a projective left R-module P is a direct summand of P.*
(5) *Every left (right) module is flat.*

Clearly, the first three condition are ring-theoretical and the last two module-theoretical. Furthermore, it is worth mentioning that the first condition quantifies only elements of a ring while the second and third ones are formulated in language of lattices of one-sided ideals.

Unfortunately, not all classes of rings defined by some natural condition on category of modules can be described by some nice ring-theoretical property. There are two reasons of such a lack. The first one is caused by our ignorance; the problem seems to be simply too hard for our imperfect tools and the goal of this thesis is to at least partially solve some problems of the described character. The second reason is fundamental, based on set theory. Such an example of a module-theoretical properties which cannot be describe in the ring structure example is the existence of Whitehead test module for projectivity. As it is proved in [96] it is independent of ZFC with Generalized Continuum Hypothesis over all right hereditary non-right perfect ring.

The main objective of this thesis is to partially summarize known results for several classes of rings determined by some categorical property. Four of the studied classes, namely of steady, strongly steady, tall, and mod-retractable rings represent typical examples of such rings, since their categorical definitions are natural an easily applicable, while other two classes of general semiartinian, and RM-rings can be defined by ring-theoretical property, nevertheless relevant structural questions coming from the context of module theory needs transfer categorical properties to the ring structure.

Let us remark that a module means right R-module over some unitary associative ring R within the whole text. For non-explained terminology we refer to standard monographs [11, 49, 90].

1. Compact objects

An object c of an abelian category closed under coproducts and products is said to be *compact* if the covariant functor $\mathrm{Hom}(c, -)$ commutes with all direct sums i.e. there is a canonical isomorphism in the category of abelian groups $\mathrm{Hom}(c, \bigoplus \mathcal{D}) \cong \bigoplus \mathrm{Hom}(c, \mathcal{D})$ for every system of objects $\mathcal{D}$. The concept of compactness presents an easy way to replace finitely generated modules in general abelian categories. Nevertheless, the clear form of the categorical definition is a reason why compact objects can be applied as a useful tool also in categories containing finitely generated objects in standard sense.

1.1. History.

The systematic research of compact objects in the context of module categories was started by Hyman Bass in 60's. His famous book [18] contains as an exercise a basic non-categorical characterization of the notion. Let us mention here the author's comment to the exercise that examples of compact objects in the category of all modules which are not finitely generated "are not easy to find" [18, p.54].

The introductory work on theory of compact modules is due to Rudolf Rentschler. However his PhD thesis [78] and the paper [79] contains a list of basic examples and several necessary and sufficient conditions of compactness, the core of these works is an attempt to answer the natural question over which rings coincide the classes of compact modules and of finitely generated ones. It should be mentioned that compact objects in categories of all modules have been studied under various terms: module of type Σ, dually slender, Σ-compact, or U-compact module. We will use the term *small* module here.

Further study of small modules has been motivated by progress of research in several different branches of algebra. One of the most important source of questions concerning smallness or more generally compactness comes from the context of representable equivalences of module categories. However $\star$-modules which appear to be an important notions in this branch of module theory were shown to be necessary finitely generated [32, 33], more general context deals with infinitely generated small modules [93, 94, 95]. Another important motivation for study of the notion has appeared in the structure theory of graded rings [75] and almost free modules [94]. Lattice theoretical approach to smallness is presented in the work [54]. Fruitful motivation of many questions in this theory comes from the dual context of so called slim and slender modules [38, 42, 45, 74], albeit the tools and ideas of research are far from being dual.

Commuting properties of functors Hom are studied in many cases only for modules from the category $\mathrm{Add}(M)$ of direct summands of direct sums of copies of some module M. Recall that a module M which is a compact object of the category $\mathrm{Add}(M)$ is called *self-small*. The notion was introduced in [12] as a tool for generalization of Baer's lemma [48, 86.5]. Self-small modules turn out to be important also in the study of splitting properties, [5, 21] and representable equivalences between subcategories of module categories in connection with tilting theory [32, 33]. The notion is very useful in structure theory of mixed abelian groups [9, 20].

The work [71] is devoted to study of compactness in stable categories, i.e. categories whose objects are all modules and groups of morphisms factorize through projective modules. Namely, it is proved that over right perfect rings compact of objects of the stable category can be represented by some standard finitely generated modules.

1.2. **Small modules.** As it is shown in [18] or in [79, 1^o], small modules can be described in natural way by language of systems of submodules.

Lemma 1.1. *The following conditions are equivalent for an arbitrary module M:*

(1) *M is small,*
(2) *if $M = \bigcup_{i<\omega} M_n$ for an increasing chain of submodules $M_n \subseteq M_{n+1} \subseteq M$, then there exists n such that $M = M_n$,*
(3) *if $M = \sum_{i<\omega} M_n$ for a system of submodules $M_n \subseteq M$, $n < \omega$, then there exists n such that $M = \sum_{i<\omega} M_n$.*

The condition (2) implies immediately that every finitely generated module is small. Moreover, it is clear from (3) that there is no infinitely countably generated small module. An another easy consequence of Lemma 1.1 is an observation that a union of strictly increasing chain of the length κ, for an arbitrary cardinal κ of uncountable cofinality, consisting of small submodules give us as well a small module. A small module can be constructed in such a way for instance as a κ-generated uniserial module, which is, indeed, the union of a chain of κ-many cyclic submodules.

This construction motivates definition of particular subclasses of small modules. For an arbitrary cardinal number λ we say that a module M is *λ-reducing* if for every submodule $N \subseteq M$ such that $gen(N) \leq \lambda$ there exists a finitely generated submodule F such that the inclusions $N \subseteq F \subseteq M$ holds.

For an arbitrary ring R let us denote by $\mathcal{SM}(R)$, $\mathcal{R}_\kappa(R)$, $\mathcal{FG}(R)$ and $\mathcal{FP}(R)$ respectively the classes of all small, κ-reducing, finitely generated and finitely presented right R-modules. It is easy to formulate the following hierarchy of these classes:

$$\mathcal{FP}(R) \subseteq \mathcal{FG}(R) \subseteq \mathcal{R}_\kappa(R) \subseteq \mathcal{R}_\lambda(R) \subseteq \mathcal{SM}(R)$$

where $\lambda < \kappa$ are infinite cardinals.

Note that all the inclusions are strict in general. Of course, every κ^+-generated ideal in valuation ring is a witness of the inequalities $\mathcal{R}_{\kappa^+}(R) \neq \mathcal{R}_\kappa(R) \neq \mathcal{FG}(R)$. Furthermore, it is proved in the paper [95, Theorem 2.8] that a ring power F^ω for each field F contains a small right ideal which is not ω-reducing. It is important to remark that classes $\mathcal{R}_\kappa(R)$ and $\mathcal{SM}(R)$ have similar class properties as the class $\mathcal{FG}(R)$. Namely, the classes $\mathcal{SM}(R)$ and $\mathcal{R}_\kappa(R)$ for each infinite cardinal are closed under taking homomorphic images, extensions and finite sums [104, Proposition 1.3].

There exist natural classes of rings over which each injective module is necessary small. Let us denote by $\mathcal{I}(R)$ the class of all injective modules over a ring R.

Theorem 1.2. [95, Theorem 1.6] *Let κ be an infinite cardinal and R a ring.*

1. *If there exists an embedding $R_R^{(\kappa)} \to R_R$, then $\mathcal{I}(R) \subseteq \mathcal{R}_\kappa(R)$.*
2. *If there exists an embedding $R_R^2 \to R_R$, then $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$.*

The hypothesis of (1) is satisfied by the endomorphism ring $End(V)$ for any κ-dimensional vector space V. Furthermore, any non-commutative domain which does not satisfy the right Ore condition (for example polynomials in two non-commuting variables $\mathbb{Z}\langle x, y\rangle$) satisfies the hypothesis of (2).

A similar observation as in Theorem 1.2 is made in the paper [33]:

Proposition 1.3. [33, Lemma 1.10] *Let R be a simple ring containing an infinite orthogonal set of idempotents. Then $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$.*

As a consequence we get that $\mathcal{I}(R) \subseteq \mathcal{R}_\omega(R)$ for every non-artinian simple von Neumann regular ring R. Indeed, it means that if all injective modules are small, then there exists a proper class of non-isomorphic small modules. Thus we have examples of rings over which small modules can be arbitrarily large.

1.3. **Steady rings.** Rings over which the class of all compact (or small) modules coincides with the class of all finitely generated ones are called *right steady.* It is well-known that the class of all right steady rings is closed under factorization [33, Lemma 1.9], finite products [94, Theorem 2.5], and Morita equivalence [43, Lemma 1.7].

Clearly, rings over which there exists proper class of non-isomorphic small modules (as those from Theorem 1.2 and Proposition 1.3) are not steady. On the other hand, several classes of rings satisfying some finiteness conditions are well-known to be right steady:

Theorem 1.4. *A ring R is right steady provided any of the following conditions holds true:*

(1) *R is right noetherian; ,*
(2) *R is right perfect;*
(3) *R is right semiartinian of finite socle length,*
(4) *R is a countable commutative ring,*
(5) *R is an abelian regular ring with countably generated ideals.*

Note that (1) has been established independently by several authors ([79, 7^0], [32, Proposition 1.9], [44, p.79], (2) is proved in [33, Corollary 1.6], (3) in [95, Theorem 1.5], (4) in [79, 11^0], and (5) in [113, Corollary 7].

As an easy consequence of the Theorem 1.4(1) we obtain a characterization of rings over which small modules are precisely finitely presented ones:

Theorem 1.5. [104, Theorem 1.4] *A ring R is right noetherian if and only if $\mathcal{SM}(R) = \mathcal{FP}(R)$.*

Although an existence of a general ring-theoretic criterion for steady rings is still an open problem, there is a construction of some kind of minimal example of an infinitely generated small module over a non-steady ring:

Theorem 1.6. [102, Theorem 1.4] *Let R be a ring, $\kappa = card(R)^+$, and denote by Simp the representative set of all simple right modules. Then R is not right steady if and only if $T = \prod_{S\in Simp} S^\kappa \oplus \bigoplus_{S\in Simp} E(S)$ contains an infinitely generated small submodule.*

Obviously Theorem1.6 can be reformulated to the claim that a ring R is right steady if and only if the module T (of cardinality bounded by $2^{2^{card(R)}}$) contains no infinitely generated small submodule.

Let us remark that for commutative regular rings a module-theoretical criterion of existence of an infinitely generated small module can be formulated in a more elegant form, that the representative class of small modules over a commutative regular ring is in general a set, and there is an estimate of the cardinality of each small module:

Theorem 1.7. [102, Theorem 2.7] *Let R be a commutative regular ring. Then R is steady if and only if the module $R^* = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ contains no infinitely generated small submodule.*

We have remarked that a ring-theoretical characterization of steadiness is an open problem, nevertheless criteria of steadiness are known for several particular classes of rings.

Remind that a ring is called *right semiartinian* if every non-zero cyclic module contains a simple submodule [72]. Large classes of examples both steady and non-steady abelian regular semiartinian rings are constructed in the paper [43]. The articles [83] and [105] characterize steadiness of abelian regular semiartinian rings [83, Theorem 3.4] and regular semiartinian rings with primitive factors artinian:

Theorem 1.8. [105, Theorem 3.5] *Let R be a regular semiartinian ring with primitive factors artinian. Then the following conditions are equivalent:*

(1) *R is right steady;*
(2) *R is left steady;*
(3) *There exists no infinitely generated small right ideal of any factor of R.*
(4) *There exists no infinitely generated small left ideal of any factor of R.*

In particular, an abelian regular semiartinian ring R is is not right steady if and only if there is an abelian regular factor-ring, $\bar{R}$, of R and a member, I, of the socle chain of $\bar{R}$ such that I is an infinitely generated dually slender right $\bar{R}$-module [83, Criterion A].

In the case of abelian regular rings the criterion of steadiness is formulated in the work [109] where $w(M) = \sup\{\dim_{R/I}(M/MI)|\ I \text{ maximal ideal}\}$:

Theorem 1.9. [109, Theorem 3.2] *Let R be an abelian regular ring. Then the following conditions are equivalent:*

(1) *R is right steady,*
(2) *$R/\bigcap_{n<\omega} I_n$ is right steady for every system of maximal ideals I_n, and there exists no small module M with finite $w(M)$ which is either ω_1-generated or contained in $\prod_{i<\omega} F_i$ where F_i are n-generated modules.*
(3) *There exists no (ω_1-generated) $\omega_1$reducing module and no infinitely generated small submodule of $\prod_{n<\omega} R/J_n$ for any system of ideals J_n.*
(4) *There exists no infinitely generated small submodule of $\prod_{n<\omega} R/J_n$ for any system of ideals J_n and every ω_1-generated module M with finite $w(M)$ contains a countable set C such that $M/\bigcap_{c\in C} MAnn(c)$ is infinitely generated.*

The fact that steady continuous regular rings are precisely semisimple rings is presented in [107, Theorem 4.7]. Furthermore, a necessary and sufficient condition of steadiness of valuation rings is given in [113, Theorem 13] and more general case of chain rings (i.e. rings with linearly ordered lattices of both right and left ideals) is characterized in the paper [103]:

Theorem 1.10. [103, Theorem 2.4] *For a chain ring R the following conditions are equivalent:*

(1) *R is right steady.*
(2) *There exists no ω_1-generated uniserial right module.*
(3) *$R/rad(R)$ contains no uncountable strictly decreasing chain of ideals, R contains no uncountably generated right ideal and for every ideal I and for every prime ideal $P \subseteq I$ there exists an ideal K such that $P \subset K \subset I$.*

However countable valuation rings are steady, it is known an example of a countable chain ring which is not right steady [103, Example 1.9]. The result of Theorem 1.10 can be generalized for a class of serial rings:

Theorem 1.11. [103, Theorem 3.5] *The following conditions are equivalent for a serial ring R with a complete set of orthogonal idempotents $\{e_i,\ i \leq n\}$:*

(1) *R is right steady,*
(2) *e_iRe_i is right steady for every $i \leq n$,*
(3) *there exists no ω_1-generated uniserial right R-module.*

It is an open problem whether some analogue of Hilbert basis theorem is valid for steadiness, i.e. whether a polynomial ring over a right steady ring is necessary right steady. It is known for example that polynomial rings in finitely many variables over right perfect ring [108, Proposition 2.6] and polynomial rings in countably many variables over commutative neotherian rings are right steady [79, 11^o], but the question whether polynomial rings in countably many variables over non-commutative neotherian rings are right steady waits for an answer. The strongest result concerning countably many variables is the following claim:

Theorem 1.12. [108, Theorem 2.7] *If X is a countable set of variables and R a right perfect ring such that $End_R(S)$ is finitely generated as a right module over its center for every simple module S, then $R[X]$ is right steady.*

On the other hand, polynomial rings in uncountably many variables are not steady as it is witnessed by the following example.

Example 1.13. [108, Example 3.1, Proposition 3.2] Let R be an arbitrary ring and consider the additive monoid $\mathbb{N}^{\omega_1}$. For every $\alpha < \beta \leq \omega_1$ define $e_{\alpha\beta} \in \mathbb{N}^{\omega_1}$ by the rule $e_{\alpha\beta}(\gamma) = 1$ whenever $\gamma \in \langle \alpha, \beta)$ and $e_{\alpha\beta}(\gamma) = 0$ elsewhere. Moreover, E denotes the submonoid of $\mathbb{N}^{\omega_1}$ generated by $\{e_{\alpha\beta} |\ \alpha < \beta \leq \omega_1\}$ and consider a monoid ring $S = R[E]$. Then the ideal $\bigcup_{\alpha<\omega} e_{0\alpha}S$ is ω_1-generated and ω_1-reducing as a right S-module, which proves that S is not right steady.

Let X be an uncountable set of variables. Since there exists a surjective map of X onto the monoid E, it can be extended to a surjective homomorphism from the free commutative monoid in free generators X to the monoid E and this homomorphism of monoids can be extended to a surjective homomorphism of the polynomial ring $R[X]$ onto $R[E]$. Thus $R[X]$ is not right steady.

1.4. **Self small modules.** Recall that a module M is self-small provided it is a compact object in the category $\mathrm{Add}(M)$. Similarly as in the case of non-small modules, non-self-small ones can be characterized by the condition that there exists a countable chain $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n \subsetneq \dots,\ n < \omega$ of submodules of M such that $M = \bigcup_{n<\omega} M_n$ and for every $n < \omega$ there exists a non-zero endomorphism $f_n : M \to M$ such that $f_n(M_n) = 0$ [12, Proposition 1.1]. It is worth mentioning that the full endomorphism ring can serve as a tool for recognizing whether a module is self-small. In particular, if for a module M either $\mathrm{End}(M)$ is countable or the finite topology on $\mathrm{End}(M)$ is discrete, then M is self-small [12, Corollaries 1.4 and 2.1]. Nevertheless, endomorphism rings cannot detect self-smallness of a module in general:

Theorem 1.14. [106, Theorem 2.9] *Let R be a non-artinian abelian regular ring. Then there exists a pair of a self-small module M and a non-self-small module N such that* $\mathrm{End}_R(M) \cong \mathrm{End}_R(N)$.

The class of all self-small modules is closed under endomorphic images and direct summands but the following example shows that it is not closed under finite direct sums:

Example 1.15. [40, Example 4] The group $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is self-small by [106, Example 2.7] as well as the group $\mathbb{Q}$. The product $\mathbb{Q} \times \prod_{p\in\mathbb{P}} \mathbb{Z}_p$ is not self-small by [40, Example 3].

Let us remark that the natural question which finite sums of self-small modules are as well self-small has an easy answer. Note that the hypothesis on Hom-groups in the condition (2) is satisfied if for example $\mathrm{Hom}(M_i, M_j) = 0$ whenever $i \neq j$.

Proposition 1.16. [40, Proposition 2.4] *The following conditions are equivalent for a finite system of self-small modules $(M_i|\ i \leq k)$:*

(1) *$\bigoplus_{i\leq k} M_i$ is not self-small*
(2) *there exist $i, j \leq k$ and a chain $N_1 \subseteq N_2 \subseteq \dots \subseteq N_n \subseteq \dots$ of proper submodules of M_i such that $\bigcup_{n=1}^{\infty} N_n = M_i$ and $\mathrm{Hom}_R(M_i/N_n, M_j) \neq 0$ for each $n \in \mathbb{N}$.*

The case of infinite products of self-small modules is much more complicated and only particular results are known.

Proposition 1.17. [106, Proposition 1.6] *Let $(M_i|\ i \in I)$ be a system of self-small modules satisfying the condition $Hom_R(\prod_{j\in I\setminus\{i\}} M_j, M_i) = 0$ for each $i \in I$. Then $\prod_{j\in I} M_j$ is a self-small module.*

It is well-known that over semisimple rings as well as over local or commutative perfect rings the classes of small, self-small and finitely generated modules coincides. On the other hand, every generic module is an example of an infinitely generated self-small module over (of course artinian) Kronecker algebras.

It motivates the definition of *right strongly steady* rings as rings over which every right self-small module is finitely generated. Note that the ring $R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ is non-singular right artinian but it is not right strongly steady since its maximal right ring of quotients $\begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ serves as an example of an infinitely generated self-small R-module [22, Example 3.11]. On the other hand, every upper triangular matrix ring over a division ring (and, in particular, over a field) is right strongly steady [22, Example 3.13].

Closure properties of strongly steady rings are similar as in the case of steady ones; they include factorization, finite products and Morita equivalence [22, Lemmas 2.1-4]. However the commutativity simplifies the situation, ring theoretical characterization of strongly steady rings is an open problem even in this case. More clear is the (important) case of right non-singular rings:

Theorem 1.18. [22, Theorem 3.9] *Let R be a right non-singular right strongly steady ring. Then R is right artinian.*

This result allows to formulate a criterion for commutative non-singular rings:

Theorem 1.19. [22, Theorem 3.10] *A non-singular commutative ring is strongly steady if and only if it is semi-simple.*

Furthermore note that every right noetherian right strongly steady ring is right artinian by [22, Proposition 3.16].

Special attention is given to study of self-small abelian groups. It is easy to see that every self-small torsion group is finite [12, Proposition 3.1], but the question which mixed abelian groups are self-small seems to be very interesting and attractive for researchers [5, 7, 20, 21]. If A is a torsion free abelian group of finite rank, then the *R-typ of* A is the quasi-isomorphism class of A/F, where F is a free subgroup of A with A/F torsion. To conclude this section recall at least one basic result about self-smallness of mixed abelian group with finite rank torsion-free part:

Proposition 1.20. [12, Proposition 3.6] *Suppose that A is a mixed abelian group and that A/tA has finite rank. Then A is self-small if and only if*

(a) *for all primes, p, $(tA)_p$ is finite and*
(b) *the R-type of A/tA is p-divisible for all primes p with $(tA)_p \neq 0$.*

1.5. **Abelian categories.** However the definition of a compact object is categorical, we have discussed results in the category of modules which can be formulated just in the language of modules. Nevertheless, some particular questions of the theory can be easily formulated in language of abelian categories. Before we try to do it, let us start with needed categorical terminology and basic tools.

A category with a zero object is called *additive* if for every finite system of objects there exist product and coproduct which are canonically isomorphic, every Hom-set has the structure of an abelian group and the composition of morphisms is bilinear. An additive category is *abelian* if there exists kernel and a cokernel for each morphism, monomorphisms are exactly kernels of some morphisms and epimorphisms cokernels. A category is said to be *complete (cocomplete)* whenever it has all limits (colimits) of small diagrams exist.

We suppose in the sequel that $\mathcal{A}$ is an abelian category closed under arbitrary coproducts and products. By the term family or system we mean any discrete diagram, which can be formally described as a mapping from a set of indexes to a set of objects. Suppose that $\mathcal{N} \subseteq \mathcal{M}$ are two families of objects of the category $\mathcal{A}$. Then a corresponding coproducts are denoted by $(\bigoplus \mathcal{M}, (\nu_M|\ M \in \mathcal{M})), (\bigoplus \mathcal{N}, (\widetilde{\nu}_N|\ N \in \mathcal{N}))$ and a products by $(\prod \mathcal{M}, (\pi_M|\ M \in \mathcal{M})), (\prod \mathcal{N}, (\widetilde{\pi}_N|\ N \in \mathcal{N}))$. Note that there exists canonical morphisms $\nu_\mathcal{N} : \bigoplus \mathcal{N} \to \bigoplus \mathcal{M}$ and $\pi_\mathcal{N} : \prod \mathcal{M} \to \prod \mathcal{N}$ given by universal properties of colimit $\bigoplus \mathcal{N}$ and limit $\prod \mathcal{N}$, which satisfies $\nu_N = \nu_\mathcal{N} \widetilde{\nu}_N$ and $\pi_N = \widetilde{\pi}_N \pi_\mathcal{N}$ for each $N \in \mathcal{N}$.

For arbitrary $\varphi = (\varphi_N|\ N \in \mathcal{N}) \in \bigoplus\{\mathcal{A}(M, N) \mid N \in \mathcal{N}\}$ let us denote by $\mathcal{F}$ a finite subsystem such that $\varphi_N = 0$ whenever $N \notin \mathcal{F}$ and let $\tau : M \to \prod \mathcal{N}$ be the morphism given by the universal property of the product $(\prod \mathcal{N}, (\pi_N), N \in \mathcal{F})$ applied on the cone $(M, (\varphi_N|\ N \in \mathcal{N}))$ (i.e. $\pi_N \circ \tau = \varphi_N$). Then

$$\Psi_\mathcal{N}(\varphi) = \nu_\mathcal{F} \circ \nu^{-1} \circ \pi_\mathcal{F} \circ \tau$$

where $\nu : \bigoplus \mathcal{F} \to \prod \mathcal{F}$ denotes the canonical isomorphism. Note that the definition $\Psi_\mathcal{N}(\varphi)$ does not depend on choice of $\mathcal{F}$. Furthermore the mapping $\Psi_\mathcal{N}$ is a monomorphism in the category of abelian groups for every family of objects $\mathcal{N}$.

Now, we are ready to formulate precise general definition of the central notion. An object M is said to be $\mathcal{C}$*-compact* if $\Psi_\mathcal{N}$ is an isomorphism for every family $\mathcal{N} \subseteq \mathcal{C}$. Note that the class of all $\mathcal{C}$-compact objects is closed under finite coproducts and cokernels since the contravariant functor $\mathcal{A}(-, \bigoplus \mathcal{N})$ commutes with finite coproducts and it is left exact.

Now we are able to formulate an elementary criterion of compact object, which generalizes Lemma 1.1:

Lemma 1.21. *If M is an object and a class of objects $\mathcal{C}$, then it is equivalent:*

(1) *M is $\mathcal{C}$-compact,*
(2) *for every $\mathcal{N} \subseteq \mathcal{C}$ and every $f \in \mathcal{A}(M, \bigoplus \mathcal{N})$ there exists finite subsystem $\mathcal{F} \subseteq \mathcal{N}$ and a morphism $f' \in \mathcal{A}(M, \bigoplus \mathcal{F})$ such that $f = \nu_\mathcal{F} \circ f'$,*
(3) *for every $\mathcal{N} \subseteq \mathcal{C}$ and every $f \in \mathcal{A}(M, \bigoplus \mathcal{N})$ there exists finite subsystem such that $\mathcal{F} \subseteq \mathcal{N}$, $f = \sum_{F \in \mathcal{F}} \nu_F \circ \rho_F \circ f$.*

Note that the commuting properties seems to play important role not only for Hom-functors. For example coherent functors introduced in [13] are characterized in the module categories in [36, Lemma 1] as exactly those covariant functors which commute with direct limits and direct products. The result was extended to locally finitely presented categories in [66, Chapter 9]. Commuting properties of covariant Ext^1-functors are studied in [24, 91, 50, 8, 86].

The defect functor $\mathrm{Dev}_\beta = \mathrm{CokerHom}(\beta, -)$ of a morphism β is a natural generalization of both covariant Hom and Ext^1 functors in an arbitrary locally finitely presented abelian category.

If $\beta : L \to P$ is a homomorphism in $\mathcal{C}$, then we have the following examples [23, Example 2]:

(1) If $\mathcal{C}$ is abelian, P is projective and β a monomorphism, then $\mathrm{Def}_\beta(-)$ is canonically equivalent to $\mathrm{Ext}^1(P/\beta(L), -)$.
(2) If $P = 0$, then $\mathrm{Def}_\beta(-)$ is canonically equivalent to $\mathrm{Hom}(L, -)$.
(3) If β is an epimorphism and $\upsilon : K \to L$ is the kernel of β then $\mathrm{Def}_\beta(-)$ represents the covariant defect functor associated to the exact sequence $0 \to K \overset{\upsilon}{\to} L \overset{\beta}{\to} P \to 0$.
(4) If R is a unital ring, $\mathcal{C} = \text{Mod-}R$, and L and P are finitely generated and projective then $\mathrm{Def}_\beta(R)$ represents the transpose of $P/\beta(L)$.

Furthermore, the following criteria are known for an arbitrary homomorphism $\beta : L \to P$ [23, Proposition 9]:

(1) Suppose that P is a compact object. The functor Def_β commutes with direct sums if and only if L is a compact object.
(2) Suppose that P is a finitely generated object. The functor Def_β commutes with direct unions if and only if L is finitely generated.
(3) Suppose that P is a finitely presented object. Then Def_β commutes with direct limits if and only if L is finitely presented.

As an analogue of [36, Lemma 1] in the case of direct unions the following result can be proven:

Theorem 1.22. [23, Theorem 10] *A functor $F : \mathcal{C} \to Ab$ commutes with respect direct products and direct unions if and only if it is naturally isomorphic to a defect functor* Def_β *associated to a homomorphism $\beta : L \to P$ with L and P finitely generated.*

As an consequence we obtain for any homomorphism $\beta : L \to P$ between projective modules equivalence of the three following properties:

(1) Def_β commutes with direct sums,
(2) Def_β commutes with direct unions
(3) Def_β commutes with direct limits

Moreover, if these conditions are valid, then $\mathrm{Def}_\beta(R)$ is a finitely presented left R-module [23, Proposition 11]. Let π_J denote the canonical projection. Then the commuting of Def_β with a direct sum of objects can be characterize in the following way:

Theorem 1.23. [23, Theorem 24] *If $\beta : L \to P$ is a homomorphism and $(M_i, i \in I)$ a family of objects, the following conditions are equivalent:*

(1) Def_β *commutes with the direct sum of* $(M_i, i \in I)$,

(2) *for every* $f \in \mathrm{Hom}(L, \bigoplus_{i\in I} M_i)$ *there exist finite subset* $F \subset I$, *and* $g \in \mathrm{Hom}(P, \bigoplus_{i\in I\setminus F} M_i)$ *such that* $\pi_{I\setminus F} f = g\beta$.

If κ is a cardinal less than the first ω-measurable cardinal and Def_β commutes with countable direct sums then Def_β commutes with direct sums of κ objects [23, Proposition 26]. Thus in the constructible universe $\mathrm{Def}_\beta(-)$ commutes with countable direct sums if and only if $\mathrm{Def}_\beta(-)$ commutes with all direct sums and, in particular, for each $M \in \mathcal{C}$ $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with countable direct sums if and only if $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with all direct sums.

2. Semiartinian rings

Recall that a module M is *semiartinian* provided each non-zero factor of M contains a simple submodule and a ring R is right semiartinian if R_R is a semiartinian module. Of course, a right semiartinian ring can be characterized by the module class conditions such that

(1) every module is semiartinian, or
(2) every non-zero module contains a simple submodule.

However the class of all semiartinian ring can be easily described by both ring-theoretical and categorical conditions, it seems to interesting the question how the structure of a semiartinian ring reflects some additional condition such as steadiness or strongly steadiness. This way of research motivates the definition the *right socle chain*, which is the uniquely defined strictly increasing chain of ideals $(S_\alpha \mid \alpha \leq \sigma+1)$ in a right semiartinian ring R satisfying $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(R/S_\alpha)$, $S_0 = 0$ and $S_{\sigma+1} = R$.

2.1. **History.** The notion generalizes the notion of a right artinian ring, which can be described precisely as a semiartinian ring with the socle chain of a finite socle length and finitely generated slices $\mathrm{Soc}(R/S_\alpha)$. Moreover, by [17, Theorem P] every non-zero module over a left perfect ring has a non-zero socle, hence every left perfect ring is right semiartinian. Basic structural results about general semiartinian rings are published in papers [26, 39, 48, 72, 85]. Furthermore, let us recall the important construction presented in the paper [16]:

Proposition 2.1. [16, Proposition 4.7] *Let* κ *be an infinite cardinal,* K *a field and* R_γ *a semiartinian* K*-algebra with primitive factors artinian and socle chain* $(S_{\alpha\gamma} \mid \alpha \leq \sigma_\gamma)$ *for each* $\gamma < \kappa$. *Let* $R = \bigoplus_{\gamma<\kappa} R_\gamma + K \subseteq \prod_{\gamma<\kappa} R_\gamma$ *and put* $\sigma = \sup_{\gamma<\kappa}\sigma_\gamma$. *If either* σ *is limit or* $\{\gamma \mid \sigma_\gamma = \sigma\}$ *is infinite, then:*

(1) $\bigoplus_{\gamma<\kappa} S_{\alpha\gamma}$ *is the* α*-th member of the socle chain* $\forall \alpha \leq \sigma$.
(2) *If each* R_γ *is right semiartinian, then* R *is right semiartinian with socle length* $\sigma+1$.
(3) R *has primitive factors artinian.*

Note that if the algebras in the construction are supposed to be (abelian) regular, then the constructed ring R is so. Actually, the classical result claims that commutative semiartinian rings are close to abelian regular ones:

Theorem 2.2. [72, Théorème 3.1, Proposition 3.2] *Let R be a ring.*

(1) *R is left semiartinian if and only if $\mathrm{J}(R)$ is right T-nilpotent and $R/\mathrm{J}(R)$ is left semiartinian.*
(2) *Let R be a commutative semiartinian ring. Then $R/\mathrm{J}(R)$ is abelian regular and semiartinian.*

Properties and constructions of semiartinan rings close to von Neumann regular ones are studied in papers [18, 15, 39, 83, 43] while papers [2, 1] are focused to correspondence between the class of semiartinian rings and other interesting classes of rings defined by some property of module categories.

2.2. **Results.** The notion of a dimension sequence plays an important role in research of regular semiartinian ring with primitive factors artinian. Nevertheless, before the definition we need to formulate the following result:

Theorem 2.3. [83, Theorem 2.1] *Let R be a right semiartinian ring and $\mathcal{L} = (S_\alpha \mid \alpha \leq \sigma+1)$ the right socle chain of R. Then the following conditions are equivalent:*

(1) *R is regular and all right primitive factor rings of R are right artinian,*
(2) *for each $\alpha \leq \sigma$ there are a cardinal λ_α, positive integers $n_{\alpha\beta}$, $\beta < \lambda_\alpha$, and skew-fields $K_{\alpha\beta}$, $\beta < \lambda_\alpha$, such that $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$, as rings without unit. The pre-image of $M_{n_{\alpha\beta}}(K_{\alpha\beta})$ coincides with the β-th homogeneous component of R/S_α and it is finitely generated as right R/S_α-module for all $\beta < \lambda_\alpha$. Moreover, λ_α is infinite if and only if $\alpha < \sigma$.*

If (1) holds true, then R is also left semiartinian, and $\mathcal{L}$ is the left socle chain of R.

Denote by $\mathcal{R}$ the class of all regular right semiartinian rings R such that all (right) primitive factor-rings of R are (right) artinian. If $R \in \mathcal{R}$, then the family

$$\mathcal{D}(R) = \{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$$

collecting data from the previous theorem is said to be the *dimension sequence* of R.

The dimension sequences of a regular semiartinian ring naturally reflects the structure of single semisimple slices. Note that structural theory of the notion is developed in [110, 112]. An application of combinatorial set theory [46, 92] allows to prove necessary conditions satisfied by this invariant:

Theorem 2.4. [110, Proposition 3.1 and Theorem 3.5] *Let $R \in \mathcal{R}$ be abelian regular, Generalized Continuum Hypothesis holds and α, δ be ordinals satisfying $\alpha + \delta \leq \sigma$. Then $|\langle\alpha, \sigma\rangle| \leq 2^{\lambda_\alpha}$. If $\mathrm{cf}(\lambda_\alpha) > \max(|\delta|, \omega)$, then $\lambda_{\alpha+\delta} \leq \lambda_\alpha$. Otherwise $\lambda_{\alpha+\delta} \leq \lambda_\alpha^+$.*

On the other hand, commutative regular semiartinian rings with a particular given rank of slices of the socle chain can be constructed:

Theorem 2.5. [110, Theorem 5.1] *Let σ be an ordinal, K a field, and $(\lambda_\alpha | \ \alpha \leq \sigma)$ a family of cardinals satisfying for every $\alpha \leq \beta \leq \sigma$ the conditions:*

(a) $\lambda_\beta \leq \lambda_\alpha^+$ *if* $\mathrm{cf}(\lambda_\alpha) = \omega$, *and* $\lambda_\beta \leq \lambda_\alpha$ *otherwise,*
(b) $\lambda_\alpha < \omega$ *iff* $\alpha = \sigma$,
(c) $|\langle \alpha, \sigma \rangle| \leq \lambda_\alpha$.

Then there exists a commutative regular semiartinian K-algebra with dimension sequence $\{(\lambda_\alpha, \{(1, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ where $K_{\alpha\beta} = K$ for all $\alpha \leq \sigma$ and $\beta < \lambda_\alpha$.

Furthermore, it is possible to generalize results on dimension sequences for a suitable subclass of regular right semiartinian rings R with primitive factors artinian, namely, for those R satisfying the condition that every ideal which is finitely generated as two-sided ideal is finitely generated as right ideal. It is proved in the paper [112, Theorem 3.4] that over these rings and under Generalized Continuum Hypothesis it holds that $\lambda_{\alpha+\delta}(n) \leq \lambda_\alpha(m)$, whenever $m \geq n$ and α, δ are ordinals such that $\alpha+\delta \leq \sigma$ and $\mathrm{cf}(\lambda_\alpha(\mathrm{n})) > \max(|\delta|, \omega)$, where $\{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ is dimension sequence and $\lambda_\alpha(n) = \mathrm{card}\{\beta < \lambda_\alpha | \ n_{\alpha\beta} \geq n\})$.

3. Tall rings

A module M which contains a non-noetherian submodule N such that the factor M/N is non-noetherian as well is studied first in the paper [84] under the term *tall*. The notion of a *right tall* ring is defined in the same paper as a ring over which every non-noetherian right module is tall. Note that this notion presents a "typical" example of a ring described by a module-class property.

3.1. History.

It is not hard to see that the class of all right tall rings is closed under factors, finite products, and Morita equivalence. Although in [84] is presented a criterion of right tall rings using the notion of Krull dimension of all modules, an existence a general ring-theoretic necessary and sufficient condition remains to be an open problem.

Theorem 3.1. [84, Theorem 2.7] *The following statements are equivalent for a ring R:*

(1) *R is right tall,*
(2) *every non-noetherian module has a proper non-noetherian submodule,*
(3) *every module with Krull dimension is noetherian.*

Since every maximal submodule of a non-noetherian module is non-noetherian, the condition (2) implies that every right max ring, over which every nonzero right module contains a maximal submodule, is necessarily right tall [31, p. 31]. Nevertheless, the following example shows that the revers implication does not hold.

Example 3.2. [76, Example 3.2] Put $I = \sum_i x_i^2 F[\mathbf{X}]$ and $R = F[\mathbf{X}]/I$ for a field F and an infinite countable set of variables $\mathbf{X} = \{x_1, x_2, \dots\}$. Let $X_i = x_i + I$ and define an ideal $J = \sum_i X_i R$. Then J is a nil ideal, since $X_i^2 = 0$ and R is commutative. As $R/J \cong F$, R is tall ring by [76, Lemma 3.1]. Moreover, J is a nil maximal ideal of R, thus it is the Jacobson radical of R. Since $X_1 \cdot \dots \cdot X_n \neq 0$ for every n, J is not T-nilpotent, hence R is not a max ring.

No general ring-theoretical criterion characterizing max rings neither correspondence between the classes of all tall and all max rings is known. Nevertheless, max rings are studied by many authors from various points of view and with different motivations [25, 27, 31, 47, 53, 63, 97]. Among another results let us recall several classical module-theoretical necessary and sufficient conditions:

Theorem 3.3. [47, 53, 63] *The following conditions are equivalent for a ring R:*

(1) *R is a right max ring;*
(2) *$R/J(R)$ is a right max ring and $J(R)$ is right T-nilpotent,*
(3) *every non-zero submodule of injective envelops $E(S)$ contains a maximal submodule for every simple module S,*
(4) *there is a cogenerator for the category of right modules whose every non-zero submodule contains a maximal submodule.*

Much more is known about both commutative max rings and commutative tall rings. The most important fact from our point of view is ring-theoretical criteria of commutative max ring:

Theorem 3.4. [47, 53, 63] *The following conditions are equivalent for a commutative ring R:*

(1) *R is a max ring;*
(2) *$R/J(R)$ is a regular ring and $J(R)$ is left T-nilpotent,*
(3) *the localization at any maximal ideal of R is a max ring,*
(4) *the localization at any maximal ideal of R is a perfect ring.*

3.2. **Results.** For description of commutative tall rings are very useful to formulate necessary structural condition of non-tall rings:

Theorem 3.5. [76, Theorem 2.6] *Let R be a commutative non-tall ring. Then there exists a maximal ideal I and a sequence of ideals $I = J_1 \supset J_2 \supset \dots$ such that*

(1) *$IJ_i \subseteq J_{i+1}$ for each i,*
(2) *R/J_i is artinian for each i,*
(3) *$\bigcap_i J_i$ is a prime ideal,*
(4) *$R/\bigcap_n I^n$ is not a tall ring.*

Note that ideals J_n from the previous theorem cannot be replaced by the powers J_1^n in general [76, Example 3.7]. On the other hand, if R is tall, then for every non-idempotent maximal ideal I such that R/I^i is artinian for each i, the intersection $\bigcap_j I^j$ is not a prime ideal [76, Proposition 2.9]. As the consequence can be formulated the following criterion:

Theorem 3.6. [76, Theorem 2.12] *The following conditions are equivalent for a commutative ring R:*

(1) *R is not tall,*
(2) *there exists a non-noetherian artinian module,*
(3) *there exists an artinian module M, elements $x \in R$ and $m_j \in M$ such that $m_{j+1}x = m_j$ and $m_{j+1} \notin m_jR$ for each j and $M = \bigcup_j m_jR$.*
(4) *there exists a sequence of ideals J_j of R and elements $x_j \in R$ such that R/J_j is artinian, $J_{j+1} \subsetneq J_j$, $x_jr \in J_{j+1}$ iff $r \in J_j$ and the length of $S_j(R/J_j)$ is equal to the length of $S_j(R/J_k)$ for each $j \leq k < \omega$.*

Finally note that the previous criterion can be expressed in a very simple form in the case of a commutative noetherian ring R, namely, R is tall if and only if R is artinian [76, Theorem 2.10].

4. Retractability and coretractability

Both the central notions of this section, i.e. completely coretractable and mod-retractable rings present examples of rings naturally determined by a categorical property. A module M such that its every nonzero submodule contains a nonzero endomorphic image of M is called *retractable* and, dually, M is called *coretractable* if there exists a nonzero homomorphism of M/K to M for every proper submodule $K \subseteq M$. For example each finitely generated module over commutative ring is retractable [41].

4.1. **History.** The importance of the notions has emerged in research of Baer modules [81, 82], endomorphism rings of nonsingular modules [61, 62], compressible modules [87, 89] and module lattices [51, 101]. The works [10, 41, 52] are devoted to rings over which every module is retractable or coretractable.

Main results of [10] describe rings over which every right module is coretractable, such rings are called *right completely coretractable.* Dually, a ring R is said to be *right mod-retractable* provided every right R-module is retractable.

It is proved in [111, Theorem 2.4] that a ring is right (left) completely coretractable if and only if it is isomorphic to a finite product of matrix rings over right and left perfect rings. Furthermore, every cyclic right and left R-module is coretractable [111, Proposition 3.2].

The papers [41, 52] started to study mod-retractable rings. Note that the class of mod-retractable rings is closed under Morita equivalence, factorization, and finite products [41]. Moreover it is known that any right mod-retractable ring is an example of a right max ring [64, Theorem 3.3].

4.2. **Results.** However mod-retractable rings are precisely rings such that all their torsion theories are hereditary, the general ring-theoretical criterion of mod-retractability is not know. The characterization is available only for several particular classes of rings and for all commutative rings:

Theorem 4.1. [64, Theorem 3.3] *Let R be a left perfect ring. Then R is right mod-retractable if and only if $R \cong \prod_{i\le k} M_{n_i}(R_i)$ for a system of a local rings R_i, $i \le k$, which are both left and right perfect.*

As an easy consequence can be shown that every commutative perfect ring is mod-retractable.

A similar criterion is proved for the class of right noetherian rings:

Theorem 4.2. [64, Theorem 3.3] *Let R be a right noetherian ring. Then R is right mod-retractable if and only if $R \cong \prod_{i\le k} M_{n_i}(R_i)$ for a local right artinian rings R_i.*

As it was mentioned a criterion of mod-retractability is known for the class of commutative rings:

Theorem 4.3. [64, Theorem 3.10] *Let R be a commutative ring. Then R is mod-retractable if and only if R is semiartinian.*

Finally note that from the previous result immediately follows that every commutative semiartinian ring is necessarily mod-retractable.

5. RM rings

First recall that a module M satisfies the *restricted minimum condition* if for every essential submodule N of M, the factor M/N is artinian. The class of all modules satisfying the restricted minimum condition is well-known to be closed under submodules, factors as well as finite direct sums. Note that a semiartinian module M satisfies the restricted minimum condition if and only if $M/\operatorname{Soc}(M)$ is artinian.

5.1. **History.** A ring R is called a right *RM-ring* if R_R satisfies restricted minimum condition as a right module. Obviously, the class of all right RM-rings contains all right artinian rings and principal ideal domains. This observation partially explains the historical motivation of research of these rings. Structure theory of RM-rings and domains was studied in the papers [28, 29, 34, 37, 73]. Among others let us recall the following result:

Theorem 5.1. [34, Theorem 1] *Let R be a noetherian domain. Then R has Krull dimension 1 if and only if it is an RM-domain*

However the definition of RM-rings has a ring-theoretical nature (it actually deals with cyclic modules), the correspondence between RM-rings and the classes of rings studied above is clarified in the context of results of the paper [6], which is devoted to structure research of classes of torsion modules over RM-domains. Namely, it seems to be interesting question here whether there exists nice categorical property which is equivalent to the ring-theoretical definition.

5.2. **Results.** Ring-theoretical results for non-commutative as well as commutative rings are proved in [113]. Recall a useful technical result which consists of several necessary conditions of modules over general RM-ring where $E(M)$ denotes the injective envelope of a module M:

Theorem 5.2. [113, Theorem 2.11] *Let R be a right RM-ring and M a right R-module.*

1. *If M is singular, then M is semiartinian.*
2. *$E(M)/M$ is semiartinian.*
3. *If M is semiartinian, then $E(M)$ is semiartinian. In particular, $E(S)$ is semiartinian for every simple module S.*

As an consequence it can be obtained the observation for a right RM-ring R that R is a nonsingular ring of finite Goldie dimension whenever $\mathrm{Soc}(R) = 0$ [113, Theorem 2.12].

For a semilocal RM-rings can be proved the following criterion:

Theorem 5.3. [113, Theorem 2.17] *Let R be a semilocal RM-ring and* $\mathrm{Soc}(R) = 0$. *Then the following conditions are equivalent:*

1. *R is noetherian,*
2. *$J(R)$ is finitely generated,*
3. *the socle length of $E(R/J(R))$ is at most ω.*

Recall characterization of commutative RM-domains from the paper [6] which motivates are research:

Theorem 5.4. [6, Theorem 6 and 9] *The following conditions are equivalent for a commutative domain R:*

1. *R is an RM-ring,*
2. *$M = \oplus_{P\in Max(R)} M_{[P]}$ for all torsion modules M,*
3. *R is noetherian and every non-zero (cyclic) torsion R-module has an essential socle,*
4. *R is noetherian and every self-small torsion module is finitely generated.*

The most important result of the article [113] describes commutative RM-ring in the language of module categories which generalizes the previous result:

Theorem 5.5. [113, Theorem 3.7] *The following conditions are equivalent for a commutative ring R:*

1. *R is an RM-ring,*
2. *$M = \oplus_{P\in Max(R)} M_{[P]}$ for all singular modules M,*

(3) $R/\operatorname{Soc}(R)$ *is Noetherian and every self-small singular module is finitely generated.*

The question whether a similar result is valid in the non-commutative case remains open.

6. Conclusion

Let us summarize the contribution of the present thesis:

(1) We describe the structure of rings belonging to classes determined by a particular categorical property. Namely, in [22], [105], [76], [64], and [65] respectively are characterized several subclasses of right strongly steady, steady, tall, mod-retractable, and RM-rings. The structural theory of abelian regular semiartinian rings is developed in [110].

(2) We answer several structural question on classes of compact objects, in particular, [106] is devoted to closure properties of the class of all self-small modules and [102] determines test modules for the class of all small modules.

(3) We contribute to the study of commuting properties of functors in [23].

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Chapter 2

Self-small modules and strongly steady rings

The main goal of this chapter is to describe structural properties of classes of self small modules. The text consists of two papers:

A. Simion Breaz, Jan Žemlička, *When every self-small module is finitely generated*, J. Algebra 315/2 (2007), 885–893.

B. Jan Žemlička, *When products of self-small modules are self-small*, Commun. Algebra 36/7 (2008), 2570–2576.

A. WHEN EVERY SELF-SMALL MODULE IS FINITELY GENERATED

SIMION BREAZ AND JAN ŽEMLIČKA

Abstract. The aim of this paper is to give necessary and sufficient conditions for rings for which every right self-small module is finitely generated. It is proved that: semi-simple rings, commutative perfect rings and right nonsingular Σ-extending rings have this property; a right nonsingular semi-prime ring has the property if and only if it is semi-simple; a commutative noetherian ring has the property if and only if it is artinian.

1. Introduction

Let R be a unital ring. A right R-module M is called *self-small* if the covariant functor $\mathrm{Hom}(M,-)$ commutes with direct sums of copies of M. This notion was introduced by Arnold and Murley in [2] in order to study some generalization of a well known Baer's Lemma, [10, Proposition 86.5]. Meanwhile, these modules were useful in the study of a large variety of properties as splitting properties, [1] and [5], properties for homomorphisms of graded modules, [11]. Maybe the most important utility of self-small modules is in the study of representable equivalences between subcategories of module categories in connection with tilting theory. In this context they are viewed as generalizations for finitely generated modules and they are used in the definition of $\star$-modules, [6]. In [18] it is proved that every $\star$-module is finitely generated. However, the property "finitely generated" is not valid for some generalizations of $\star$-modules, [19]. These two results had lead to the following problem: "Characterize the rings for which every small right module is finitely generated". These rings are called *right steady* and they are studied in [7], [9], [20], [21], [22]. We recall that a right R-module M is *small* if the covariant functor $\mathrm{Hom}(M,-)$ commutes with arbitrary direct sums.

In the present paper we consider the problem: "Characterize the rings for which every self-small right module is finitely generated". We call these rings *right strongly steady*. The class of right steady rings is larger than the class of right strongly steady rings. To see this it is enough to consider the ring $\mathbb{Z}$ of all integers. This ring is right steady but it is not a right strongly steady ring. For example $\mathbb{Q}$ is a self-small $\mathbb{Z}$-module which is not finitely generated.

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In the second section we present some basic facts. It is proved that a ring is right strongly steady if it is semi-simple (Proposition 2.5) or perfect and commutative (Theorem 2.9). We mention here that in [7, Corollary 1.6] it is proved that every right perfect ring is right steady. This is not the case for the strongly steady property. We show in Example 3.11 that a right artinian ring which is not right strongly steady does exist.

In Section 3 we consider right non-singular rings which are right strongly steady. The study of this case is in fact a part of a more general problem in module theory which asks to find correspondences between properties of a ring S and a unital subring $R \leq S$ such that S is finitely generated as a right R-module, [4], [13, Section 3]. If R is a right strongly steady right non-singular ring then it is right artinian (Theorem 3.9). Moreover, a right nonsingular ring which is commutative or semi-prime is right strongly steady if and only if it is semi-simple (Theorem 3.8 and Theorem 3.12). In Example 3.13 we proved that there exist right nonsingular strongly steady rings which are not semi-simple. Finally it is shown in Proposition 3.16 that every right noetherian strongly steady ring is right artinian. Using these we prove in Theorem 3.17 that a commutative noetherian ring is strongly steady if and only if it is artinian.

In this paper every ring is associative and unital. If R is a ring, then J denotes the Jacobson radical of R. A module (or R-module) means right module over a ring R. All unexplained notions and notations can be founded in [13], [14] and [16].

2. Basic facts

We start with some closure properties for the class of right strongly steady rings.

Lemma 2.1. *Strongly steadiness is preserved by factorization.*

Proof. Let R be a right strongly steady ring and I an ideal of R. The ring R/I is right strongly steady as a consequence of the equality $\mathrm{Hom}_R(M,N) = \mathrm{Hom}_{R/I}(M,N)$ for all right R/I-modules M and N. □

Lemma 2.2. *Every infinite product of rings contains a self-small non-finitely generated ideal.*

Proof. By [17, Lemma 2.4] any infinite product of rings contains non-finitely generated ideal which is small as a right and a left ideal. □

Lemma 2.3. *Let R_i, $i \in I$ be rings. $\prod_{i\in I} R_i$ is right strongly steady if and only if I is finite and R_i is right strongly steady for every $i \in I$.*

Proof. (⇒) By Lemma 2.2, the set I is finite. Any ring R_i is right strongly steady since it is a factor of $\prod_{i\in I} R_i$.

(⇐) Let M be a self-small module. Then every right R-module M is isomorphic to a finite direct product $M \cong \prod_{i\in I} M_i$, where each M_i is a right R_i-module. It is not hard to observe that M is a self-small right R-module if and only if every M_i

is self-small as an R_i-module. Since M_i is finitely generated for each $i \in I$, M is finitely generated as well. □

Since Morita equivalences preserve (and reflect) finitely generated modules as well as self-small modules, we deduce immediately the following result.

Lemma 2.4. *The property "to be right strongly steady" is preserved by Morita equivalences.*

Proposition 2.5. *Every semi-simple ring is right strongly steady.*

Proof. It follows immediately from Lemma 2.3 and Lemma 2.4. □

Lemma 2.6. *Let R be a right perfect ring. If $0 \neq M$ is a right R-module, then M has a non-zero simple quotient.*

Proof. The Jacobson radical of R is right T-nilpotent, hence $MJ \neq M$ by [14, Theorem 23.16]. Moreover, M/MJ can be viewed as a right R/J-module. Since R/J is semi-simple, it has a non-zero simple summand (as an R/J-module). But every simple R/J-module is simple as an R-module, and the proof is complete. □

For the convenience of the reader we recall a characterization of self-small modules which was proved in [2, Proposition 2.1].

Proposition 2.7. *A right R-module M is not self-small if and only if there exists a countable chain $M_0 \subseteq M_1 \subseteq \dots \subseteq M_n \subseteq \dots$, $n < \omega$ of submodules of M such that $M = \bigcup_{n<\omega} M_n$ and for every $n < \omega$ there exists a non-zero endomorphism $f_n : M \to M$ such that $f_n(M_n) = 0$.*

Remark 2.8. *We can suppose that the chain of submodules in the previous proposition is strictly increasing. We recall that a right R module is not small if and only if it is a union of a strictly increasing countable chain of submodules.*

Proposition 2.9. *Every commutative perfect ring is (right) strongly steady.*

Proof. A commutative perfect ring is product of finitely many local perfect rings, hence it is enough to prove the property for the local case.

Let R be a local commutative perfect ring and $S = R/J$, the simple R-module. Fix an R-module M which is not finitely generated. Since R is steady by [7, Corollary 1.6], M is not a small module. Therefore, we have a strictly increasing chain of submodules $M_n \subsetneq M_{n+1} \subsetneq M$ such that $M = \bigcup_{n<\omega} M_n$. If $n < \omega$, since $M/M_n \neq 0$, there exists an epimorphism $M/M_n \to S$ by Lemma 2.6. But M has a submodule isomorphic to S as a consequence of Bass Theorem. Therefore, for every $n < \omega$ there is a non-zero homomorphism $f_n : M \to M$ such that $f(M_n) = 0$, which proves M is not self-small. □

3. Right nonsingular and noetherian rings

Suppose that $\mathfrak{G}$ is a Gabriel topology on the ring R and M is an R-module. We will denote, as in [16], by $M_{\mathfrak{G}}$ the module of quotients of M with respect $\mathfrak{G}$.

Lemma 3.1. *Let R be a right non-singular ring, $\mathfrak{G}$ a Gabriel topology on R and N a $\mathfrak{G}$-torsion-free right R-module. If K is an R-submodule of N and M is the $R_{\mathfrak{G}}$-submodule of $N_{\mathfrak{G}}$ which is generated by K then M/K is a $\mathfrak{G}$-torsion R-module.*

Proof. We consider $K_{\mathfrak{G}}$ as a submodule of $N_{\mathfrak{G}}$ and we construct M as a submodule of $K_{\mathfrak{G}}$. Then M/K is a submodule of the $\mathfrak{G}$-torsion R-module $K_{\mathfrak{G}}/K$, hence M/K is $\mathfrak{G}$-torsion since the torsion theory induced by $\mathfrak{G}$ is hereditary. □

Recall a useful observation made in [8, Lemma 12.4] and [16, Exercise IX.15]):

Lemma 3.2. *Let R be a ring and $\mathfrak{G}$ a Gabriel topology on R. If M and N are $R_{\mathfrak{G}}$-modules such that N a $\mathfrak{G}$-torsion-free as an R-module then the canonical map $\operatorname{Hom}_{R_{\mathfrak{G}}}(M,N) \to \operatorname{Hom}_R(M,N)$ is an isomorphism.*

Lemma 3.3. *Let R be a right non-singular ring and $\mathfrak{G}$ a Gabriel topology on R.*

1. *If M is a $\mathfrak{G}$-torsion-free small R-module then $M_{\mathfrak{G}}$ is self-small as an R-module.*
2. *If M is a self-small $R_{\mathfrak{G}}$-module which is $\mathfrak{G}$-torsion-free as an R-module then M is self-small as an R-module.*

Proof. (1) Let $f : M_{\mathfrak{G}} \to M_{\mathfrak{G}}^{(I)}$ be an R-homomorphism. Since M is small, there exists a finite subset $F \subseteq I$ such that $f(M) \subseteq M_{\mathfrak{G}}^{(F)}$. Therefore the R-homomorphism $\overline{f} : M_{\mathfrak{G}}/M \to M_{\mathfrak{G}}^{(I\setminus F)}$, $\overline{f}(x+M) = \pi_{I\setminus F} f(x)$ is well defined ($\pi_{I\setminus F} : M_{\mathfrak{G}}^{(I)} \to M_{\mathfrak{G}}^{(I\setminus F)}$ denotes the canonical projection). But $\operatorname{Hom}_R(M_{\mathfrak{G}}/M, (M_{\mathfrak{G}}^{(I\setminus F)}) = 0$ since $M_{\mathfrak{G}}/M$ is $\mathfrak{G}$-torsion and $M_{\mathfrak{G}}$ is $\mathfrak{G}$-torsion free, hence $\overline{f} = 0$. It follows that $f(M_{\mathfrak{G}}) \subseteq M_{\mathfrak{G}}^{(F)}$.

(2) If I is a set then $M^{(I)}$ is $\mathfrak{G}$-torsion-free as a right R-module, hence every R-homomorphism $f : M \to M^{(I)}$ is an $R_{\mathfrak{G}}$-homomorphism by Lemma 3.2, and it follows that there exists a finite subset $F \subseteq I$ such that $\operatorname{Im}(f) \subseteq M^{(F)}$. □

As usual, we denote by $Q_{\max}$ the maximal right ring of quotients of a ring R. Since $Q_{\max}$ is torsion-free over any nonsingular ring, we may conclude the following consequence of Lemma 3.3 (1):

Corollary 3.4. *If R is a right non-singular ring, then $Q_{\max}$ is a self-small R-module.*

For the proof of the next proposition we use an idea presented in [9, Theorem 2.8]. We also need some technical results.

For an arbitrary cardinal κ we say that a module M is *κ-reducing* if for every submodule $N \subseteq M$ such that $gen(N) \leq \kappa$ there exists a finitely generated submodule F for which $N \subseteq F \subseteq M$.

Remark 3.5. *Note that a κ-reducing module is small by* [9, Lemma 1.5]*, hence self-small, for every infinite cardinal κ.*

Lemma 3.6. *Let R be a subring of a ring Q such that Q is finitely generated as a right R-module.*

(1) *Every non-finitely generated ω-reducing Q-module is non-finitely generated ω-reducing as an R-module.*
(2) *If Q is simple regular which is not right artinian, every injective Q-module is small as an R-module.*

Proof. (1) An easy observation.

(2) Since Q is a non-artinian regular ring, it contains a set of non-zero orthogonal idempotents $\{e_n | n < \omega\}$, i.e. we have a right ideal $\bigoplus_{n<\omega} e_n Q \subseteq Q$.

Now, we modify the proof of [7, Lemma 1.10]. Let P be an injective Q-module and assume that $P = \bigcup_{n<\omega} P_n$ for a strictly increasing chain of R-submodules $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subsetneq \cdots$. Note that $Pe_nQ = P$ because Q is a simple ring, so there exists $p_n \in P$ such that $p_n e_n Q \nsubseteq P_n$. As P is Q-injective, the Q-homomorphism $\varphi : \bigoplus_{n<\omega} e_n Q \to P$ defined by the rule $\varphi(e_n r) = p_n e_n r$ may be extended to a homomorphism $\varphi : Q \to P$. Hence there exists $p \in P$ such that $\sum_{n<\omega} p_n e_n Q \subseteq pQ = \mathrm{Im}\varphi$. Since pQ is a finitely generated R-module, there is $k < \omega$ for which $pQ \subseteq P_k$, so $\sum_{n<\omega} p_n e_n Q \subseteq P_k$, a contradiction. □

Proposition 3.7. *If R is a right non-singular right strongly steady ring then its maximal ring of quotients $Q_{\max}$ is semi-simple.*

Proof. First, note that $Q_{\max}$ is a finitely generated R-module by Corollary 3.4 and $Q_{\max}$ is a regular and self-injective ring by [16, Proposition XII.2.2 and Corollary XII.2.3]. Therefore, from [12, Theorem 10.22], $Q_{\max}$ is a direct product of rings $Q_{\max} = Q_1 \times Q_2 \times Q_3$ such that Q_1 is of type I_f, Q_2 is of type II_f, and Q_3 is a purely infinite ring. We view all these factors of $Q_{\max}$ as two-sided ideals of $Q_{\max}$.

Suppose $Q_2 \neq 0$. Then for every maximal two-sided ideal M which contains Q_1 and Q_3 we deduce, using [12, Theorem 10.29], that $Q_{\max}/M$ is of type II_f. It follows that $Q_{\max}/M$ is not artinian by [12, Theorem 10.26]. Then $R \cap M$ is a two-sided ideal of R and $R/R \cap M$ can be embedded into $Q_{\max}/M$ such that $Q_{\max}/M$ is finitely generated as a right $R/R\cap M$-module. Then $R/R\cap M$ is not right steady as a consequence of Lemma 3.6 (2), a contradiction, hence $Q_{\max} = Q_1 \times Q_3$. Moreover, $Q_{\max}$ has no purely infinite factors. Note that every injective module over a purely infinite ring is ω-reducing by [12, Proposition 5.8] and [17, Example 2.8], so any non-finitely generated injective module over Q_3 is non-finitely generated ω-reducing as an R-module by Lemma 3.6(1). Hence by Lemma 2.2 $Q_{\max}$ is directly finite of type I_f. Then $Q_{\max} \cong \prod_{n\in I} M_n(S_n)$ where S_n, $n < \omega$ are abelian regular rings and I is a countable set by [12, Theorem 10.24]. Assume I is infinite. Then $Q_{\max}$ contains a non-finitely generated self-small right ideal, which is torsion-free as an R-module, so it is a non-finitely generated self-small R-module by Lemma 3.3(2). Then I is a finite set. Using again Lemma 3.3(2) we deduce that for every $n \in I$ every small ideal of $M_n(S_n)$ is finitely generated. Then every small ideal of S_n is finitely generated since S_n and $M_n(S_n)$ are Morita equivalent rings. As a consequence of the proof of [9, Theorem 2.8] we deduce that S_n are semi-simple rings for all $n \in I$ and the proof is complete. □

From this result we obtain a first characterization theorem.

Theorem 3.8. *The following conditions are equivalent for a non-singular commutative ring R:*

a) *R is artinian;*
b) *R is semi-simple;*
c) *R is (right) strongly steady.*

Proof. a) $\Rightarrow$ b) It suffices to prove that $J = 0$. If $s \in J \cap Soc(R)$, then $Soc(R) \subseteq Ann(s)$. As R is non-singular, $s = 0$, hence $J = 0$. The implication b) $\Rightarrow$ a) is obvious.

b) $\Rightarrow$ c) By Proposition 2.5.

c) $\Rightarrow$ b) By Proposition 3.7 and [13, Corollary 3.97]. □

In fact, the conclusion "R is artinian" is valid for all right non-singular rings.

Theorem 3.9. *Let R be a right non-singular right strongly steady ring. Then R is right artinian.*

Proof. Applying the Proposition 3.7 and [16, Theorem XII 2.5] we deduce that R is of finite rank. Therefore $(Q_{\max})_R \cong E(R) \cong \bigoplus_{i\leq m} E(S_i)$ for uniform R-submodules (right ideals) S_i of R, $i \leq m$. Here $E(S_i)$ denotes the injective envelope of S_i. Note that every $E(S_i)$ is simple as a right $Q_{\max}$-module.

We show that every non-zero R-submodule N of $E(S_i)$ is self-small, so finitely generated. Let $N = \bigcup_{n<\omega} N_i$ for an strictly increasing chain of non-zero submodules. Since S_i is essential in $E(S_i)$, $S_i \cap N_n \neq 0$ for all n. For an arbitrarily $n < \omega$, take $\varphi \in End_R(N)$ such that $\varphi(N_n) = 0$. Then φ can be extended to a homomorphism $\varphi \in End_R(E(S_i))$. Since $E(S_i)$ is a torsion-free module (concerning maximum Gabriel topology) and $E(S_i) \cong S_i \otimes Q_{\max}$, we have a natural bijection $End_R(E(S_i)) \to End_{Q_{\max}}(E(S_i))$ by Lemma 3.2. Hence φ is a $Q_{\max}$-endomorphism of a simple $Q_{\max}$-module which is not mono, so $\varphi = 0$. Applying Proposition 2.7, we obtain that N is a self-small module. Then $E(S_i)$ are noetherian R-modules for all $i \leq m$, hence $Q_{\max}$ is a noetherian R-module since it is a finite direct sum of noetherian modules.

Therefore R is a right noetherian ring. Since R is a right strongly steady ring, $Q_{\max} \cong E(R)$ is finitely generated as a right R-module. Then R is right artinian as a consequence of [4, Corollary 5.1]. □

Corollary 3.10. *Let R be a right nonsingular ring which is strongly steady. Then its maximal ring of right quotients is artinian and noetherian as a right R-module.*

The conditions R is right artinian does not imply R is right strongly steady.

Example 3.11. *The ring* $R = \begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$ *is non-singular right artinian but it is not right strongly steady.*

Proof. It is not hard to see that the maximal right ring of quotients of R is $Q_{\max} = \begin{pmatrix} \mathbb{R} & \mathbb{R} \\ \mathbb{R} & \mathbb{R} \end{pmatrix}$ which is self small as an R-module by Corollary 3.4. Since $Q_{\max}$ is an infinitely generated R-module, R is not right strongly steady. □

Theorem 3.12. *The following conditions are equivalent for a non-singular semi-prime ring R:*

a) *R is semi-simple;*
b) *R is right strongly steady.*

Proof. a)⇒b) By Proposition 2.5.

b)⇒a) We apply Proposition 3.7 to obtain that the maximal ring of quotient is semi-simple. Applying [16, Proposition XV.3.3], $Q_{\max}$ is the classical ring of right quotients of R. Using [14, Exercise II.16] we deduce that R and its classical ring of quotients coincide and the proof is complete. □

The following example shows that the hypothesis "R is semi-prime" is not superfluous. Moreover, it follows that another important class of (non-singular) rings is included in the class of strongly steady rings.

Example 3.13. *Every upper triangular matrix ring $T_n(K)$ over a division ring K is strongly steady.*

Proof. The proof is by induction on n. If $n = 1$ the property is obvious. Supposing that $T_n(K)$ is a strongly steady ring, we will prove that $T_{n+1}(K)$ is strongly steady. We observe that the right socle of $T_{n+1}(K)$ is the ideal S which consists in all matrices with 0 on the first n columns. Moreover S is the smallest essential ideal of $T_{n+1}(K)$, hence every singular $T_{n+1}(K)$-module is annihilated by S. Therefore a singular $T_{n+1}(K)$-module M is a $T_{n+1}(K)/S$-module and M is self-small as a $T_{n+1}(K)$-module if and only if it is self-small as a $T_{n+1}(K)/S$-module. Since $T_{n+1}(K)/S \cong T_n(K)$, a singular self-small $T_{n+1}(K)$-module is finitely generated by the hypothesis. Moreover, every nonsingular right $T_{n+1}(K)$-module is projective (see the proof of [8, 12.21]), hence it is a direct sum of projective cyclic ideals since $T_{n+1}(K)$ is an exchange ring, and it follows that a nonsingular self-small $T_{n+1}(K)$-module is finitely generated. To close the proof, it is enough to observe that every $T_{n+1}(K)$-module is a direct sum of a singular module and a non-singular module by [16, Section VI.8]. □

Using Lemma 2.4 and [8, 12.21], we obtain

Corollary 3.14. *If R is a non-singular and Σ-extending ring then every self-small R-module is finitely generated.*

Using Corollary 3.4 we can deduce directly that for a right nonsingular ring which is right strongly steady ring the right R-module $Q_{\max}$ is finitely generated. In the following example we show that the condition "$Q_{\max}$ is finitely generated" does not imply that the right nonsingular ring is strongly steady.

Example 3.15. *There exists a right nonsingular ring R with simple maximal ring of quotients $Q_{\max}$ such that $Q_{\max}$ is finitely generated as a right R-module and R is not strongly steady. (There exists a right nonsingular ring R which is not right artinian with simple maximal ring of quotients $Q_{\max}$ such that $Q_{\max}$ is finitely generated as a right R-module).*

Proof. We consider the ring $R = \begin{pmatrix} \mathbb{R} & \mathbb{R} & \mathbb{R} \\ 0 & \mathbb{Q} & \mathbb{R} \\ 0 & 0 & \mathbb{R} \end{pmatrix}$. It is not hard to see that R is not right artinian. If we calculate a cyclic right ideal generated by an element of R we observe that it contains one of the ideals which have only 0 on the first two columns and two 0 and one $\mathbb{R}$ on the last column. Therefore, the right socle of R is $S = Soc(R_R) = \begin{pmatrix} 0 & 0 & \mathbb{R} \\ 0 & 0 & \mathbb{R} \\ 0 & 0 & \mathbb{R} \end{pmatrix}$.

Using [13, Lemma 7.2] we deduce that R is right nonsingular. Moreover, by direct calculations it follows that S is a dense ideal of R and it is minimal (since every proper R-submodule of S is not essential in R, [13, Examples 8.3]). Hence by [13, Theorem 13.22] we deduce $Q_{\max} = Mat_{3\times 3}(\mathbb{R})$.

The right R-module $Q_{\max}(R)$ is generated by E_{11}, E_{12} and E_{13},where E_{ij} denotes the matrix which have 1 on the position (i, j) and 0 on the other positions. □

Now we can complete Theorem 3.9.

Proposition 3.16. *Every right noetherian right strongly steady ring is right artinian.*

Proof. As $R/N(R)$ contains no nilpotent ideal, $R/N(R)$ is a right non-singular strongly steady ring by [16, Lemma II.2.5], and it is semi-prime. Therefore it is semi-simple by Theorem 3.12. By [16, Corollary XV.1.3] $N(R) \subseteq J(R)$, and $J(R)/N(R) \subseteq J(R/N(R)) = 0$ from which it follows $N(R) = J(R)$. Applying [16, Lemma XV.1.4] we deduce that $J(R)$ is nilpotent, which implies that R is a right artinian ring as a consequence of Hopkins-Levitzki Theorem [14, Theorem 4.15]. □

Theorem 3.17. *For a commutative noetherian ring R, the following properties are equivalent:*

a) *R is artinian;*
b) *R is (right) strongly steady.*

Proof. The implication b)⇒a) follows by Proposition 3.16 and the reverse one holds true by Proposition 2.9. □

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B. WHEN PRODUCTS OF SELF-SMALL MODULES ARE SELF-SMALL

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Abstract. A module M is called self-small if the functor $\mathrm{Hom}(M, -)$ commutes with direct sums of copies of M. The main goal of the present paper is to construct a non-self-small product of self-small modules without non-zero homomorphisms between distinct ones and to correct an error in a claim about products of self-small modules published by Arnold and Murley in a fundamental paper on this topic. The second part of the paper is devoted to the study of endomorphism rings of self-small modules.

The notion of a compact object of a category, i.e. an object c for which the covariant functor $\mathrm{Hom}(c, -)$ commutes with all direct sums, has appeared as a natural tool in many branches of module theory. *Small* modules, which are precisely compact objects of the category of all modules over a ring, have been useful in the study of the structure theory of graded rings [8] and almost free modules [5]. The most recent motivation of this topic comes from the context of representable equivalences of module categories [3, 4]. The central notion of the present paper, a *self-small* module, which can be defined as a compact object c of the category of direct summands of all direct sums of copies of c, was introduced in [1] as a tool for generalization of Baer's lemma [6, 86.5]. Nevertheless, self-small modules, similarly to small modules, turn out to be important in the study of generalization of Morita equivalence [3, 4].

The first two sections of the work [1] are devoted to describe basic properties of general self-small modules. Among correct and classical structure results about self-small modules there is one [1, Corollary 1.3] which has a too weak hypothesis (and a too quick proof) as we show in Proposition 1.4 and Example 1.5 in the present paper. The construction of Proposition 1.4 is a consequence of a more general fact that a product of a representable set of all simple modules over a non-steady abelian regular ring is not a self-small module, but $Hom(S, T) = 0$ for an arbitrary pair of non-isomorphic simple modules S and T (Corollary 1.3). A correct version of [1, Corollary 1.3] is proved in Proposition 1.6 and in the rest of the section we investigate several classes of examples for which the hypothesis of Proposition 1.6 is satisfied, i.e. when the conclusion of [1, Corollary 1.3] holds true. The second section of this paper illustrates the limitations of using endomorphism rings to

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detect whether a module is self-small. Theorems 2.5, and 2.9 respectively show that over a non-trivial commutative principal ideal domain with zero Jacobson radical, and over a non-artinian abelian regular rings respectively there exists a pair of a self-small module and a non-self-small module with the same isomorphism rings.

We consider rings as unitary and associative, and a module means a right R-module over an arbitrary ring R. Let M be a module and N its submodule, then define $V_M(N) = \{f \in \mathrm{End}(M) \mid f(N) = 0\}$. Recall that M is a self-small module iff for each increasing chain of submodules $N_n \subsetneq N_{n+1}$ of M for which $M = \bigcup_{n<\omega} N_n$ there exists n such that $V_M(N_n) = 0$ [1, Proposition 1.1 (b)]. Moreover, a small module M can be characterized in similar way: if $M = \bigcup_{n<\omega} N_n$ for an increasing chain of submodules of N, then there exists n such that $N_n = M$. Note that a finitely generated module is a compact object of any full subcategory of the category of all modules. Thus the class of all self-small modules contains the class of all small modules and it contains all finitely generated modules. Both inclusions are strict in general, as it is shown in [1, 10], which leads to the definitions of a *right strongly steady* (*right steady*) ring as a ring over which every right self-small (small) module is necessary finitely generated. Several necessary and sufficient conditions of strongly steady rings are proved in [2], properties of steady rings are investigated in [5, 9, 10, 11].

Let M be a module. Jacobson radical of M is denoted by $J(M)$, and for $\xi \in M$ or $\xi = M$, the annihilator of ξ in a ring R is denoted by $Ann_R(\xi)$. Recall that an *abelian regular* ring has every principal right and left ideal generated by a central idempotent. For basic properties of abelian regular rings we refer to [7, Chapter III].

1. Products

Denote by $\mathcal{S}_R$ a representative set of simple modules over a ring R. Before we start to construct an example of a non-self-small product of self-small modules without non-zero homomorphisms between distinct modules, we prove the following criterion.

Theorem 1.1. *Let R be an abelian regular ring and M a module containing a copy of every simple module as a submodule. Then M is self-small iff it is small.*

Proof. First, note that every small module is self-small in general, so it is enough to prove the direct implication. Suppose that M is not small and let $M = \bigcup_n M_n$ for a strictly increasing chain of submodules $M_n \subsetneq M_{n+1} \subseteq M$. As every simple module over an abelian regular ring is injective by [7, Proposition 6.18], every R-module has zero Jacobson radical. Thus M/M_n has a simple quotient for each n. Since M contains a copy of every simple submodule, there exists a non-zero homomorphism

$g: M/M_n \to M$. Finally, put $f_n = g\pi_n$ where π_n is the natural projection of M onto M/M_n. For every n we have found a non-zero endomorphism f_n such that $f_n(M_n) = 0$, i.e. M is not self-small by [1, Proposition 1.1 (b)]). □

As $\bigoplus_{S \in \mathcal{S}_R} S \subseteq \prod_{S \in \mathcal{S}_R} S$, we may formulate the following consequences:

Corollary 1.2. *Let R be an abelian regular ring. Then $\prod_{S \in \mathcal{S}_R} S$ is a self-small module iff it is small.*

Corollary 1.3. *Let R be an abelian regular ring which is right steady and such that $\prod_{S \in \mathcal{S}_R} S$ is infinitely generated. Then $\prod_{S \in \mathcal{S}_R} S$ is not self-small.*

Proposition 1.4. *Let R be a countable non-artinian abelian regular ring. Then $\prod_{S \in \mathcal{S}_R} S$ is not self-small.*

Proof. Since $\mathcal{S}_R$ is infinite, the module $\prod_{S \in \mathcal{S}_R} S$ is not countable, hence it is infinitely generated. As R is right steady by [11, Theorem 9] we may apply Corollary 1.3. □

Example 1.5. Let R be the subring of the ring $\mathbf{Q}^\omega$ generated by $\mathbf{Q}^{(\omega)}$ and by all constant functions $c_q \in \mathbf{Q}^\omega$ (i.e. R is formed by the eventually constant functions, cf. the constructions in [5] and [9]). Note that R is a countable abelian regular ring containing the infinitely generated ideal $\mathbf{Q}^{(\omega)}$, hence $\prod_{S \in \mathcal{S}_R} S$ is not self-small by Proposition 1.4. Since every simple module is self-small and $Hom_R(S, T) = 0$ for every different modules $S, T \in \mathcal{S}_R$, we have constructed a counterexample to [1, Corollary 1.3].

[1, Corollary 1.3] can be corrected in the following way:

Proposition 1.6. *Let $(M_i \mid i \in I)$ be a system of self-small modules satisfying the condition $Hom_R(\prod_{j \in I \setminus \{i\}} M_j, M_i) = 0$ for each $i \in I$. Then $\prod_{j \in I} M_j$ is a self-small module.*

Proof. Put $M = \prod_{j \in I} M_j$ and suppose $M = \bigcup_{n < \omega} N_n$ for a chain of submodules $N_n \subseteq N_{n+1}$. Denote by π_i and ν_i respectively the natural projection of M onto M_i and the natural injection of M_i into M respectively.

First, note that there exists a minimal n_i such that $V_{M_i}(\pi_i(N_{n_i})) = 0$ for every $i \in I$. If $\{n_i \mid i \in I\}$ is not bounded, fix for every $n < \omega$ some i_n so that $V_{M_{i_n}}(\pi_{i_n}(N_n)) \neq 0$, i.e. $\pi_{i_n}(N_n) \neq M_{i_n}$ and we can take $m \in M$ for which $\pi_{i_n}(m) \notin \pi_{i_n}(N_n)$. Hence $m \notin N_n$ for every n, a contradiction (this part of the proof works as in the proof of [1, Corollary 1.3]).

Thus $\{n_i\}$ is bounded, so there exists n such that $V_{M_i}(\pi_i(N_n)) = 0$ for each $i \in I$. Note that $m - \nu_i\pi_i(m) \in \prod_{j \in I \setminus \{i\}} M_j$, where we consider $\prod_{j \in K} M_j$ in natural way as a submodule of $\prod_{j \in I} M_j$ whenever $K \subset I$. Hence $\pi_i\varphi(m - \nu_i\pi_i(m)) = 0$ for every $i \in I$, $m \in M$ and $\varphi \in End(M)$ by the premise. Let $\varphi \in End(M)$ such that $\varphi(N_n) = 0$. Then $0 = \pi_i\varphi(p) = \pi_i\varphi(\nu_i\pi_i(p))$ for arbitrary $p \in N_n$. Since

$V_{M_i}(\pi_i(N_n)) = 0$ and $\pi_i\varphi\nu_i \in End(M_i)$ we get $\pi_i\varphi(m) = \pi_i\varphi(\nu_i\pi_i(m)) = 0$ for every $m \in M$ and i, which implies that $\varphi = 0$ and $V_M(N_n) = 0$. □

Applying Proposition 1.6 we will show that we can produce new examples of self-small modules as products of self-small modules (which was an original aim of [1, Corollary 1.3]).

Lemma 1.7. *Let R be a commutative principal ideal ring and let $T \in \mathcal{S}_R$. Then*

$$Hom_R(\prod_{S\in\mathcal{S}_R\setminus\{T\}} S, T) = 0.$$

Proof. Put $iR = Ann_R(T)$ for a suitable $i \in R$. Fix an arbitrary simple module S non-isomorphic to T. Since $iR \nsubseteq Ann_R(S)$, $Si \neq 0$ and so $Si = S$. Hence $(\prod_{S\in\mathcal{S}_R\setminus\{T\}} S)i = \prod_{S\in\mathcal{S}_R\setminus\{T\}} S$. If we take a homomorphism $\varphi : \prod_{S\in\mathcal{S}_R\setminus\{T\}} S \to T$, it holds true that $\prod_{S\in\mathcal{S}_R\setminus\{T\}} S = (\prod_{S\in\mathcal{S}_R\setminus\{T\}} S)Ann_R(T) \subseteq ker\ \varphi$, i.e. $\varphi = 0$. □

Corollary 1.8. *If R is a commutative principal ideal ring, $\prod_{S\in\mathcal{S}_R} S$ is a self-small module.*

As it is shown in the sequel, the stronger version of the hypothesis of [1, Corollary 1.3] can be easily verified for direct summands of rings. First, we make an observation which follows from the fact that $Hom_R(M, eR) = 0$ for every central idempotent e contained in the annihilator of M:

Lemma 1.9. *Let $\{e_i|\ i \in I\}$ be an orthogonal set of central idempotents of a ring R. Then $Hom_R(e_nR, e_iR) = 0$ and $Hom_R(\prod_{j\neq i} e_jR, e_iR) = 0$ for every $i \neq n$.*

Proposition 1.10. *$\prod_{i\in I} e_iR$ is a self-small module for a ring R and every orthogonal set of non-zero central idempotents $\{e_i|\ i \in I\}$.*

Proof. Since $Hom_R(\prod_{j\neq i} e_jR, e_iR) = 0$ by Lemma 1.9 and every e_iR is self-small, we may apply Proposition 1.6. □

We finish this section with a characterization of strongly steady abelian regular rings.

Proposition 1.11. *Over every non-artinian abelian regular ring there exists an infinitely generated self-small module.*

Proof. As R is a non-artinian abelian regular ring, it contains an infinite orthogonal set of non-zero central idempotents, $\{e_i|\ i < \omega\}$. If $\prod_{i<\omega} e_iR$ is an infinitely generated module, it is an example of an infinitely generated self-small module by Proposition 1.10.

Now, suppose that $S = \prod_{i<\omega} e_iR$ is finitely generated. As $\prod_{i\in K} e_iR$ is a direct summand, it is a finitely generated module as well for every $K \subset \omega$. Let $\mathcal{F}$ is a filter on ω and define $\varphi(\mathcal{F}) = \{s \in S|\exists X \in \mathcal{F} : \pi_i(s) = 0\ \forall i \in X\}$, a submodule of S, where π_i is the natural projection onto e_iR. If $\mathcal{F}$ is a non-principal ultrafilter, then the same argument as in [10, Lemma 2.4 (ii)] shows that $\varphi(\mathcal{F})$ is a small, so self-small module. Moreover, $\varphi(\mathcal{F})$ is an infinitely generated R-module since it is an infinitely generated S-module by [10, Lemma 2.2 (ii)] where $S = \prod_{i<\omega} e_i R$ is considered as a ring. We have found an infinitely generated self-small module. □

By [2, Proposition 2.5 (ii)] and since every artinian regular ring is semisimple, we obtain:

Corollary 1.12. *An abelian regular ring is strongly steady iff it is semisimple.*

2. Endomorphism-rings

As it is shown already in [1], an endomorphism ring of a module can serve as a useful tool when we ask whether the module is self-small. So for example M is a self-small module if $\mathrm{End}(M)$ is countable [1, Corollary 1.4]. We show in this section that the structure of an endomorphism ring of a module need not allow to decide whether the module is self-small over a general ring. Nevertheless, note first that the self-smallness of a module over a commutative artinian ring may be recognized in the structure of its endomorphism ring.

Proposition 2.1. *Let R be a commutative artinian ring and let M be a module. Then M is self-small, iff $End_R(M)$ is a left artinian ring.*

Proof. Let M be a self-small module and put $E = End_R(M)$. Then M is finitely generated by [2, Proposition 2.9], i.e. $M = \sum_{i\le n} m_i R$. Since we may suppose that M is faithful over R and identify elements of R with endomorphisms, we may consider the ring R as a subring of (the center of) E. As $M = \sum_{i\le n} m_i R = \sum_{i\le n} E m_i$ is an artinian R-module, it has the structure of an artinian left E-module. Hence every left cyclic E-module $Em_i \cong_E E/Ann_E(m_i)$ is artinian as well. Since R is commutative, $\bigcap_{i\le n} Ann_E(m_i) = \{\varphi \in E |\ \varphi(m_i) = 0\ \forall i \le n\} = \{\varphi \in E |\ \varphi(M) = 0\} = 0$. It shows the natural E-homomorphism of E into $\prod_{i\le n} E/Ann_E(m_i)$ is a monomorphism, hence E is a left artinian ring.

The reverse implication follows immediately from [1, Proposition 1.1.(d)]. □

Now, we find a set of modules such that the endomorphism ring of the direct sum is isomorphic to the endomorphism ring of the product over this set.

Lemma 2.2. *Let $(M_i |\ i \in I)$ be a system of modules such that $Hom_R(M_j, M_i) = 0$ whenever $i \ne j$. Then $End_R(\bigoplus_{i\in I} M_i) \cong \prod_{i\in I} End_R(M_i)$. Moreover, if $Hom_R(\prod_{j\in I\setminus\{i\}} M_j, M_i) = 0$ for each $i \in I$, then $End_R(\prod_{i\in I} M_i) \cong \prod_{i\in I} End_R(M_i)$.*

Proof. Define mappings $\Psi_1 : End_R(\bigoplus_{i\in I} M_i) \to \prod_{i\in I} End(M_i)$ and $\Psi_2 : End_R(\prod_{i\in I} M_i) \to \prod_{i\in I} End(M_i))$ by the formula $\Psi_k(\varphi) = (\pi_i \varphi \nu_i |\ i \in I)$, $k \in \{1, 2\}$, where π_i is the natural projection onto M_i and ν_i denotes the natural injection of M_i. Obviously, Ψ_k is in the both cases a surjective $\mathbf{Z}$-homomorphism and $\Psi_k(\mathbf{1}) = \mathbf{1}$.

Since $\varphi\nu_i(M_i) \subseteq M_i$ ($\subseteq \bigoplus_{i\in I} M_i$ or $\subseteq \prod_{i\in I} M_i$) by the hypothesis, we obtain that $\nu_i\pi_i\varphi\nu_i = \varphi\nu_i$, hence $[\Psi_k(\varphi_1\varphi_2)](i) = \pi_i\varphi_1\varphi_2\nu_i = (\pi_i\varphi_1\nu_i)(\pi_i\varphi_2\nu_i) = [\Psi_k(\varphi_1)\Psi_k(\varphi_2)](i)$ for every $i \in I$, $k \in \{1,2\}$. We have proved Ψ_1 and Ψ_2 surjective ring homomorphisms.

It is obvious that Ψ_1 is injective and so it is an isomorphism. To conclude that Ψ_2 is an isomorphism it remains to prove that $ker\ \Psi_2 = 0$. Fix any non-zero $\varphi \in End_R(\prod_{i\in I} M_i)$. Then there exists $i \in I$ such that $\pi_i\varphi \neq 0$. Now, it follows by the hypothesis that $\prod_{j\in I\setminus\{i\}} M_i \subseteq ker\ \pi_i\varphi$, hence $[\Psi_2(\varphi)](i) = \pi_i\varphi\nu_i \neq 0$. □

Corollary 2.3. *$End_R(\bigoplus_{S\in\mathcal{S}_R} S) \cong \prod_{S\in\mathcal{S}_R} End_R(S)$, for an arbitrary ring R.*

Proposition 2.4. *Let R be a commutative non-artinian principal ideal domain with $J(R) = 0$. Then $\mathcal{S}_R$ is infinite and $End_R(\prod_{S\in\mathcal{S}_R} S) \cong \prod_{S\in\mathcal{S}_R} End_R(S)$.*

Proof. Note that $\mathcal{S}_R$ is infinite since R contains infinitely many maximal ideals (otherwise R would be artinian). Moreover, the hypothesis of Lemma 2.2 is satisfied for a system of all simple modules $\mathcal{S}_R$ by Lemma 1.7. □

Theorem 2.5. *Let R be a commutative non-artinian principal ideal domain with $J(R) = 0$. Then $M = \prod_{S\in\mathcal{S}_R} S$ is a self-small module, $N = \bigoplus_{S\in\mathcal{S}_R} S$ is not self-small and $End_R(M) \cong End_R(N)$.*

Proof. M is self-small by Corollary 1.8 and N is not self-small clearly. Using Proposition 2.4 and Corollary 2.3 we get the ring isomorphisms $End_R(\prod_{S\in\mathcal{S}_R} S) \cong \prod_{S\in\mathcal{S}_R} End_R(S) \cong End_R(\bigoplus_{S\in\mathcal{S}_R} S)$. □

Corollary 2.6. *Let R be a commutative ring such that $J(R)$ is not maximal and $R/J(R)$ is principal ideal domain. Then there exists a pair of a self-small module M and a non-self-small module N such that $End_R(M) \cong End_R(N)$.*

Example 2.7. Denote by $\mathbb{P}$ the set of all prime numbers. By Theorem 2.5 $M = \prod_{p\in\mathbb{P}} \mathbf{Z}_p$ is a self-small abelian group, $N = \bigoplus_{p\in\mathbb{P}} \mathbf{Z}_p$ is not self-small and $End_R(M) \cong End_R(N)$.

Another class of examples can be constructed over rings containing an infinite set of central orthogonal idempotents.

Proposition 2.8. *Let R be a ring and $\{e_i|\ i < \omega\}$ an orthogonal set of non-zero central idempotents. Then $End_R(\prod_{i<\omega} e_iR) \cong \prod_{i<\omega} e_iR$.*

Proof. Since $Hom_R(\prod_{j\neq i} e_jR, e_iR) = 0$ for every i by Lemma 1.9, we may apply Lemma 2.2 on the system of modules $\{e_iR|\ i < \omega\}$. □

Theorem 2.9. *Let R be a non-artinian abelian regular ring. Then there exists a pair of a self-small module M and a non-self-small module N such that $End_R(M) \cong End_R(N)$.*

Proof. Since R is not artinian, it contains an infinite set of orthogonal idempotents which are obviously central, take such a set $\{e_i \mid i < \omega\}$ and put $M = \prod_{i<\omega} e_i R$ and $N = \bigoplus_{i<\omega} e_i R$. Note that M is self-small by Proposition 1.10, and it is obvious that N is not self-small. Finally, applying Lemma 2.2 and Proposition 2.8 we get that $End_R(M) \cong \prod_{i<\omega} e_i R \cong End_R(N)$. □

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Chapter 3

Small modules and steady rings

The chapter presents two particular solutions of general problem on ring-theoretical description of steady rings. The first text provides instead a ring-theoretical property an idea of a test module of steadiness and the second one gives a ring-theoretical characterization in the particular class of regular semiartinian rings with primitive factors artinian:

C. Jan Žemlička, *Steadiness is tested by a single module*, in: Kelarev, A. V. (ed.) et al., Abelian groups, rings and modules. Proceedings of the AGRAM 2000 conference, Perth, Australia, July 9-15, 2000. American Mathematical Society (AMS), Providence, RI. (2001), Contemp. Math. 273, 301–308.

D. Jan Žemlička, *Steadiness of regular semiartinian rings with primitive factors artinian*, J. Algebra 304/1 (2006), 500–509.

C. STEADINESS IS TESTED BY A SINGLE MODULE

JAN ŽEMLIČKA

DEDICATED TO PROFESSORS LÁSZLÓ FUCHS AND LADISLAV PROCHÁZKA IN HONOUR OF THEIR 75-TH AND 70-TH BIRTHDAY

ABSTRACT. The main goal of the paper is to provide a module-theoretic criterion of the steadiness of a ring. We show that the existence of an infinitely generated dually slender module depends on the existence of an infinitely generated dually slender submodule of a single module. In particular, steadiness is tested over a commutative regular ring R by the module R^*.

Dually slender modules are the modules M for which the covariant functor $\mathrm{Hom}(M,-)$ commutes with direct sums. In particular, finitely generated modules are dually slender. Countably generated modules are not dually slender. The notion of dually slender module dualizes the well known notions of slender modules and slim modules [EM]. Bass noticed that a module is dually slender if and only if it is not the union of a countable strictly increasing chain of submodules [B]. From this characterization it simply follows that the class of all dually slender modules is closed under homomorphic images. For some types of rings dually slender modules form a class much larger than the finitely generated ones. For instance, it is proved in [CT] that the class of all dually slender modules over the endomorphism ring of an infinitely generated free module contains the class of all injective modules.

Rings for which the dually slender modules coincide with the finitely generated modules are called *right steady* rings. Rings satisfying various finiteness conditions are known to be steady. For instance, right noetherian [R, CT], right perfect [CT], and semiartinian rings of countable socle length [T2] belong to the class of right steady rings. It is proved in [RTZ] that commutative semiartinian rings over which no cyclic module contains an infinitely generated dually slender submodule are steady as well.

Although a general ring-theoretic characterization of steady rings has not been found as yet, and so it is still an open problem, we present a characterization that uses a single module. Assume that R is a ring. It is shown here that a ring is right steady if and only if a single module of cardinality bounded by $2^{2^{card(R)}}$ contains no infinitely generated dually slender submodule. It is also proved that any representative class of dually slender modules over a commutative regular ring is a set, and we give an estimate of the cardinality of each dually slender module.

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Furthermore, we present a more precise characterization of steady commutative regular rings. A commutative regular ring R is steady if and only if R^* contains no infinitely generated dually slender submodule.

In the rest of the paper module means a right R-module. An ideal is a two-sided ideal. A regular ring means a von Neumann regular ring. A regular ring is called abelian regular provided all of its idempotents are central. A ring is said to be semilocal if its Jacobson radical is the intersection of finitely many maximal right ideals.

A representative set of simple modules over a ring will be denoted by *Simp*. R^* will denote the right R-module $R^* = ({}_R R)^* = \mathrm{Hom}_{\mathbf{Z}}({}_R R, \mathbf{Q}/\mathbf{Z})$.

Let M be a module and I be a right ideal. The set of all maximal submodules of M will be denoted by $Max(M)$. $J(M)$ will denote the Jacobson radical of M and $E(M)$ the injective envelope of M. The minimal cardinality of sets of generators of M will be denoted by $gen(M)$. Finally, $Ann_I(M)$ will stand for the right annihilator of M in I, i.e. $Ann_I(M) = \{i \in I;\ Mi = 0\}$.

For further notation we refer to [AF] and [G].

1. General case

Definition 1.1. Let R be a ring and $\kappa = card(R)^+$. Define the modules

$$T_1 = \prod_{S \in Simp} S^\kappa \quad \text{and} \quad T_2 = \prod_{S \in Simp} S^{(\omega)}.$$

Finite rings are noetherian, so they are right and left steady. Note that every dually slender module over a skew-field is a finitely generated semisimple module.

Lemma 1.2. *Let R be a ring and M be an infinitely generated dually slender module such that $J(M) = 0$. Then there exists a homomorphism $\phi\colon M \to T_1$ such that $\phi(M)$ is infinitely generated. Moreover, if the ring R is commutative, then M is embeddable in T_2.*

Proof. The ring R is not right steady, so it is infinite.

Let ρ_N denote the natural projection $M \to M/N$ for each $N \in Max(M)$. Since $J(M) = 0$, the product $h : M \to \prod_{N \in Max(M)} M/N$ of the homomorphisms ρ_N is a monomorphism. Define a set $X \subset M$ by letting X be an arbitrary subset of M of cardinality κ if $card(M) \geq \kappa$, and $X = M$ if $card(M) < \kappa$ (recall that $\kappa = card(R)^+$). Clearly, the cardinality of the module XR ($\subseteq M$) is at most κ.

For each $a \in XR$, $a \neq 0$, fix a module $N_a \in Max(M)$ for which $a \in M \setminus N_a$. Put $U = \{N_a;\ a \in XR,\ a \neq 0\} \subseteq Max(M)$. Obviously, $card(U) \leq card(XR) \leq \kappa$. Let ρ be the natural projection of $\prod_{N \in Max(M)} M/N$ onto $\prod_{N \in U} M/N$. Then $\rho h\restriction_{(XR)}$ is a monomorphism. If $X = M$, ρh is a monomorphism as well. If $card(X) = \kappa$, then $card(\rho h(M)) \geq card(\rho h(XR)) \geq \kappa > card(R^{(\omega)}) = card(R)$, because R is an infinite ring. Hence $\rho h(M)$ is an infinitely generated module. As $card(U) \leq \kappa$ and $\rho h(M)$ is a submodule of $\prod_{N \in U} M/N$, $\rho h(M)$ is embeddable in T_1.

Assume that R is a commutative ring. Then for every $N \in Max(M)$ there exists an $I \in Max(R)$ such that $MI \subseteq N$. Since M is a dually slender R-module, M/MI is a dually slender R/I-module, and so $M/MI \cong (R/I)^{(n_I)}$ for each $I \in Max(R)$ and for a suitable finite number n_I. As $\bigcap_{I \in Max(R)} MI = 0$, the product $h\colon M \to \prod_{I \in Max(R)} (R/I)^{(n_I)}$ of the projections $\rho_I\colon M \to M/MI$, $I \in Max(R)$, is a monomorphism. Obviously, there is a natural injection $i : \prod_{I \in Max(R)} (R/I)^{(n_I)} \to T_2$, so ih embeds M into T_2. □

Note that in the commutative case all finitely generated modules with zero Jacobson radical are embeddable in T_2; the same holds for infinitely generated dually slender modules with zero Jacobson radical.

Lemma 1.3. *Let R be a ring and M be a non-zero module such that $J(M) = M$. Then there exists a homomorphism $\phi\colon M \to E(S)$ for a simple module S such that $\phi(M)$ is infinitely generated.*

Proof. Fix an arbitrary non-zero element $m \in M$. Then there exists a non-zero homomorphism $\phi^*\colon mR \to E(S)$ for a suitable $S \in Simp(R)$. As $E(S)$ is an injective module, we can extend ϕ^* to a homomorphism $\phi\colon M \to E(S)$. Since both M and $\phi(M)$ are non-zero and have no maximal submodule, $\phi(M)$ is not finitely generated. □

Theorem 1.4. *Let R be a ring. If R is commutative, put $T = T_2$, otherwise put $T = T_1$. Then the following conditions are equivalent:*

1. *R is right steady.*
2. *Each dually slender submodule of T and each dually slender submodule of R^* is finitely generated.*
3. *Each dually slender submodule of T and each dually slender submodule of every $E(S)$, $S \in Simp$, is finitely generated.*

Proof. (1)→(2) Obvious.

(2)→(3) It is well known that R^* is an injective cogenerator, see for instance [S, Chapter I, Proposition 9.3]. Thus both S and $E(S)$ are embeddable in R^* for each $S \in Simp$.

(3)→(1) Let R be non-steady and M be an infinitely generated dually slender module. If $M/J(M)$ is infinitely generated , T contains an infinitely generated dually slender submodule by Lemma 1.2. On the other hand, assume that $M = F + J(M)$ for a finitely generated module F. Then $J(M/F) = M/F$. Applying Lemma 1.3 we get a simple module S such that $E(S)$ contains an infinitely generated dually slender submodule. □

Corollary 1.5. *Let R be a semilocal ring. Then R is right steady if and only if $E(R/J(R))$ contains no infinitely generated dually slender submodule.*

Proof. Since R is semilocal, $R/J(R)$ is semisimple. So the module T is semisimple as well. Hence T contains no infinitely generated dually slender submodule and the assertion follows immediately from Theorem 1.4. □

Proposition 1.6. *Let R be a commutative regular ring. Then each dually slender module is embeddable in T_2.*

Proof. Let M be a dually slender module. It is well known that every module over a commutative abelian regular ring has zero Jacobson radical. Hence, from Lemma 1.2 it follows that M embeds in T_2. □

It is an immediate consequence of the last proposition that a representative class of dually slender modules over a commutative regular ring is a set.

Proposition 1.7. *Let R be a commutative ring. If R^* contains no infinitely generated dually slender submodule, then a representative class of dually slender modules is a set.*

Proof. Suppose that a representative class of dually slender modules is proper. Obviously, R is non-steady, so R is infinite. Thus there exists a dually slender module M such that $gen(M) > card(T_2)$. By Lemma 1.2 $M/J(M)$ embeds in T_2, so there is a submodule N of M such that $gen(N) \leq card(T_2)$ and $N + J(M) = M$. Hence M/N is an infinitely generated dually slender module containing no maximal submodule. Applying Lemma 1.3 we get an infinitely generated dually slender submodule of R^*, a contradiction. □

In [T2] it is shown that the class of all dually slender modules over the endomorphism ring of an infinitely generated free module contains the class of all injective modules. Thus any representative class of dually slender modules over this ring is a proper class.

2. Steadiness of commutative regular rings

Definition 2.1. Let M be a module and I be a two-sided ideal which is maximal as a right ideal. Then define

$$d_I(M) = dim_{R/I}((M + MI)/MI).$$

Clearly, the notion is well defined because R/I is a skew-field.

Note that $d_I(N) = dim_{R/I}(N + MI/MI)$ for all modules M and N over an abelian regular ring, $N \subseteq M$. Indeed, $N \cap NI = N \cap MI$ over each abelian regular ring, so $(N + NI)/NI \cong N/(N \cap NI) = N/(N \cap MI) \cong (N + MI)/MI$.

If M is a dually slender module, $d_I(M) < \omega$ for each maximal ideal I.

Lemma 2.2. *Let R be an abelian regular ring and M be a dually slender module.*

(1) *Assume that N is a submodule of M such that $d_I(N) = d_I(M)$ for each $I \in Max(R)$. Then $M = N$.*

(2) *Assume that* J *is an ideal satisfying* $d_I(M) = 0$ *for each* $I \in Max(R)$ *such that* $J \subseteq I$. *Then* $M = MJ$.

Proof. Each module over an abelian regular ring has zero Jacobson radical. In particular, every non-zero dually slender module over an abelian regular ring contains a maximal submodule.

(1) Let I be an arbitrary maximal ideal. Since $(N + MI)/MI = M/MI$, we get following isomorphisms: $0 = M/(N + MI) \cong (M/N)/((N + MI)/N) \cong (M/N)/((M/N)I)$. Hence $d_I(M/N) = 0$ for each maximal ideal I, so the module M/N contains no maximal submodule. From the hypothesis it follows that M/N is the zero-module, and so $M = N$.

(2) Assume that $J \subseteq I$. Then $(MJ + MI)/MI = 0$. Therefore $d_I(MJ) = 0 = d_I(M)$.

On the other hand, assume that $J \not\subseteq I$. Since I is maximal, $J + I = R$. Thus $MJ + MI = M(J + I) = M$, and so $d_I(MJ) = d_I(M)$ for every maximal ideal I. Hence (1) implies the assertion. □

It follows from Proposition 1.6 that the cardinality of each dually slender module over an infinite commutative regular ring is bounded by $2^{2^{card(R)}}$. The following assertion improves on the estimate of the cardinality.

Corollary 2.3. *Let* R *be an infinite commutative regular ring. Let* M *be a dually slender module. Then* $gen(M) \leq card(M) \leq 2^{card(R)}$.

Proof. Since M is a dually slender module, $d_I(M) < \omega$ for each $I \in Max(R)$. Hence there exist finitely generated modules $F_I \subseteq M$ such that $F_I + MI = M$. Thus, by Lemma 2.2 (1), $\sum_{I \in Max(R)} F_I = M$. As the cardinality of $Max(R)$ is bounded by $2^{card(R)}$ and the cardinality of each F_I is bounded by $card(R)$, $card(M) \leq 2^{card(R)}$. □

Lemma 2.4. *Assume that* M *is a module and* $M = \sum_{i<\omega} M_i$, *where the* M_i, $i < \omega$, *are submodules of* M *such that no factor-module of* M_i *contains an infinitely generated dually slender submodule. Then no factor-module of* M *contains an infinitely generated dually slender submodule.*

Proof. This is an easy generalization of [ZT, Lemma 5] where the statement is proved for $M_i = m_iR$, $i < \omega$ and $M = \sum_{i<\omega} m_iR$. □

Let R be an abelian regular ring. Fix a module P and an element $x \in R$. Note that $PxR = PeR = Pe$ for a suitable central idempotent $e \in R$ (moreover, $xR = eR$). Thus PxR is a homomorphic image of P (a homomorphism is defined as multiplication by the central element e). Consequently, PxR is dually slender , if P is a dually slender module.

Lemma 2.5. *Let* R *be an abelian regular ring such that no cyclic module contains an infinitely generated dually slender submodule. Let* N *be an infinitely generated* *dually slender module. Define the set*

$$S = \{r \in R;\ gen(NrR) < \omega\}.$$

Then S is an ideal. In addition, either N/NS is an infinitely generated module or there exists a finitely generated submodule F such that $(N/F)S = N/F$ and $S/Ann_S(N/F)$ is infinitely generated.

Proof. Since each factor of R contains no infinitely generated dually slender ideal, finitely generated modules contain no finitely generated dually slender module (Lemma 2.4).

First, we will prove that S is an ideal. Since R is abelian regular, it is sufficient to show that S is a right ideal.

Fix two elements $r, s \in S$. We have observed that $N(r+s)R$ is a homomorphic image of N, hence $N(r+s)R$ is a dually slender module. Moreover, $N(r+s)R$ is a submodule of the finitely generated module $NrR + NsR$. It follows from Lemma 2.4 that $N(r+s)R$ is finitely generated. Thus $r+s \in S$. Now fix elements $i \in S$ and $r \in R$. Similarly, $NirR$ is a dually slender submodule of the finitely generated module NiR, hence $NirR$ is finitely generated as well. Therefore, $ir \in S$. This proves that S is a right ideal.

Assume that N/NS is finitely generated. Then there exists a finitely generated module $F \subseteq N$ for which $F + NS = N$. Thus $N/F = (N/F)S$. If $S/Ann_S(N/F)$ were finitely generated, there would exist a central idempotent $e \in S$ such that $S = eR + Ann_S(N/F)$. Obviously, $Ne + F = N$, so N would be finitely generated, a contradiction. Thus $S/Ann_S(N/F)$ is infinitely generated. □

Lemma 2.6. *Let R be a non-steady abelian regular ring. Then:*

(1) *There exist an infinitely generated dually slender module M and an ideal J such that $MJ = M$ and $gen(J/Ann_J(M)) \geq \omega$.*
(2) *There exist an infinitely generated dually slender module M and a strictly increasing chain of ideals $(J_\alpha;\ \alpha < \kappa)$ for an uncountable cardinal $\kappa = cf(\kappa)$ such that $M = \bigcup_{\alpha<\kappa} MJ_\alpha$ and $M \neq MJ_\alpha$ for each $\alpha < \kappa$.*

Proof. (1) If there exists a factor-ring of R which contains an infinitely generated dually slender ideal I, put $M = I$ and let J be a lifting of I to the ring R. Clearly, $MJ = II = I = M$ since R is a regular ring.

Let no factor of R contain an infinitely generated dually slender ideal. Since R is non-steady, there exists an infinitely generated dually slender module N. Now applying Lemma 2.5 we get an ideal S such that either N/NS is an infinitely generated module or there exists a finitely generated submodule F such that $(N/F)S = N/F$ and $S/Ann_S(N/F)$ are infinitely generated ideals.

Assume that N/NS is a finitely generated module. Put $M = N/F$ and $J = S$. Obviously, $MJ = M$ and $J/Ann_J(M)$ is infinitely generated.

On the other hand, let $\overline{N} = N/NS$ be an infinitely generated module.

Assume that there exists a central idempotent $e \in R \setminus S$ for which $gen(\overline{N}e) < \omega$. ¿From the definition of S it follows that $gen(Ne) \geq \omega$. Moreover, there is a finitely generated module $F \subseteq Ne$ for which $Ne + NS = F + NS$. Since R is an abelian regular ring, we get that $Ne \subseteq NSe + Fe \subseteq NeS + F$. Hence $(Ne/F)S = Ne/F$. If $S/Ann_S(Ne/F)$ were finitely generated, there would exist a central idempotent $f \in S$ such that $Nef + F = Ne$, so Ne would be finitely generated, a contradiction. Now put $M = Ne/F$ and $J = S$. We have proved that $M = MJ$, and $J/Ann_J(M)$ is infinitely generated.

Finally, assume that there exists no idempotent $e \in R \setminus S$ such that $gen(\overline{N}e) < \omega$. If $\overline{N}x = 0$ for an element $x \in R$, $0 = \overline{N}xR = \overline{N}e$ for a central idempotent such that $xR = eR$. Then $e \in S$ and $xR \subseteq S$. Hence $\overline{N} = N/NS$ is faithful over $\overline{R} = R/S$. Since $\overline{R}$ is also non-steady (indeed, N/NS is an infinitely generated dually slender module over $\overline{R}$), it is not semisimple. Thus there exists an infinitely generated maximal ideal in $\overline{R}$; let us denote it by I. Moreover, there is a finitely generated module K for which $K + \overline{N}I = \overline{N}$. Now both $\overline{N}$ and $M = \overline{N}/K$ are faithful over $\overline{R}$, because finitely generated modules contain no infinitely generated dually slender module (Lemma 2.4). Let the ideal J be a lifting of the ideal I to the ring R. Obviously, $MJ = M$ and $gen(J/Ann_J(M)) \geq \omega$.

(2) Applying (1) we get an infinitely generated dually slender module M and an ideal J such that $MJ = M$ and $gen(J/Ann_J(M)) \geq \omega$. Fix M and J for which $gen(J/Ann_J(M))$ is minimal (but infinite). W.l.o.g. we can suppose that M is faithful (i.e. $Ann_J(M) = Ann_R(M) = 0$). Note that $MjR \neq M$ for each $j \in J \neq R$, since $jR = eR \neq R$ for a suitable central idempotent $1 \neq e \in jR$ and $Mjr = Me \neq Me \oplus M(1-e) = M$. Let $(J_\alpha;\ \alpha < \kappa)$ be an arbitrary filtration of J by submodules such that $cf(\kappa) = \kappa$ and $card(J_\alpha) < card(J)$ for each $\alpha < \kappa$. By the minimality of $gen(J)$, $MJ_\alpha \neq M$ for each $\alpha < \kappa$. Since M is a dually slender module (i.e. M is not the union of any countable chain of submodules), κ is not countable. □

Now we are ready to characterize the steadiness of a commutative regular ring in terms of the module R^* (it is easy to generalize our result to abelian regular rings).

Theorem 2.7. *Let R be a commutative regular ring. Then R is steady if and only if R^* contains no infinitely generated dually slender submodule.*

Proof. Let R be a non-steady commutative regular ring. Applying Lemma 2.6 we get an infinitely generated dually slender module M and a strictly increasing chain of ideals $(J_\alpha;\ \alpha < \kappa)$ such that $M = \bigcup_{\alpha<\kappa} MJ_\alpha$ and $M \neq MJ_\alpha$ for each $\alpha < \kappa$. We will define a sequence of maximal ideals $(I_\beta;\ \beta < \kappa)$ and a strictly increasing sequence of ordinals $(\alpha_\beta;\ \beta < \kappa)$ such that $d_{I_\beta}(M) \neq 0$, $J_{\alpha_\beta} \subseteq I_\beta$ and $J_{\alpha_{\beta+1}} \not\subseteq I_\beta$ via transfinite induction.

Fix an arbitrary maximal ideal I_0 such that $d_{I_0}(M) \neq 0$ and let $\alpha_0 = 0$.

Assume that both I_β and α_β are defined. Since $d_{I_\beta}(M) \neq 0$, $M \not\subseteq MI_\beta$. Moreover, $M = \bigcup_{\alpha<\kappa} MJ_\alpha$, hence there exists $\alpha > \alpha_\beta$ such that $MJ_\alpha \not\subseteq MI_\beta$. Now put $\alpha_{\beta+1} = \alpha$. Obviously, $J_{\alpha_{\beta+1}} \not\subseteq I_\beta$. Applying Lemma 2.2(2) (for $J = J_{\alpha_{\beta+1}}$), we get $I_{\beta+1} \in Max(R)$ for which $J_{\alpha_{\beta+1}} \subseteq I_{\beta+1}$ and $d_{I_{\beta+1}}(M) \neq 0$.

If γ is a limit ordinal, put $\alpha_\gamma = sup(\alpha_\beta;\ \beta < \gamma)$ and define I_γ in the same way as in the non-limit step.

Since $\kappa = cf(\kappa)$ is an uncountable cardinal, there exists $n < \omega$ such that there is a cofinal subset C of κ satisfying $d_{I_\beta}(M) = n$ for each $\beta \in C$. Hence w.l.o.g we can assume that $d_{I_\beta}(M) = n$ for each $\beta < \kappa$.

Now we are ready to find a suitable infinitely generated factor of M with the essential socle.

Denote by ρ_β the natural projection of M onto $M/(MI_\beta)$, and let $\rho\colon M \to \prod_{\beta<\kappa} M/(MI_\beta)$ be the product of the homomorphisms ρ_β, $\beta < \kappa$. Note that $\bigcap_{\beta<\kappa} \rho(M)I_\beta = \bigcap_{\beta<\kappa} \rho(MI_\beta) = \bigcap_{\beta<\kappa} \rho(ker \rho_\beta) = 0$. As the module M is dually slender and the module M/MI_β is semisimple, $M/MI_\beta \cong (R/I_\beta)^{(n)}$. Since $J_{\alpha_\beta} \subseteq I_\gamma$ for each $\gamma \geq \beta$, $MJ_{\alpha_\beta} \subseteq MI_\gamma$. ¿From this it follows that $\rho_\gamma(MJ_{\alpha_\beta}) = 0$ and $\rho(MJ_{\alpha_\beta}) \neq \rho(M)$ for each $\gamma \geq \beta$. Moreover, $\rho(M) = \bigcup_{\alpha<\kappa} \rho(MJ_{\alpha_\beta}) = \bigcup_{\beta<\kappa} \rho(M)J_{\alpha_\beta}$, because $M = \bigcup_{\alpha<\kappa} MJ_\alpha$. So $\rho(M)$ is the union of a strictly increasing chain of submodules (a suitable subchain of $\rho(M)J_{\alpha_\beta}$). Consequently, $\rho(M)$ is an infinitely generated dually slender module.

Fix an arbitrary non-zero element $m \in \rho(M)$. Let β be the minimal ordinal such that $mR \not\subseteq \rho(M)I_\beta$ (it exists because $\bigcap_{\beta<\kappa} \rho(M)I_\beta = 0$). By the construction of sequences $(\alpha_\beta;\ \beta < \kappa)$ and $(I_\beta;\ \beta < \kappa)$ there exists a central idempotent $e \in J_{\alpha_{\beta+1}} \setminus I_\beta$. As I_β is a maximal ideal, $eR + I_\beta = R$, so $meR + mI_\beta = mR$. Hence $d_{I_\beta}(meR) = d_{I_\beta}(mR) \neq 0$. Since $\rho_\gamma(meR) \subseteq \rho_\gamma(mR) = 0$ for each $\gamma < \beta$ (minimality of β) and $\rho_\gamma(meR) = 0$ for each $\gamma > \beta$ (indeed, $eR \subseteq J_{\alpha_{\beta+1}} \subseteq I_\gamma$), $0 \neq meR \subseteq \rho_\beta(M)$. As $\rho_\beta(M)$, $\beta < \kappa$, is semisimple, $Soc(\rho(M))$ is essential in $\rho(M)$.

In addition, $Soc(\rho(M))$ is a submodule of $\bigoplus_{\beta<\kappa} M/MI_\beta \cong \bigoplus_{\beta<\kappa} (R/I_\beta)^{(n)}$. So $Soc(\rho(M))$ is embeddable in $(R^*)^{(n)}$. Since $(R^*)^{(n)} \cong (R^*)^n$ is an injective module, $\rho(M)$ is embeddable in $(R^*)^n$ as well. Hence, from Lemma 2.4 it follows that R^* contains an infinitely generated dually slender submodule. □

Theorem 2.7 shows that the implication of Proposition 1.7 cannot be reversed. Indeed, there exist examples of non-steady commutative regular rings [EGT]. From Proposition 1.6 it follows that any representative class of dually slender modules is a set, but the module R^* contains infinitely generated dually slender submodules.

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D. STEADINESS OF REGULAR SEMIARTINIAN RINGS WITH PRIMITIVE FACTORS ARTINIAN

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Abstract. We provide a ring-theoretic criterion of steadiness which applies to all regular semiartinian rings with primitive factors artinian.

In the 60's, Hyman Bass remarked that the covariant functor $Hom(M, -)$ commutes with direct sums if and only if the module M is not a union of a countably infinite increasing chain of proper submodules [3]. Such a module has been known and studied under various terms (Σ-compact, of type Σ, U-compact). We called it *dually slender* according to [5]. As the functor $Hom(M, -)$ is well-known to commute with direct sums for each finitely generated module M, every finitely generated module is dually slender. Although the class of all finitely generated modules coincides with the class of all dually slender ones for some important classes of rings (such as right neotherian or perfect rings), several constructions of dually slender modules that are not finitely generated were described. In fact, only three ways of producing these modules are known: constructions of rings containing infinitely generated dually slender (right) ideals ([5], [10], [8]), constructions of a directed system of (right) ideals whose direct limit is an infinitely generated dually slender module ([11], [13]) and constructions of rings over which all injective modules (obviously including infinitely generated ones) are dually slender ([4], [9]).

A general ring-theoretical characterization of rings over which there does not exist any infinitely generated dually slender right module, i.e. of *right steady* rings is still not done. As we have noted, the question is solved only for some classes of rings. In the present paper, we focus on the class of all regular semiartinian rings with primitive factors artinian (for basic properties concerning this class of rings see [2], [7], [8], [12]). The examples appeared in paper [5] show that the classes of both steady and non-steady semiartinian regular rings are non-empty. And it is proved in [8, Theorem 3.4.] that a semiartinian abelian regular ring R is steady if and only if no factor of R contains an infinitely generated dually slender ideal. The present work generalizes this characterization for regular semiartinian rings with primitive factors artinian.

Using the notion of a homogeneous ideal and Kaplansky's idea concerning regular rings with primitive factors artinian [6, Theorem 7.14] we show that the structure of a non-steady regular semiartinian ring with primitive factors artinian is not far

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from the structure of an abelian regular semiartinian ring (Proposition 2.7 and Theorem 2.8). Namely, we find "enough" central idempotents in a suitable factor of every non-steady ring which generate an infinitely generated dually slender ideal (Lemma 3.3). This allows to generalize the proof of the characterization of steadiness which is done in [8, Lemmas 3.1-3.]. Finally, Theorem 3.5 says that a regular semiartinian ring with primitive factors artinian is right steady if and only if no factor of the ring contains any infinitely generated dually slender right (left) ideal.

In the sequel, a *ring* means an associative ring with unit and a *module* is a right module. A ring R is (von Neumann) *regular* provided that each $x \in R$ has a pseudo-inverse element (i.e. there is a $y \in R$ satisfying $xyx = x$). A regular ring R is *abelian regular* if all idempotents of R are central. Define by induction the Loewy chain of a module M: $M_0 = 0$, $M_{\alpha+1}/M_\alpha = Soc(M/M_\alpha)$ and $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for a limit ordinal α. The Loewy length is the least ordinal σ such that $M_\sigma = M_{\sigma+1}$. We say that M is *semiartinian* if $M = M_\sigma$, i.e if each non-zero factor-module of M contains a simple submodule. Recall that a ring R is *right semiartinian* if R_R is a semiartinian module. A *primitive factor* of a ring is a factor modulo the annihilator of a simple module. A submodule $N \subseteq M$ is *essential* in M if $N \cap K \neq 0$ for every nonzero submodule $K \subseteq M$. Denote by $gen(M)$ the least cardinality of a set of generators of M. Note that we identify cardinals with the least ordinals of the given cardinality.

Finally, the term ideal means a two-sided ideal. We say that an ideal is finitely (or infinitely) generated if it is finitely (or infinitely) generated as a two-sided ideal.

1. Dually slender modules

We start this section by recalling some general facts concerning dually slender modules and steady rings.

Theorem 1.1. *(1) The class of all dually slender modules contains all finitely generated ones and it is closed under taking homomorphic images, extensions and finite sums.*

(2) The class of all right steady rings is closed under taking homomorphic images and finite direct products.

(3) Every countably generated dually slender module is finitely generated.

Proof. (1) See [4, Corollary 1.2].

(2) See [4, Lemma 1.9 and Lemma 2.1].

(3) See [9, Lemma 1.2] □ □

It is well known (see for example [6, Chapter 6]) that both the class of all regular semiartinian rings and the class of all rings with primitive factors artinian are closed under taking homomorphic images and finite direct products. Hence the assertion of Theorem 1.1 (2) is also true for the class of all right steady regular rings with primitive factors artinian.

Notation. According to [8] we denote by $\mathcal{R}$ the class of all regular semiartinian rings with primitive factors artinian. Recall that every right module over a right semiartinian ring is semiartinian. Take a ring $R \in \mathcal{R}$ and let $\mathcal{L} = (S_\alpha \mid \alpha \leq \sigma + 1)$ be the right Loewy chain of R. Recall that for each $\alpha \leq \sigma$ there are a cardinal λ_α, positive integers $n_{\alpha\beta}$, $\beta < \lambda_\alpha$, and skew-fields $K_{\alpha\beta}$, $\beta < \lambda_\alpha$, such that $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$ (as rings without unit). The pre-image of $M_{n_{\alpha\beta}}(K_{\alpha\beta})$ coincides with the β-th *homogeneous component* of R/S_α and it is finitely generated as right R/S_α-module for all $\beta < \lambda_\alpha$ (cf. [8, Theorem 2.1]). The system $\mathcal{D}(R) = \{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ is called the *dimension sequence* of the ring R.

Finally, note that each primitive factor of R is isomorphic to a matrix ring over a skew-field and so the set of all primitive ideals coincides with the set of all maximal ideals (cf. [8, Proposition 2.6]).

Lemma 1.2. *Let $R \in \mathcal{R}$ and let M be an R-module. Then $M = 0$ if and only if $MI = M$ for every maximal ideal I (cf.* [6, Lemma 6.14]*).*

Proof. If $M \neq 0$, there exists a simple module $S \subseteq M$. Then $I = Ann(S)$ is a maximal ideal. Since R is regular, $MI \cap S = 0$. Thus $MI \neq M$. □ □

Lemma 1.3. *Let $R \in \mathcal{R}$ and let M be a dually slender R-module. Then M/MI is a finitely generated semisimple module for every maximal ideal I.*

Proof. M/MI is a dually slender module over a semisimple ring R/I. Hence M/MI is semisimple and dually slender and so it is a finitely generated module. □ □

Let R be a ring. Following [6, p. 71], we define the *index* of an ideal J (denote it by $i(J)$) as the supremum of the indexes of all nilpotent elements $x \in J$, i.e. of all minimal natural numbers n such that $x^n = 0$.

Obviously, $i(S) \leq i(R)$ whenever S is either a subring or an ideal of R. Let us remark that the index of a matrix ring $M_n(T)$, where T is an abelian regular ring, is equal to n. Moreover, $i(\prod_{\alpha<\kappa} R_\alpha) = sup\{i(R_\alpha) | \; \alpha < \kappa\}$ for any arbitrary system of rings R_α, $\alpha < \kappa$.

Notation. Let M be a module and $\mathcal{J} \subseteq \mathcal{M}(R)$ where $\mathcal{M}(R)$ denotes the set of all maximal ideals of R. Denote by $\pi_{\mathcal{J}M} : M \to \prod_{I\in\mathcal{J}} M/MI$ the product of the natural projections $\pi_I : M \to M/MI$. Note that the ring-homomorphism $\pi_{\mathcal{J}R}$ maps $R/\bigcap_{I\in\mathcal{J}} I \cong \pi_{\mathcal{J}R}(R)$ into the ring $\prod_{I\in\mathcal{J}} R/I$. Moreover, $\pi_{\mathcal{J}M}(M)$ is a right $\pi_{\mathcal{J}R}(R)$-module. Put $\mathcal{I}_n = \{I \in \mathcal{M}(R) | \; i(R/I) = n\}$, $p_{nM} = \pi_{\mathcal{I}_n M}$ and $\pi_n = \pi_{\mathcal{I}_n R}$. Since R is a semiartinian ring with primitive factors artinian, $\mathcal{M}(R) = \bigcup_{n<\omega} \mathcal{I}_n$. Finally, note that $\pi_{nM}(M)$ has a naturally defined structure of $\pi_n(R)$-module.

Lemma 1.4. *Let $R \in \mathcal{R}$.*

(1) Let $\mathcal{D} = \{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \le \sigma\}$ be a dimension sequence of the ring R. Then $i(R) = sup\{n_{0\beta} | \ \beta < \lambda_0\}$.

(2) If $\pi_n(R) \neq 0$, then $i(\pi_n(R)) = n$.

Proof. (1) By [6, Proposition 7.7. and Corollary 7.8] the index of $Soc(R)$ is equal to $sup\{n_{0\beta} | \ \beta < \lambda_0\}$. As $Soc(R)$ is essential in R, $i(R) = i(Soc(R)) = sup\{n_{0\beta} | \ \beta < \lambda_0\}$ by [6, Corollary 7.5].

(2) Obviously, $i(\pi_n(R)) \le n$. Since there exists an ideal $I \in \mathcal{I}_n$ such that the homomorphism $\pi_I : R \to R/I$ is onto, the reverse inequality holds true. □ □

Proposition 1.5. *If $R \in \mathcal{R}$ is not a right steady ring, there exists an ideal K and an integer n such that $i(R/K) = n$ and R/K is not right steady.*

Proof. Let M be a dually slender module. Assume that $p_{nM}(M)$ is a finitely generated $\pi_n(R)$-module for every n. Then there exist finitely generated submodules $F_n \subseteq M$, $n < \omega$, such that $p_{nM}(F_n) = p_{nM}(M)$. Hence $\pi_I(M/\sum_{n<\omega} F_n) = 0$ for every $I \in \mathcal{M}(R)$ and $M = \sum_{n<\omega} F_n$ by Lemma 1.2. We have proved that $gen(M) \le \omega$. As M is a dually slender module, it is finitely generated by Theorem 1.2 (3). Thus there exists an n such that $p_{nM}(M)$ is an infinitely generated dually slender $\pi_n(R)$-module whenever M is infinitely generated. Finally, note that $i(\pi_n(R)) = n$ by Lemma 1.4 (2) and put $K = Ker\ \pi_n$. □ □

2. 2-Dually slender ideals

An ideal J is said to be *2-dually slender* provided J is not a union of a countably infinite strictly increasing chain of ideals, i.e. $J = \bigcup_{n<\omega} J_n$ implies there exists an n such that $J_n = J$ for every increasing chain of ideals J_n, $n < \omega$, contained in J.

As we proceed to show in the following lemma, the interval $\langle 0, J\rangle$ of the lattice of all two-sided ideals for every 2-dually slender ideal J has the same properties as the lattice of all submodules of a dually slender module.

Lemma 2.1. *Let J be a 2-dually slender ideal of a ring R.*

1. *Let S be a ring and $\pi : R \to S$ be a surjective homomorphism of rings. Then $\pi(J)$ is a 2-dually slender ideal of S.*
2. *Let J be countably generated as a two-sided ideal. Then J is finitely generated as a two-sided ideal.*

Proof. (1) Let $\pi(J) = \bigcup_{n<\omega} K_n$ for an increasing chain of ideals K_n, $n < \omega$. Then $J = J \cap (\bigcup_{n<\omega} \pi^{-1}(K_n)) = \bigcup_{n<\omega}(J \cap \pi^{-1}(K_n))$. As J is 2-dually slender, $J = J \cap \pi^{-1}(K_n)$ for some n, hence $\pi(J) = K_n$.

(2) Let $\{j_i | \ i < \omega\}$ be a set of generators of J. Since $J = \bigcup_{n<\omega} \sum_{i\le n} Rj_iR$, we get that $J = \sum_{i\le n} Rj_iR$ for a suitable n, i.e. J is a finitely generated ideal. □ □

Lemma 2.2. *Let J be an ideal generated by central elements. Then the following conditions are equivalent:*

(1) *J is 2-dually slender.*
(2) *J is dually slender as a right ideal.*
(3) *J is dually slender as a left ideal.*

Proof. An easy exercise. □ □

In the sequel, we investigate basic properties of 2-dually slender ideals of rings belonging to the class $\mathcal{R}$. It is well known that every finitely generated ideal of a regular ring is generated by one (idempotent) element. Therefore we focus our attention on infinitely generated 2-dually slender ideals.

Lemma 2.3. *Let $R \in \mathcal{R}$ and let J be an infinitely generated 2-dually slender ideal. Then there exists a natural number n such that $\pi_n(J)$ is an infinitely generated 2-dually slender ideal.*

Proof. The proof works similarly as the proof of Proposition 1.5. If $\pi_n(J)$ was finitely generated for every $n < \omega$, J would be countably generated, hence by Lemma 2.1 (2) J would be finitely generated which contradicts the hypothesis. □
□

An ideal J is said to be *homogeneous of index n* provided that for every $I \in \mathcal{M}(R)$ the ideal $\pi_I(J)$ is either isomorphic to a matrix ring $M_n(K)$ for a suitable skew-field K or it is equal to zero. We say that J is *homogeneous* if there exists a natural number n such that J is homogeneous of index n.

Lemma 2.4. *Let A and B be ideals of a regular ring R such that $B \subseteq A$ and let n be a natural number. Then A is a homogeneous ideal of index n if and only if both B and A/B are homogeneous ideals of index n.*

Proof. Note that $(A/B)/(A/B)I = (A/B)/(AI+B/B) \cong A/(AI+B)$ and $B/BI = B/(B \cap AI) \cong (B + AI)/AI$ for every $I \in \mathcal{M}(R)$. From this immediately follows the direct implication. Since $(B + AI)/AI$ is an ideal of R/AI contained in A/AI and AI is a maximal ideal in A, the reverse implication holds true. □ □

The following lemma generalizes the idea of the proof of [6, Theorem 7.14].

Lemma 2.5. *Let $R \in \mathcal{R}$, J a homogeneous ideal and $x \in J$. Then RxR is generated by a central idempotent.*

Proof. Fix an element $x \in J$ and put $n = i(J)$. Note that RxR is homogeneous by Lemma 2.4.

Let us first suppose that there exist elements $e, y, r_i, s_i, g_{ij} \in R$ for $i, j \leq n$ such that the following system of equations holds true:

$$e = \sum_{i \leq n} r_i x s_i, \ ey = x, \ ee = e, \ \sum_{i \leq n} g_{ii} = e, \ g_{ij} g_{jl} = g_{il}, \ g_{jj} g_{kk} = 0$$

whenever $j \neq k$. Obviously, e is an idempotent and $ReR = RxR$. We need to prove that e is a central element.

Fix an $I \in \mathcal{M}(R)$ and suppose $\pi_I(e) \neq 0$. Then there exists an i such that $\pi_I(g_{ii}) \neq 0$. Moreover $\pi_I(g_{jj}) \neq 0$ as well, since $g_{ii} = g_{ij}g_{jj}g_{ji}$, for every j. As $\{\pi_I(g_{ii}) | \ i \leq n\}$ forms an orthogonal set of non-zero idempotents, $\pi_I(e)$ is an idempotent matrix of rank n. Hence it is the identity matrix, i.e. a central idempotent. The element 0 is a central idempotent as well, so all $\pi_I(e)$, $I \in \mathcal{M}(R)$, are central idempotents generating $\pi_I(RxR)$.

Let us consider the ring homomorfism $\pi : R \to \prod_{I \in \mathcal{M}(R)} R/I$ defined by the formula $\pi(r) = (\pi_I(r) | I \in \mathcal{M}(R))$. As $Ker\ \pi = 0$ by Lemma 1.2, e is central if $\pi(e)$ is central, which is true since $\pi_I(e)$ is central for every $I \in \mathcal{M}(R)$. We have proved that e is a central idempotent generating the ideal RxR.

It remains to solve the system of equations. According to [6, Lemma 6.9] it suffices to solve the equations in all primitive factors. The element $\pi_I(x)$ is either equal to zero or $\pi_I(x)$ is a non-zero element of a ring of matrices of the size precisely equal to n over a skew-field, $I \in \mathcal{M}(R)$. Since the system of the equations is easily solvable in both the cases we get the required central idempotent. □ □

Lemma 2.6. *Let n be a natural number and let J be an ideal of a ring $R \in \mathcal{R}$. Suppose that $\pi_j(J) = 0$ whenever $j \neq n$. Then J is homogeneous of index n.*

Proof. Take an arbitrary maximal ideal $I \in \mathcal{M}(R)$. Suppose that $\pi_I(J)$ is non-zero, i.e. $\pi_I(J) = R/I$ and put $k = i(R/I)$. Note that as the module $\pi_I(J)$ is a non-zero subfactor of $\pi_k(J)$, $\pi_k(J) \neq 0$. By the premise we get that $k = n$, so J is homogeneous of index n. □ □

Proposition 2.7. *Let $R \in \mathcal{R}$ and let J be an infinitely generated 2-dually slender ideal. Then there exists a factor ring S of R (denote the natural projection $R \to S$ by p) such that $p(J)$ is an infinitely generated 2-dually slender homogeneous ideal.*

Proof. By Lemma 2.3 there exists a natural number n such that $\pi_n(J)$ is an infinitely generated ideal. Take n minimal. Then $i(\pi_n(R)) = n$ by Lemma 1.4 (2). Now, $\pi_j(J)$ is a finitely (in fact one-) generated ideal for every $j < n$, hence there exists an element $y \in J$ such that $\pi_j(J) = \pi_j(RyR)$ for each $j < n$. Thus the ring $S = \pi_n(R/RyR)$ contains the infinitely generated ideal $K = \pi_n(J/RyR)$ which is 2-dually slender by Lemma 2.1 (1). Moreover, $p_{jS}(S) = 0$ and so $p_{jS}(K) = 0$ if $j > n$ because $i(S) \leq n$. Finally, $p_{jS}(K) = 0$ if $j < n$, since $\pi_j(J) = \pi_j(RyR)$. Applying Lemma 2.6 we obtain the required result. □ □

Theorem 2.8. *Let R be a regular semiartinian ring with primitive factors artinian such that at least one factor of R contains an infinitely generated 2-dually slender ideal. Then R is neither right nor left steady. Moreover, there exists a factor of R containing an infinitely generated dually slender right (left) ideal.*

Proof. By Proposition 2.7 there exists a factor ring S of R containing an infinitely generated 2-dually slender ideal J which is generated by central idempotents. From

Lemma 2.2 it follows that J is infinitely generated and dually slender as both a right and a left ideal. □ □

3. Steadiness

Let us first recall and extend [8, Definition 2.7]. Let $\mathcal{L} = (S_\alpha \mid \alpha \leq \sigma + 1)$ be the right Loewy chain of R and let $M \in ModR$. Let $\mathcal{P} = \{P_{\alpha\beta} \mid \alpha \leq \sigma, \beta < \lambda_\alpha\}$ be a representative set of the isomorphism classes of all simple modules.

(1) For each $\alpha \leq \sigma$, $MS_{\alpha+1}/MS_\alpha$ is called the *α-th slice* of M.
(2) An ordinal α is said to be the *height* of M (denote it by $h_R(M)$) if it is the minimal $\alpha \leq \sigma + 1$ such that $MS_\alpha = M$.
(3) M is said to be *saturated* provided that $P_{\alpha\beta}$ is a subfactor of M for all $\alpha < h_R(M)$ and all $\beta < \lambda_\alpha$. Let J be an ideal. Then M is said to be *J-saturated* if $P_{\alpha\beta}$ is a subfactor of M for those $\alpha < h_R(M)$ and $\beta < \lambda_\alpha$ for which $P_{\alpha\beta}$ is a subfactor of J.

Suppose $(J_\alpha \mid \alpha \leq \rho)$ is the right Loewy chain of an ideal J. Note that by regularity $J_\alpha = J \cap S_\alpha = JS_\alpha$, i.e. the α-th slice of J is isomorphic to $J_{\alpha+1}/J_\alpha$. Moreover, the α-slice of any module MJ is isomorphic to $MJ_{\alpha+1}/MJ_\alpha$.

Lemma 3.1. *Let $R \in \mathcal{R}$ be of uncountable socle length. Let $\mathcal{I}$ be the class of all infinitely generated dually slender modules. Assume $\mathcal{I} \neq \emptyset$ and take $M \in \mathcal{I}$ such that the ordinal $h_{\bar{R}}(M)$ is minimal, where $\bar{R} = R/Ann_R(M)$. Then $h_{\bar{R}}(M)$ is a limit ordinal of cofinality $\geq \omega_1$.*

Proof. See [8, Lemma 3.1] □ □

Lemma 3.2. *Let $R \in \mathcal{R}$, M, a dually slender module and J, an ideal such that $M = MJ$. Suppose that M is J-saturated. Then J is 2-dually slender.*

Proof. Let $J = \bigcup_{n<\omega} I_n$ for an increasing chain of ideals I_n. We will follow the proof of [8, Lemma 3.3] where we replace S_ρ by J.

As $M = MJ = \bigcup_{n<\omega} MI_n$ is dually slender, there exists an $m < \omega$ such that $MI_m = M$. By [8, Proposition 2.6], if $I_m \neq J$ then there are $\alpha < h_R(M)$ and $\beta < \lambda_\alpha$ such that $I_m \subseteq Ann(P_{\alpha\beta})$ and $P_{\alpha\beta}$ is a subfactor of J. Moreover, $P_{\alpha\beta}$ is not a subfactor of M by [8, Lemma 2.8(i)] which contradicts the assumption of M being J-saturated. Hence $I_m = J$. □ □

Lemma 3.3. *Suppose that $R \in \mathcal{R}$ is not right steady. Then there exist a factor S of the ring R, an infinitely generated dually slender S-module M and a homogeneous ideal $J \subseteq S$ such that $MJ = M$.*

Proof. Fix an infinitely generated dually slender module M such that the ordinal $\sigma = h_{\bar{R}}(M)$ is minimal, where $\bar{R} = R/Ann_R(M)$. By Proposition 1.5 we may suppose that R is of a bounded index and $\bar{R} = R$. Denote by L the σ-th member of the Loewy chain of R. Then $ML = M$. Take a minimal n such that $p_{nM}(M)$ is an infinitely generated module. We may suppose that $p_{kM}(M) = 0$ for every $k < n$. Indeed, since $p_{kM}(M)$ is finitely generated for every $k < n$, there exists a finitely generated module F such that $p_{kM}(M) = p_{kM}(F)$, hence $p_{kM/F}(M/F) = 0$ for every $k < n$. As $p_{nM}(M)$ is a $\pi_n(R)$-module we may replace the module M by $p_{nM}(M)$ and the ring R by $\pi_n(R)$. Clearly, $i(R) = n$. Put $J = L \cap (\bigcap_{k<n} Ker\ \pi_k)$

Note that $\pi_k(R) = 0$ for every $k \geq n$ and $\pi_k(J) = 0$ for every $k < n$. Thus the ideal J is homogeneous of index n by Lemma 2.6. Moreover, $Soc(L/J)$ contains no homogeneous component of index n by Lemma 2.4. Hence $i(Soc(L/J)) < n$. Applying [6, Corollary 7.5] we get $i(L/J) = i(Soc(L/J)) < n$. Put $\bar{M} = M/MJ$ and $\bar{L} = L/J$. As $i(\bar{L}) < n$, $p_{nR/J}(\bar{L}) = 0$. Thus $p_{n\bar{M}}(\bar{M}) = p_{n\bar{M}}(\bar{M}\bar{L}) = p_{n\bar{M}}(\bar{M})\pi_{nR/J}(\bar{L}) = 0$. Moreover, $p_{k\bar{M}}(\bar{M}\bar{L}) = 0$ for each $k < n$, hence $\bar{M} = 0$ and $M = MJ$. Recall that J is homogeneous which finishes the proof. □ □

Lemma 3.4. *Assume $R \in \mathcal{R}$ is not right steady. Then there exist a factor ring S of R, an infinitely generated ideal $J \subseteq S$ and a J-saturated infinitely generated dually slender S-module M such that $MJ = M$.*

Proof. By Lemma 3.3 there exists an infinitely generated dually slender S-module M and a homogeneous ideal $J \subseteq S$ such that $MJ = M$ where S is a suitable factor-ring of R. Moreover, we may suppose that $Ann(M) = 0$ and that the height of M (and so, the Loewy length of J) is minimal. Denote it by ρ. Let $(J_\alpha | \ \alpha \leq \rho)$ be the right Loewy chain of J. Since J is homogeneous, every central idempotent generating a homogeneous component modulo J_α can be lifted to a central idempotent by Lemma 2.5. Now, following the proof of [8, Lemma 3.2] we show by induction on $\alpha < \rho$ that a simple module P embeds into the α-th slice of M if P embeds into J/J_α.

Put $A = Ann(Soc(M))$. As $M \in ModR$ is faithful, $MA \cap Soc(M) = 0$, so $A = 0$. Hence $Soc(M)$ contains a copy of each simple module $P \subseteq J_0 = Soc(J)$ and the assertion is true for $\alpha = 0$.

Take $0 < \alpha < \rho$ and assume that a simple submodule P of the α-th slice of J does not embed into the α-th slice of M. We have proved that there is a central idempotent $e \in R$ such that $e + J_\alpha$ generates homogeneous component corresponding to P.

If α is non-limit, we apply [8, Proposition 2.6] to get that $M' = (M/MJ_{\alpha-1})e$ is a dually slender submodule of the $(\alpha-1)$-th slice of M. As M' is finitely generated, there is an idempotent $f \in J_\alpha$ such that $M'f = M'$. Thus $Me(1-f)R \subseteq MJ_{\alpha-1}$. By the induction premise for $\alpha-1$ and by [8, Proposition 2.6], we get $e(1-f) \in J_{\alpha-1}$, whence $e \in J_\alpha$, a contradiction.

If α is a limit ordinal, we have $Me \subseteq MJ_\alpha$, whence $Me = MeJ_\alpha$. Since Me is dually slender, the minimality assumption implies that Me is finitely generated, so $Me \subseteq MJ_\gamma$ for some $\gamma < \alpha$. By the induction premise for γ and by [8, Proposition 2.6], we get $e \in J_\gamma$, a contradiction. □ □

Theorem 3.5. *Let $R \in \mathcal{R}$. Then the following conditions are equivalent:*

(i) *R is not right steady;*

(ii) *R is not left steady;*

(iii) *There exists an infinitely generated (as a two-sided ideal) 2-dually slender ideal of a suitable factor-ring of R.*

(iv) *There exists an infinitely generated dually slender right ideal of a suitable factor-ring of R.*

(v) *There exists an infinitely generated dually slender left ideal of a suitable factor-ring of R.*

Proof. The condition (iii) is left-right symmetric, therefore it suffices to prove the equivalence of conditions (i), (iii) and (iv).

(i)→ (iii) It follows from Lemmas 3.1, 3.2 and 3.4.

(iii)→ (iv) It is proved in Theorem 2.8.

(iv)→ (i) Trivial. □ □

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Chapter 4

The defect functor of homomorphisms and direct unions

This chapter is constituted by the single article devoted to commuting properties of the defect functor which generalizes the notions of Hom and Ext functors:

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E. THE DEFECT FUNCTOR OF A HOMOMORPHISM AND DIRECT UNIONS

SIMION BREAZ AND JAN ŽEMLIČKA

ABSTRACT. We will study commuting properties of the defect functor $\mathrm{Def}_\beta = \mathrm{Coker}\,\mathrm{Hom}_{\mathcal{C}}(\beta, -)$ associate to a homomorphism β in a finitely presented category. As an application, we characterize objects M such that $\mathrm{Ext}^1_{\mathcal{C}}(M, -)$ commutes with direct unions (i.e. direct limits of monomorphisms), assuming that $\mathcal{C}$ has a generator which is a direct sum of finitely presented projective objects.

1. Introduction

Commuting properties of some canonical functors defined on some categories play important roles in the study of various mathematical objects. For instance, finitely presented objects in a category with directed colimits are defined by the condition that the induced covariant Hom-functor commutes with all directed colimits. In the case of module categories the equivalence between the property used in this definition and the classical notion of finitely presented module was proved by Lenzing in [26]. In that paper it is also proved that there are strong connections between commuting properties of covariant Hom-functors and commuting properties of tensor product functors with respect to direct products. These connections were extended to the associated derived functors in [9] and [12]. Moreover, Drinfeld proposed in [15] to use flat Mittag-Leffler modules in order to construct a theory for infinite dimensional vector bundles. Recent progresses in this directions were obtained in [8], [17] and [18]. Auslander introduced in [6] the class of coherent functors, and W. Crawley-Boevey characterized (in the case of module categories) these functors as those covariant functors which commute with direct limits and direct products, [14, Lemma 1]. This result was extended to locally finitely presented categories by H. Krause, [24, Chapter 9]. The influence of these functors is presented in [14] and [20].

Brown [12] and Strebel [32] used commuting properties of covariant $\mathrm{Ext}^1_{\mathcal{C}}$-functors with respect to direct limits in order to characterize groups of type (FP). In module theory an important ingredient used in the study of tilting classes (e.g. [19, Lemma 5.2.18 and Theorem 5.2.20]) is a homological characterization, [19, Theorem 4.5.6],

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of the closure $\varinjlim \mathcal{C}$, where $\mathcal{C}$ is a class of FP_2-modules. This is based on the fact that $\mathrm{Ext}^1_R(M,-)$ commutes with direct limits whenever M is an FP_2-module, [19, Lemma 3.1.6]. In the case of Abelian groups, commuting properties of Ext^1 functors with respect to particular direct limits were also studied in [4] and [31].

In this paper we will focus on commuting properties with respect to direct limits for the defect functor associated to a homomorphism in a locally finitely presented abelian category. Let us introduce basic notions which we will use in the sequel. Let M be an object in an additive category $\mathcal{C}$ with directed colimits, and $G : \mathcal{C} \to Ab$ a covariant functor. Furthermore suppose that $\mathfrak{F} = (M_i, v_{ij})_{i,j\in I}$ is a directed system of objects in $\mathcal{C}$ such that there exists $\varinjlim M_i$ and let $v_i : M_i \to \varinjlim M_i$ be the canonical homomorphisms. Then $(G(M_i), G(v_{ij}))$ is also a direct system, and we denote by $\varinjlim G(M_i)$ its direct limit. Moreover, we have a canonical homomorphism

$$\Gamma_{\mathfrak{F}} : \varinjlim G(M_i) \to G(\varinjlim M_i)$$

induced by the homomorphisms $G(v_i) : G(M_i) \to G(\varinjlim M_i)$, $i \in I$.

We say that G *commutes with* $\mathfrak{F}$ if $\Gamma_{\mathfrak{F}}$ is an isomorphism. The functor G *commutes with direct limits* (*direct unions*, resp. *direct sums*) if the homomorphisms $\Gamma_{\mathfrak{F}}$ are isomorphisms for all directed systems $\mathfrak{F}$ (such that all v_{ij} are monomorphisms, resp. all direct sums).

Let $\mathcal{C}$ be an additive category with direct limits. We recall from [1] and [2] that an object M is *finitely presented* (*finitely generated*) respectively if and only if $\mathrm{Hom}_{\mathcal{C}}(M,-)$ commutes with direct limits (of monomorphisms), i.e. the canonical homomorphisms

$$\Psi^M_{\mathfrak{F}} : \varinjlim \mathrm{Hom}_{\mathcal{C}}(M, M_i) \to \mathrm{Hom}_{\mathcal{C}}(M, \varinjlim M_i)$$

are isomorphisms for all direct systems $\mathfrak{F} = (M_i, v_{ij})$ (such that all v_{ij} are monomorphisms). The category $\mathcal{C}$ is *finitely accessible* if $\mathcal{C}$ has directed colimits and every object is a direct limit of finitely presented objects. A cocomplete finitely accessible category $\mathcal{C}$ is a *locally finitely presented category*.

The notion of defect functor associated to a homomorphism extends the defect functor of an exact sequence used in [7]. This functor represents generalizations for the following canonical functors: the Hom-covariant functor induced by an object, the Pext-covariant functor induced by an object, respectively the Ext^1-covariant functor in the case when $\mathcal{C}$ is a functor category.

In Section 2 we introduce the defect functor $\mathrm{Def}_\beta : \mathcal{C} \to Ab$ associated to a homomorphism β, and we establish some basic properties for this functor. In Theorem 3 we show that the canonical decomposition of β induces a short exact sequence of defect functors. Since Def_β commutes with direct products, we can apply [14] and [25] to manage the case when Def_β commutes with all direct limits. Therefore we will focus our study to commuting properties with respect to particular direct limits.

In Section 3 we study when the natural homomorphism $\Phi^{\beta}_{\mathfrak{F}}:\varinjlim \mathrm{Def}_{\beta}(M_i)\to \mathrm{Def}_{\beta}(\varinjlim M_i)$, where $\mathfrak{F}=(M_i,v_{ij})_{i,j\in I}$ is a directed family in $\mathcal{C}$, is an epimorphism. It is proved that $\Phi^{\beta}_{\mathfrak{F}}$ is an epimorphism for all directed family $\mathfrak{F}$ (of monomorphisms) if and only if β is a section in the quotient category of $\mathcal{C}$ modulo the ideal of all homomorphisms which factorizes through a finitely presented (generated) object. We apply these results in Sections 4 and 5 in order to characterize the homomorphisms β such that Def_{β} commutes with direct unions or direct sums. Assuming that there is no ω-measurable cardinal we prove that it is enough to consider only commuting of Def_{β} with countable direct sums (Proposition 26).

In Section 6 (this section includes the results proved in the unpublished manuscript [11]) we apply the previous results to characterize objects M in a functor category with the property that the functor $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with direct unions (Theorem 44). These are exactly the direct summands in direct sums of projective objects and finitely presented objects. In [15, Section 6] the author used these objects (called, *2-almost projective modules*) in order to study various kind of objects, e.g. differentially nice k-schemes are defined using 2-almost projective modules. These objects are also studied in [21] for the case of quasivarieties, cf. [21, Proposition 4.3].

For the case of coherent categories these are exactly those objects such that the induced $\mathrm{Ext}^1_{\mathcal{C}}$-covariant functor commutes with direct limits (Corollary 47). We mention that in fact the structure of these objects can be very complicated. For such an example we refer to [27, Lemma 4.3].

Furthermore, Theorem 34 gives a description of objects M for which $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with direct sums using some splitting properties of projective presentations of M. We close the paper with a discussion about steadiness relative to Ext^1, i.e. the condition when commuting of $\mathrm{Ext}^1(M,-)$ with direct sums implies commuting of $\mathrm{Ext}^1(M,-)$ with direct unions.

In this paper $\mathcal{C}$ will denote an *locally finitely presented abelian category*, i.e. $\mathcal{C}$ is a Grothendieck category with a generating set of finitely presented objects. Therefore, an object is finitely generated iff it is an epimorphic image of a finitely presented object [1, Proposition 1.69], and the structural homomorphisms associated to direct unions are monomorphisms by [1, Proposition 1.62].

2. The defect functor associated to a homomorphism

In order to define the defect functor Def_{β} associated to a homomorphism β it is useful to consider, as in [24], the big category $(\mathcal{C},Ab)$ of all additive covariant functors from $\mathcal{C}$ into the category of all abelian groups. Albeit $(\mathcal{C},Ab)$ is not a category we can construct pointwise all notions which define abelian categories (kernels, cokernels, direct sums etc.), and the universal properties associated to these notions can be transfered from Ab to $(\mathcal{C},Ab)$. For instance, if $\eta:F\to G$ is a natural transformation then we can define a functor $\mathrm{Coker}(\eta)$ and a natural transformation $\mu : G \to \mathrm{Coker}(\eta)$ in the following way: For all $X \in \mathcal{C}$ we define $\mathrm{Coker}(\eta)(X) = \mathrm{Coker}(\eta_X) = G(X)/\mathrm{Im}(\eta_X)$, and for every $\alpha : X \to Y$ we define $\mathrm{Coker}(\eta)(\alpha) : \mathrm{Coker}(\eta)(X) \to \mathrm{Coker}(\eta)(Y)$ is the unique map which make the diagram

$$\begin{array}{ccccccc} F(X) & \xrightarrow{\eta_X} & G(X) & \xrightarrow{\mu_X} & G(X)/\mathrm{Im}(\eta_X) & \longrightarrow & 0 \\ \downarrow{\scriptstyle F(\alpha)} & & \downarrow{\scriptstyle G(\alpha)} & & \downarrow{\scriptstyle \mathrm{Coker}(\eta)(\alpha)} & & \\ F(Y) & \xrightarrow{\eta_Y} & G(Y) & \xrightarrow{\mu_Y} & G(Y)/\mathrm{Im}(\eta_Y) & \longrightarrow & 0 \end{array}$$

commutative, where $\mu_X : G(X) \to \mathrm{Coker}(\eta)(X)$ and $\mu_Y : G(Y) \to \mathrm{Coker}(\eta)(Y)$ are the canonical epimorphisms. It is not hard to see that $\mathrm{Coker}(\eta)$ is a functor, and the collection μ_X define a natural transformation $G \to \mathrm{Coker}(\eta)$ which has the same universal property as those which defines the classical cokernel in additive categories.

Definition 1. Suppose that $\beta : L \to P$ is a homomorphism in $\mathcal{C}$. Then β induces a natural transformation $\mathrm{Hom}(\beta, -) : \mathrm{Hom}(P, -) \to \mathrm{Hom}(L, -)$. The functor

$$\mathrm{Def}_\beta(-) = \mathrm{Coker}(\mathrm{Hom}(\beta, -))$$

will be called *the defect functor associated to* β.

It is clear from the previous observation that Def_β is characterized by the conditions:

(i) $\mathrm{Def}_\beta(X) = \mathrm{Hom}(L, X)/\mathrm{Im}(\mathrm{Hom}(\beta, X))$ for each object X and
(ii) $\mathrm{Def}_\beta(\gamma)(\alpha + B_X) = \gamma\alpha + B_Y$ for each objects X, Y and homomorphisms $\gamma \in \mathrm{Hom}(X, Y)$, $\alpha \in \mathrm{Hom}(K, X)$ where $B_X = \mathrm{Im}(\mathrm{Hom}(\beta, X))$ and $B_Y = \mathrm{Im}(\mathrm{Hom}(\beta, Y))$.

In fact, if $f : X \to Y$ is a homomorphism then we have a commutative diagram:

$$\begin{array}{ccccccc} \mathrm{Hom}(P, X) & \longrightarrow & \mathrm{Hom}(L, X) & \longrightarrow & \mathrm{Def}_\beta(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathrm{Hom}(P, Y) & \longrightarrow & \mathrm{Hom}(L, Y) & \longrightarrow & \mathrm{Def}_\beta(Y) & \longrightarrow & 0. \end{array}$$

Here are some examples:

Example 2. Let $\beta : L \to P$ be a homomorphism in $\mathcal{C}$.

(1) If $\mathcal{C}$ is abelian, P is projective and β a monomorphism, then $\mathrm{Def}_\beta(-)$ is canonically equivalent to $\mathrm{Ext}^1(P/\beta(L), -)$.
(2) If $P = 0$, then $\mathrm{Def}_\beta(-)$ is canonically equivalent to $\mathrm{Hom}(L, -)$.
(3) If β is an epimorphism and $\upsilon : K \to L$ is the kernel of β then $\mathrm{Def}_\beta(-)$ represents the covariant defect functor associated to the exact sequence $0 \to K \overset{\upsilon}{\to} L \overset{\beta}{\to} P \to 0$, [23].
(4) If R is a unital ring, $\mathcal{C} = \text{Mod-}R$, and L and P are finitely generated and projective then $\mathrm{Def}_\beta(R)$ represents the transpose of $P/\beta(L)$.

In the following we will prove some general properties of defect functors. Since in the category of all abelian groups the direct products are exact, it is easy to see that Def_β commutes with direct products. Moreover, in many situations the study of these functors can be reduced to the study of defect functors associated to monomorphisms or to epimorphisms.

Theorem 3. *Let $\beta : L \to P$ be a homomorphism in the abelian category $\mathcal{C}$. If $i_K : K \to L$ is the kernel of β, $\pi_K : L \to L/K$ is the canonical epimorphism, and $\overline{\beta} : L/K \to P$ is the homomorphism induced by β then there exists a canonical exact sequence of functors and natural transformations*

$$0 \to \mathrm{Def}_{\overline{\beta}} \to \mathrm{Def}_\beta \to \mathrm{Def}_{\pi_K} \to 0.$$

Proof. Starting with the exact sequence

$$0 \to K \stackrel{\iota_K}{\to} L \stackrel{\beta}{\to} P \to M \to 0,$$

where M is the cokernel of β, we obtain the short exact sequences

$$0 \to K \stackrel{\iota_K}{\to} L \stackrel{\pi_K}{\to} L/K \to 0$$

and

$$0 \to L/K \stackrel{\overline{\beta}_K}{\to} L \to M \to 0.$$

Passing to the Hom covariant functors induced by the objects involved in the previous exact sequences we obtain, using the Ker-Coker Lemma, the following commutative diagram of functors and natural transformations:

$$\begin{array}{ccccccccccc}
 & & & & & & 0 & & 0 & & \\
 & & & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M,-) & \longrightarrow & (P,-) & \xrightarrow{\overline{\beta}^*} & (L/K,-) & \longrightarrow & \mathrm{Def}_{\overline{\beta}}(-) & \longrightarrow & 0 \\
 & & \| & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (M,-) & \longrightarrow & (P,-) & \xrightarrow{\beta^*} & (L,-) & \longrightarrow & \mathrm{Def}_{\beta}(-) & \longrightarrow & 0 \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & \mathrm{Def}_{\pi_K} & = & \mathrm{Def}_{\pi_K} & & \\
 & & & & & & \downarrow & & \downarrow & & \\
 & & & & & & 0 & & 0 & & ,
\end{array}$$

hence the statement of the theorem is proved. □

Proposition 4. *If $\beta : L \to P$ is a homomorphism in $\mathcal{C}$, the following statements are true:*

(1) *If P is projective and $M = \mathrm{Coker}(\beta)$ then every exact sequence*

$$0 \to X \to Y \to Z \to 0$$

induces an exact sequence

$$0 \to (M,X) \to (M,Y) \to (M,Z) \to \mathrm{Def}_\beta(X) \to \mathrm{Def}_\beta(Y) \to \mathrm{Def}_\beta(Z).$$

(2) *If L is projective then Def_β preserves the epimorphisms.*
(3) *If L and P are projective then Def_β is a right exact functor.*

Proof. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence. Applying the Hom-functors we obtain the following commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (M,X) & \longrightarrow & (M,Y) & \longrightarrow & (M,Z) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (P,X) & \longrightarrow & (P,Y) & \longrightarrow & (P,Z) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & (L,X) & \longrightarrow & (L,Y) & \longrightarrow & (L,Z) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \mathrm{Def}_\beta(X) & \longrightarrow & \mathrm{Def}_\beta(Y) & \longrightarrow & \mathrm{Def}_\beta(Z) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array},$$

and the statements are obvious. □

Remark 5. Recently the defect functor associated to a homomorphism between projective object was involved in the study of silting modules, [5]: a homomorphism $\beta : L \to P$ with L and P projective objects is a *silting module* if $\mathrm{Gen}(P/\beta(L)) = \mathrm{Ker}(\mathrm{Def}_\beta)$.

3. The defect functor and direct limits

Throughout the section we suppose that $L \overset{\beta}{\to} P \overset{\alpha}{\to} M \to 0$ is an exact sequence in $\mathcal{C}$, $\mathfrak{F} = (M_i, v_{ij})_{i,j\in I}$ is a direct system of objects in $\mathcal{C}$, and $v_i : M_i \to \varinjlim M_i$ are the canonical homomorphisms. Furthermore, we denote by

$$\Phi^\beta_{\mathfrak{F}} : \varinjlim \mathrm{Def}_\beta(M_i) \to \mathrm{Def}_\beta(\varinjlim M_i)$$

the natural homomorphisms induced by the families $\mathrm{Def}_\beta(v_{ij})$, $i, j \in I$, and $\mathrm{Def}_\beta(v_i)$, $i \in I$. Following the general definition considered in Section 1, we say that $\mathrm{Def}_\beta(-)$ *commutes with* $\mathfrak{F}$ if $\Phi^\beta_{\mathfrak{F}}$ is an isomorphism. The functor $\mathrm{Def}_\beta(-)$ *commutes with direct limits* (*direct unions*, resp. *direct sums*) if the homomorphisms $\Phi^\beta_{\mathfrak{F}}$ are isomorphisms for all directed systems $\mathfrak{F}$ (such that all v_i are monomorphisms, resp. all direct sums).

We have the following useful commutative diagram

(D1)

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \varinjlim(M,M_i) & \longrightarrow & \varinjlim(P,M_i) & \longrightarrow & \varinjlim(L,M_i) & \xrightarrow{\varinjlim \xi_i} & \varinjlim \mathrm{Def}_\beta(M_i) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Psi^M_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Psi^P_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Psi^L_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Phi^\beta_{\mathfrak{F}}} & & \\ 0 & \longrightarrow & (M,\varinjlim M_i) & \longrightarrow & (P,\varinjlim M_i) & \xrightarrow{\beta^*} & (L,\varinjlim M_i) & \xrightarrow{\xi} & \mathrm{Def}_\beta(\varinjlim M_i) & \longrightarrow & 0 \end{array}$$

whose rows are exact, where the natural homomorphisms $\Psi^X_{\mathfrak{F}}$ are defined in Section 1.

Using this diagram we have the following simple consequences:

Corollary 6. (1) *If L is finitely presented and $\beta : L \to P$ is a homomorphism, then for every direct family $\mathfrak{F}$ the canonical homomorphism $\Phi^\beta_{\mathfrak{F}}$ is an epimorphism.*

(2) *If L is finitely generated and $\beta : L \to P$ is a homomorphism, then for every direct family of monomorphisms $\mathfrak{F}$ the canonical homomorphism $\Phi^\beta_{\mathfrak{F}}$ is an epimorphism.*

Example 7. There exists a homomorphism $\beta : L \to P$ and a direct family $\mathfrak{F}$ (of monomorphisms) such that L is finitely presented (generated) and the canonical homomorphism $\Phi^\beta_{\mathfrak{F}}$ is not an isomorphism.

Proof. Let $\mathcal{C}$ be the category of all abelian groups. If p is a prime number we denote by $\mathbb{Z}_p = \{\frac{m}{p^k} \mid m \in \mathbb{Z},\ k \in \mathbb{N}\} \leq \mathbb{Q}$. If $\beta : \mathbb{Z} \to \mathbb{Z}_p$ is the canonical inclusion in the category of all abelian groups then for every torsion-free abelian group A we have a natural isomorphism

$$\mathrm{Def}_\beta(A) \cong A/D_p(A),$$

where $D_p(A)$ is the maximal p-divisible subgroup of A.

We can write the abelian group $\mathbb{Q}$ as a union of a chain of cyclic subgroups $F_n = \frac{1}{n!}\mathbb{Z}$, $n \in \mathbb{N}^*$, where the connecting homomorphisms $u_{m,n} : F_m \to F_n$, $m < n$, are the inclusion maps. Since $\mathrm{Hom}(\mathbb{Z}_p, F_n) = 0$ for all $n > 0$, it follows that we can identify $\mathrm{Def}_\beta(F_n) = F_n$ and $\mathrm{Def}_\beta(u_{m,n}) = u_{m,n}$ for all $m, n \in \mathbb{N}^*$. Then $\varinjlim \mathrm{Def}_\beta(F_n) = \mathbb{Q}$. But $\mathrm{Def}_\beta(\varinjlim F_n) = \mathrm{Def}_\beta(\mathbb{Q}) = 0$, hence $\Phi^\beta_{\mathfrak{F}} : \mathbb{Q} \to 0$ is not a monomorphism. □

We will use the following lemma:

Lemma 8. *An object M is finitely generated if and only if there exists an exact sequence $0 \to L \to P \to M \to 0$ with P a finitely presented object.*

Consequently, if M is finitely generated then for every direct system $\mathfrak{F}$ the natural homomorphism $\Psi^M_{\mathfrak{F}}$ is a monomorphism. Moreover, M is finitely presented if and only if L is finitely generated.

Proof. The first part is proved in [1, Proposition 1.69], while for the other statements we can apply Ker-Coker Lemma on diagram (D1). □

Applying the above definitions to the diagram (D1), it is not hard approach that case when $\mathrm{Hom}(P,-)$ commutes with direct sums, direct unions, respectively direct limits. We recall that P is called *small* if $\mathrm{Hom}(P,-)$ commutes with direct sums.

Proposition 9. *Let $\beta : L \to P$ be a homomorphism.*

1. *Suppose that P is a small object. The functor Def_β commutes with direct sums if and only if L is a small object.*
2. *Suppose that P is a finitely generated object. The functor Def_β commutes with direct unions if and only if L is finitely generated.*
3. *Suppose that P is a finitely presented object. Then Def_β commutes with direct limits if and only if L is finitely presented.*

Proof. (1) Let $\mathfrak{F} = (M_i)_{i\in I}$ be a family of objects in $\mathcal{C}$. We construct a diagram (D1) induced by the direct sum of $\mathfrak{F}$. Since the class of small objects is closed with respect to epimorphic images, $\Psi^M_{\mathfrak{F}}$ and $\Psi^P_{\mathfrak{F}}$ are isomorphisms. Therefore $\Phi^\beta_{\mathfrak{F}}$ is an isomorphism if and only if $\Psi^L_{\mathfrak{F}}$ is an isomorphism. The conclusion is now obvious.

(2) The proof follows the same steps as for (1), using this time a direct system $\mathfrak{F} = (M_i, \nu_{ij})_{i,j\in I}$ such that all ν_{ij} are monomorphisms, and the fact that the class of finitely generated objects is closed with respect to epimorphic images.

(3) Suppose that Def_β commutes with direct limits. By what we just proved L is finitely generated, hence M is finitely presented. Therefore, for every direct system $\mathfrak{F} = (M_i, \nu_{ij})_{i,j\in I}$ the homomorphisms $\Psi^M_{\mathfrak{F}}$ and $\Psi^P_{\mathfrak{F}}$ are isomorphisms. Therefore $\Psi^L_{\mathfrak{F}}$ is an isomorphism, hence L is finitely presented.

Conversely, the objects L, M, and P are finitely presented, hence the first three vertical maps in diagram (D1) are isomorphisms. Then $\Phi^\beta_{\mathfrak{F}}$ is also an isomorphism. □

Using the statement (2) in the above proposition we can reformulate the characterization presented in [14, Lemma 1] for the case of direct unions. Since the proof is *verbatim* to Crawley-Boevey's proof, it is omitted.

Theorem 10. *A functor $F : \mathcal{C} \to Ab$ commutes with respect direct products and direct unions if and only if it is naturally isomorphic to a defect functor Def_β associated to a homomorphism $\beta : L \to P$ with L and P finitely generated.*

Using the same techniques as in [10], it is not hard to see that when L and P are projective the three commuting properties considered in the Proposition 9 are equivalent.

Proposition 11. *Let $\beta : L \to P$ be a homomorphism between projective right R-modules. Then the following are equivalent:*

1. *Def_β commutes with direct limits;*

(2) Def_β *commutes with direct unions;*
(3) Def_β *commutes with direct sums;*
(4) Def_β *commutes with direct sums of copies of* R*;*

Under these conditions $\mathrm{Def}_\beta(R)$ *is a finitely presented left* R*-module.*

Proof. (4)$\Rightarrow$(1) From Proposition 4 and from the proof of Watts's theorem [34, Theorem 1], we obtain that $\mathrm{Def}_\beta(-)$ is naturally isomorphic to $- \otimes_R \mathrm{Def}_\beta(R)$. Therefore it preserves direct limits.

Moreover, if these equivalent conditions are satisfied the functor $- \otimes_R \mathrm{Def}_\beta(R)$ preserves direct products. This is true exactly if the left R-module $\mathrm{Def}_\beta(R)$ is finitely presented. $\square$

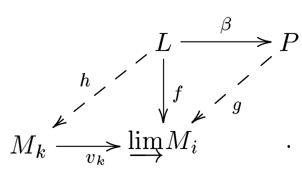
It is well known that if $G : \mathcal{C} \to Ab$ is an additive functor then for every family $\mathfrak{F} = (M_i)_{i\in I}$ the natural homomorphism $\oplus_{i\in I} G(M_i) \to G(\oplus_{i\in I} M_i)$ is a monomorphism. Therefore in the above proposition it is enough to verify whether the natural homomorphisms $\Phi^\beta_{\mathfrak{F}}$ are epimorphisms.

In the following we will study the case when the natural homomorphisms $\Phi^\beta_{\mathfrak{F}}$ are epimorphisms.

Lemma 12. *Let* $\beta : L \to P$ *be a homomorphism,* $\mathfrak{F} = (M_i, v_{ij})_{i,j\in I}$ *a direct system, and let* $f : L \to \varinjlim M_i$ *be a homomorphism. Using the same notations as in diagram* (D1)*, the following are equivalent:*

(1) $\xi(f) \in \mathrm{Im}(\Phi^\beta_{\mathfrak{F}})$*;*
(2) *there exists* $k \in I$*,* $h : L \to M_k$ *and* $g : P \to \varinjlim M_i$ *such that* $f = g\beta + v_k h$*.*

Proof. The homomorphisms from (2) can be represented in the following diagram



(1)$\Rightarrow$ (2) If we look at the commutative diagram (D1), we observe that $\xi(f) \in \mathrm{Im}(\Phi^\beta_{\mathfrak{F}})$ if and only if there is an element $x \in \varinjlim \mathrm{Hom}_{\mathcal{C}}(L, M_i)$ such that $\xi(f) = \Phi^\beta_{\mathfrak{F}}(\varinjlim \xi_i)(x) = \xi\Psi^L_{\mathfrak{F}}(x)$. Then $f - \Psi^L_{\mathfrak{F}}(x) = \beta^*(g) = g\beta$ for some element $g \in \mathrm{Hom}_{\mathcal{C}}(P, \varinjlim M_i)$.

Since $x \in \varinjlim \mathrm{Hom}_{\mathcal{C}}(L, M_i)$, there exist $k \in I$ and $h \in \mathrm{Hom}_{\mathcal{C}}(L, M_k)$ such that $x = \overline{v}_k(h) \in \mathrm{Im}\overline{v}_k$, where $\overline{v}_k : \mathrm{Hom}(L, M_k) \to \varinjlim \mathrm{Hom}_{\mathcal{C}}(L, M_i)$ is the structural homomorphism associated to the direct limit. Since $\Psi^\beta_{\mathfrak{F}}\overline{v}_k = \mathrm{Hom}_{\mathcal{C}}(L, v_k)$, it follows that $\Psi^\beta_{\mathfrak{F}}(x) = \mathrm{Hom}_{\mathcal{C}}(L, v_k)(h) = v_k h$. Thus $f = g\beta + v_k h$.

(2)$\Rightarrow$(1) If $f = g\beta + v_k h$ then $f - v_k h \in \mathrm{Im}(\beta^*)$, hence

$$\xi(f) = f + \mathrm{Im}(\beta^*) = v_k h + \mathrm{Im}(\beta^*) = \xi\Psi^L_{\mathfrak{F}}(\overline{v}_k(h)) = \Phi_{\mathfrak{F}}(\varinjlim \xi_i)(\overline{v}_k(h)),$$

and the proof is complete. $\square$

In the following $\mathcal{FP}$ will be the ideal in $\mathcal{C}$ consisting of those homomorphisms which factorize through a finitely presented object, i.e. $\mathcal{FP}$ represents the collection of subgroups $\mathcal{FP}(A,B) \leq \mathrm{Hom}_{\mathcal{C}}(A,B)$, $A,B \in \mathcal{C}$, of those homomorphisms $A \to B$ which factorize through a finitely presented object. Then $\mathcal{C}/\mathcal{FP}$ will denote the quotient category which has the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{FP}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(A,B)/\mathcal{FP}(A,B).$$

In the following assertion, if $f : A \to B$ and $h : A \to C$ are homomorphisms, we will denote by $(f,h)^t : A \to B \oplus C$ the canonical homomorphism induced by f and h.

Theorem 13. *Let $\beta : L \to P$ be a homomorphism. The following are equivalent:*

(1) *for every direct system $\mathfrak{F}$ the map $\Phi^{\beta}_{\mathfrak{F}}$ is an epimorphism;*
(2) *there exists $g : P \to L$ such that $1_L - g\beta$ factorizes through a finitely presented object;*
(3) *the induced homomorphism $\overline{\beta}$ in $\mathcal{C}/\mathcal{FP}$ is a section;*
(4) *there exists a homomorphism $h : L \to F$ such that F is a finitely presented object and the induced map $(\beta,h)^t : L \to P \oplus F$ is a splitting monomorphism.*

Proof. (1)⇒(2) We can write L as a direct limit of finitely presented objects, $L = \varinjlim L_i$. Then an application of Lemma 12 for $f = 1_L$ gives us the conclusion.

(2)⇒(1) Since $1_L - g\beta$ factorizes through a finitely presented object, there exists a finitely presented object F and two homomorphisms $h_1 : L \to F$, $h_2 : F \to L$ such that $1 - g\beta = h_2 h_1$.

Since F is finitely presented, for every direct limit $\varinjlim M_i$ and every homomorphism $f : L \to \varinjlim M_i$ we can find an index i and a homomorphism $f_i : F \to M_i$ such that $fh_2 = v_i f_i$. It follows that $f(1 - g\beta) = fh_2h_1 = v_i f_i h_1$. Then there exists $g' = fg : P \to \varinjlim M_i$ and $h = f_i h_1 : L \to M_i$ such that $f = g'\beta + v_i h$, and we apply Lemma 12 to complete the proof.

(2)⇔(3) This is obvious.

(2)⇒(4) Let g be as in (2) and $h = 1_L - g\beta$. There exists a finitely presented object F and two maps $h_1 : L \to F$, $h_2 : F \to L$ such that $h = h_2h_1$. Then the map $(\beta, h_1)^t : L \to P \oplus F$ induced by β and h_1 is a splitting monomorphism, and a left inverse is the homomorphism $(g, h_2) : P \oplus F \to L$ induced by g and h_2.

(4)⇒(2) Let $g' : P \oplus F \to L$ be a left inverse for $(\beta,h)^t$. Then $1_L = g_{|P}\beta + g_{|F}h$, hence $1_L - g_{|P}\beta$ factorizes through a finitely presented object. □

4. Commuting with direct unions

Recall from [1, Proposition 1.62] that in our hypotheses the structural maps $v_i : M_i \to \varinjlim M_i$ of a direct union are monomorphisms.

Since the class of finitely generated objects is closed with respect to epimorphic images, we will prove that Theorem 13 can be improved to characterize the commuting of Def_β with direct unions.

Theorem 14. *Let $\beta : L \to P$ be a homomorphism. The following are equivalent:*

(1) *for every direct system $\mathfrak{F}$ of monomorphisms the induced homomorphism $\Phi^\beta_\mathfrak{F}$ is an epimorphism;*
(2) *there exists $g : P \to L$ such that $1_L - g\beta$ factorizes through a finitely generated object;*
(3) *if $\mathcal{FG}$ is the ideal of all homomorphisms which factorize through a finitely generated object then the induced homomorphism $\overline{\beta}$ in $\mathcal{C}/\mathcal{FG}$ is a section;*
(4) *there exists a homomorphism $h : L \to M$ such that h factorizes through a finitely generated object and the induced map $(\beta, h)^t : L \to P \oplus M$ is a splitting monomorphism.*
(5) *there exists a homomorphism $h : L \to F$ such that F is a finitely generated object and the induced map $(\beta, h)^t : L \to P \oplus F$ is a splitting monomorphism,*
(6) *there exists a finitely generated subobject $H \le L$ such that the induced homomorphism $\overline{\beta} : L/H \to P/\beta(H)$ is a split mono and there exists a left inverse for $\overline{\beta}$ which can be lifted to a homomorphism $P \to L$.*

Proof. It is enough to prove the equivalence (2)⇔(6) since for the other equivalences we can repeat the arguments of the proof of Theorem 13, using the fact that L can be written as a direct union of its finitely generated subobjects.

(2)⇒(6) By (2) we know that there exists a homomorphism $g : P \to L$ such that $1_L - g\beta$ factorizes through a finitely generated object. Therefore there exists a finitely generated subobject $H \le L$ such that $\mathrm{Im}(1_L - g\beta) \subseteq H$. If $h : H \to L$ is the embedding of H in L then there exists a homomorphism $\gamma : L \to H$ such that $1_L - g\beta = h\gamma$.

Since $h = g\beta h + h\gamma h$, we have $\mathrm{Im}(g\beta h) \le \mathrm{Im}(h)$, hence $g\beta(H) \le H$. Therefore there are canonical homomorphisms $\overline{\beta} : L/H \to P/\beta(H)$ and $\overline{g} : P/\beta(H) \to L/H$ which are induced by β, respectively g, and the diagram

$$\begin{array}{ccccc} L & \xrightarrow{\beta} & P & \xrightarrow{g} & L \\ \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi_{\beta(H)}} & & \downarrow{\scriptstyle \pi_H} \\ L/H & \xrightarrow{\overline{\beta}} & P/\beta(H) & \xrightarrow{\overline{g}} & L/H \end{array}$$

is commutative, where the vertical arrows are the canonical epimorphisms.

Moreover, $\pi_H h = 0$, hence $\overline{g}\overline{\beta}\pi_H = \pi_H g\beta = \pi_H(1_L - h\gamma) = \pi_H$. Since π_H is an epimorphism we have $\overline{g}\overline{\beta} = 1_{K/K_k}$, hence $\overline{\beta}$ is a splitting monomorphism.

(6)⇒(2) Let $g : P \to L$ be a homomorphism such that $g(\beta(H)) \subseteq H$ and the induced homomorphism $\overline{g} : P/\beta(H) \to L/H$ satisfies the equality $\overline{g}\overline{\beta} = 1_{L/H}$. Then $\mathrm{Im}(1_L - g\beta) \subseteq H$, and the proof is complete. □

If P is projective the lifting condition stated in (6) is always satisfied. This is not the case if P is not projective.

Example 15. Let $\mathcal{C}$ be the category of all abelian groups, and let $\mathbb{Z}_p$ be the subgroup of $\mathbb{Q}$ defined in Example 7.

If $\beta : \mathbb{Z}_p \to \mathbb{Q}$ is the inclusion map then the induced homomorphism $\overline{\beta} : \mathbb{Z}_p/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ is split mono. But $\mathrm{Hom}(\mathbb{Q}, \mathbb{Z}_p) = 0$, so the left inverse of $\overline{\beta}$ (in this case this left inverse is unique) cannot be lifted to a homomorphism $\mathbb{Q} \to \mathbb{Z}_p$.

We obtain the following interesting characterization of pure-projective objects. Let us recall that an exact sequence $0 \to A \to B \to C \to 0$ is pure (and $B \to C$ is a pure epimorphism) if all finitely presented objects are projective with respect to it, and an object is projective if and only if it is projective with respect to all pure exact sequences. It is not hard to see that an object is pure-projective iff it is a direct summand of a direct sum of finitely presented objects. As in the standard homological algebra we can define the functor $\mathrm{Pext}^1(M, -)$ as Def_β, where $\beta : P \to M$ is a pure epimorphism such that P is pure-projective. Remark that M is pure-projective iff $\mathrm{Pext}^1(M, -) = 0$. For more details we refer to [22, Appendix A].

Proposition 16. *The following are equivalent for an object $M \in \mathcal{C}$:*

1. *The functor* $\mathrm{Pext}^1(M, -)$ *commutes with direct limits;*
2. $\mathrm{Pext}^1(M, -)$ *commutes with direct unions;*
3. M *is pure projective.*

Proof. It is enough to prove that (2)⇒(3).

Let M be an object such that $\mathrm{Pext}^1(M, -)$ commutes with direct unions. Since M is a direct limit of finitely presented objects, there exists a pure exact sequence

$$0 \to L \stackrel{\beta}{\to} \oplus_{i\in I} P_i \to M \to 0$$

such that all P_i are finitely presented objects. Hence $\mathrm{Pext}^1(M, -) = \mathrm{Def}_\beta$, and we apply Theorem 14. Therefore there exists a finitely generated subobject $K \leq L$ such that the induced homomorphism $\overline{\beta} : L/K \to \oplus_{i\in I} P_i/\beta(K)$ is a splitting monomorphism. But $\mathrm{Coker}(\overline{\beta}) \cong M$, hence M is isomorphic to a direct summand of $\oplus_{i\in I} P_i/\beta(K)$. Since $\beta(K)$ is finitely generated we can view $\beta(K)$ as a subobject of a subsum $\oplus_{i\in J} P_i/\beta(K)$, were J is a finite subset of I. Since $\oplus_{i\in J} P_i$ is finitely presented, it follows that $\oplus_{i\in J} P_i/\beta(K)$ is also finitely presented, hence M is a direct summand of a direct sum of finitely presented objects. Then M is pure-projective. □

The next observation allows us to prove that, in order to study the commuting properties with respect to direct unions, it is enough to restrict to defect functors associated to the homomorphisms which appear in the canonical decomposition of β.

Proposition 17. *Let $\beta : L \to P$ be an epimorphism. Then for every direct system $\mathfrak{F} = (M_i, v_{ij})_{i,j\in I}$ of monomorphisms, the canonical map $\Phi^{\beta}_{\mathfrak{F}}$ is a monomorphism.*

Proof. Let $x \in \mathrm{Ker}(\Phi^{\beta}_{\mathfrak{F}})$. Using the notations from diagram (D1), there exists $y \in \varinjlim \mathrm{Hom}(L, M_i)$ such that $x = \varinjlim \xi_i(y)$ and $\Psi^{L}_{\mathfrak{F}}(y)$ factorizes through β. There exists $i \in I$ and $\alpha_i : L \to M_i$ such that $y = \overline{v}_i(\alpha_i)$, where $\overline{v}_i$ denotes the canonical map $\overline{v}_i : \mathrm{Hom}(L, M_i) \to \varinjlim \mathrm{Hom}(L, M_i)$. Then $\Psi^{\beta}_{\mathfrak{F}}\overline{v}_i(\alpha_i) = v_i\alpha_i$ factorizes through β. Let $\gamma : P \to \varinjlim M_i$ be a homomorphism such that $v_i\alpha_i = \gamma\beta$.

Let $K = \mathrm{Ker}(\beta)$ and $\iota_K : K \to L$ be the canonical homomorphism. Then $v_i\alpha_i\iota_K = \gamma\beta\iota_K = 0$. Since the structural homomorphisms v_i are monomorphisms we obtain $\alpha_i\iota_K = 0$, hence α_i factorizes through β. Then $x = 0$, and the proof is complete. □

Corollary 18. *Let $\beta : L \to P$ be an epimorphism. The following are equivalent:*

(1) *the functor Def_β commutes with direct unions;*
(2) *for every direct family $\mathfrak{F}$ of monomorphisms the induced homomorphism $\Phi^{\beta}_{\mathfrak{F}}$ is an epimorphism;*

Using Theorem 3 and Proposition 17 we have the following result:

Corollary 19. *Suppose that $\beta : L \to P$ is a homomorphism in the abelian category $\mathcal{C}$, $i_K : K \to L$ is the kernel of β, $\pi_K : L \to L/K$ is the canonical epimorphism, and $\overline{\beta} : L/K \to P$ is the homomorphism induced by β. Then $\Phi^{\beta}_{\mathfrak{F}}$ is an isomorphism (epimorphism) for a direct family of monomorphisms $\mathfrak{F}$ if and only if $\Phi^{\overline{\beta}}_{\mathfrak{F}}$ and $\Phi^{\pi_K}_{\mathfrak{F}}$ are isomorphisms (epimorphisms).*

Proof. In order to prove the equivalence, let us remark, using the fact that direct limits are exact in $\mathcal{C}$, that for every direct family $\mathfrak{F}$ we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varinjlim \mathrm{Def}_{\overline{\beta}}(M_i) & \longrightarrow & \varinjlim \mathrm{Def}_{\beta}(M_i) & \longrightarrow & \varinjlim \mathrm{Def}_{\pi_K}(M_i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Phi^{\overline{\beta}}_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Phi^{\beta}_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Phi^{\pi_K}_{\mathfrak{F}}} & & \\
0 & \longrightarrow & \mathrm{Def}_{\overline{\beta}}(\varinjlim M_i) & \longrightarrow & \mathrm{Def}_{\beta}(\varinjlim M_i) & \longrightarrow & \mathrm{Def}_{\pi_K}(\varinjlim M_i) & \longrightarrow & 0,
\end{array}$$

and $\Phi^{\pi_K}_{\mathfrak{F}}$ is monic by Proposition 17. Now the equivalence stated in this corollary is obvious. □

The condition $\Phi^{\pi_K}_{\mathfrak{F}}$ is an epimorphism can be replaced by a factorization condition:

Lemma 20. *Let $\beta : L \to P$ be a homomorphism and $\mathfrak{F} = (M_i, v_{ij})_{i,j\in I}$ a direct system. Consider the following statements:*

(1) *$\Phi^{\beta}_{\mathfrak{F}}$ is an epimorphism;*
(2) (a) *if $\iota_K : K \to L$ is the kernel of β, then for every homomorphism $f : L \to \varinjlim M_i$ there exists $i \in I$ and $h : L \to M_i$ such that $f\iota_K = v_ih\iota_K$ (i.e. the restriction of f to factorizes through the canonical map v_i),*

(b) *if* $\overline{\beta} : L/K \to P$ *is induced by* β *then* $\Phi_{\mathfrak{F}}^{\overline{\beta}}$ *is an epimorphism.*

Then (2) $\Rightarrow$ (1)*. If all* v_i *are monomorphisms we have* (1) $\Leftrightarrow$ (2)*.*

Proof. (2) $\Rightarrow$ (1). Let $f : L \to \varinjlim M_i$ be a homomorphism. Using (a) we can find $i \in I$ and $h : L \to M_i$ such that $f\iota_K = v_i h\iota_K$. Then $(f - v_i h)\iota_K = 0$, hence there exists $\overline{\delta} : L/K \to \varinjlim M_i$ such that $\overline{\delta}\pi = f - v_i h$.

Using (b) and Lemma 12, we can find $j \in I$, $g : P \to \varinjlim M_i$ and $\gamma : L/K \to M_j$ such that $\overline{\delta} = g\overline{\beta} + v_j\gamma$. We can suppose $i = j$. Then $f - v_i h = g\overline{\beta}\pi + v_i\gamma\pi$, hence $f = g\beta + v_i(\gamma\pi + h)$. Another application of Lemma 12 completes the proof.

(1) $\Rightarrow$ (2) Let $f : L \to \varinjlim M_i$ be a homomorphism. Using Lemma 12 there exist $h : L \to M_i$ and $g : P \to \varinjlim M_i$ which are homomorphisms such that $f = g\beta + v_i h$. Then $f\iota_K = v_i h\iota_K$, hence (a) is valid.

The condition (b) follows from Corollary 19. □

Remark 21. In fact the condition (b) in the above lemma can be proved directly. In order to do this, let us consider a homomorphism $\overline{f} : L/K \to \varinjlim M_i$. If $\pi : L \to L/K$ is the canonical projection then we can find $i \in I$ and two homomorphisms $h : L \to M_i$, $g : P \to \varinjlim M_i$ such that $\overline{f}\pi = g\beta + v_i h$. Then $v_i h(K) = 0$. Since v_i is a monomorphism, we have $h(K) = 0$. It follows that there exists $\overline{h} : L/K \to M_i$ such that $h = \overline{h}\pi$. Then $(g\overline{\beta} + v_i\overline{h})\pi = \overline{f}\pi$, hence $g\overline{\beta} + v_i\overline{h} = \overline{f}$, and the proof is complete.

In fact the case when $\Phi_{\mathfrak{F}}^{\beta}$ is an epimorphism for all direct systems of monomorphisms can be characterized in the following way:

Theorem 22. *Let* $\beta : L \to P$ *be a homomorphism in* $\mathcal{C}$*. The following are equivalent:*

(1) *for every direct system* $\mathfrak{F} = (M_i, v_{ij})_{i,j\in I}$ *of monomorphisms* $\Phi_{\mathfrak{F}}^{\beta}$ *is an epimorphism;*
(2) (a) *if* $\iota_K : K \to L$ *is the kernel of* β*, then* K *can be embedded in a finitely generated subobject* $H \leq L$*, and*
    (b) *if* $\overline{\beta} : L/K \to P$ *is induced by* β *then* $\Phi_{\mathfrak{F}}^{\overline{\beta}}$ *is an epimorphism for all direct systems of monomorphisms* $\mathfrak{F} = (M_i, v_{ij})_{i,j\in I}$*.*

Proof. (1)$\Rightarrow$(2) We apply Lemma 20 to obtain (b). For (a), we apply Theorem 14 to find a homomorphism $g : P \to L$ such that $\mathrm{Im}(1_L - g\beta)$ can be embedded in a finitely generated subobject H of L. Then K can be also embedded in H.

(2)$\Rightarrow$(1) It is enough to prove that for every direct system of monomorphisms and for every $f : L \to \varinjlim M_i$ there exists $i \in I$ and $h : L \to M_i$ such that $f\iota_K = v_i h\iota_K$.

Let $f : L \to \varinjlim M_i$ be a homomorphism. By (a) there exists a factorization $\iota_K = \iota_H\iota_{KH}$. Since H is finitely generated there exists an index $i \in I$ such that $i_H f$ factorized through v_i. Therefore there exists $h : L \to M_i$ such that $f\iota_H = v_i h$, hence $f\iota_k = v_i h\iota_K$. □

In the end of this section we come back to the general case, in order to characterize the functor Def_β associated to a monomorphism $\beta : L \to P$ for the case when we can find a subobject $H \leq L$ such that the induced homomorphism $\overline{\beta} : L/H \to P/\beta(H)$ is split mono.

Proposition 23. *Let $\beta : L \to P$ be a monomorphism and H a subobject of L. If $\overline{\beta} : L/H \to P/\beta(H)$ is the homomorphism induced by β then we have an exact sequence of functors*

$$(P/\beta(H), -) \xrightarrow{\overline{\beta}^*} (L/H, -) \to \mathrm{Def}_\beta \to \mathrm{Def}_{\iota_{\beta(H)}} \to \mathrm{Def}_{\iota_H} \to 0,$$

and the following are equivalent:

(1) *the induced homomorphism $\overline{\beta} : L/H \to P/\beta(H)$ is a splitting monomorphism;*
(2) *the induced sequence of functors*

$$0 \to \mathrm{Def}_\beta \to \mathrm{Def}_{\iota_{\beta(H)}} \to \mathrm{Def}_{\iota_H} \to 0$$

is exact.

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & H & \xrightarrow{\beta'} & \beta(H) & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle \iota_H} & & \downarrow{\scriptstyle \iota_{\beta(H)}} & & & & \\
0 & \longrightarrow & L & \xrightarrow{\beta} & P & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_H} & & \downarrow{\scriptstyle \pi_{\beta(H)}} & & \| & & \\
0 & \longrightarrow & L/H & \xrightarrow{\overline{\beta}} & P/\beta(H) & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

with exact sequences, which induces a the solid part of the following commutative diagram of functors and natural transformations

$$
\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \to & (M,-) & \to & (P/\beta(H),-) & \xrightarrow{\overline{\beta}^*} & (L/H,-) & \dashrightarrow & \mathrm{Def}_\beta(-) & & \\
 & & \| & & \downarrow{\scriptstyle \pi^*_{\beta(H)}} & & \downarrow{\scriptstyle \pi^*_H} & & \| & & \\
0 & \to & (M,-) & \to & (P,-) & \xrightarrow{\beta^*} & (L,-) & \xrightarrow{\xi_\beta} & \mathrm{Def}_\beta(-) & \to & 0 \\
 & & & & \downarrow{\scriptstyle \iota^*_{\beta(H)}} & & \downarrow{\scriptstyle \iota^*_H} & & \downarrow{\scriptstyle 0} & & \\
 & & 0 & \to & (\beta(H),-) & \xrightarrow{\beta'^*} & (H,-) & \to & 0 & & \\
 & & & & \downarrow{\scriptstyle \xi_{\iota_{\beta(H)}}} & & \downarrow{\scriptstyle \xi_{\iota_H}} & & \| & & \\
 & & & & \mathrm{Def}_{\iota_{\beta(H)}}(-) & \to & \mathrm{Def}_{\iota_H}(-) & \to & 0 & & \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}
$$

in which all lines and columns are exact sequences. Applying the snake lemma we obtain the natural transformation $(L/H,-) \dashrightarrow \mathrm{Def}_\beta(-)$ such that the sequence

$$(P/\beta(H),-) \xrightarrow{\overline{\beta}^*} (L/H,-) \dashrightarrow \mathrm{Def}_\beta \to \mathrm{Def}_{\iota_{\beta(H)}} \to \mathrm{Def}_{\iota_H} \to 0$$

is exact.

Now the equivalence (1)⇔(2) is obvious since $\overline{\beta}$ is split mono iff the natural homomorphisms $\mathrm{Hom}_{\mathcal{C}}(\overline{\beta}, X)$ are epimorphisms for all $X \in \mathcal{C}$.

□

5. Commuting with direct sums

Let $\mathfrak{F} = (M_i)_{i\in I}$ be a family of objects in $\mathcal{C}$ and $\nu_i : M_i \to \bigoplus_i M_i$ the canonical monomorphisms. Recall that

$$\Phi^\beta_{\mathfrak{F}} : \bigoplus_i \mathrm{Def}_\beta(M_i) \to \mathrm{Def}_\beta(\bigoplus_i M_i)$$

denotes the natural homomorphisms induced by the family $\mathrm{Def}_\beta(\nu_i)$, $i \in I$. It is easy to see that $\Phi^M_{\mathfrak{F}}$ is a monomorphism, since $(\Psi^M_{\mathfrak{F}})^{-1}(\mathrm{Im}(\mathrm{Hom}(\beta, \bigoplus_i M_i) = \bigoplus_i \mathrm{Im}(\mathrm{Hom}(\beta, M_i))$. Moreover, Def_β commutes with finite direct sums (it is additive).

For every family of objects $(M_i, i \in I)$ and for $J \subset I$ denote by π_J the canonical projection $\bigoplus_{i\in I} M_i \to \bigoplus_{i\in J} M_i$.

Theorem 24. *If $\beta : L \to P$ is a homomorphism and $(M_i, i \in I)$ a family of objects, the following are equivalent:*

(1) Def_β *commutes with direct sum of* $(M_i, i \in I)$.

(2) *For every homomorphism* $f : L \to \bigoplus_{i\in I} M_i$ *there exist a finite subset* $F \subset I$, *and a homomorphism* $g : P \to \bigoplus_{i\in I\setminus F} M_i$ *such that* $\pi_{I\setminus F} f = g\beta$.

Proof. Since $\bigoplus_{i\in I} M_i$ is a direct limit of the system $(\bigoplus_{i\in F} M_i, F \in I^{<\omega})$ with canonical inclusions, $\Phi^{\beta}_{\mathfrak{F}}$ is a monomorphism. Now it remains to apply Lemma 12. □

If I is a set, $X \subseteq I$, and M_i, $i \in I$, is a family of objects, we denote by

$$\Pi^I_X : \mathrm{Def}_\beta(\bigoplus_{i\in I} M_i) \to \mathrm{Def}_\beta(\bigoplus_{i\in X} M_i)$$

the canonical epimorphism which is induced by the canonical map $\bigoplus_{i\in I} M_i \to \bigoplus_{i\in X} M_i$. Note that Π^I_X is a splitting epimorphism of abelian groups.

Using a standard set-theoretical argument under assumption $(V = L)$ we prove that commuting of the functor Def_β with countable direct sums is equivalent to commuting with arbitrary direct sums. First, we make an easy observation:

Lemma 25. *Let* M_i, $i \in I$, *be a family of objects. Then* Def_β *commutes with* $\bigoplus_{i\in I} M_i$ *if and only if for every* $\epsilon \in \mathrm{Def}_\beta(\bigoplus_{i\in I} M_i)$ *there is a finite subset* $F \subseteq I$ *such that* $\Pi^I_{I\setminus F}(\epsilon) = 0$.

A cardinal $\lambda = |I|$ is *ω-measurable* if it is uncountable and there exists a countably-additive, non-trivial, $\{0, 1\}$-valued measure μ on the power set of I such that $\mu(I) = 1$ and $\mu(\{x\}) = 0$ for all $x \in I$. We recall that if such a cardinal exists then there exists a smallest ω-measurable cardinal μ and all cardinals $\lambda \geq \mu$ are also ω-measurable.

Proposition 26. *Let* κ *be a cardinal less than the first* ω*-measurable cardinal. If* Def_β *commutes with countable direct sums then* Def_β *commutes with direct sums of* κ *objects.*

Proof. Let K_i, $i \in I$, be a family of modules such that I is of cardinality κ and $\epsilon \in \mathrm{Def}_\beta(\bigoplus_{i\in I} K_i)$ is a fixed extension. By Lemma 25 it is enough to prove that there is a finite subset $F \subseteq I$ such that $\Pi^I_{I\setminus F}(\epsilon) = 0$. Let consider the set

$$\mathcal{I}(I) = \{X \subseteq I \mid \text{there is a finite subset } F \subseteq X \text{ such that } \Pi^I_{X\setminus F}(\epsilon) = 0\}.$$

Suppose that $I \notin \mathcal{I}(I)$. We claim that there exists a subset $Y \subseteq I$ such that $\mathcal{I}(Y)$ is a non-principal ω_1-complete ideal. Let us observe that for every subsets $F \subseteq X \subseteq I$ we have $\Pi^I_{X\setminus F} = \Pi^X_{X\setminus F}\Pi^I_X$. Furthermore, it is not hard to see that $\mathcal{I}(I)$ contains $\emptyset$ and it is closed with respect to subsets and finite unions. In order to complete the proof of our claim it is remains to prove that if X_n, $n \in \mathbb{N}$, is a countable set of pairwise disjoint subsets of I then there exists $n_0 \in \mathbb{N}$ such that $\bigcup_{n\geq n_0} X_n \in \mathcal{I}(I)$.

Let X_n, $n \in \mathbb{N}$, be a family of pairwise disjoint subsets of I, and $X = \bigcup_{n\in\mathbb{N}} X_n$. Since Def_β commutes with countable direct sums, the canonical homomorphism

$$\bigoplus_{n\in\mathbb{N}} \mathrm{Def}_\beta(\bigoplus_{i\in X_n} K_i) \to \mathrm{Def}_\beta(\bigoplus_{n\in\mathbb{N}}(\bigoplus_{i\in X_n} K_i)) = \mathrm{Def}_\beta(\bigoplus_{i\in X} K_i)$$

is an isomorphism. Therefore there is a positive integer n_0 such that

$$\Pi^I_{\bigcup_{n\geq n_0} X_n}(\epsilon) = \Pi^X_{\bigcup_{n\geq n_0} X_n}(\Pi^I_X(\epsilon)) = 0,$$

hence $\bigcup_{n\geq n_0} X_n \in \mathcal{I}(I)$.

Now we claim that there exists $Y \notin \mathcal{I}(I)$ such that for every subset $Z \subset Y$ with $Z \notin \mathcal{I}(I)$ we have $Y \setminus Z \in \mathcal{I}(I)$.

Suppose by contradiction that such a Y does not exists. It follows that for every $Y \notin \mathcal{I}(I)$ there exists a nonempty subset $Z \subseteq Y$ such that $Z, Y \setminus Z \notin \mathcal{I}(I)$. Since $\emptyset \neq I \notin \mathcal{I}(I)$ we can find a partition $I = Z_0 \cup Y_1$ such that $Z_0, Y_1 \notin \mathcal{I}(I)$. Now $\emptyset \neq Y_1 \notin \mathcal{I}(I)$, hence there exists a partition $Y_1 = Z_1 \cup Y_2$ such that $Z_1, Y_2 \notin \mathcal{I}(I)$. We continue inductively this kind of choice: if Y_n is constructed then exists a partition $Y_n = Z_n \cup Y_{n+1}$ such that $Z_n, Y_{n+1} \notin \mathcal{I}(I)$. Therefore we obtain a countable sequence of sets $Z_n \notin \mathcal{I}(I)$, and it is not hard to see that these sets are pairwise disjoint. But, by what we proved so far, there exists n_0 such that $\bigcup_{n\geq n_0} Z_n \in \mathcal{I}(I)$, a contradiction.

Then there exists a subset $Y \subseteq I$ such that $Y \notin \mathcal{I}(I)$ and for every subset $Z \subset Y$ with $Z \notin \mathcal{I}(I)$ we have $Y \setminus Z \in \mathcal{I}(I)$. It is easy to see that we can define an ω-additive $\{0,1\}$-valuated map μ on the power-set of I via the rule $\mu(U) = 1$ if $U \cap Y \notin \mathcal{I}(I)$, and $\mu(U) = 0$ otherwise. It follows that I is ω-measurable, a contradiction. □

Corollary 27. *Assume* $(V = L)$. *If* $\mathrm{Def}_\beta(-)$ *commutes with countable direct sums then* $\mathrm{Def}_\beta(-)$ *commutes with all direct sums.*

It is well-known that $\mathrm{Hom}(M, -)$ commutes with countable direct sums iff it commutes with all direct sums. Furthermore, as a consequence of the previous result and Example 2 we obtain

Corollary 28. *Let* $(V = L)$ *and* $M \in \mathcal{C}$. *Then* $\mathrm{Ext}^1_{\mathcal{C}}(M, -)$ *commutes with countable direct sums if and only if* $\mathrm{Ext}^1_{\mathcal{C}}(M, -)$ *commutes with all direct sums.*

Remark 29. We don't know what it happens if we remove the set theoretic assumption $(V = L)$. For the case of abelian groups it can be proved, as in [4, Theorem 5.3] that if M is an abelian group such that $\mathrm{Ext}^1_{Ab}(M, -)$ commutes with countable direct sums then M is a Whitehead group. On the other side, the same result shows us that if $\mathrm{Ext}^1_{Ab}(M, -)$ commutes with all direct sums then M is free. The interested reader can find some similar phenomena in [3, Section 2].

The following results characterize when Def_β commutes with countable direct sums. It generalizes a classical characterization of small modules proved by Rentschler in [29].

Proposition 30. *Let* $\beta : L \to P$ *be a homomorphism. The functor* Def_β *commutes with countable direct sums if and only if for every countable chain of subobjects*

$$(D) : L_0 \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \dots$$

such that L is a direct union of (D) there exists n for which the induced map $\beta' : L/L_n \to P/\beta(L_n)$ is a splitting monomorphism.

Proof. ($\Rightarrow$) For each $i \in \mathbb{N}$ denote by $\iota_i : L_i \to L$ the canonical monomorphism and put $A_i = \mathrm{Coker}(\iota_i) \cong L/L_i$. Suppose that $\sigma : L \to \bigoplus_i A_i$ is the morphism defined by direct sum of the canonical epimorphisms $\rho_i : L \to A_i$, i.e. $\pi_i\sigma = \rho_i$, where $\pi_i : \bigoplus_i A_i \to A_i$ is the canonical projection. By the hypothesis, there exist a finite subset $F \subseteq \mathbb{N}$ and a homomorphism $g : P \to \bigoplus_{i\in\mathbb{N}\setminus F} A_i$ such that $\pi_{\mathbb{N}\setminus F}\sigma = g\beta$. Let $n \notin F$. If $\rho_n : L \to A_n$ represents the canonical epimorphism, we obtain $\rho_n = \pi_n g\beta$, hence $\pi_n g\beta(L_n) = 0$. Then $\pi_n g$ factorizes through the canonical epimorphism $\mu_n : P \to P/\beta(L_n)$, so $\pi_n g = g'\mu_n$ and $g' : P/\beta(L_n) \to A_n$.

We obtain $g'\beta'\rho_n = g'\mu_n\beta = \pi_n g\beta = \rho_n$, hence $g'\beta' = 1_{A_n}$, and the proof is complete.

($\Leftarrow$) Let $\sigma \in \mathrm{Hom}(L, \bigoplus_{i<\omega} A_i)$ and denote by $\pi_{\geq n}$ the canonical epimorphisms $\bigoplus_i A_i \to \bigoplus_{i\geq n} A_i$. Obviously, the family $L_n = \mathrm{Coker}(\pi_{\geq n}\alpha)$ with canonical monomorphisms forms an increasing chain such that L is its direct union.

By the hypothesis there exists n such that $\beta' : L/L_n \to P/\beta(L_n)$ has a left inverse $g' : P/\beta(L_n) \to L/L_n$.

If we put $F = \{1, \dots, n\}$ and $g : P \to \bigoplus_{i>n} A_i$, $g = g'\mu_n$, where $\mu_n : P \to P/\beta(L_n)$ is the canonical epimorphism, then we can apply Theorem 24 to obtain the conclusion. □

We will say that the homomorphism $\beta : L \to P$ is *κ-splitting small*, where κ is a cardinal, if for every system of objects $(A_i, i < \kappa)$ and for every homomorphism $\sigma : L \to \bigoplus_{i<\kappa} A_i$ there exists a finite subset $F \subset \kappa$ such that the cokernel homomorphism ρ in the pushout diagram

(D2)
$$\begin{array}{ccccccc}
L & \xrightarrow{\beta} & P & \longrightarrow & U & \longrightarrow & 0 \\
{\scriptstyle \pi_{\kappa\setminus F}\sigma}\downarrow & & \downarrow & & \| & & \\
\bigoplus_{i\in\kappa\setminus F} A_i & \longrightarrow & X & \xrightarrow{\rho} & U & \longrightarrow & 0
\end{array}$$

splits.

Now, we make an elementary observation.

Lemma 31. *Consider a commutative diagram with exact rows and columns*

(D3)
$$\begin{array}{ccccccc}
A_0 & \xrightarrow{\nu_0} & B_0 & \xrightarrow{\pi_0} & C & \longrightarrow & 0 \\
{\scriptstyle \alpha_0}\downarrow & & {\scriptstyle \beta_0}\downarrow & & \| & & \\
A_1 & \xrightarrow{\nu_1} & B_1 & \xrightarrow{\pi_1} & C & \longrightarrow & 0
\end{array} .$$

If $\overline{\pi_0} : B_0/\mathrm{Ker}\beta_0 \to C \to 0$ induced by (D3) *splits, then π_1 splits as well.*

Proof. Let $\overline{\beta}_0 : B_0/\mathrm{Ker}\beta_0 \to B_1$ the homomorphism induced by β_0. Since there exists $\rho : C \to B_0/\mathrm{Ker}\beta_0$ such that $\mathrm{id}_C = \overline{\pi_0}\rho = \pi_1\overline{\beta_0}\rho$, the homomorphism π_1 splits. □

Proposition 32. *The homomorphism β is ω-splitting small if and only if* Def_β *commutes with countable direct sums.*

Proof. ($\Rightarrow$) We consider a countable chain of subobjects

$$(D) : L_0 \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \dots$$

such that L is a direct union of (D).

For each i denote by $\iota_i : L_i \to L$ the canonical monomorphism and put $A_i = \mathrm{Coker}(\iota_i) \cong L/L_i$. Suppose that $\sigma : L \to \bigoplus_i A_i$ is the homomorphism defined by direct sum of the canonical projections. By the hypothesis, there exists a finite subset F such that the homomorphism ρ in the pushout diagram (D2) splits. Let $n \notin F$. Clearly, the homomorphism $\overline{\rho}$ in the pushout diagram

(D4)
$$\begin{array}{ccccccc} \bigoplus_{i\in\kappa\setminus F} A_i & \longrightarrow & X & \xrightarrow{\rho} & U & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \| & & \\ A_n & \longrightarrow & Y & \xrightarrow{\overline{\rho}} & U & \longrightarrow & 0 \end{array}$$

splits. Since π_n is an epimorphism and the composition of the pushout diagrams (D2) and (D4) is so, the diagram

$$\begin{array}{ccccccc} L & \xrightarrow{\beta} & P & \longrightarrow & U & \longrightarrow & 0 \\ {\scriptstyle\pi_n}\downarrow & & {\scriptstyle\tau}\downarrow & & \| & & \\ A_n & \longrightarrow & Y & \xrightarrow{\overline{\rho}} & U & \longrightarrow & 0 \end{array}$$

commutes and τ is an epimorphism. As $\overline{\rho}$ splits, it remains to observe that $\mathrm{Ker}\tau = \beta(L_n)$.

($\Leftarrow$) Let $\sigma \in \mathrm{Hom}(L, \bigoplus_{i<\omega} A_i)$ and denote by $\pi_{\geq n}$ the canonical epimorphisms $\bigoplus_i A_i \to \bigoplus_{i\geq n} A_i$. Obviously, the family $L_n = \mathrm{Coker}(\pi_{\geq n}\alpha)$ with canonical monomorphisms forms an increasing chain such that L is its direct union.

By the hypothesis there exists n such that $\beta(L)/\beta(L_n)$ is a direct summand of $P/\beta(L_n)$, hence it remains to put $F = \{1, \dots, n\}$ and to apply Lemma 31 on the pushout diagram (D2). □

Now we will apply the above results in order to see when $\mathrm{Ext}^1_{\mathcal{C}}(M, -)$ commutes with direct sums.

Lemma 33. *Let κ be a cardinal, and consider an exact sequence $L \xrightarrow{\beta} P \to M \to 0$. Then β is κ-splitting small if and only if* Def_β *commutes with direct sums of κ objects.*

Proof. ($\Rightarrow$) Let A_i, $i \leq \kappa$ be a family of objects. Let $\epsilon \in \mathrm{Def}_\beta(\bigoplus_{i<\kappa} A_i)$, and consider a homomorphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(L, \bigoplus_{i<\kappa} A_i)$ whose coset modulo the image of $\mathrm{Hom}(\beta, \bigoplus_{i<\kappa} A_i)$ is ϵ. We construct the pushout diagram

$$\text{(D5)}\qquad \begin{array}{ccccccc} L & \xrightarrow{\beta} & P & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\ \sigma\downarrow & & \rho\downarrow & & \| & & \\ \bigoplus_{i<\kappa} A_i & \xrightarrow{\nu} & X & \xrightarrow{\pi} & M & \longrightarrow & 0. \end{array}$$

As β is κ-splitting small, there exists a finite set $F \subset \kappa$ such that the second row of the pushout diagram

$$\text{(D6)}\qquad \begin{array}{ccccccc} L & \xrightarrow{\beta} & P & \xrightarrow{\alpha} & M & \longrightarrow & 0 \\ \pi_{\kappa\setminus F}\sigma\downarrow & & \downarrow & & \| & & \\ \bigoplus_{i\in\kappa\setminus F} A_i & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0 \end{array}.$$

splits. Thus $\Pi^{\kappa}_{\kappa\setminus F}(\epsilon) = 0$. Now the assertion follows from Lemma 25.

($\Leftarrow$) Fix $\sigma : L \to \bigoplus_{i<\kappa} A_i$. Then there exist an object X and homomorphisms ρ and ν such that (D5) forms a pushout diagram. Since Def_β commutes with the direct sums of family $(A_i, i < \kappa)$, Lemma 25 imply that there exists a finite subset $F \subset \kappa$ such that the second row of the pushout diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bigoplus_{i<\kappa} A_i & \xrightarrow{\nu} & X & \xrightarrow{\pi} & M & \longrightarrow & 0 \\ & & \pi_{\kappa\setminus F}\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \bigoplus_{i\in\kappa\setminus F} A_i & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

splits. Thus we get pushout diagram (D6) where the second row splits, so β is κ-splitting small. □

Theorem 34. *Let κ be a cardinal less than the first ω-measurable cardinal. The following are equivalent for a homomorphism $L \overset{\beta}{\to} P$ in $\mathcal{C}$:*

(1) *The functor Def_β commutes with direct sums of κ objects,*
(2) *β is κ-splitting small,*
(3) *β is ω-splitting small.*

Proof. (1) $\Leftrightarrow$ (2) is the assertion of Lemma 33, (2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (1) follows from Proposition 26 and Lemma 33. □

We can apply the last assertion to see when an Ext^1-covariant functor preserves direct sums.

Corollary 35. *Suppose that $\mathcal{C}$ has projective strong generator which is a direct sum of finitely presented objects. Let κ be a cardinal less than the first ω-measurable cardinal. The following are equivalent for a projective presentation $0 \to L \overset{\beta}{\to} P \to M \to 0$ of $M \in \mathcal{C}$:*

(1) *The functor $\mathrm{Ext}^1_{\mathcal{C}}(M, -)$ commutes with direct sums of κ objects,*
(2) *β is κ-splitting small,*
(3) *β is ω-splitting small.*

Corollary 36. *Let* $(V = L)$*, and suppose that* $\mathcal{C}$ *has projective strong generator which is a direct sum of finitely presented objects. If* $M \in \mathcal{C}$ *then* $\mathrm{Ext}^1(M,-)$ *commutes with all direct sums if and only if there exists a projective presentation* $0 \to L \xrightarrow{\beta} P \to M \to 0$ *for* M *such that* β *is* ω*-splitting small.*

In these conditions for all projective presentations $0 \to L' \xrightarrow{\beta'} P' \to M \to 0$ *and for all cardinals* κ *the homomorphism* β' *is* κ*-splitting small.*

We close this section with an application of Proposition 9 to the study of the covariant Ext^1-functor.

Lemma 37. *Let* $0 \to L \to P \to M \to 0$ *be an exact sequence such that* P *is projective. If* L *and* M *are small objects, then* M *and* P *are finitely generated and* $\mathrm{Ext}^1(M,-)$ *commutes with direct sums.*

Proof. Since the class of small objects is closed with respect to extensions, P is small, hence finitely generated. Note that a direct sum $\bigoplus_{i<\kappa} A_i$ is precisely direct union of the diagram $\mathfrak{F} = (\bigoplus_{i\in F} A_i, \nu_{FG} \mid F \subseteq G \in \kappa^{<\omega})$ where ν_{FG} are the canonical inclusions $\bigoplus_{i\in F} A_i \to \bigoplus_{i\in G} A_i$. As all homomorphisms $\Psi_{\mathfrak{F}}$ from the diagram (D1) are isomorphisms, $\Phi^M_{\mathfrak{F}}$ is isomorphism as well. □

Applying Proposition 9(1) we obtain the following

Corollary 38. *Suppose that* $\mathcal{C}$ *has projective strong generator which is a direct sum of finitely presented objects. Let* M *be a finitely generated object, and let* $0 \to L \xrightarrow{\beta} P \to M \to 0$ *be a projective presentation such that* P *is finitely generated. Then* $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ *commutes with direct sums if and only if for* L *is small.*

6. The covariant Ext^1-functor and direct unions

In this and the next sections $\mathcal{C}$ *will be an abelian category with a projective strong generator which is a direct sum of finitely presented objects.*

Let us fix an object M in $\mathcal{C}$. We will apply the previous results to study the commuting properties for the covariant functor $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$. In order to do this, we fix a projective presentation

$$0 \to L \xrightarrow{\beta} P \to M \to 0,$$

and we can apply the previous results for the functor Def_β.

In order to simplify our presentation we will say, as in [10], that the object M is an fg-Ω^1*-object* (respectively fp-Ω^1*-object*) if there is a projective resolution

$$(\mathbf{P}): \ \cdots \to P_2 \to P_1 \xrightarrow{\alpha_1} P_0 \to M \to 0$$

such that $\Omega^1(\mathbf{P}) = \mathrm{Im}(\alpha_1)$ is finitely generated (respectively, finitely presented), i.e. there is a projective resolution for M such that the first syzygy associated to this resolution is finitely generated (finitely presented). The object M is an FP_n-*object* if it has a projective resolution such that P_i are finitely presented for all $i \in \{0, \ldots, n\}$.

It is proved in [19, Lemma 3.1.6] that if M is an FP_2 object then $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with direct limits. Using Proposition 9 and Example 2(1) for the kernel of a projective presentation $P \to M \to 0$, it is easy to see that for the finitely generated objects this hypothesis is sharp. We recall that in our hypothesis every finitely generated projective object is finitely presented.

Corollary 39. *Let M be a finitely generated object.*

(1) $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ *commutes with direct unions if and only if M is finitely presented.*
(2) $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ *commutes with direct limits if and only if M is an FP_2-object.*

Inductively, using the dimension shifting formula we obtain a version, of [12, Theorem 2] and [32, Theorem A]:

Corollary 40. *The following are equivalent for an object M:*

(1) *M has a projective resolution*
$$(\mathbf{P}):\ P_n \to \cdots \to P_2 \to P_1 \stackrel{\alpha_1}{\to} P_0 \to M \to 0$$
such that P_i are finitely presented for all $0 \le i \le n$;
(2) *The functors $\mathrm{Ext}^i_{\mathcal{C}}(M,-)$ commute with direct unions for all $0 \le i \le n$.*

Remark 41. Corollary 39 can be reformulated in the following way: *the functor $\mathrm{Hom}_{\mathcal{C}}(M,-)$ commutes with direct limits if and only if the functors $\mathrm{Hom}_{\mathcal{C}}(M,-)$ and $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commute with direct unions.* The proof presented here uses the existence of the strong generator $\mathcal{U}$ which is a direct sum of finitely presented projective objects. It is an open question if this result is valid in more general settings, e.g. for general Grothendieck categories without enough projectives. For instance this equivalence is valid for the category of all Abelian p-groups (p is a fixed prime), which is a Grothendieck category without non-trivial projective objects, as a consequence of [31, Theorem 5.4].

In order to prove the main result of this section, we say that a covariant functor $F : \mathcal{C} \to Ab$ is *isomorphic to a direct summand* of a functor $G : \mathcal{C} \to Ab$ if we can find two natural transformations $\rho : F \to G$ and $\pi : G \to F$ such that $\pi\rho = 1_F$.

Lemma 42. *Let $F, G : \mathcal{C} \to Ab$ be additive covariant functors such that F is isomorphic to a direct summand of G. If $\mathfrak{F} = (M_i)_{i\in I}$ is a direct family such that the canonical homomorphism $\Phi_G : \varinjlim G(M_i) \to G(\varinjlim M_i)$ is monic (epic) then the canonical homomorphism $\Phi_F : \varinjlim F(M_i) \to F(\varinjlim M_i)$ is monic (epic).*

Proof. If $\rho : F \to G$ and $\pi : G \to F$ are natural transformations such that $\pi\rho = 1_F$, we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varinjlim F(M_i) & \xrightarrow{\varinjlim \rho_{M_i}} & \varinjlim G(M_i) & \xrightarrow{\varinjlim \pi_{M_i}} & \varinjlim F(M_i) & \longrightarrow & 0 \\
 & & \Phi_F \downarrow & & \Phi_G \downarrow & & \Phi_F \downarrow & & \\
0 & \longrightarrow & F(\varinjlim M_i) & \xrightarrow{\rho_{\varinjlim M_i}} & G(\varinjlim M_i) & \xrightarrow{\pi_{\varinjlim M_i}} & F(\varinjlim M_i) & \longrightarrow & 0
\end{array},$$

and the conclusion is now obvious. □

Corollary 43. *Let M be an object such that $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with the colimit of a direct system $\mathfrak{F}$. Then every direct summand N of M has the same property.*

Now we are ready to characterize when the covariant $\mathrm{Ext}^1_{\mathcal{C}}$ functors commute with direct unions. We recall that M is 2-*almost projective* if it is a direct summand of a direct sum $P \oplus F$ with P a projective object and F a finitely presented object, [15]. For reader's convenience we include a proof for the following characterization.

Theorem 44. *The following are equivalent for an object M in $\mathcal{C}$:*

(1) *The functor $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with direct unions;*
(2) *M is a direct summand of an fg-Ω^1-object.*
(3) *M is a 2-almost projective object.*

Proof. (1) ⇒ (2) We consider a projective resolution $0 \to L \overset{\beta}{\to} P \to M \to 0$. By Theorem 14 there exists a finitely generated object $H \leq L$ such that the induced homomorphism $\overline{\beta} : L/H \to P/\beta(H)$ is split mono. Since $\mathrm{Coker}(\overline{\beta}) \cong \mathrm{Coker}(\beta) \cong M$, it follows that M is isomorphic to a direct summand of the fg-Ω^1-object $P/\beta(H)$.

(2)⇒(3) It is enough to assume that M is an fg-Ω^1-object. If M is such an object then we can consider the diagram (D1) with P projective and L finitely generated. If U is an object such that $P \oplus U$ is a direct sum of copies of some objects from $\mathcal{U}$, we consider the induced exact sequence $0 \to L \overset{\beta}{\to} P \oplus U \overset{\alpha \oplus 1_U}{\longrightarrow} M \oplus U \to 0$. Let $P \oplus U = \oplus_{i \in I} P_i$, where all objects P_i are finitely presented and projective. Since L is finitely generated, there is a finite subset $J \subseteq I$ such that $\beta(L) \subseteq \oplus_{i \in J} P_i$. Therefore $M \oplus U \cong (\oplus_{i \in I \setminus J} P_i) \oplus (\oplus_{i \in J} P_i)/\beta(L)$ is a direct sum of a projective object and a finitely presented object.

(3)⇒(1) In view of Corollary 43, we can assume that M is finitely presented. Then we apply Corollary 39. □

For arbitrarily direct limits, we have not a general answer. However, for some particular cases, including coherent categories ($\mathcal{C}$ is *coherent* if every finitely generated subobject of a projective object is finitely presented), we can apply the previous result. In order to do this, let us state the following

Proposition 45. *Let M be an fg-Ω^1-object. The following are equivalent:*

(1) *M is an fp-Ω^1-object;*
(2) *$\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with direct limits.*

Proof. (1)⇒(2) Suppose that M is an fp-Ω^1-object. As in the proof for (2)⇒(3) in the previous theorem, we observe that there is a projective object L such that $M \oplus L$ is a direct sum of an FP_2-object and a projective object. Therefore we can suppose that M is FP_2. For this case the result is well known (see [19, Lemma 3.1.6]).

(2)⇒(1) Let $0 \to K \to P \to M \to 0$ be an exact sequence such that K is finitely generated. Using again the proof of (2) ⇒ (3) in the previous theorem, there is a projective object K such that $M \oplus K = N \oplus U$, $N = P'/K$, where P' is a finitely generated projective object and U is projective such that $P' \oplus U = P \oplus K$. Then $\mathrm{Ext}^1_{\mathcal{C}}(N,-) \cong \mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with direct limits. Since P' is finitely presented we can use Lemma 9, and we conclude that K is finitely presented. □

From this proposition and its proof we obtain some useful corollaries. First of them allows us to construct examples of objects M such that $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with respect to direct unions, but it does not commute with direct limits.

Corollary 46. *Suppose that M is an* fp-Ω^1*-object and $0 \to L \to P \to M \to 0$ is an exact sequence such that P is finitely generated projective. Then L is finitely presented.*

Consequently, if for every finitely presented object M the functor $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ *commutes with direct limits then $\mathcal{C}$ is coherent.*

Proof. We consider an exact sequence $0 \to L_1 \to P_1 \to M \to 0$ such that P_1 is finitely presented projective and L_1 is finitely presented. By Schanuel's lemma we have $L_1 \oplus P \cong L \oplus P_1$, and now the conclusion is obvious. □

In fact, for coherent categories (in particular for modules over coherent rings or for the category of modules over the category mod-R) the functor $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ commutes with direct limits if and only it it commutes with respect direct unions:

Corollary 47. *Suppose that M has a projective resolution* $(\mathbf{P})$ *such that* $\Omega^1(\mathbf{P})$ *is a direct union of finitely presented subobjects. The following are equivalent:*

(1) $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ *commutes with direct limits;*
(2) $\mathrm{Ext}^1_{\mathcal{C}}(M,-)$ *commutes with direct unions;*
(3) *M is a direct summand of an* fp-Ω^1*-object.*

Proof. In the proof of Theorem 44 we can choose $\mathfrak{F}$ such that all M_i are finitely presented. □

As each countably generated object is a direct union of a chain of finitely generated modules, Theorem 44, Example 2(1) and the previous result implies the following consequence:

Corollary 48. *Let $0 \to L \to P \to M \to 0$ be a short exact sequence with L countably generated. If* $\mathrm{Ext}^1(M,-)$ *commutes with direct sums then M is 2-almost projective, hence* $\mathrm{Ext}^1(M,-)$ *commutes with direct unions.*

Moreover, for coherent categories we obtain from Theorem 44 a generalization of [13, Theorem A]:

Theorem 49. *Suppose that $\mathcal{C}$ is a coherent category. The following are equivalent for an object M and a positive integer n:*

(1) $\mathrm{Ext}^n_{\mathcal{C}}(M,-)$ *commutes with direct limits (unions);*
(2) *if* $m \geq n$ *is an integer then then* $\mathrm{Ext}^m_{\mathcal{C}}(M,-)$ *commutes with direct limits (unions).*

Proof. (1)⇒(2) By dimension shifting formula we can assume $n = 1$, and it is enough to prove that $\mathrm{Ext}^2_{\mathcal{C}}(M,-)$ commutes with direct unions.

Let $0 \to L \xrightarrow{\beta} P \to M \to 0$ be an exact sequence such that P is projective. By Theorem 14(6) there exists a finitely generated subobject $H \leq L$ such that the induced homomorphism $\beta' : L/H \to P/\beta(H)$ is split mono. Using Theorem 44 we obtain that $\mathrm{Ext}^1_{\mathcal{C}}(L/H,-)$ commutes with direct limits. Moreover, we can view H as a finitely generated subobject of P, hence H is finitely presented. Therefore for every directed family $\mathfrak{F} = (M_{ij}, v_{ij})$, in the commutative diagram

$$\begin{array}{ccccccc}
\varinjlim(H, M_i) & \longrightarrow & \varinjlim \mathrm{Ext}^1_{\mathcal{C}}(L/H, M_i) & \longrightarrow & \varinjlim \mathrm{Ext}^1_{\mathcal{C}}(L, M_i) & \longrightarrow & \varinjlim \mathrm{Ext}^1_{\mathcal{C}}(H, M_i) \\
\downarrow{\scriptstyle \Psi^H_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Phi^{L/H}_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Phi^L_{\mathfrak{F}}} & & \downarrow{\scriptstyle \Phi^H_{\mathfrak{F}}} \\
(H, \varinjlim M_i) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}}(L/H, \varinjlim M_i) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}}(L, \varinjlim M_i) & \longrightarrow & \mathrm{Ext}^1_{\mathcal{C}}(H, \varinjlim M_i)
\end{array}$$

the homomorphisms $\Psi^H_{\mathfrak{F}}$, $\Phi^{L/H}_{\mathfrak{F}}$ and $\Phi^H_{\mathfrak{F}}$ are isomorphisms. Then $\Phi^L_{\mathfrak{F}}$ is also an isomorphism, and the proof is complete.

(2)⇒(1) is obvious. □

7. Ext-steady rings

Recall that $\mathcal{C}$ means again an abelian category with a projective strong generator which is a direct sum of finitely presented objects and denote by $\mathcal{U}$ the corresponding set of finitely presented objects. We say that the category $\mathcal{C}$ is *finite ext-steady* if for every finitely generated object M such that $\mathrm{Ext}^1(M,-)$ commutes with all direct sums it holds that $\mathrm{Ext}^1(M,-)$ commutes with all direct unions.

Proposition 50. *The following conditions are equivalent:*

(1) $\mathcal{C}$ *is finite ext-steady,*
(2) *every small subobject of every projective object is finitely generated,*
(3) *every small subobject of every finitely generated projective object is finitely generated.*

Proof. (1)⇒(2) Let L be a small subobject of a projective object P, i.e. there exist a cardinal κ, a family $P_i \in \mathcal{U}$, $i < \kappa$, and a monomorphism $L \to \bigoplus_{i<\kappa} P_i$. Moreover, as L is small, there exists $F \subset \kappa$ and a monomorphism $\iota : L \to \bigoplus_{i \in F} P_i$. Put $M = \mathrm{Coker}(\iota)$. By Lemma 37, the functor $\mathrm{Ext}^1(M,-)$ commutes with direct sums, hence it commutes with direct unions by the hypothesis. Thus L is finitely generated by Theorem 44.

(2)⇒(3) Clear.

(3)⇒(1) Let M be a finitely generated object such that $\mathrm{Ext}^1(M,-)$ commutes with direct sums. Then homomorphisms $\Psi^M_{\mathfrak{F}}$, $\Psi^P_{\mathfrak{F}}$ and $\Phi^M_{\mathfrak{F}}$ from the diagram (D1) are isomorphisms, $\Phi^L_{\mathfrak{F}}$ is isomorphism as well, hence L is small. By the hypothesis L is finitely generated. Thus $\mathrm{Ext}^1(M, -)$ commutes with direct unions by Theorem 44(4). $\square$

We say that a unital ring R is right finite ext-steady if the category of all right R-modules is finite ext-steady.

Example 51. It is proved in [33] that every infinite product of unital rings contains an infinitely generated small ideal, hence it is not finite ext-steady.

As every finitely generated projective module is a direct summand of a finitely generated free module we obtain a consequence of the last proposition:

Corollary 52. *A ring R is a right finite ext-steady if and only if every small right ideal is finitely generated.*

Proof. By Proposition 50 it is enough to prove that every small submodule I of every finitely generated free module R^n is finitely generated. Proceed by induction, the claim is clear for $n = 1$ hence suppose that $n > 1$ and denote by $\pi : R^n \to R$ the canonical projection and by $\nu : R \to R^n$ the canonical injection on the first coordinate. As $\pi(I)$ is small so finitely generated submodule of R, and the small module $I + \nu(R)/\nu(R) \cong I/(I \cap \nu(R))$ is embeddable into R^{n-1}, the module $I/(I \cap \nu(R))$ is finitely generated by the induction hypothesis. Moreover, $I/\nu\pi(I)$ and $\nu\pi(I)$ are finitely generated as well because $I \cap \nu(R) \subset \nu\pi(I)$, hence I is finitely generated. $\square$

Recall that a ring is *right steady* provided every small right module is necessarily finitely generated, [16]. However general structural ring-theoretical characterization of right steady rings is still an open problem, various classes of rings are known to be right steady (noetherian and perfect [29], semiartinian of countable Loewy chain [16]). Let us remark here that the criterion of steadiness for commutative semiartinian rings [30] and for regular semiartinian rings with primitive factors artinian [36] has a similar form as Corollary 52 since steadiness is in this cases equivalent to the condition that every small ideal of every factor-ring is finitely generated.

Corollary 53. *The category of right modules over right steady or countable ring is right finite ext-steady.*

Since there are known countable non-steady rings, as it is illustrated in the following example, the inclusion of classes of steady rings and finite ext-steady rings is strict.

Example 54. Let F be a countable field and $\mathbb{X}$ is a infinite countable set. Then it is is well known that over the polynomial ring $F\langle\mathbb{X}\rangle$ in noncommuting variables $\mathbb{X}$ every injective module is small. Thus $F\langle\mathbb{X}\rangle$ is a non-steady countable ring.

Remark 55. From Proposition 50 and [35, Example 14] we deduce that the ext-steadiness property is not left-right symmetric.

We conclude the section with generalization of Corollary 48 in the case of modules over perfect rings.

Proposition 56. *Let R be a right perfect ring and M a right R-module such that* $\mathrm{Ext}^1(M,-)$ *commutes with direct sums. Then M is fg-Ω^1.*

Proof. Let $0 \to L \to P \to M \to 0$ be a projective presentation for M. Denote by J the Jacobson radical of R. Since R is right perfect, L/LJ is semisimple, hence $L/LJ \cong \bigoplus_{i\in I} S_i$ for a family of simple modules $(S_i, i \in I)$. We consider $f : L \to \bigoplus_{i\in I} S_i$ as the canonical projection. Then by Theorem 24 there exists a finite set $F \subset I$, and $g : P \to \bigoplus_{i\in I\setminus F} S_i$ such that $\pi_{I\setminus F} f = g\beta$.

Let $\rho : Q \to \bigoplus_{i\in I\setminus F} S_i$ be a projective cover of $\bigoplus_{i\in I\setminus F} S_i$ which exists because R is right perfect. As ρ is surjective, there exists a homomorphisms $\tau : P \to Q$ such that $\rho\tau = g$. Note that $\mathrm{Ker}\rho = QJ$ is superfluous in Q and $\rho\tau\beta = g\beta = \pi_{I\setminus F} f$, thus $\tau\beta(L) = Q$ where Q is projective. Clearly, there exists a homomorphism $\varphi : Q \to P$ such that $\tau\beta\varphi = \mathrm{id}_Q$, hence $L = \varphi(Q) \oplus \mathrm{Ker}\tau\beta$ and $P = \beta\varphi(Q) \oplus \mathrm{Ker}\tau$. Since the factorization by $\varphi(Q)$ induces a short exact sequence

$$0 \to L/\varphi(Q) \to P/\beta\varphi(Q) \to M \to 0$$

and $P/\beta\varphi(Q)$ is projective, it remains to prove that $L/\varphi(Q)$ is finitely generated. This follows from the observations that $V = \mathrm{Ker}\tau\beta \cong L/\varphi(Q)$ and $V/VJ \cong \bigoplus_{i\in F} S_i$. □

Finally we summarize the results about connections between possible commuting properties of a functor $\mathrm{Ext}^1(M,-)$:

Corollary 57. *For a right R-module M we consider the following possible properties:*

(DS) $\mathrm{Ext}^1(M,-)$ *commutes with direct sums;*
(DU) $\mathrm{Ext}^1(M,-)$ *commutes with direct unions;*
(DL) $\mathrm{Ext}^1(M,-)$ *commutes with direct limits.*

Then the following are true:

(1) *If R is hereditary then* (DS)⇔(DU)⇔(DL).
(2) *R is right coherent if and only if* (DU)⇔(DL) *for all right R-modules M.*
(3) *If R is right perfect then* (DS)⇔(DU).

Proof. (1) is a consequence of Proposition 11 (this is also proved in [32]).
(2) is proved in Theorem 49 and Corollary 46.
(3) is a consequence of Proposition 56. □

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Chapter 5

Reflection of categorical properties to a ring structure

This chapter summarizes results on four classes of rings with similar correspondence between ring structure and categories of modules. The work was originally published as the following four papers:

F. Jan Žemlička, *Socle chains of abelian regular semiartinian rings*, J. Pure Appl. Algebra 217/6 (2013), 1018–1025.

G. Tomáš Penk, Jan Žemlička, Commutative tall rings, J. Algebra Appl., 13/4 (2014).

H. M. Tamer Kosan, Jan Žemlička, Mod-retractable rings, Commun. Algebra 42/3 (2014) 998–1010.

I. M. Tamer Kosan, Jan Žemlička, On modules and rings with restricted minimum condition, Colloq. Math., 140,1 (2015), 75–86.

F. SOCLE CHAINS OF ABELIAN REGULAR SEMIARTINIAN RINGS

JAN ŽEMLIČKA

ABSTRACT. Let R be an abelian regular and semiartinian ring with socle chain $(S_\alpha \mid \alpha \leq \sigma)$. If λ_α denotes the rank of the semisimple module $S_{\alpha+1}/S_\alpha$, for every $\alpha < \sigma$, then the *dimension sequence* $(\lambda_\alpha \mid \alpha < \sigma)$ is an invariant for R. By applying classical results of combinatorial set theory we prove necessary conditions satisfied by this invariant. On the other hand, we present constructions of commutative regular semiartinian rings with given ranks of slices of socle chain. In some particular cases we prove a necessary and sufficient condition under which there exists an abelian regular semiartinian ring with given ranks of slices.

Given a right module M over some ring R, the socle chain of M is the increasing chain of submodules $(S_\alpha \mid \alpha \geq 0)$ defined by the following rules: set $S_0 = 0$ and, recursively, $S_{\alpha+1}/S_\alpha = \mathrm{Soc}(M/S_\alpha)$ (we denote by Soc(M) the socle of M) for each ordinal α and $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ if α is a limit ordinal. The first ordinal σ such that $S_\sigma = S_{\sigma+1}$ is called the socle length of M and one says that M is semiartinian if $S_\sigma(M) = M$. This notion was examined first by László Fuchs in [8], however the idea to study it in the particular case of ideals in commutative noetherian rings goes back to Wolfgang Krull [12].

A ring R is said to be *right semiartinian* provided it is semiartinian as a right R-module. It is easy to see that every right artinian module as well as every right module over a right artinian (semiartinian) ring is semiartinian. Moreover, the classical Bass's result [3, Theorem P, (1)→(7)] asserts that every non-zero right module over a left perfect ring has a non-zero socle, hence every left perfect ring is right semiartinian. Basic structural results about semiartinian rings were presented by Constantin Năstăsescu and Nicolae Popescu in [13]. They proved that a ring R is right semiartinian if and only if its Jacobson radical $J(R)$ is left T-nilpotent and $R/J(R)$ is right semiartinian [13, Proposition 3.2]. The works [1, 2, 5, 14, 18] are focused on the important case of semiartinian rings whose Jacobson radical is zero, note that such rings are von Neumann regular if they are commutative by [13, Théorème 3.1]. Various constructions of semiartinian rings and modules of given socle lengths are presented in [4, 5, 6, 8, 15] and properties of semiartinian rings close to commutative has been studied in [1, 14, 18].

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The dimension sequence of a regular semiartinian ring whose all primitive factors are artinian, which reflects the structure of single semisimple slices, was introduced in [14]. Dimension sequences appear to be a useful tool for investigation of the global structure of these rings and of corresponding module categories, however they do not completely describe categorical properties of classes of all modules. Pairs of examples of semiartinian rings R_σ, T_σ with non-equivalent module categories which have the same dimension sequences are presented in [6] for every non-limit uncountable socle length σ. The class of all modules over R_σ contains an infinitely generated small module while the ring T_σ does not satisfy this property.

The present paper is motivated by the question of which sequences can be represented as the dimension sequence of some abelian regular semiartinian ring, or more precisely, which ranks of semisimple socle slices are possible. Proposition 3.1 gives a first estimation of relationships between ranks of slices of an abelian regular semiartinian ring. Theorem 3.5, which is based on results of combinatorial set theory, enhances Proposition 3.1 under the Generalized Continuum Hypothesis. Specifically, let R be an abelian regular, semiartinian ring with dimension sequence $(\lambda_\alpha \mid \alpha < \sigma)$. Then we prove that $\lambda_{\alpha+\delta} \leq \lambda_\alpha$ if $\mathrm{cf}(\lambda_\alpha) > \max(|\delta|, \omega)$ and $\lambda_{\alpha+\delta} \leq \lambda_\alpha^+$ otherwise. On the other hand, if a family of cardinals λ_α, $\alpha \leq \sigma$, satisfies conditions (a) $\lambda_\beta \leq \lambda_\alpha^+$ whenever $\mathrm{cf}(\lambda_\alpha) = \omega$, and $\lambda_\beta \leq \lambda_\alpha$ otherwise; (b) $|\{\beta \mid \alpha \leq \beta \leq \sigma\}| \leq \lambda_\alpha$, (c) $\lambda_\alpha < \omega$ iff $\alpha = \sigma$, we construct examples of commutative regular semiartinian rings with the rank of the α-slice of the socle chain equal to λ_α (Theorem 5.1). Note that our constructions are inspired by the works [4] and [6]. Criteria on dimension sequences in particular cases are given in Theorem 5.2 and Corollary 5.3. Examples 4.7 and 5.4 illustrate boundaries of our tools and dependency of the results on set theoretical axioms.

1. Preliminary

Throughout the paper a *ring* (*K-algebra*) R means an associative unitary ring (K-algebra) and a *module* M means a right R-module. Recall that R is *regular* if for each $x \in R$ there exists $y \in R$ satisfying $xyx = x$ and R is *abelian regular* provided it is a regular ring whose all idempotents are central. It is well known that every finitely generated (right) ideal of an abelian regular ring is generated by some central idempotent. For further properties of abelian regular and regular rings we refer to the monograph [10].

The socle of M, i.e. the sum of all simple submodules of M, is denoted by $\mathrm{Soc(M)}$ and M is said to be *semisimple* provided that $M = \mathrm{Soc(M)}$. If M is semiartinian with the socle chain $(S_\alpha |\ \alpha \leq \sigma)$, the ordinal σ is called the *socle length* of M and the semisimple module $S_{\alpha+1}/S_\alpha$ is the *α-th slice* of M in this paper. Note that every finitely generated semiartinian module is of a non-limit socle length and every module over a right semiartinian ring of socle length σ is semiartinian of socle length less or equal to σ. The least cardinality of any set of generators of M is denoted by gen(M). If M is semisimple and infinitely generated, gen(M) will be called *rank* of M, and the rank of a finitely generated semisimple module is defined as its uniform dimension. We refer to [9] for a survey and generalizations of the theory of semiartinian modules.

Let X be a set and κ a cardinal. Cardinality of X is denoted by $|X|$, and the symbol κ^+ means the successor of κ, i.e. the least cardinal greater then κ. Note that we identify cardinals with the least ordinals of given cardinality, in particular, ω means the first infinite cardinal (ordinal) and the first uncountable cardinal (ordinal) is denoted by ω_1. The symbol $[X]^\kappa$ ($[X]^{<\kappa}$) means the set of all subsets of X of cardinality κ (less than κ), and $\langle\alpha,\beta\rangle$ stands for the interval $\{\gamma \mid \alpha \leq \gamma \leq \beta\}$ for every ordinals $\alpha \leq \beta$. As our results depend on a model of the set theory, note that we work in Zermelo-Fraenkel system with the axiom of choice. For further properties of set theoretical notions including elementary cardinal arithmetics we refer to [11].

We recall a natural description of slices of regular semiartinian rings with primitive factors artinian.

Theorem 1.1. [14, Theorem 2.1] *Let R be a right semiartinian ring and let $\mathcal{L} = (S_\alpha \mid \alpha \leq \sigma + 1)$ be the right socle chain of R. Then the following conditions are equivalent:*

1. *R is regular and all right primitive factor rings of R are right artinian,*
2. *for each $\alpha \leq \sigma$ there are a cardinal λ_α, a positive integer $n_{\alpha\beta}$ and a skew field $K_{\alpha\beta}$ for each $\beta < \lambda_\alpha$ such that $S_{\alpha+1}/S_\alpha \cong \bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$, as rings without unit. The pre-image of $M_{n_{\alpha\beta}}(K_{\alpha\beta})$ coincides with the β-th homogeneous component of R/S_α and it is finitely generated as a right R/S_α-module for all $\beta < \lambda_\alpha$. Moreover, λ_α is infinite if and only if $\alpha < \sigma$.*

If (1) holds true, then R is also left semiartinian, and $\mathcal{L}$ is the left socle chain of R.

In general, for any regular ring the condition of being semiartinian is right/left symmetrical, as the right and left socle chains coincide (see [1, Proposition 2.3]).

Suppose that R is a regular semiartinian ring of socle length $\sigma + 1$ such that all right primitive factors of R are right artinian. Following [14] we define

$$\mathcal{D}(R) = \{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$$

as the *dimension sequence* of R; it collects invariants determined by Theorem 1.1. The structural question about relationships between single slices $S_{\alpha+1}/S_\alpha$ may be formulated in the language of Theorem 1.1. Namely, we ask (and partially answer in this paper) which sequences are representable as the dimension sequence of some regular semiartinian rings with primitive factors artinian.

Let $\alpha < \sigma$. Since each homogeneous component of R/S_α is injective by [10, Proposition 6.18], it is a ring direct summand of R/S_α. Thus Theorem 1.1 induces a surjective ring-homomorphism $\pi_{\alpha\beta} : R/S_\alpha \to M_{n_{\alpha\beta}}(K_{\alpha\beta})$ for each $\beta < \lambda_\alpha$ and the family $(\pi_{\alpha\beta})_{\beta<\lambda_\alpha}$ induces the canonical map $\varphi_\alpha : R/S_\alpha \to \prod_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$.

Finally, denote by $\mathcal{E}(R)$ the set of all central idempotents of R and put $\mathcal{E}(I) = I \cap \mathcal{E}(R)$ whenever $I \subset R$.

First, we prove basic observations concerning dimension sequences and central idempotents.

Lemma 1.2. *Let κ be a cardinal, R_γ a ring for every $\gamma < \kappa$, and S a subring of $\prod_{\gamma<\kappa} R_\gamma$ containing $\bigoplus_{\gamma<\kappa} R_\gamma$. Suppose that $e \in S$. Then e is a central idempotent of S iff e is a central idempotent of $\prod_{\gamma<\kappa} R_\gamma$.*

Proof. It is enough to prove that each central element e of S commutes with all elements of $\prod_{\gamma<\kappa} R_\gamma$, which follows immediately from the fact that e commutes with every element of every ring summand $R_\gamma \subset \prod_{\gamma<\kappa} R_\gamma$, $\gamma < \kappa$. □

Lemma 1.3. *Suppose that R is a regular and right semiartinian ring with all primitive factors right artinian, let $(S_\alpha \mid \alpha \leq \sigma + 1)$ be its socle chain and $\{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ its dimension sequence. Then the following hold:*

(1) *For every $\alpha \leq \sigma$, the canonical map φ_α induces an isomorphism from R/S_α onto a subring of $\prod_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$ which contains $\bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$.*
(2) $|\mathcal{E}(R)| \leq 2^{\lambda_0}$.

Proof. (1) Let $\alpha \leq \sigma$. Since the socle of R/S_α is isomorphic as a ring without unit to $\bigoplus_{\beta<\lambda_\alpha} M_{n_{\alpha\beta}}(K_{\alpha\beta})$ by Theorem 1.1 and since R/S_α is a semiartinian ring, we get that $\mathrm{Soc(R/S_\alpha)} \cap \mathrm{Ker}(\varphi_\alpha) = 0$. Hence φ_α is an injective homomorphism.

(2) Using the notation established above, $\pi_{0\beta}(e)$ is a central idempotent of $M_{n_{0\beta}}(K_{0\beta})$ for any central idempotent $e \in R$, hence $\pi_{0\beta}(e)$ is either the zero or unit matrix. Applying (1) we get that e is a central idempotent iff $\varphi_0(e)$ is a central idempotent of $S = \prod_{\beta<\lambda_0} M_{n_{0\beta}}(K_{0\beta})$. Now, $|\mathcal{E}(R)| \leq |\mathcal{E}(S)| = 2^{\lambda_0}$ by Lemma 1.2. □

Lemma 1.4. *Let R be a regular semiartinian ring whose all right primitive factors are right artinian. Suppose that R is not semisimple, $I \subseteq \mathrm{Soc(R)}$ is an ideal, and $e \in \mathcal{E}(R)$.*

(1) *$\mathcal{E}(I)$ is finite $\Leftrightarrow$ $I = fR$ for some $f \in \mathcal{E}(I)$,*
(2) *if $\mathcal{E}(I)$ is infinite and $e \in \mathcal{E}(I)$, then $|\mathcal{E}(I)| = \mathrm{gen(I)} = |\mathcal{E}(\mathrm{(1-e)I})|$,*
(3) *$\mathcal{E}(eR)$ is finite $\Leftrightarrow$ $\mathcal{E}(e\mathrm{Soc(R)})$ is finite $\Leftrightarrow$ $e \in \mathrm{Soc(R)}$.*

Proof. (1) is a direct consequence of [10, Corollary 6.8], while (2) and (3) are straightforward consequences of Lemma 1.3 and the fact that I is a direct sum of homogeneous components of Soc(R), as it follows from [2, Corollary 1.12] □

We will use freely the following consequence of Lemma 1.4(2).

Corollary 1.5. *Let R be a regular semiartinian ring whose all right primitive factors are right artinian. If $\{(\lambda_\alpha, \{(n_{\alpha\beta}, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ is the dimension* *sequence and* $(S_\alpha \mid \alpha \le \sigma+1)$ *is the socle chain of* R, *then* $\lambda_\alpha = \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha) = |\mathcal{E}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)|$ *for every* $\alpha < \sigma$.

It is well-known that the class of all semiartinian modules is closed under sums, submodules, factors and extensions and that every module over a semiartinian ring is semiartinian. We conclude this section by the several technical observations about socle chains of direct sums of semiartinian modules and rings, which we will use frequently in our constructions. The first two are easy and the third is a corrected version of [1, Proposition 4.7].

Lemma 1.6. *Let* κ *be a cardinal and* $(M_\gamma \mid \gamma < \kappa)$ *a family of semiartinian modules. Denote by* σ_γ *the socle length of* M_γ *and by* $(M_{\alpha\gamma} \mid \alpha \le \sigma_\gamma)$ *the socle chain of* M_γ *for every* $\gamma < \kappa$. *Then* $(\bigoplus_{\gamma<\kappa} M_{\alpha\gamma} \mid \alpha \le \sigma)$ *is the socle chain of* $M = \bigoplus_{\gamma<\kappa} M_\gamma$, *where* $\sigma = \sup_{\gamma<\kappa} \sigma_\gamma$ *and* $M_{\alpha\gamma} = M_\gamma$ *for every* $\gamma < \kappa$ *and every* $\alpha > \sigma_\gamma$.

Corollary 1.7. *Let* n *be a natural number and* $\{(\lambda_\alpha, \{(1, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \le \sigma\}$ *the dimension sequence of an abelian regular semiartinian ring* R. *Then* R^n *is an abelian regular semiartinian ring of socle length* $\sigma+1$. *Moreover, if* $\{(\lambda'_\alpha, \{(1, K'_{\alpha\beta}) \mid \beta < \lambda'_\alpha\}) \mid \alpha \le \sigma\}$ *is the dimension sequence of* R^n, *then* $\lambda'_\sigma = n \cdot \lambda_\sigma$ *and* $\lambda'_\alpha = \lambda_\alpha$ *for each* $\alpha < \sigma$.

Proposition 1.8. *Let* κ *be an infinite cardinal,* K *a field and* R_γ *a* K*-algebra with socle length* σ_γ *and socle chain* $(S_{\alpha\gamma} \mid \alpha \le \sigma_\gamma)$ *for each ordinal* $\gamma < \kappa$. *Let us consider the* K*-subalgebra* $R = \bigoplus_{\gamma<\kappa} R_\gamma + K$ *of the direct product* $\prod_{\gamma<\kappa} R_\gamma$ *and put* $\sigma = \sup_{\gamma<\kappa} \sigma_\gamma$. *If either* σ *is limit or the set* $\{\gamma \mid \sigma_\gamma = \sigma\}$ *is infinite, then:*

1. *For every ordinal* $\alpha \le \sigma$, *the* α*-th member of the socle chain of* R *is* $\bigoplus_{\gamma<\kappa} S_{\alpha\gamma}$.
2. *If each* R_γ *is right semiartinian, then* R *is right semiartinian with socle length* $\sigma+1$.
3. R *is regular, unit regular, directly finite, has right primitive factor rings artinian or is a right* V*-ring if and only if* R_γ *have the corresponding property for all* $\gamma < \kappa$.

Proof. The proof works as in [1, Proposition 4.7] where only the argument $K \not\subseteq \bigoplus_{\gamma<\kappa} S_{\sigma\gamma}$ is missing. Clearly, if all R_γ are right semiartinian, R is right semiartinian of a non-limit socle length equal either to σ or $\sigma+1$. Thus the socle length of R is $\sigma+1$ whenever σ is limit. If $\{\gamma \mid \sigma_\gamma = \sigma\}$ is an infinite set, then $\sigma = \rho + 1$ is a non-limit ordinal and the socle of $R/\bigoplus_{\gamma<\kappa} S_{\rho\gamma}$ is infinitely generated, hence $R \ne \bigoplus_{\gamma<\kappa} S_{\sigma\gamma}$. □

2. Some combinatorial set theory

Before we apply tools of combinatorial set theory to families of idempotents generating single slices of abelian regular semiartinian rings, we recall basic notation and several corresponding results. Note that we follow formulations from [17], which is more convenient for our purpose than the original version of the results.

Recall that a subset A of a well-ordered set $(X, \leq)$ is *cofinal* if for every $x \in X$ there exists $a \in A$ such that $x \leq a$. The *cofinality* of an ordinal (cardinal) κ is the least cardinality of a cofinal subset of $(\kappa, \leq)$, we denote this cardinal by $\mathrm{cf}(\kappa)$. Let X be a set and κ be a cardinal. A family $\mathcal{S}$ of subsets of X is called *almost disjoint* with *degree of disjunction* κ, provided $|A \cap B| < \kappa$ for every different sets $A, B \in \mathcal{S}$. Finally, recall that the Generalized Continuum Hypothesis (GCH) is the statement $2^\kappa = \kappa^+$ for every infinite cardinal κ [17, p.x], [11, p.55].

The following classical combinatorial result due to Tarski [16] plays an important role for improving the necessary condition which will be expressed in Proposition 3.1(4).

Theorem 2.1. [17, Theorem 1.1.6] *Suppose that GCH holds. Let λ and κ be cardinals such that* $\mathrm{cf}(\lambda) < \mathrm{cf}(\kappa)$. *If $\mathcal{S}$ is an almost disjoint family of subsets of κ such that $|A| \geq \lambda$ for every $A \in \mathcal{S}$ and degree of disjunction of $\mathcal{S}$ is at most λ, then $|\mathcal{S}| \leq \kappa$.*

Suppose X is a set, κ, γ, η cardinals and m a natural number. Let us call a *decomposition* of X a collection of (possibly empty) subsets of X with union X. A decomposition $\mathbf{\Delta} = \{\Delta_\alpha | \ \alpha < \gamma\}$ is a *partition* provided $\Delta_\alpha \cap \Delta_\beta = \emptyset$ and $\Delta_\beta \neq \emptyset$ for every $\alpha < \beta < \gamma$. The *ordinary partition symbol* $\kappa \to (\eta)^m_\gamma$ means that for every set X with $|X| = \kappa$ and for every decomposition $\mathbf{\Delta} = \{\Delta_\alpha | \ \alpha < \gamma\}$ of $[X]^m$ into γ parts, there exist an ordinal $\alpha < \gamma$ and a subset H of X such that $|H| = \eta$ and $[H]^m \subseteq \Delta_\alpha$.

As the original results about ordinary partition symbol proved by Erdős and Rado in [7] are formulated for partitions, we recall an easy observation [17, p. 25]:

Lemma 2.2. *Let κ, γ, η be infinite cardinals, m a natural number, and $\gamma \leq \kappa$. Then $\kappa \to (\eta)^m_\gamma$ iff for every partition $\mathbf{\Delta} = \{\Delta_\alpha | \ \alpha < \gamma\}$ of $[\kappa]^m$ into γ parts, there exist an ordinal $\alpha < \gamma$ and a subset H of κ such that $|H| = \eta$ and $[H]^m \subseteq \Delta_\alpha$.*

Proof. We need to prove only the "if" part and we may assume that $X = \kappa$. Let $\{\Delta_\alpha \mid \alpha < \gamma\}$ be a decomposition of $[\kappa]^m$ Without loss of generality we may assume that $\bigcup_{\delta<\alpha} \Delta_\delta \neq [\kappa]^m$ for every $\alpha < \gamma$. Let us choose an element $x_0 \in [\kappa]^m$ and, inductively, an element $x_\alpha \in [\kappa]^m \setminus \bigcup_{\delta<\alpha}(\Delta_\delta \cup \{x_\delta\})$ for every α with $0 < \alpha < \gamma$. If we set $\Delta'_\alpha = \{x_\alpha\} \cup (\Delta_\alpha \setminus \bigcup_{\delta<\alpha}(\Delta_\delta \cup \{x_\delta\}))$ for all $\alpha < \gamma$, then it is not difficult to check that $\{\Delta'_\alpha | \ \alpha < \gamma\}$ is a partition of $[\kappa]^m$. By the hypothesis there exist $H' \subseteq \kappa$ and $\alpha < \gamma$ such that $|H'| = \eta$ and $[H']^m \subseteq \Delta'_\alpha$. By taking $H = H' \setminus x_\alpha$ we see that $|H| = |H'|$, $x_\alpha \notin [H]^m$ and therefore $[H]^m \subseteq \Delta_\alpha$, as wanted. □

As we prefer decompositions to partitions, we formulate the result due to Erdős and Rado in the language of the monograph [17]:

Theorem 2.3. [17, Theorem 2.2.5] $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ *for all infinite cardinals κ.*

Obviously, under GCH we have $\kappa^{++} \to (\kappa^+)^2_\kappa$ and, moreover, $\kappa^{++} \to (\kappa^+)^2_\lambda$ for every $\lambda \leq \kappa$.

Finally, we prove an easy technical combinatorial lemma.

Lemma 2.4. *If κ is an infinite cardinal of countable cofinality, then there exists an almost disjoint family $\mathcal{M} \subset [\kappa]^\omega$ with degree of disjunction ω such that $|\mathcal{M}| > \kappa$.*

Proof. Since $\kappa \cdot \omega = \kappa$, there exists a disjoint family $\mathcal{D} \subset [\kappa]^\omega$ of cardinality κ such that $\bigcup_{Y \in \mathcal{D}} Y = \kappa$. Using Zorn's Lemma, we may extend $\mathcal{D}$ to a maximal (with respect to inclusion) almost disjoint family $\mathcal{M} \subset [\kappa]^\omega$ with degree ω. Assume that $|\mathcal{M}| = \kappa$.

As $\mathrm{cf}(\kappa) = \omega$, there is an increasing chain of families $\mathcal{M}_n$ such that $\mathcal{M} = \bigcup_{n<\omega} \mathcal{M}_n$ and $|\mathcal{M}_n| < \kappa$. Since all sets from $\mathcal{M}$ are countable, $|\bigcup_{Y \in \mathcal{M}_n} Y| < \kappa \cdot \omega = \kappa$, hence there exists $x_n \in \kappa \setminus \bigcup_{Y \in \mathcal{M}_n} Y$ for each $n < \omega$. Now, put $A = \{x_n |\ n < \omega\}$ and fix an arbitrary $Z \in \mathcal{M}$. Obviously, A is an infinite countable set. As there exists n_0 for which $Z \in \mathcal{M}_{n_0}$ and so $x_n \notin Z$ for each $n \geq n_0$, we obtain that $|A \cap Z| < \omega$. Hence we have proved that $A \notin \mathcal{M}$ and $\mathcal{M} \cup \{A\}$ forms an almost disjoint family of infinite countable sets with degree ω, in contradiction with the maximality of $\mathcal{M}$. Thus $|\mathcal{M}| > \kappa$. □

3. Necessary conditions

In this section R denotes an abelian regular semiartinian ring of socle length $\sigma + 1$ with socle chain $(S_\alpha \mid \alpha \leq \sigma + 1)$ throughout this section. Moreover we put $\lambda_\alpha = \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)$ for each $\alpha < \sigma$ (cf. Corollary 1.7) and λ_σ is defined as the (finite) rank of the semisimple ring R/S_σ. As all primitive factors of R are simple, cardinality of the set of all idempotents of R is upper bounded by $2^{\mathrm{gen}(\mathrm{Soc(R)})}$ by Lemma 1.3(2) and Corollary 1.5. Clearly, the dimension sequence is determined in such case by the cardinals λ_α, $\alpha \leq \sigma$, and by simple modules, which have the structure of skew fields. Applying Lemma 1.3(2) again we can describe a basic correspondence between ranks of single slices of R.

Proposition 3.1. *Suppose $\alpha \leq \beta \leq \sigma$.*

(1) $|\mathrm{gen(I)}| \leq 2^{\lambda_0}$ *for each ideal I of R,*
(2) $|\langle \alpha, \sigma \rangle| \leq 2^{\lambda_\alpha}$*, in particular* $|\sigma| \leq 2^{\lambda_0}$*,*
(3) $\lambda_\alpha \leq 2^{\lambda_0}$*,*
(4) $\lambda_\beta \leq 2^{\lambda_\alpha}$*.*

Proof. (1) Since every ideal of an abelian regular ring is generated by a suitable set of central idempotents, the assertion follows immediately from Lemma 1.3(2).

(2) Suppose that $I = \bigcup_{\gamma<\nu} I_\gamma$ for some strictly increasing chain of ideals I_γ and some infinite ordinal ν. If we fix $e_\gamma \in \mathcal{E}(I_{\gamma+1}) \setminus \mathcal{E}(I_\gamma)$ for each $\gamma < \nu$, then $|\nu| = |\{e_\gamma |\ \gamma < \nu\}| \leq 2^{\lambda_0}$ by Lemma 1.3(2). Now, it remains to apply this observation to the ideal $I = \bigcup_{\alpha \leq \gamma \leq \sigma} S_\gamma / S_\alpha$ of the factor ring R/S_α.

(3) $\lambda_\alpha = \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha) \leq \mathrm{gen}(\mathrm{S}_{\alpha+1}) \leq 2^{\lambda_0}$ by (1).

(4) Replacing R by R/S_α the assertion follows from (3). □

Before we start to examine relationship between sets of idempotents $\mathcal{E}(S_{\alpha+1}/S_\alpha)$ and $\mathcal{E}(S_{\alpha+\delta+1}/S_{\alpha+\delta})$ for $\alpha+\delta\leq\sigma$ from a combinatorial point of view, we make an elementary observation about orthogonal idempotents.

Lemma 3.2. *If $\delta\leq\sigma$ is an ordinal, then there exists $E_1\subset\mathcal{E}(S_{\delta+1})\setminus\mathcal{E}(S_\delta)$ such that $|E_1|=\lambda_\delta$ and $S_{\delta+1}/S_\delta=\bigoplus_{e\in E_1}e(S_{\delta+1}/S_\delta)=\bigoplus_{e\in E_1}(eR+S_\delta)/S_\delta$.*

By applying Theorem 2.1 we improve the estimate of Proposition 3.1(3) for $\alpha=1$.

Lemma 3.3. *Suppose that GCH holds. If $\mathrm{cf}(\lambda_0)>\omega$, then $\lambda_1\leq\lambda_0$.*

Proof. According to Lemma 3.2 we may consider a set $E_1\subset\mathcal{E}(S_2)\setminus\mathcal{E}(S_1)$ of cardinality λ_1 such that $S_2/S_1=\bigoplus_{e\in E_1}(eR+S_1)/S_1$. Note that if f and g are different elements of E_1, then $fg\in S_1$, hence $\mathcal{E}(fgS_1)=\mathcal{E}(fS_1\cap gS_1)=\mathcal{E}(fS_1)\cap\mathcal{E}(gS_1)$ is finite by Lemma 1.4(3). Moreover, $\mathcal{E}(fS_1)$ is infinite for each $f\in E_1$, thus $\mathcal{E}=\{\mathcal{E}(eS_1)|\ e\in E_1\}$ forms an almost disjoint family with degree of disjunction ω. By applying Corollary 1.5 and Theorem 2.1 with $\kappa=|\mathcal{E}(S_1)|$ and $\lambda=\omega$, we get $\lambda_1=|\mathcal{E}|\leq\kappa=\lambda_0$. □

The previous assertion applied to R/S_α yields that $\lambda_{\alpha+1}\leq\lambda_\alpha$ for an arbitrary $\alpha<\sigma$.

If we fix a set E of non-zero orthogonal idempotents from the δ-th slice of R, as it is allowed by Lemma 3.2, for every ideal J contained in S_δ we may consider the set $\Delta_J=\{\{f_1,f_2\}\in[E]^2|\ f_1f_2\in J\}$ of subsets of $[E]^2$. By constructing a suitable decomposition and using Theorem 2.3 we prove a limit version of Lemma 3.3.

Lemma 3.4. *Suppose that GCH holds and $\delta\leq\sigma$ is a limit ordinal. If $\mathrm{cf}(\delta)<\mathrm{cf}(\lambda_0)$ and $\lambda_\alpha=\lambda_0$ for every $\alpha<\delta$, then $\lambda_\delta\leq\lambda_0$.*

Proof. Assume that $\mathrm{gen}(\mathrm{S}_{\delta+1}/\mathrm{S}_\delta)=\lambda_\delta>\lambda_0$. By applying Lemma 3.2 we get a subset E_1 of $\mathcal{E}(S_{\delta+1})\setminus\mathcal{E}(S_\delta)$ for which $|E_1|=\lambda_\delta$ and $S_{\delta+1}/S_\delta=\bigoplus_{e\in E_1}(eR+S_\delta)/S_\delta$. By using Corollary 1.5 we see that for every $\alpha<\delta$ there exists an increasing chain of ideals $(I_{\alpha\beta}|\ \beta<\mathrm{cf}(\lambda_0))$ of R/S_α such that $S_{\alpha+1}/S_\alpha=\bigcup_{\beta<\mathrm{cf}(\lambda_0)}I_{\alpha\beta}$ and $|\mathcal{E}(I_{\alpha\beta})|=\mathrm{gen}(\mathrm{I}_{\alpha\beta})<\lambda_\alpha=\lambda_0$. Obviously, for each $e\in E_1$ and for each $\alpha<\delta$ there exists an ordinal $\beta_{e\alpha}<\mathrm{cf}(\lambda_0)$ for which $eI_{\alpha\beta_{e\alpha}}\neq 0$. Let us consider a strictly increasing cofinal transfinite sequence $\{\alpha_\gamma|\ \gamma<\mathrm{cf}(\delta)\}$ of the ordinal δ, so that $\delta=\sup\{\alpha_\gamma|\ \gamma<\mathrm{cf}(\delta)\}$, and put $\beta_e=\sup\{\beta_{e\alpha_\gamma}|\ \gamma<\mathrm{cf}(\delta)\}$ for every $e\in E_1$. Since $\mathrm{cf}(\delta)<\mathrm{cf}(\lambda_0)$, we get $\beta_e<\mathrm{cf}(\lambda_0)$ by [11, Lemmas 3.6, 3.7, 3.8]. Thus $eI_{\alpha_\gamma\beta_e}\neq 0$ for every $e\in E_1$ and $\gamma<\mathrm{cf}(\delta)$. As $|E_1|>\lambda_0=\lambda_0^2$, there exists $\beta_0<\mathrm{cf}(\lambda_0)$ such that $|\{e\in E_1\mid \beta_0=\beta_e\}|>\lambda_0$. Set $E_2=\{e\in E_1\mid\beta_0=\beta_e\}$. Recall that $|E_2|>\lambda_0$ and $eI_{\alpha_\gamma\beta_0}\neq 0$ for every $\gamma<\mathrm{cf}(\delta)$ and every $e\in E_2$. Now, put $\mu=\sup\{\mathrm{gen}(\mathrm{I}_{\alpha_\gamma\beta_0})|\ \gamma<\mathrm{cf}(\delta)\}$. Since $\mathrm{cf}(\delta)<\mathrm{cf}(\lambda_0)$ and $\mathrm{gen}(\mathrm{I}_{\alpha_\gamma\beta_0})<\lambda_0$, it follows that $\mu<\lambda_0$.

Note that we intend to apply Theorem 2.3 to the cardinal $\kappa = \max(\mu, \mathrm{cf}(\delta))$. Since $\mathrm{cf}(\delta) < \mathrm{cf}(\lambda_0) \leq \lambda_0$, it follows that $\kappa < \lambda_0 < |E_2|$, hence there exists a subset E of E_2 such that $|E| = \kappa^{++}$. For every $\gamma < \mathrm{cf}(\delta)$ Let us define the set $\Delta_\gamma = \{\{f_1, f_2\} \in [E]^2 |\ f_1 f_2 \in S_{\alpha_\gamma}\}$ for every $\gamma < \mathrm{cf}(\delta)$ and $\Delta_\gamma = \emptyset$ whenever $\gamma \geq \mathrm{cf}(\delta)$. Obviously, for every $\{f_1, f_2\} \in [E]^2$ there exists $\gamma < \mathrm{cf}(\delta)$ such that $f_1 f_2 \in S_{\alpha_\gamma}$, hence $[E]^2 = \bigcup_{\gamma < \mathrm{cf}(\delta)} \Delta_\gamma = \bigcup_{\gamma < \kappa} \Delta_\gamma$. By applying Theorem 2.3 to the decomposition $\{\Delta_\gamma |\ \gamma < \kappa\}$ we obtain that there are an ordinal $\gamma < \kappa$ and a subset H of E of cardinality κ^+ such that $[H]^2 \subseteq \Delta_\gamma$. In other words, we have proved that $f_1 f_2 \in S_{\alpha_\gamma}$ for every different $f_1, f_2 \in H$. This implies that $\bigoplus_{f \in H} f I_{\alpha_\gamma \beta_0} \subseteq S_{\alpha_\gamma + 1}/S_{\alpha_\gamma}$. Finally, as $f I_{\alpha_\gamma \beta_0} \neq 0$ for each $f \in H$, we have $\kappa \geq \mathrm{gen}(\mathrm{I}_{\alpha_\gamma \beta_0}) = |\mathcal{E}(\mathrm{I}_{\alpha_\gamma \beta_0})| \geq |\mathrm{H}| = \kappa^+$, a contradiction. □

Note that the decomposition $\{\Delta_\gamma |\ \gamma < \mathrm{cf}(\delta)\}$ from the last proof is very far from being a partition, as it is a chain. Using the argument of Lemma 2.2, we can see that Theorem 2.3 proves about the set E much more than we actually need.

Now we can sum up the previous results.

Theorem 3.5. *Suppose that GCH holds and α, δ are ordinals satisfying $\alpha + \delta \leq \sigma$. If $\mathrm{cf}(\lambda_\alpha) > \max(|\delta|, \omega)$, then $\lambda_{\alpha+\delta} \leq \lambda_\alpha$. Otherwise $\lambda_{\alpha+\delta} \leq \lambda_\alpha^+$.*

Proof. First, note that $\lambda_{\alpha+\delta} \leq \lambda_\alpha^+$ follows under GCH from Proposition 3.1. We prove that $\lambda_{\alpha+\beta} \leq \lambda_\alpha$ for $\mathrm{cf}(\lambda_\alpha) > \max(|\delta|, \omega)$ by transfinite induction on $\beta \leq \delta$.

The base step is trivial; $\lambda_{\alpha+0} = \lambda_\alpha$. Suppose that $\lambda_{\alpha+\gamma} \leq \lambda_\alpha$ for all $\gamma < \beta$. If there exists $\gamma < \beta$ for which $\lambda_{\alpha+\gamma} < \lambda_\alpha$, then $\lambda_{\alpha+\beta} \leq \lambda_{\alpha+\gamma}^+ \leq \lambda_\alpha$ by Proposition 3.1(4) and GCH. Now, suppose that $\lambda_{\alpha+\gamma} = \lambda_\alpha$ for all $\gamma < \beta$. If β is non-limit, i.e. $\beta = \gamma_0 + 1$, then $\lambda_{\alpha+\beta} \leq \lambda_{\alpha+\gamma_0} = \lambda_\alpha$ by by Lemma 3.3 applied to the ring $R/S_{\alpha+\gamma_0}$ since $\mathrm{cf}(\lambda_{\alpha+\gamma_0}) = \mathrm{cf}(\lambda_\alpha) > \omega$ by the hypothesis. Finally, if β is a limit ordinal, the assertion follows immediately from Lemma 3.4 applied to R/S_α. □

Recall that index of a (general) ring is the supremum of the index of nilpotence of its all nilpotent elements. Since every semiartinian ring with primitive factors artinian is a subring of a countable product of semiartinian rings of bounded index by [18, Lemma 1.4] which have similar properties as abelian regular semiartinian rings (cf. [10, Chapter 7] and [18, Section 2]), we conjecture that results of the present section remains true (and can be refined) for that larger class of semiartinian rings.

4. Constructions

First, we make an easy observation on subalgebras generated by ideals.

Lemma 4.1. *Let K be a field and I a proper ideal of a K-algebra V. If A is the K-subalgebra of V generated by I, then I is an ideal of A and $A/I \cong K$.*

Proof. As $A = K + I$ and $I \neq A$, we get $K \cap I = 0$, hence $A/I \cong K/(K \cap I) \cong K$. □

We intend to construct commutative semiartinian algebras over a field. Throughout this section K denotes a given field, κ a cardinal number and $(R_\gamma|\ \gamma < \kappa)$ a family of commutative, regular and semiartinian K-algebras. For each $\gamma < \kappa$, we denote with $\sigma_\gamma + 1$ and $(S_{\alpha\gamma}|\ \alpha \leq \sigma_\gamma + 1)$ respectively the socle length and the socle series of R_γ. We denote with π_β the β-th projection from the direct product $\prod_{\gamma<\kappa} R_\gamma$ onto R_β.

The following technical lemmas are based on the tools developed in the paper [6] and on Lemma 1.6.

Lemma 4.2. *Let κ be an infinite cardinal. Put $V = \prod_{\gamma<\kappa} R_\gamma$, $I = \bigoplus_{\gamma<\kappa} R_\gamma$ and $\sigma = \sup_{\gamma<\kappa}(\sigma_\gamma + 1)$. Suppose that A is a K-subalgebra of V containing I.*

(1) *I is a semiartinian ideal of A with the socle chain $(\bigoplus_{\gamma<\kappa} S_{\alpha\gamma}|\ \alpha \leq \sigma)$.*
(2) *If A/I is a regular ring, A is regular as well.*
(3) *Suppose that for every $a \in A \setminus I$ and $\alpha < \sigma$ there are infinitely many $\gamma < \kappa$ such that $\pi_\gamma(a) \notin S_{\alpha\gamma}$. If A/I is regular semiartinian, then A is regular semiartinian as well and $\bigoplus_{\gamma<\kappa} S_{\alpha\gamma}$ is the α-th member of the socle chain of A for every $\alpha \leq \sigma$.*

Proof. (1) Since I is a V-module, I is an ideal of A, . Note that R_γ as a V-module has the same structure as an R_γ-module, hence the lattices of A-submodules and R_γ-submodules of R_γ coincide. In particular, R_γ is a semiartinian A-module with the socle chain $(S_{\alpha\gamma}|\ \alpha \leq \sigma_\gamma + 1)$. Thus I is a semiartinian A-module with the socle chain $(\bigoplus_{\gamma<\kappa} S_{\alpha\gamma}|\ \alpha \leq \sigma)$.

(2) It follows directly from [10, Lemma 1.3].

(3) A is commutative regular by (2) and, obviously, it is semiartinian. Applying (1) it suffices to show for every $a \in A\setminus I$ and every $\alpha < \sigma$ that $a + T_\alpha \notin \mathrm{Soc_A(A/T_\alpha)}$, where T_α is the α-th member of the socle chain of I, i.e. $T_\alpha = \bigoplus_{\gamma<\kappa} S_{\alpha\gamma}$.

Given $a \in A \setminus I$, let us consider the set $G = \{\gamma < \kappa|\ \pi_\gamma(a) \notin S_{\alpha\gamma}\}$. Choose idempotents $e \in A$ and $e_\gamma \in I$ such that $aA = eA$ and $e_\gamma A = R_\gamma$ for each $\gamma \in G$ (where R_γ is considered as a principal ideal of A). Note that $ee_\gamma \notin T_\alpha$, since $\pi_\gamma(a) \notin S_{\alpha\gamma}$ for every $\gamma \in G$. Hence $\{ee_\gamma + T_\alpha|\ \gamma \in G\}$ forms an infinite subset of $\mathcal{E}(e(A/T_\alpha))$, which implies that $e + T_\alpha$ and so $a + T_\alpha$ is not an element of $\mathrm{Soc_A(A/T_\alpha)}$ by Lemma 1.4(3). □

We construct a semiartinian K-algebra of socle length $\sigma + 1$ for a limit ordinal σ, if there exists a suitable chain of rings of lower socle lengths.

Proposition 4.3. *Let κ be an infinite cardinal, $\sigma = \sup_{\gamma<\kappa}(\sigma_\gamma)$ and $\lambda_{\alpha\gamma} = \mathrm{gen(S_{\alpha+1\gamma}/S_{\alpha\gamma})}$ for every $\gamma < \kappa$ and $\alpha < \sigma_\gamma$. If $\sigma > \sigma_\gamma$ for each $\gamma < \kappa$, then there exists a commutative regular semiartinian K-algebra A of socle length $\sigma + 1$ with socle chain $(T_\alpha|\ \alpha \leq \sigma + 1)$ such that $\dim_K(T_{\sigma+1}/T_\sigma) = 1$ and $\mathrm{gen(T_{\alpha+1}/T_\alpha)} = \lambda_{\alpha\gamma}$ for every $\gamma < \kappa$ and $\alpha < \sigma_\gamma$.*

Proof. Applying Proposition 1.8 to the family $(R_\gamma|\ \gamma < \kappa)$ we obtain a commutative regular semiartinian K-algebra A of socle length $\sigma + 1$ and with socle chain $(T_\alpha|\ \alpha \leq$

$\sigma+1)$ such that $T_{\sigma+1}/T_\sigma \cong K$ and $T_\alpha = \bigoplus_{\gamma<\kappa} S_{\alpha\gamma}$ for every $\alpha \le \sigma$. Hence $\mathrm{gen}(\mathrm{T}_{\alpha+1}/\mathrm{T}_\alpha) = \mathrm{gen}(\bigoplus_{\beta<\kappa} \mathrm{S}_{\alpha+1\beta}/\mathrm{S}_{\alpha\beta}) = \sum_{\beta<\kappa,\alpha\le\sigma_\beta} \lambda_{\alpha\beta} = \lambda_{\alpha\gamma}$ for every $\gamma<\kappa$ and $\alpha<\sigma$ by the hypothesis and by [11, Lemma 5.8]. □

Suppose in the rest of the section that we have constructed a commutative regular semiartinian K-algebra R with socle chain $(S_\alpha|\ \alpha\le\sigma+1)$. A similar argument as above allows us to construct a semiartinian K-algebra of socle length $\sigma+2$ which has the same ranks of all but the last two slices.

Proposition 4.4. *If λ is an infinite cardinal and $\lambda \le \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)$ for every $\alpha<\sigma$, there exists a commutative regular semiartinian K-algebra A of socle length $\sigma+2$ with socle chain $(T_\alpha|\ \alpha\le\sigma+2)$ such that* $\dim_K(T_{\sigma+2}/T_{\sigma+1}) = 1$, $\mathrm{gen}(\mathrm{T}_{\sigma+1}/\mathrm{T}_\sigma) = \lambda$, *and* $\mathrm{gen}(\mathrm{T}_{\alpha+1}/\mathrm{T}_\alpha) = \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)$ *for every $\alpha<\sigma$.*

Proof. Put $\lambda_\alpha = \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)$ for each $\alpha<\sigma$. If we apply Proposition 1.8 to the ideal $R^{(\lambda)}$ of the algebra $V = R^\lambda$, i.e. $\kappa=\lambda$ and $R_\gamma = R$ for all γ, we obtain a commutative regular semiartinian K-algebra A of socle length $\sigma+2$ whose socle chain $(T_\alpha|\ \alpha\le\sigma+2)$ satisfies the conditions $T_{\sigma+2}/T_{\sigma+1}\cong K$ and $T_\alpha = S_\alpha^{(\lambda)}$ for every $\alpha\le\sigma+1$. Hence $\mathrm{gen}(\mathrm{T}_{\sigma+1}/\mathrm{T}_\sigma) = 1\cdot\lambda = \lambda$, and $\mathrm{gen}(\mathrm{T}_{\alpha+1}/\mathrm{T}_\alpha) = \mathrm{gen}((\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)^{(\lambda)}) = \lambda\cdot\lambda_\alpha = \lambda_\alpha$ for every $\alpha<\sigma$, since $\lambda\le\lambda_\alpha$. □

The final step of our construction uses the combinatorial argument of Lemma 2.4.

Lemma 4.5. *Let κ be an infinite cardinal of countable cofinality, $V = R^\kappa$ and $\lambda_\alpha = \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha) \ge \kappa$ for every $\alpha<\sigma$. Then there exists a commutative regular semiartinian K-subalgebra A of V with socle chain $(T_\alpha|\ \alpha\le\sigma+2)$ such that* $\dim_K(T_{\sigma+2}/T_{\sigma+1}) = 1$, $\mathrm{gen}(\mathrm{T}_{\sigma+1}/\mathrm{T}_\sigma) = \kappa^+$ *and $T_\alpha = S_\alpha^{(\kappa)}$ for each $\alpha\le\sigma$.*

Proof. Put $I = R^{(\kappa)}$. For every $C\subset\kappa$ denote by R^C the naturally defined principal ideal of the ring R^κ and by e_C the idempotent generating R^C, i.e. $\pi_\gamma(e_C)=1$ if $\gamma\in C$ and $\pi_\gamma(e_C)=0$ otherwise. According to Lemma 2.4 there exists an almost disjoint family $\mathcal{S}\subset[\kappa]^\omega$ with degree of disjunction ω such that $|\mathcal{S}| = \kappa^+$. Now, define A as a K-subalgebra of V generated by $I\cup\{e_C|\ C\in\mathcal{S}\}$. Note that I is a semiartinian ideal of A by Lemma 4.2(1) and, obviously, $A = I + K + \sum_{C\in\mathcal{S}} e_C K$. Since we intend to apply Lemma 4.2(3), we have to verify that A/I is regular semiartinian and for every $a\in A\setminus I$, $\alpha\le\sigma$ there are infinitely many $\gamma<\kappa$ such that $\pi_\gamma(a)\notin S_\alpha$.

Fix two distinct sets $C, D\in\mathcal{S}$. Note that $\pi_\gamma(i)=0$ for all but finitely many $\gamma<\kappa$ whenever $i\in I$, hence $e_C\notin I$. Moreover, since $C\cap D$ is finite $e_C\cdot e_D\in R^{C\cap D}\subseteq I$, which implies that $(e_CA+I)/I \cong_A (e_CK+I)/I \cong_A K$, so $(e_CA+I)/I$ is a simple A-module. Thus $(e_C+I|\ C\in\mathcal{S})$ forms a set of non-zero orthogonal idempotents of the factor ring A/I such that $J = \bigoplus_{C\in\mathcal{S}} e_C(A/I) \subseteq \mathrm{Soc(A/I)}$. Obviously, J is infinitely generated, hence A/I is not semisimple. Moreover, $(A/I)/J$ is an 1-dimensional K-vector space by Lemma 4.1. Thus $J = \mathrm{Soc(A/I)}$ and we have shown that A/I is semiartinian. Note that A/I is embeddable into the commutative regular K-algebra $\prod_{C\in\mathcal{S}} e_C(A/I) \cong K^{|\mathcal{S}|}$ and we have $J \cong K^{(|\mathcal{S}|)}$ by Lemma 1.3(1) Applying Lemma 4.2(2) for an ideal $K^{(|\mathcal{S}|)}$ of the algebra $K^{|\mathcal{S}|}$ we obtain that A/I is regular.

Let $a \in A \setminus I$. There exist an integer n, pairwisely distinct sets $C_1, \dots, C_n \in \mathcal{S}$, elements $k_0, k_1, \dots, k_n \in K$ and $i \in I$ such that $a = i + k_0 \cdot 1 + \sum_{i=1}^{n} k_i e_{C_i}$ and $k_j \neq 0$ for at least one j. Put $b = a - i$. As the set $\{\gamma < \kappa | \ \pi_\gamma(i) \neq 0\}$ is finite and $b \notin I$, we obtain that the set $G = \{\gamma < \kappa | \ \pi_\gamma(b) \neq 0,\ \pi_\gamma(i) = 0\}$ is infinite. Observe that $\pi_\gamma(a) = \pi_\gamma(b)$ is an invertible element of R, so $\pi_\gamma(b) \notin S_\alpha$ for every $\gamma \in G$ and $\alpha \leq \sigma$. Now, Lemma 4.2(3) yields that A is semiartinian and $T_\alpha = S_\alpha^{(\kappa)}$ for $\alpha \leq \sigma$ where $(T_\alpha | \ \alpha \leq \rho)$ is the socle chain of A. Moreover, $\mathrm{gen}(\mathrm{T}_{\sigma+1}/\mathrm{T}_\sigma) = |\mathcal{S}| = \kappa^+$, since $T_{\sigma+1}/T_\sigma = \mathrm{Soc}(\mathrm{A/I}) = \bigoplus_{\mathrm{C}\in\mathcal{S}} \mathrm{e_C(A/I)}$. Finally, $T_{\sigma+2}/T_{\sigma+1} \cong (A/I)/\mathrm{Soc(A/I)} \cong \mathrm{K}$. □

Now we are able to improve the construction of Proposition 4.4, in the case $\mathrm{cf}(\lambda) = \omega$.

Proposition 4.6. *If λ is an infinite cardinal of countable cofinality and $\lambda \leq \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)$ for every $\alpha < \sigma$, then there exists a commutative regular semiartinian K-algebra A of socle length $\sigma + 2$ with socle chain $(T_\alpha | \ \alpha \leq \sigma + 2)$ such that $\dim_K(T_{\sigma+2}/T_{\sigma+1}) = 1$, $\mathrm{gen}(\mathrm{T}_{\sigma+1}/\mathrm{T}_\sigma) = \lambda^+$, and $\mathrm{gen}(\mathrm{T}_{\alpha+1}/\mathrm{T}_\alpha) = \mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)$ for every $\alpha < \sigma$.*

Proof. By applying Lemma 4.5 for $\kappa = \lambda$ we get a commutative regular semiartinian K-algebra A of socle length $\sigma + 2$ and with socle chain $(T_\alpha | \ \alpha \leq \sigma + 2)$ such that $T_{\sigma+2}/T_{\sigma+1} \cong K$, $\mathrm{gen}(\mathrm{T}_{\sigma+1}/\mathrm{T}_\sigma) = \lambda^+$ and $T_\alpha = S_\alpha^{(\lambda)}$ for every $\alpha < \sigma$. Finally, note that $\mathrm{gen}(\mathrm{T}_{\alpha+1}/\mathrm{T}_\alpha) = \mathrm{gen}((\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)^{(\lambda)}) = \lambda \cdot \mathrm{gen}((\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha)) = \mathrm{gen}((\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha))$ for each $\alpha < \sigma$. □

Example 4.7. By [11, Lemma 9.21] there exists an almost disjoint family on ω of cardinality 2^ω. Using the argument of the proof of Lemma 4.5 for such an almost disjoint family, we have got a commutative regular semiartinian algebra of socle length 3 such that $\mathrm{gen}(\mathrm{S}_1) = \omega$ and $\lambda_1 = \mathrm{gen}(\mathrm{S}_2/\mathrm{S}_1) = 2^\omega$. We have shown that the estimation of Proposition 3.1(3) cannot be improved in general, since we have $\lambda_1 = 2^{\lambda_0}$. Finally, let us stress that the construction need neither GCH nor the Continuum Hypothesis.

5. Main results

The constructions of Propositions 4.3, 4.4 and 4.6 allow us to prove the main representation theorem of this paper. Before we state it, let us remark that Proposition 4.3, which is our only tool to construct semiartinian algebras of the socle length $\sigma + 1$ for a limit ordinal σ, does not enable to produce an algebra with socle length greater then the dimension of the socle whenever $\mathrm{cf}(\sigma) = |\sigma|$, since the construction needs the sum of at least $\mathrm{cf}(\sigma)$ algebras. More generally, it is a gap between the estimation $|\langle\alpha, \sigma\rangle| \leq 2^{\lambda_\alpha}$ of Proposition 3.1(2) and the condition $|\langle\alpha, \sigma\rangle| \leq \lambda_\alpha$ of the following assertion, which needs the hypothesis of Proposition 4.3. It remains an open question whether either the estimation or the construction can be improved.

Theorem 5.1. *Let σ be an ordinal, K a field and $(\lambda_\alpha |\ \alpha \leq \sigma)$ a family of cardinals satisfying for every $\alpha \leq \beta \leq \sigma$ the conditions:*

(a) $\lambda_\beta \leq \lambda_\alpha^+$ *if* $\mathrm{cf}(\lambda_\alpha) = \omega$, *and* $\lambda_\beta \leq \lambda_\alpha$ *otherwise,*
(b) $\lambda_\alpha < \omega$ *iff* $\alpha = \sigma$.
(c) $|\langle\alpha, \sigma\rangle| \leq \lambda_\alpha$,

Then there exists a commutative regular semiartinian K-algebra with dimension sequence $\{(\lambda_\alpha, \{(1, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ where $K_{\alpha\beta} = K$ for all $\alpha \leq \sigma$ and $\beta < \lambda_\alpha$.

Proof. By Corollary 1.7 it is sufficient to show that, for every ordinal δ, there exists a commutative, regular and semiartinian K-algebra R_δ such that, if $(S_{\alpha\delta}|\alpha \leq \delta+1)$ is its socle chain, then $\lambda_\alpha = \mathrm{gen}(\mathrm{S}_{\alpha+1,\delta}/\mathrm{S}_{\alpha\delta})$ for $\alpha < \delta$, $\dim_K(S_{\delta+1\delta}/S_{\delta\delta}) = 1$ and, in addition, the conditions (a), (b) and (c) are satisfied. We proceed by transfinite induction on δ.

First, put $R_0 = K$.

Given an ordinal δ such that $0 < \delta \leq \sigma$, suppose that, for every $\beta < \delta$, an algebra R_β with the required properties exists.

Assume firstly that δ is limit and put $\kappa = \mathrm{cf}(\delta)$. Then there exists a strictly increasing transfinite sequence of ordinals $\{\sigma_\gamma |\ \gamma < \kappa\}$ such that $\sup_{\gamma<\kappa}(\sigma_\gamma) = \delta$. Note that $\kappa = \mathrm{cf}(\delta) \leq |\langle\alpha, \delta\rangle| \leq |\langle\alpha, \sigma\rangle| \leq \lambda_\alpha$ for every $\alpha < \sigma$ by the condition (c) of the hypothesis, hence the existence of R_δ follows immediately from Proposition 4.3.

Now, let $\delta = \epsilon + 1$ for some ordinal ϵ. Note that λ_ϵ is infinite by (b). If $\lambda_\epsilon \leq \lambda_\alpha$ for every $\alpha < \epsilon$, then an algebra R_δ exists by Proposition 4.4 where $\lambda = \lambda_\epsilon$ and $R = R_\epsilon$. Finally, suppose that there exists $\alpha < \epsilon$ for which $\lambda_\epsilon > \lambda_\alpha$. Then $\lambda_\epsilon = \lambda_\alpha^+$, $\mathrm{cf}(\lambda_\alpha) = \omega$ and $\lambda_\beta = \lambda_\alpha$ whenever $\beta < \epsilon$ and $\lambda_\epsilon > \lambda_\beta$ by (a). Moreover, $\lambda_\alpha \leq \lambda_\beta$ for each $\beta < \epsilon$, otherwise we would have $\lambda_\epsilon > \lambda_\alpha > \lambda_\beta$, so $\lambda_\epsilon > \lambda_\beta^+$, which contradicts (a). Hence we may apply Proposition 4.6 for $\lambda = \lambda_\alpha$ and $R = R_\epsilon$. □

Note that if we have a family of fields $\{K_\alpha |\ \alpha \leq \sigma\}$ such that K_α is a subfield of K_β whenever $\alpha > \beta$, we may modify the construction of Theorem 5.1 such that every inductively constructed ring R_γ is K_γ-algebra (and so K_β-algebra for each $\beta > \gamma$), i.e. $K_{\alpha\beta}$ in the dimension sequence of A may be replaced by K_α. Finally, note that the family $\{K_\alpha |\ \alpha \leq \sigma\}$ can be chosen as strictly decreasing. For instance, take K_0 as the field of rational functions, over a field K, in indeterminates x_α for $\alpha < \sigma$ and, if $\beta < \sigma$, take K_β as the subfield of K_0 generated by K and the set $\{x_\alpha |\ \beta < \alpha < \sigma\}$.

We are able to state the following criterion, which is a consequence of our main results Theorem 3.5 and Theorem 5.1.

Theorem 5.2. *Suppose that GCH holds. Let σ be an ordinal, and $(\lambda_\alpha |\ \alpha \leq \sigma)$ a family of cardinals such that either $\mathrm{cf}(\lambda_\alpha) > |\langle\alpha, \sigma\rangle|$ for each $\alpha < \sigma$ or $|\sigma| \leq \omega$.*

Then there exists an abelian regular semiartinian ring R with dimension sequence $\{(\lambda_\alpha, \{(1, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$ for some fields $K_{\alpha\beta}$ iff the conditions (a) and (b) of Theorem 5.1 are satisfied for the family $(\lambda_\alpha | \ \alpha \leq \sigma)$.

Proof. Let R be an abelian regular semiartinian ring with the dimension sequence $\{(\lambda_\alpha, \{(1, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\}$. Then (b) follows immediately from Theorem 1.1. Suppose that $\mathrm{cf}(\lambda_\alpha) > \omega$ and $\beta = \alpha + \delta \leq \sigma$. Then either $|\delta| \leq |\langle \alpha, \sigma \rangle| < \mathrm{cf}(\lambda_\alpha)$ or $|\langle \alpha, \sigma \rangle| \leq \omega < \mathrm{cf}(\lambda_\alpha)$ by the hypothesis. Thus we have shown that the hypothesis $\mathrm{cf}(\lambda_\alpha) > \max(|\delta|, \omega)$ of Theorem 3.5 is satisfied in both cases, hence $\lambda_\beta \leq \lambda_\alpha$. Finally, if $\mathrm{cf}(\lambda_\alpha) = \omega$ then $\lambda_\beta \leq \lambda_\alpha^+$ follows from Theorem 3.5 again, which proves that (a) holds .

To prove the reverse implication it remains to show that the condition (c) of Theorem 5.1 holds true, which follows from $|\langle \alpha, \sigma \rangle| < \mathrm{cf}(\lambda_\alpha) \leq \lambda_\alpha$ if $\mathrm{cf}(\lambda_\alpha) > |\langle \alpha, \sigma \rangle|$ and it is obvious if $|\sigma| \leq \omega$. □

By applying the previous theorem and Corollaries 1.5 and 1.7 for abelian regular semiartinian rings of countable socle lengths we obtain the final consequence.

Corollary 5.3. *Suppose that GCH holds. The following conditions are equivalent for a countable ordinal σ and a family of cardinals $(\lambda_\alpha | \ \alpha \leq \sigma)$:*

(1) *There exists an abelian regular semiartinian ring of socle length $\sigma + 1$ such that if $(S_\alpha | \ \alpha \leq \sigma + 1)$ is its socle chain, then $\mathrm{gen}(\mathrm{S}_{\alpha+1}/\mathrm{S}_\alpha) = \lambda_\alpha$ for each $\alpha \leq \sigma$,*

(2) *there exists a field $K_{\alpha\beta}$ for every $\alpha \leq \sigma$, $\beta < \lambda_\alpha$ and an abelian regular semiartinian ring of socle length $\sigma + 1$ with dimension sequence*

$$\{(\lambda_\alpha, \{(1, K_{\alpha\beta}) \mid \beta < \lambda_\alpha\}) \mid \alpha \leq \sigma\},$$

(3) *$(\lambda_\alpha | \ \alpha \leq \sigma)$ satisfies the conditions (a) and (b) of Theorem 5.1.*

We conclude the paper with an example illustrating the fact that we cannot omit GCH from the hypothesis of our results.

Example 5.4. Suppose the negation of the Continuum Hypothesis, i.e. $\omega_1 < 2^\omega$. Denote by A_1 the commutative regular semiartinian ring constructed in Example 4.7 and by A_2 a commutative regular semiartinian K-subalgebra of K^{ω_1} generated by $K^{(\omega_1)}$. Since the socle length of A_2 is equal to 2 and $\mathrm{gen}(\mathrm{Soc}(\mathrm{A}_2)) = \omega_1$ by Lemmas 4.1 and 4.2, $A = A_1 \times A_2$ is a commutative regular semiartinian ring for which $\mathrm{gen}(\mathrm{Soc}(\mathrm{A})) = \omega_1$ and $\mathrm{gen}(\mathrm{Soc}(\mathrm{A}/\mathrm{Soc}(\mathrm{A}))) = 2^\omega > \omega_1$ by Lemma 1.6. Since $\mathrm{cf}(\omega_1) \neq \omega$, we have shown that the assertion of Lemma 3.3 is not true without the hypothesis $\omega_1 = 2^\omega$.

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G. COMMUTATIVE TALL RINGS

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Abstract. A ring is right tall if every non-noetherian right module contains a proper non-noetherian submodule. We prove a ring-theoretical criterion of tall commutative rings. Besides other examples which illustrate limits of proven necessary and sufficient conditions, we construct an example of a tall commutative ring that is not max.

Many conditions of the category of all modules over a ring can be easily expressed in the language of the ring theory. A typical example of such correspondence between classes of modules and the ring structure is presented by perfect rings, which are characterized in Bass' paper [2] by both ring-theoretic and module-theoretic conditions. In particular, recall that every nonzero right module over a right perfect ring contains a maximal submodule. Rings over which every nonzero right module contains a maximal submodule are said to be *right max* (or almost noetherian) and they are widely studied by many authors from various points of view and with different motivations [3, 4, 5, 6, 7, 8, 11, 15]. Nevertheless, a ring-theoretic criterion of max rings is available only for a few interesting classes of rings (e.g. for p.i.-rings in [11]).

The notion of a *tall* module goes back to paper [13] and it is defined as a module M which contains some submodule N such that both M/N and N are non-noetherian. A ring is called *right tall* if every non-noetherian right module is tall. Although [13, Theorem 2.7] presents a nice characterization of right tall rings using the notion of Krull dimension of all modules, a general ring-theoretic necessary and sufficient condition hasn't been known yet. It is not hard to see that every right max ring is right tall, and John Clark in [5] asked if there was a difference between the classes of all tall and all max rings.

In this paper we present several necessary conditions as well as several sufficient conditions for commutative tall rings, which are easily applicable to many natural classes of rings. Namely, if $\bigcap_i J_i$ is not a prime ideal for every countable decreasing chain of ideals $\{J_i\}$ of a commutative ring R such that J_1 is maximal, $J_1 J_i \subseteq J_{i+1}$, and R/J_i is artinian for each i, we prove in Theorem 2.6 that R is tall. On the other hand, if R is tall, then for every non-idempotent maximal ideal I such that R/I^i is artinian for each i, the intersection $\bigcap_j I^j$ is not a prime ideal (Proposition 2.9). As a consequence we prove a ring-theoretic criteria of tallness for general commutative (Theorem 2.12) and for noetherian commutative rings (Proposition 2.10). The last

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section contains examples illustrating the limits of both necessary and sufficient conditions (Examples 3.4, 3.7) and an example of a tall non-max commutative ring (Example 3.2).

1. Preliminaries

Throughout the paper, a *ring* R means an an associative ring with unit and a *module* means a right R-module. For a module M denote by $\mathcal{E}(M)$ an injective envelope, $J(M)$ the Jacobson radical, $\mathrm{Ann}(M)$ the annihilator and $\mathrm{Soc}(M)$ the socle of M.

Given a module M, the *Loewy* (or socle) chain of M means an increasing chain of submodules $(S_\alpha(M) \mid \alpha \geq 0)$ defined by the following rules: set $S_0(M) = 0$ and, recursively, $S_{\alpha+1}(M)/S_\alpha(M) = \mathrm{Soc}(M/S_\alpha(M))$ for each ordinal α and $S_\alpha(M) = = \bigcup_{\beta<\alpha} S_\beta(M)$ if α is a limit ordinal. The first ordinal σ such that $S_\sigma(M) = S_{\sigma+1}(M)$ is called the *socle length* of M and one says that M is *semiartinian* if $S_\sigma(M) = M$. Note that a submodule and a factor-module of a semiartinian module M are semiartinian and $S_\alpha(N) = S_\alpha(M) \cap N$ for each α and each submodule N of M. A ring is called *right semiartinian* if it is semiartinian as a right module. For other basic properties of semiartinian modules and rings we refer to [10]. Recall that the *length* of a module M is the length of a composition series of M, i.e. n such that $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ where M_{i+1}/M_i is simple. Obviously, if M is of length n, then it is semiartinian of socle length $\leq n$.

We recall the general criterion of tall rings proven by Sarath:

Theorem 1.1. [13, Theorem 2.7] *The following conditions are equivalent for a ring R:*

1. *R is right tall,*
2. *every non-noetherian module has a proper non-noetherian submodule,*
3. *every module with Krull dimension is noetherian.*

As a consequence we obtain (cf. [5, p. 31]):

Corollary 1.2. *Every right max ring is right tall.*

Note that closure properties of the class of all tall rings are similar to those of the class of all max rings:

Lemma 1.3. *The class of all right tall rings is closed under factors, finite products, and Morita equivalence.*

Proof. Since a non-tall module over a factor ring R/I has a natural structure of a non-tall module over the ring R, we see that a factor of every tall ring is tall.

Let $R = \prod_{i=1}^n R_i$ be a product of rings which is not right tall. Denote by $e_i \in R$ the central idempotent satisfying $(e_i)_j = \delta_{ij}$ and fix a non-noetherian module M for which every proper submodule is noetherian. Such a module exists by Theorem 1.1.

Then $M = \bigoplus_{i=1}^n Me_i$ and there exists j for which Me_j is not noetherian, hence $Me_j = M$ and R_j is not right tall again by Theorem 1.1.

As the existence of a non-tall module implies the existence of a non-tall module over a Morita equivalent ring, the last closure property is clear. □

Moreover, if R is a tall subring of a commutative ring S and S is finitely generated as an R-module, then S is tall by [13, Theorem 2.11].

Note that every right non-tall ring is not max, and over a right non-max ring there exists a non-zero module which contains no maximal submodule. Now we recall several easily verifiable properties of such modules, which we shall apply later.

Lemma 1.4. *Let M be a module over a commutative ring R, N a proper submodule of M and $r \in R \setminus \mathrm{Ann}(M)$. If $J(M) = M$, then*

(1) *$J(M/N) = M/N$ is not noetherian,*
(2) *$J(rM) = rM$ is not noetherian,*
(3) *if M is not tall then M/N and Mr are not tall modules*

2. Ring structure

We begin this section by an easy observation that for each prime number p the Prüffer p-group $\mathbf{Z}_{p^\infty}$ forms a natural example of a non-tall $\mathbf{Z}$-module. The following example due to Sarath [13, Proposition 2.8] shows that a similar construction of non-tall modules works also over an arbitrary polynomial commutative ring:

Example 2.1. Let R be a commutative ring and I a maximal ideal of R. Then $F = R/I$ is a field and $F[x]$ has a natural structure of a cyclic module over the polynomial ring $R[x]$. For every n put $C_n = F[x]/x^nF[x]$ and note that $xC_{n+1} \cong C_n$, hence we have a natural embedding $i_n : C_n \to C_{n+1}$. It is easy to see that the inverse limit C of $C_1 \overset{i_1}{\hookrightarrow} C_2 \overset{i_2}{\hookrightarrow} \dots$ is a uniserial artinian module which is not noetherian, hence C presents an example of a non-noetherian non-tall $R[x]$-module.

We make an easy technical observation about semiartinian modules including the previous examples.

Lemma 2.2. *Let M be a semiartinian module of an infinite socle length such that every proper submodule of M is of finite length. Then M is artinian, non-noetherian and non-tall.*

Proof. M is not noetherian because $\{S_j(M) \mid j < \omega\}$ forms a strictly increasing chain of submodules. If $M_0 \supsetneq M_1 \supsetneq \dots$ is a strictly decreasing chain of submodules, then M_1 is artinian by the hypothesis, and hence the sequence terminates after finitely many steps. Finally, N is non-tall since every proper submodule is noetherian. □

The previous observation about infinitely generated semiartinian modules allows us to generalize the construction of the uniserial examples of non-tall modules:

Lemma 2.3. *If M is a non-noetherian semiartinian module for which every Loewy chain member is artinian, then there exists a non-noetherian artinian submodule of M which is not tall.*

Proof. Note that every submodule of a semiartinian module is semiartinian and that $S_\omega(M)$ is a non-noetherian submodule of M. Put

$$\mathcal{N} = \{N \subseteq S_\omega(M) |\ S_{j+1}(N)/S_j(N) \neq 0\ \forall j < \omega\}.$$

Obviously, $S_\omega(M) \in \mathcal{N}$, hence $\mathcal{N} \neq \emptyset$. Suppose $(N_\nu, \nu < \kappa)$ is a decreasing chain of modules from $\mathcal{N}$. Let $j < \omega$. Since $S_j(N_\nu) = S_j(M) \cap N_\nu$ is an artinian module for each $\nu < \kappa$ by the hypothesis, there exists $\nu_j < \kappa$ such that $S_k(N_\mu) = S_k(N_{\nu_j})$ for every $k \leq j+1$ and $\mu > \nu_j$, hence $S_{j+1}(N_\mu)/S_j(N_\mu) = S_{j+1}(N_{\nu_j})/S_j(N_{\nu_j}) \neq 0$ for each $\mu > \nu_j$. It implies that $S_{j+1}(N_{\nu_j}) \subseteq \bigcap_\nu N_\nu$ and $S_{j+1}(\bigcap_\nu N_\nu)/S_j(\bigcap_\nu N_\nu) \neq 0$ for every j and so $\bigcap_\nu N_\nu \in \mathcal{N}$.

Using a Zorn's lemma argument we obtain a minimal submodule N with respect to inclusion such that $S_{j+1}(N)/S_j(N)$ are nonzero for all j. Clearly, N has an infinite socle length. If P is a proper submodule of N, then there exists j such that $S_{j+1}(P)/S_j(P) = 0$ by minimality of N, thus $P \subseteq S_j(M)$ is artinian. By applying Lemma 2.2 we get that N is artinian, non-noetherian and non-tall. □

As every artinian module is semiartinian and submodules of artinian modules are artinian as well, we obtain

Corollary 2.4. *Every non-noetherian artinian module contains a non-tall submodule.*

The following technical lemma shows that over a non-tall commutative ring there exists a module possessing a similar structure of the lattice of submodules as $\mathbf{Z}_{p^\infty}$.

Lemma 2.5. *If R is a commutative non-tall ring, then there exists a non-tall module M such that*

1. $\operatorname{Ann}(M)$ *is a prime ideal,*
2. $S = \operatorname{Soc}(M)$ *is a simple essential submodule of M and for every $j < \omega$ there exists a natural number n_j such that $S_{j+1}(M)/S_j(M) \cong S^{n_j}$,*
3. *M is semiartinian of socle length ω and every proper submodule of M has finite length,*
4. *M is non-noetherian and artinian,*
5. *there are elements $x \in R$ and $s_j \in M$, $j < \omega$, such that $s_{j+1}x = s_j$ for each j and $M = \bigcup_j s_j R$,*
6. *there are elements $m_j \in M$, $j < \omega$, such that $m_j R \subseteq m_{j+1}R$ for each j, $M = \bigcup_j m_j R$ and $S_j(m_j R) = S_j(M)$.*

Proof. By Theorem 1.1 there exists a non-noetherian module N such that every proper submodule of N is noetherian. Since every nonzero factor of N contains no maximal submodule by Lemma 1.4, it is a non-tall infinitely generated module.

Pick $0 \neq u \in N$ and let I be a maximal ideal containing $\operatorname{Ann}(u)$. Then the natural epimorphism $uR \to R/I$ can be extended to a nonzero homomorphism $\pi\colon N \to \mathcal{E}(R/I)$. Observe that $M = \pi(N)$ is a non-tall infinitely generated module and $\operatorname{Soc}(M) = \{m \mid mI = 0\} \cong R/I$. Moreover, note that $aR \cong R/\operatorname{Ann}(a)$ is a noetherian module, hence $R/\operatorname{Ann}(a)$ is a noetherian ring for every $a \in M$.

(1) Let $rs \in \operatorname{Ann}(M)$. Assuming that $r \notin \operatorname{Ann}(M)$ then $Mr \subseteq M$ is non-noetherian by Lemma 1.4, so $Mr = M$ and $0 = M(rs) = (Mr)s = Ms$.

(2) and (3) Put $M_j = \{m \in M \mid mI^j = 0\}$. Obviously, $S_0(M) = M_0 \subsetneq M_1 = S_1(M)$ and $M_j \subseteq S_j(M)$. We will prove by induction on $j > 0$ that $S_{j-1}(M) \subsetneq S_j(M) \subsetneq M$ and that $M_j = S_j(M)$ is a module of finite length:

As $S_j(M)$ is of finite length by the induction hypothesis and since M is not noetherian, $S_j(M) \neq M$. Moreover, $S_{j+1}(M)/S_j(M)$ is a finitely generated semisimple module, hence $S_{j+1}(M)$ is of finite length as well.

We have noted that for every $a \in M \setminus M_j$ the ring $R/\operatorname{Ann} a$ is noetherian, which implies that there exists a maximal ideal J with respect to inclusion satisfying the condition $\exists b \in aR \setminus M_j\colon bJ \subseteq M_j$. Fix such b and J. Let $i \in I$. We will show that $i \in J$. Assume that $i \notin J$ which implies that $bi \in M \setminus M_j$. By the induction hypothesis M_j/M_{j-1} is essential in M_{j+1}/M_{j-1}, hence there exists $r \in R$ for which $bir \in M_j \setminus M_{j-1}$. Note that $br \notin M_j$ since $bri \notin M_{j-1}$. Moreover, $br(J + iR) = bJr + briR \subseteq M_j$. By the maximal property of J we see that $J + iR = J$, a contradiction. Hence $i \in J$. This proves $I \subseteq J$ where I is maximal and J is a proper ideal of R, so $I = J$. Since $b \in S_{j+1}(M) \setminus M_j$, we obtain that $S_j(M) \subsetneq S_{j+1}(M)$.

Now, if $\overline{m} = m + M_j \in \operatorname{Soc}(M/M_j) = S_{j+1}(M)/M_j$, then $\operatorname{Ann}(\overline{m})$ is a maximal ideal and $m \operatorname{Ann}(\overline{m}) \subseteq M_j$, thus $\operatorname{Ann}(\overline{m}) = I$ and $m \in M_{j+1}$.

We have proven that $S_j(M) \subsetneq M_{j+1} = S_{j+1}(M) \neq M$ and M_{j+1} has finite length for each j. Since $M_0 \subset M_1 \subset \dots$ forms a strictly increasing chain of submodules and M contains no proper non-noetherian submodule, we get $\bigcup_j M_j = M$. Finally, if P is a proper submodule of M, then P is noetherian, so there exists j such that $P \subseteq M_j$, which proves that P is of finite length.

(4) By (3) we may apply Lemma 2.2.

(5) Let s_0 be a generator of $\operatorname{Soc}(M)$ and $x \in I \setminus \operatorname{Ann}(M)$. Since $Mx = M$ by (3) and Lemma 1.4, we can construct a sequence $s_n \in M$ such that $s_k = s_{k+1}x$. It is easy to see that $s_kR \subseteq s_{k+1}I \neq s_{k+1}R$, hence $\bigcup_k s_kR$ is a non-noetherian module and so $\bigcup_k s_kR = M$.

(6) As M_k is noetherian, there exist indices $k_j > k_{j-1}$ for which $M_j \subseteq s_{k_j}R$. Now it remains to put $m_j = s_{k_j}$. □

By applying Lemma 2.5 we are ready to formulate sufficient conditions for the ring structure of commutative tall rings.

Theorem 2.6. *Let R be a commutative non-tall ring. Then there exists a maximal ideal I and a sequence of ideals $I = J_1 \supset J_2 \supset \dots$ such that*

(1) $IJ_i \subseteq J_{i+1}$ *for each* i,
(2) R/J_i *is artinian for each* i,
(3) $\bigcap_i J_i$ *is a prime ideal,*
(4) $R/\bigcap_n I^n$ *is not a tall ring.*

Proof. Set $J_i = \operatorname{Ann}(S_i(M))$ where M is from Lemma 2.5.

(1) $IJ_i \subseteq J_{i+1}$, because $S_{i+1}(M)I \subseteq S_i(M)$ by Lemma 2.5(2).

(2) For $g_1, \dots, g_k$ generators of $S_i(M)$ define the homomorphism $f\colon R \to \prod_j g_j R$ by $f(r) = (g_1 r, \dots, g_k r)$. Then $\operatorname{Ker}(f) = J_i$ and $R/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$ is artinian since $\prod_j g_j R$ is artinian.

(3) From $M = \bigcup_i S_i(M)$ we have $Mr = 0$ iff $S_i(M)r = 0$ for all i. Hence $\operatorname{Ann}(M) = \bigcap_i \operatorname{Ann}(S_i(M)) = \bigcap_i J_i$ is a prime ideal.

(4) Since $\bigcap_n I^n \subseteq \bigcap_i J_i$, M has the structure of an $R/\bigcap_n I^n$-module. □

We can formulate the following consequence of Theorem 2.6(4) and Lemma 1.3:

Corollary 2.7. *A commutative ring* R *is tall iff* $R/\bigcap_n I^n$ *is tall for every maximal ideal* I.

Now, we prove several necessary conditions for commutative tall rings.

Lemma 2.8. *If* R *is a commutative ring and* I *is a maximal ideal such that* I^{j-1}/I^j *is nonzero artinian for each* j, *then* $\mathcal{E}(R/I)$ *contains a non-noetherian semiartinian module* M *for which every Loewy chain member is artinian.*

Proof. Set $E_1 = R/I = \mathcal{E}(R/I_{R/I})$, $E_j = \mathcal{E}((E_{j-1})_{R/I^j}) = \mathcal{E}(R/I_{R/I^j})$ and $M = \bigcup_j E_j$. Notice that R/I^j is local with I/I^j the unique maximal ideal and that the socle length of R/I^j is equal to j by the hypothesis. Hence $S_j(M) = \{m \in M \mid mI^j = 0\}$ and $E_j \subsetneq E_{j+1}$. Evidently $E_j I^j = 0$, $E_j \subseteq S_j(M)$ and $S_1(M) = R/I$. Because $\mathcal{E}(R/I_R)$ is injective we have $E_j \subseteq \mathcal{E}(R/I_R)$ and, consequently, $M \subseteq \mathcal{E}(R/I_R)$. From $S_1(M) \trianglelefteq S_j(M)$, $S_1(M) \subseteq E_j$ and R/I^j-injectivity of E_j we obtain an embedding $S_j(M) \hookrightarrow E_j$. The length of E_j is at least j ($E_0 \subsetneq E_1 \dots \subsetneq E_{j-1} \subsetneq E_j$) and finite by [9, Corollary(3.85)] therefore $E_j = S_j(M)$ is artinian for all $0 < j < \omega$ and M is not noetherian. □

Proposition 2.9. *Let* R *be a commutative ring and* I *a maximal ideal. If* R/I^j *is artinian for each* j *and* $\bigcap_j I^j$ *is a non-maximal prime ideal then* R *is not tall.*

Proof. Since $\bigcap_j I^j$ is prime but not maximal, $R/\bigcap_j I^j$ is an integral domain but not a field and it follows easily from this that $I^j \supsetneq I^{j+1}$ for each j. As there exists by Lemma 2.8 a non-noetherian semiartinian module such that every member of its Loewy chain is artinian we obtain that R is non-tall by Lemma 2.3. □

As an easy consequence of the last proposition we get the well-known fact that there exist non-noetherian artinian modules over every noetherian commutative domain which is not a field. Combining this fact with the Krull intersection theorem gives us a criterion of tallness for the class of all commutative noetherian rings.

Proposition 2.10. *Let R be a commutative noetherian ring. Then the following conditions are equivalent:*

(1) *R is tall,*
(2) *R contains no non-maximal prime ideal,*
(3) *R is artinian.*

Proof. (1)→(2) If R contains a non-maximal prime ideal P and $I \supsetneq P$ is maximal, then according to the Krull intersection theorem [12, Theorem 8.10(ii)] we have $\bigcap_j (I/P)^j = P$ and R/I^j is artinian for all j by Hopkins' theorem. Then R/P is not tall by Proposition 2.9.

(2)→(3) Denote by $N(R)$ the nilradical of R, i.e. the intersection of all prime ideals and note that $N(R)$ is nilpotent since R is commutative noetherian and that $N(R) = J(R)$ by the hypothesis. Moreover, there exist maximal ideals $I_1, \dots, I_n$ such that $J(R) = N(R) = \bigcap I_j$ by [12, Theorem 6.5], hence $R/J(R)$ is semisimple. The conclusion follows by Hopkins' theorem.

(3)→(1) This follows from Corollary 1.2 since every artinian ring is max. □

A ring is called *valuation* provided it is commutative and its lattice of ideals is linearly ordered.

Since $R/\bigcap_n I^n$ is necessarily noetherian for every valuation ring R and its maximal ideal I, we get an easy consequence of Corollary 2.7 and Proposition 2.10:

Corollary 2.11. *A valuation ring is tall iff it is artinian or it has infinitely generated Jacobson radical.*

The following criterion shows that the Prüffer p-group is a typical example of a non-tall module.

Theorem 2.12. *The following conditions are equivalent for a commutative ring R:*

(1) *R is not tall,*
(2) *there exists a non-noetherian artinian module,*
(3) *there exists an artinian module M, elements $x \in R$ and $m_j \in M$ such that $m_{j+1}x = m_j$ and $m_{j+1} \notin m_j R$ for each j and $M = \bigcup_j m_j R$.*
(4) *there exists a semiartinian module M and a sequence of elements $m_j \in M$, $j < \omega$, such that $S_j(m_j R) = S_j(M)$ is finitely generated, $m_j R \subsetneq m_{j+1} R$ for each j, and $M = \bigcup_j m_j R$,*
(5) *there exists a sequence of ideals J_j of R and elements $x_j \in R$ such that R/J_j is artinian, $J_{j+1} \subsetneq J_j$, $x_j r \in J_{j+1}$ iff $r \in J_j$ and the length of $S_j(R/J_j)$ is equal to the length of $S_j(R/J_k)$ for each $j \leq k < \omega$.*

Proof. (1)→(3) follows from Lemma 2.5(4),(5).

(3)→(2) Since every submodule of M is artinian, there exists a non-noetherian artinian module by Lemma 2.3.

(2)→(1) This is an immediate consequence of Corollary 2.4

(1)→(4) is proved in Lemma 2.5(3),(6).

(4)→(2) This is shown in Lemma 2.3.

(4)→(5) Put $J_j = \mathrm{Ann}(m_j)$ and fix x_j such that $m_{j+1}x_j = m_j$. Then $R/J_j \cong m_jR$ is artinian and $J_{j+1} = \mathrm{Ann}(m_{j+1}) \subseteq \mathrm{Ann}(m_{j+1}x_j) = \mathrm{Ann}(m_j) = J_j$. Moreover, $x_jr \in J_{j+1} = \mathrm{Ann}(m_{j+1})$ iff $rm_j = x_jrm_{j+1} = 0$ iff $r \in \mathrm{Ann}(m_j) = J_j$. Note that $S_j(m_jR) = S_j(M)$ and $S_j(m_jR) \subseteq S_j(m_kR) \subseteq S_j(M)$ for every $k \geq j$, hence $S_j(R/J_j) \cong S_j(R/J_k)$.

(5)→(4) Since the map $i_j : R/J_j \to R/J_{j+1}$ defined by the rule $i_j(r + J_j) = xr + J_{j+1}$ is a monomorphism, the inverse limit of $R/J_1 \overset{i_1}{\hookrightarrow} R/J_2 \overset{i_2}{\hookrightarrow} \dots$ forms a module M where $m_j \in M$ are the images of the elements $1 + J_j \in R/J_j$. □

3. Examples

Recall that a ring R is (Von Neumann) *regular* if for each $x \in R$ there is $y \in R$ for which $xyx = x$. It is easy to see that every prime ideal of a regular ring is necessarily maximal. Now, we generalize [13, Example (1), p.335]:

Lemma 3.1. *Let R be a commutative ring and J an ideal. If R/J is regular and every element of J is nilpotent, then R is tall.*

Proof. Since R contains no non-maximal prime ideal, the conclusion follows immediately from Theorem 2.6. □

Recall that by Corollary 1.2 every max ring is tall. We show that the converse is not true, which answers negatively the question of John Clark [5, Section 4].

Example 3.2. Let F be a field and $\mathbf{X} = \{x_1, x_2, \dots\}$ an infinite countable set of variables. Put $I = \sum_i x_i^2 F[\mathbf{X}]$ and $R = F[\mathbf{X}]/I$. Denote $X_i = x_i + I$ and define an ideal $J = \sum_i X_iR$. Then J is an ideal in which every element is nilpotent, since $X_i^2 = 0$ and R is commutative. As $R/J \cong F$, the hypothesis of Lemma 3.1 is satisfied, hence R is tall.

Moreover, J is a nil maximal ideal of R, thus it is the Jacobson radical of R. Since $X_1 \cdot \ldots \cdot X_n \neq 0$ for every n, J is not T-nilpotent, hence R is not a max ring by [1, Remark 28.5].

Since the ring of p-adic integers is a noetherian valuation domain, it is not tall by Corollary 2.11. By using a similar argument, we show that infinite products of tall rings need not be tall.

First, recall that a system $\mathcal{F}$ of subsets of a set X is called an ultrafilter if $\emptyset \notin \mathcal{F}$, it is closed under finite intersections and all oversets, and for each $G \subset X$ either $G \in \mathcal{F}$ or $X \setminus G \in \mathcal{F}$. We make some observations about ring products using an idea of [14, Lemma 2.2]:

Lemma 3.3. *Let R be a commutative ring, $\mathcal{F}$ an ultrafilter on ω, and mR a principal maximal ideal of R such that $R_n = R/m^nR$ is a valuation ring. Put $M = \{f \in \prod_{n<\omega} R_n | \ (\exists F \in \mathcal{F})(\forall n \in F) : f_n \in mR_n\}$. Then M is a maximal ideal* *of* $\prod_n R_n$*, the ring* $(\prod_n R_n)/M^k$ *is valuation for each* k *and* $\bigcap_k M^k$ *is a prime ideal.*

Proof. First, we show that M^{k-1}/M^k is a simple module for an arbitrary k. For simplicity of the notation we will denote by m the elements $m + m^n R \in R/m^n R$ for each n, as well as the element $m \cdot 1 \in \prod_n R_n$. Moreover, put $M^0 = R$.

Let $r \in M^{k-1} \setminus M^k$. Then $G = \{n < \omega | \ r_n \in m^k R_n\} \notin \mathcal{F}$ and there exists $F \in \mathcal{F}$ such that $r_n \in m^{k-1} R_n$ for each $n \in F$. Hence $H = F \cap (\omega \setminus G) \in \mathcal{F}$ and $\omega \setminus H \notin \mathcal{F}$ since $\mathcal{F}$ is an ultrafilter. Note that there exists an invertible $t_n \in R_n$ such that $r_n = m^{k-1} t_n$ for each $n \in H$, and define $a, b \in \prod_n R_n$ by the rules $a_n = 0$ and $b_n = t_n^{-1}$ if $n \in H$ and $a_n = m^{k-1}$ and $b_n = 0$ whenever $n \notin H$. Clearly, $a + rb = m^{k-1}$ and $a \in M^k$, so we have proved that $M^k + r \prod_n R_n = M^{k-1}$. Hence M^{k-1}/M^k is a simple module and M is a maximal ideal. It implies that $(\prod_n R_n)/M^k$ is a valuation ring. Finally, if $a, b \notin \bigcap_k M^k$, then there is a k for which $a, b \notin M^k$, hence $ab \notin M^{2k}$ because $(\prod_n R_n)/M^{2k}$ is valuation. □

Example 3.4. Let p be a prime number, $\mathbf{Z}_{p^n}$ denote the cyclic group of p^n elements and $S = \prod_{n<\omega} \mathbf{Z}_{p^n}$. Then by Lemma 3.3 the ring S contains a maximal ideal I such that S/I^n is valuation, hence artinian for every n. By applying Proposition 2.9 we obtain that S is not tall, however every ring $\mathbf{Z}_{p^n}$ is tall.

Nevertheless, the product of artinian rings can give us new examples of tall modules.

Proposition 3.5. *Let* κ *be a cardinal and* R_α *commutative artinian rings,* $\alpha < \kappa$*. If there exists* $n < \omega$ *such that* $J(R_\alpha)^n = 0$ *for each* $\alpha < \kappa$*, then* $\prod_{\alpha<\kappa} R_\alpha$ *is tall.*

Proof. Since $(\prod_\alpha J(R_\alpha))^n = 0$ and $\prod_\alpha R_\alpha / \prod_\alpha J(R_\alpha) \cong \prod_\alpha R_\alpha / J(R_\alpha)$ is a regular ring, $\prod_{\alpha<\kappa} R_\alpha$ is tall by Lemma 3.1. □

Note that $\prod_{\alpha<\kappa} R_\alpha$ from the previous proposition is even max by [5, Theorem 3.4].

However, it is not hard to describe which localization of non-tall rings are not tall. We present only a few examples:

Example 3.6. Localization of the ring $\mathbf{Z}$ in the maximal ideal $p\mathbf{Z}$ for each prime p and the localization of the polynomial ring $F[x]$ over any field F in the maximal ideal $xF[x]$ are noetherian domains, so they are not tall by Proposition 2.10.

Finally, we illustrate that ideals J_n from Theorem 2.6 cannot be replaced by J_1^n in general.

Example 3.7. Consider the maximal ideal $M = \sum_n x_n F[\mathbf{X}]$ of the polynomial ring $F[\mathbf{X}]$ in countably many variables $\mathbf{X} = \{x_1, x_2, \dots\}$ over a field F and let R denote the localization of $F[\mathbf{X}]$ in M. Put $P_A = \sum_{j\in A} x_j R$ and note that P_A is a prime ideal for each $A \subseteq \omega$ and P_ω is the unique maximal ideal of R. Obviously, R is not noetherian, and $R/P_{\omega\setminus\{0\}} \cong F[x_0]_{(x_0 F[x_0])}$. As the localization $F[x_0]_{(x_0 F[x_0])}$

is non-tall by Example 3.6 and every factor of a tall ring is tall by Lemma 1.3, R is not tall. However, it is easy to find ideals $J_i = \sum_{j=0}^{n} x_j^i R + \sum_{j>i} x_j R$ ensured by Theorem 2.6. All factors M^j/M^{j+1} are infinitely generated modules, hence we cannot use Proposition 2.9 to construct a non-tall module.

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H. MOD-RETRACTABLE RINGS

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Abstract. A right module M over a ring R is said to be retractable if $\mathrm{Hom}_R(M, N) \neq 0$ for each nonzero submodule N of M and the ring R is (finitely) mod-retractable if every (finitely generated) right R-module is retractable. We proved for every finite group G that $M \otimes_R RG$ is a retractable RG-module iff M_R is retractable. Some comparisons between max rings, semiartinian rings, perfect rings, noetherian rings, nonsingular rings and mod-retractable rings are investigated. In particular, we prove ring-theoretical criteria of right mod-retractability for classes of all commutative, left perfect and right noetherian rings.

1. Introduction

A module M is called *retractable* if there exists a nonzero homomorphism into every nonzero submodule $N \subseteq M$, i.e., $Hom_R(M, N) \neq 0$ for every nonzero submodule N of M. The notion of a retractable module was introduced by Khuri in [11] and it have been studied extensively by many authors (see for example, [8], [12], [13], [14],[19],[20], [21], [22], [23], [27]). Recently, Ecevit and Koşan [4] and independently Haghany, Karamzadeh, and Vedadi [9] introduced the concept of *right (finitely) mod-retractable* rings defined as rings over which every (finitely generated) right module is retractable.

The natural notion of the group module over a group ring was introduced and studied by Koşan-Lee-Zhou in [10]. It is easy to show (Lemma 2.1) that the class of all group modules by group G over a ring R coincides with the image of the tensor functor $- \otimes_R RG$. Note that this subcategory plays an important role in the category of all RG-modules. Section 2 of this note deals with the transfer of properties of retractable modules between a right R-module and its group RG-module. It is shown that MG is a retractable RG-module if and only if M is a retractable R-module.

In Section 3, we investigate some comparisons between max rings, semiartinian rings, perfect rings, noetherian rings, nonsingular rings and mod-retractable rings. We characterize mod-retractable rings as rings whose all torsion theories are hereditary. As a consequence, we prove that a commutative ring is mod-retractable if and only if it is semiartinian. Moreover, we show that a left perfect ring is mod-retractable if and only if it is isomorphic to the ring $\prod_{i \leq k} M_{n_i}(R_i)$ for a finite system of both left and right perfect local rings R_i, $i \leq k$. This result illustrates

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limits of the construction of new examples of mod-retractable rings proved in [4, Theorem 8] as finite products of matrix rings over mod-retractable rings. Namely, applying this procedure on the class of two-sided perfect local rings we get all examples of left perfect right mod-retractable rings. In case R is a right noetherian ring, then it is shown that R is right mod-retractable if and only if $R \cong \prod_{i\leq k} M_{n_i}(R_i)$ for a system of a local right artinian rings R_i, $i \leq k$.

Throughout this paper, we assume that R is an associative ring with unity, M is a unital right R-module and G is a group. In the following the symbols, "$\leq$" will denote a submodule, "$\leq_d$" a module direct summand and "$\trianglelefteq$" an essential submodule. The notations $J(M)$ means Jacobson radical of a module M, $N(R)$ is the prime radical of R and $E(M)$ means an injective envelope of M. The group ring of G over R is denoted by RG. We will refer to [1] and [24] for all undefined notations used in the text.

2. Retractability for group modules

In this section we start to investigate retractable modules over group rings. As the first step, which allows us to produce new examples of retractable modules, we define the notion of a group module.

Hereafter G is a group and M is a module over a ring R.

Let MG denote the set all formal linear combinations of the form $\sum_{g\in G} m_g g$ where $m_g \in M$ and $m_g = 0$ for almost all g.

For elements $\sum_{g\in G} m_g g, \sum_{g\in G} n_g g \in MG$ and $\sum_{g\in G} r_g g \in RG$;

$$\sum_{g\in G} m_g g = \sum_{g\in G} n_g g \quad \text{iff} \quad m_g = n_g \text{ for all } g \in G$$

$$\sum_{g\in G} m_g g + \sum_{g\in G} n_g g = \sum_{g\in G} (m_g + n_g) g$$

$$(\sum_{g\in G} m_g g)(\sum_{g\in G} n_g g) = \sum_{g\in G} (k_g) g \text{ where } \ k_g = \sum_{hh'=g} m_h r'_h.$$

Under the operations defined above, MG has a structure of a right RG-module and it is said to be the *group module* over the group G (see [10]). If we identify every element $m \in M$ with $m \cdot 1 \in MG$, we obtain M as an R-submodule of MG, where 1 hereafter denotes the identity element of G. Recall that 1 of G may be identified with the identity element of RG in natural way.

The following claim shows that the class of all group RG-modules for a ring R and a group G forms precisely the image of the tensor functor $-\otimes_R RG$. We will use this natural description of category of group modules to construct new large classes of retractable modules.

Lemma 2.1. *If MG is the group module, then $MG \cong_{RG} M \otimes_R RG$.*

Proof. Clearly, there exists an RG-homomorphism $\varphi : M \otimes_R RG \to MG$ satisfying the rule $\varphi(\sum_i m_i \otimes \sum_g r_{gi} g) = \sum_{g,i} (m r_{gi}) g = \sum_g (\sum_i m r_{gi}) g$. Now, it is easy to see that φ is onto MG. Finally, φ is injective since RG is a free left R-module. □

The map $MG \to M$, $\sum m_g g \to \sum m_g$, is an R-homomorphism and is denoted by ε_M. The kernel of ε_M is denoted by $\triangle(M)$. Thus, $\varepsilon_R : RG \to R$ is the usual *augmentation map.*

Lemma 2.2. *Let MG be the group module of G by M over RG. Then for any $x \in MG$ and any $\alpha \in RG$, $\varepsilon_M(x\alpha) = \varepsilon_M(x)\varepsilon(\alpha)$. In particular, ε_M is an R-homomorphism and ε_R is a ring homomorphism.*

Proof. Write $x = \sum_{g \in G} m_g g$ and $\alpha = \sum_{g \in G} r_g g$. Then,

$$\begin{aligned} \varepsilon_M(x\alpha) &= \sum_{g \in G} \left(\sum_{hh'=g} m_h r_{h'} \right) \\ &= \left(\sum m_g \right) \left(\sum r_g \right) \\ &= \varepsilon_M(x)\varepsilon(\alpha). \end{aligned}$$

□

As ε_R is onto R and classes of (finite) mod-retractable rings are closed under taking factors, we get the following observations:

Corollary 2.3. *If RG is a (finitely) mod-retractable ring then R is (finitely) mod-retractable.*

Proposition 2.4. *Let $R \subseteq S$ be an extension of rings and S a finitely generated free left R-module. Suppose that N_R is a submodule of a module M_R. Then $\mathrm{Hom}_R(M, N) \neq 0$ iff $\mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) \neq 0$.*

Proof. First we note that

$$\begin{aligned} \mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) &\cong \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, N \otimes_R S)) \\ &\cong \mathrm{Hom}_R(M, N \otimes_R S) \end{aligned}$$

by [24, Proposition I.9.2]. Since ${}_R S$ is a free module, there exists a natural number n such that ${}_R S \cong R^n$. Now

$$\begin{aligned} \mathrm{Hom}_R(M, N \otimes_R S) &\cong \mathrm{Hom}_R(M, N \otimes_R R^n) \\ &\cong \mathrm{Hom}_R(M, N \otimes_R R)^n \\ &\cong \mathrm{Hom}_R(M, N)^n, \end{aligned}$$

which implies that $\mathrm{Hom}_S(M \otimes_R S, N \otimes_R S) \neq 0$ if and only if $\mathrm{Hom}_R(M, N) \neq 0$. □

Recall that $MG \cong_{RG} M \otimes_R RG$ by Lemma 2.1. As the group ring RG is free as both left and right R-modules, we get the following easy consequence of Proposition 2.4.

Corollary 2.5. *Let N be a submodule of a right R-module M. Then* $\mathrm{Hom}_R(M,N) \neq 0$ *if and only if* $\mathrm{Hom}_{RG}(MG,NG) \neq 0$.

Lemma 2.6. *Let $R \subseteq S$ be an extension of rings, S a finitely generated free left R-module, M an S-module, and N an R-submodule of $M \otimes_R S$. Then every nonzero $\alpha \in \mathrm{Hom}_R(M,N)$ can be extended to a nonzero S-homomorphism of $M \otimes_R S$ into NS.*

Proof. Let $\alpha \in \mathrm{Hom}_R(M,N)$ and $\alpha \neq 0$. Obviously, $\alpha \otimes S$ is an S-homomorphism of $M \otimes_R S$ into $N \otimes_R S$. Moreover, a mapping $\psi : N \otimes_R S \to NS$ defined by the rule $\psi(\sum_i n_i \otimes s_i) = \sum_i n_i s_i$ is an S-homomorphism as well, hence $\phi = \psi\alpha \otimes S$ is the required homomorphism of $M \otimes_R S$ into NS. Finally, we can easily see that $\phi(m \otimes 1) = \psi(\alpha(m) \otimes 1) = \alpha(m)$ for each $m \in M$, i.e. $\psi \neq 0$. □

Theorem 2.7. *Suppose that R is a subring of a ring S, M an R-module and $\{e_1, \dots, e_n\}$ a free base of S as a right R-module such that $e_i r = re_i$ for all i and $r \in R$. Then $M \otimes_R S$ is a retractable S-module if and only if M is retractable.*

Proof. Note that $\{e_1, \dots, e_n\}$ is also a free base of S as a left R-module. As ${}_RS$ is a projective module, the functor $- \otimes_R S$ is exact, hence the direct implication follows immediately from Proposition 2.4.

Suppose that M is a retractable R-module and fix a nonzero S-submodule P of $M \otimes_R S$. We have to show that $\mathrm{Hom}_{RG}(M,P) \neq 0$. For each $\mu \in M \otimes_R S$ define

$$\sigma(\mu) = \{F \subset \{1, \dots, n\} \mid \exists m_1, \dots m_n \in M : \mu = \sum_{i \in F} m_i \otimes e_i\}.$$

It is easy to see that $\sigma(\mu) \neq \emptyset$, hence we may put $s(\mu) = \min\{|F| \mid F \in \sigma(\mu)\}$. Now, we can choose a nonzero element $\mu \in P$ with a minimal (nonzero) value of $s(\mu)$. Thus there exist different numbers $i_1, \dots, i_{s(\mu)} \leq n$ and nonzero elements $m_1, \dots, m_{s(\mu)} \in M$ such that $\mu = \sum_{j=1}^{k} m_j \otimes e_{i_j}$. If there are $r \in R$ and $j \leq s(\mu)$ for which $m_j \otimes e_{i_j} r = m_j r \otimes e_{i_j} = 0$, then $s(\mu r) < s(\mu)$, hence $\mu r = 0$ due to the minimality of $s(\mu)$. Thus the annihilators of all $m_1, \dots, m_{s(\mu)}$ coincide and cyclic R-modules μR and $m_i R$ are R-isomorphic. As M is retractable, there exists a nonzero R-homomorphism of M into μR. Now, applying Lemma 2.6 for $N = \mu R$, we obtain a nonzero S-homomorphism of $M \otimes_R S$ to $\mu S \subseteq P$, which finishes the proof. □

Recall that the group ring RG is right perfect iff R is a right perfect ring and G is a finite group by [26, Theorem]. As an immediate consequence of Theorem 2.7 we have the following similar result:

Corollary 2.8. *Let M be an R-module and G a finite group. Then MG is a retractable RG-module iff M_R is retractable.*

However every tensor product $M \otimes S$ of a retractable R-module M and a ring extension satisfying the hypothesis of Theorem 2.7 has to be retractable over S, the following example shows that such an extension of mod-retractable rings is not necessarily mod-retractable.

Example 2.9. Let F be a field and put $S = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. Then matrices $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ commutes with the subring $R = FE_1 \cong F$ of S and they produce a free base of S a right and left R-module. However every R-module M is retractable, hence every S module $M \otimes S$ is retractable as well by Theorem 2.7, the S-module E_2S is not retractable.

3. Mod-retractable rings

The general case and max rings

First, we prove a general module-theoretic criterion for Mod-retractable rings.

Proposition 3.1. *R is a right mod-retractable ring iff for every nonzero module M and every $m \in M$ such that $mR \trianglelefteq M$ there exists a nonzero homomorphism $M \to mR$.*

Proof. It suffices to show the reverse implication. Fix an arbitrary nonzero module M and its nonzero submodule N. Let $n \in N \setminus \{0\}$. Then an identity mapping on nR may be extended to the homomorphism $\nu : M \to E(nR)$. Note that $nR \trianglelefteq \nu(M) \subseteq E(nR)$, hence by the hypothesis there exists a nonzero homomorphism $\nu(M) \to nR \subseteq N$, which finishes the proof. □

We recall that

• a ring is called *right max* provided every nonzero right module contains a maximal submodule,

• an ideal $I \subset R$ is right T-nilpotent, provided for every sequence $a_1, a_2, \dots \in I$ there exist n such that $a_n a_{n-1} \dots a_1 = 0$.

Lemma 3.2. *If R is a right max ring, then $J(R)$ is right T-nilpotent.*

Proof. It is well known (see for example [1, Remark 28.5] or [25, Proposition 1.8]). □

Theorem 3.3. *If R is a right mod-retractable ring, then R is right max.*

Proof. Assume that $0 \neq M$ contains no maximal submodule, fix $0 \neq m \in M$ and an arbitrary maximal submodule N of mR. Then M/N contains no maximal submodule and so there exists no nonzero homomorphism M/N into a simple mR/N, i.e. M/N is not retractable. □

As an immediate consequence of the previous results we obtain

Corollary 3.4. *Jacobson radical of every right mod-retractable ring is right T-nilpotent.*

Recall that a *torsion theory* $\tau = (\mathcal{T}, \mathcal{F})$ is a pair of classes of modules closed under isomorphic images such that $\mathcal{T} \cap \mathcal{F} = 0$, $\mathcal{T}$ is closed under taking factors, $\mathcal{F}$ is closed under submodules and for every module M there exists a submodule $\tau(M)$ for which $\tau(M) \in \mathcal{T}$ and $M/\tau(M) \in \mathcal{F}$. Moreover, a torsion theory is hereditary if $\mathcal{T}$ is closed under submodules.

For a class of right R-modules $\mathcal{C}$, we consider the following *annihilator classes*:

$$^{\circ}\mathcal{C} = \{M \in \text{Mod-}R \mid \text{Hom}_R(M, \mathcal{C}) = 0\}$$

and

$$\mathcal{C}^{\circ} = \{M \in \text{Mod-}R \mid \text{Hom}_R(\mathcal{C}, M) = 0\}.$$

We notice that the annihilator classes of the form $^{\circ}\mathcal{C}$ for some $\mathcal{C} \subseteq \text{Mod-}R$ coincide with the torsion classes of modules, and the annihilator classes of the form $\mathcal{C}^{\circ}$ coincide with the torsionfree classes of modules.

Theorem 3.5. *A ring R is mod-retractable if and only if every torsion theory on* Mod-R *is hereditary.*

Proof. Suppose that R is mod-retractable and $\tau = (\mathcal{T}, \mathcal{F})$ is a torsion theory. For $M \in \mathcal{T}$ and $N \leq M$, let $\tau(N)$ be the torsion part of N. Then $M/\tau(N) \in \mathcal{T}$, while $N/\tau(N) \in \mathcal{F}$. Then $\text{Hom}(M/\tau(N), N/\tau(N)) = 0$. Since $N/\tau(N)$ is a submodule of $M/\tau(N)$ and $M/\tau(N)$ is retractable, it follows that $N/\tau(N) = 0$. Hence $N \in \mathcal{T}$.

Conversely, suppose that M is an R-module and $0 \neq N \leq M$. If $\text{Hom}(M, N) = 0$, then $N \notin {}^{\circ}(M^{\circ})$. This implies that the torsion theory $({}^{\circ}(M^{\circ}), M^{\circ})$ is not hereditary. □

A chain $(Y_\alpha | \alpha \leq \sigma)$ is called a *strictly decreasing continuous chain* of submodules of Y provided that $Y_0 = Y$, $Y_\alpha \supset Y_{\alpha+1}$ for each $\alpha < \sigma$, $Y_\alpha = \bigcup_{\beta<\alpha}$ for each limit ordinal $\alpha \leq \sigma$, and $Y_\sigma = 0$.

The following result was proved in [3, Lemma 3].

Lemma 3.6. *Let R be a ring and let X and Y be nonzero R-modules. Then the following are equivalent:*

(1) $^\circ X \subseteq {}^\circ Y$;
(2) *there exists a strictly decreasing continuous chain $(Y_\alpha | \alpha \leq \sigma)$ of submodules of Y and R-homomorphisms $\varphi_\alpha : Y_\alpha \to X$, $\alpha < \sigma$, such that $Y_{\alpha+1} = Ker(\varphi_\alpha)$ for all $\alpha < \sigma$.*

Theorem 3.7. *The following are equivalent for a ring R:*

(1) *R is mod-retractable*
(2) *If $X \trianglelefteq Y$ then $^\circ X = {}^\circ Y$*
(3) *For every module X, $^\circ X = {}^\circ E(X)$*

Proof. (1)⇒(2) The inclusion $^\circ Y \subseteq {}^\circ X$ is obvious. In order to prove the converse inclusion, we will apply Lemma 3.6. So we construct a strictly decreasing continuous chain $(Y_\sigma | \sigma \leq \tau)$ of submodules of Y and a family of nonzero homomorphisms $f_\sigma : Y_\sigma \to X$ for all $\sigma < \tau$.

We put $Y_0 = Y$. Since R is mod-retractable, there is a nonzero homomorphism $f_0 : Y_0 \to X$. Suppose that the submodules Y_ρ and the nonzero homomorphisms $f_\rho : Y_\rho \to X$ are constructed for all $\rho < \sigma$. If $\sigma = \rho + 1$ we denote $Y_\sigma = \mathrm{Ker}(f_\rho)$, and for σ a limit ordinal we put $Y_\sigma = \bigcap_{\rho<\sigma} Y_\rho$. If $Y_\sigma = 0$ then the construction is finished. If $Y_\sigma \neq 0$ then $Y_\sigma \cap X \neq 0$, hence there is a nonzero homomorphism $f_\sigma : Y_\sigma \to X$ such that $f_\sigma(Y_\sigma) \subseteq Y_\sigma \cap X$.

Since for cardinality reasons there is τ with $Y_\tau = 0$, we can apply Lemma 3.6, and the proof is complete.

(2)⇒(3) This is obvious.

(3)⇒(1) Let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. If $X \in \mathcal{F}$ then $\mathcal{T} \subseteq {}^\circ X = {}^\circ E(X)$, hence $E(X) \in \mathcal{F}$. Then $\mathcal{F}$ is closed with respect injective envelopes, hence τ is hereditary. □

Commutative rings

To obtain a relation between mod-retractable rings and semiartinian rings in the commutative case, we state the main theorem of Ohtake in [17, Theorem 8].

Theorem 3.8. *Let R be a commutative ring. Then the following are equivalent.*

(1) *Every torsion theory in* Mod-R *is of simple type.*
(2) *Every torsion theory in* Mod-R *is hereditary.*
(3) *R is a semiartinian, max ring.*

Now, we are able to prove a criterion of mod-retractability for commutative rings. Note that the the straightforward proof of the converse implication is presented in [9, Theorem 2.8].

Theorem 3.9. *Let R be a commutative ring. Then R is mod-retractable if and only if R is semiartinian.*

Proof. Suppose that R is commutative semiartinian. Then $J(R)$ is T-nilpotent by [16, Proposition 3.2] and $R/J(R)$ is von Neumann regular by [16, Theoreme 3.1]. Hence R is mod-retractable by [7, Theorem 3] and Theorem 3.8.

The converse is clear from Theorems 3.5 and 3.8. □

Example 3.10. Let λ be an ordinal and F a field. Put $\kappa = \max(\text{card } \lambda, \omega)$. Examples of commutative semiartinian regular F-subalgebras of the algebra F^κ of the socle length $\lambda + 1$, which are mod-retractable by Theorem 3.9, is constructed in [5, Theorem 2.6].

Perfect rings

Let M be an R-module. Recall that a submodule N of M is said to be a *superfluous* in M, denoted by $N \ll M$, whenever $L \leq M$ and $M = N + L$ then $M = L$.

Lemma 3.11. *Let M be a nonzero semiartinian module, N its superfluous submodule, and S_i, $i \in I$, simple modules. If $M/N \cong \bigoplus_{i\in I} S_i$ and there exists a simple subfactor of N which is not isomorphic to any S_i, $i \in I$, then there exists a non-retractable factor of M.*

Proof. Let T be a simple submodule of N/X where X is a submodule of N that is not isomorphic to any S_i. Since $N/X \ll M/X$ we may suppose that $X = 0$.

Assume that there exists a nonzero homomorphism $M \to T$. Then there exists a maximal submodule $Y \subset M$ such that $M/Y \cong T$. If $N \not\subseteq Y$, we get $N \neq N + Y = M$ and $Y = M$ because $N \ll M$, a contradiction. Thus $N \subseteq Y$, which implies that $T \cong M/Y$ is a direct summand of $M/N \cong \bigoplus_{i\in I} S_i$. Hence there exists $j \in I$ such that $S_i \cong T$, which contradicts to the hypothesis. We have proved that $\operatorname{Hom}(M, T) = 0$. □

Proposition 3.12. *Every local ring which is both right and left perfect is right mod-retractable.*

Proof. Assume that R is a local right and left perfect ring. Let M be a right module over R and N a nonzero submodule of M. Applying [1, Theorem 28.4], we get that $M/MJ(R) \cong R/J(R)^{(\kappa)}$ is nonzero semisimple since R is right perfect and N is a nonzero semiartinian module because R is left perfect. Thus there exists a surjective homomorphism $M \to M/MJ(R) \to R/J(R)$ and $\operatorname{Soc}(N)$ is a nonzero submodule of N isomorphic to a direct power of $R/J(R)$, which proved the existence of a nonzero homomorphism $M \to N$. □

The following example shows that the assumption "right perfect and left perfect" in Proposition 3.12 is not superfluous, i.e., there exits local, right perfect, but not left perfect rings, which is neither right nor left mod-retractable.

Example 3.13. Let F be a field and V_F be an infinite dimensional vector space with a countable ordered basis $\{v_n \mid n \in \mathbb{N}\}$, so that every endomorphism of V_k can be described by a column finite $\mathbb{N} \times \mathbb{N}$ matrix with entries in k. We denote by I he identity matrix and by $e_{n,m}$ the unit matrices for every $n, m \in \mathbb{N}$. Consider the F-subalgebra R of $\mathrm{End}(V_F)$ generated by I and all the matrices $e_{n,m}$ with $n, m \in \mathbb{N}$ and $n < m$. Note that $R = \{a_0 I + \sum_{i=1}^{k} a_i e_{n_i,m_i} | \ \in \mathbb{N}, a_0, \dots, a_k \in F\}$, i.e. R is the ring of all the $\mathbb{N}\times\mathbb{N}$ upper triangular matrices over the field F that are constant on the diagonal and have only finitely many nonzero entries off the diagonal, all of them over the diagonal. If f is a F-linear combination of finitely many $e_{n,m}$ with $n < m$ is strictly upper triangular matrix with finitely many entries, hence f nilpotent. We can see that all matrices with zero on the diagonal form an ideal M of R, and every $r \in R \setminus M$ is invertible. Thus R is a local ring with maximal ideal M. For every $n > 0$, we have that $e_{0,1}e_{1,2}e_{2,3}...e_{n-1,n} = e_{1,n} \neq 0$, so that M is not left T-nilpotent. It remains to show that M is right T-nilpotent. Take any sequence $a_1, a_2, \dots$ in M. Write a_1, as a linear combination of the $e_{n,m}$ with $n < m$: $a_1 = \sum_{i=1}^{t} \lambda_i e_{n_i,m_i}$. We can suppose $m_1 \leq m_2 \leq \cdots \leq m_t$. It is now easy to verify that $a_{m_t+1}a_{m_t} \dots a_2 a_1 = 0$. Hence M is right T-nilpotent. Note that R contains no simple right ideal, however it is left semiartinian.

Finally, we will show that R is neither right nor left mod-retractable. First recall that $J(R)$ is not left T-nilpotent, hence R is not left mod-retractable by Corollary 3.4.

Suppose that I is an essential right ideal of $J(R)$ and fix $i \in \mathbb{N}$. Then there exists $m \in M$ such that $e_{ii+1}m = \sum_j a_j e_{ii_j} \in I$ for some pairwise distinct i_j and some $a_j \neq 0$. It implies $e_{ii_1+s} = \sum_j a_j e_{ii_j} a_1^{-1} e_{i_1 s} = e_{ii+1} m(a_1^{-1} e_{i_1 s}) \in I$ for each $s > i_1$, hence $e_{ii+1}M^s \in I$. Since R/M^s is a semiartinian module, we have proved that R/I is semiartinian as well. Now, let $\varphi : E(R_R) \to J(R) \subset E(R_R)$ be a homomorphism. Since $J(R)$ contains no idempotent, it contains no nonzero injective submodule (cf. Lemma 3.16), hence kernel of φ is essential in $E(R)$. Thus for every $x \in E(R)$ there exists an essential right ideal I of R such that $\varphi(xR) \cong R/I$, which implies that $\varphi(xR)$ is semiartinian submodule of $J(R)$. Since R contains no simple right ideal, $\varphi = 0$, hence $\mathrm{Hom}(E(R), J) = 0$. We have proved that $E(R)$ is not retractable and so R is not right mod-retractable.

Recall that the ring $\prod_{i\in\mathcal{I}} R_i$ is right mod-retractable if and only if R_i is right mod-retractable for each $i \in \mathcal{I}$, where $\mathcal{I}$ is finite, by [4, Theorem 8] . For perfect rings we prove more precise structural result.

Theorem 3.14. *Let R be a right and left perfect ring. Then the following conditions are equivalent:*

(1) *R is right mod-retractable;*
(2) *R is right finitely mod-retractable;*
(3) *$R \cong \prod_{i \leq k} M_{n_i}(R_i)$ for a system of a local right and left perfect rings R_i, $i \leq k$.*

Proof. (1)⇒(2) is clear.

(2)⇒(3) We may suppose without lost of generality that R is an indecomposable ring. Denote by $\{e_i,\ i \leq n\}$ a complete set of orthogonal idempotents of R. Assume that there exists $i, j \leq n$ such that $e_iR/e_iJ(R) \ncong e_jR/e_jJ(R)$ and put $I = \{s \leq n |\ e_sR/e_sJ(R) \cong e_iR/e_iJ(R)\}$. Define an idempotent $e = \sum_{j \in I} e_i$ and note that

$$\mathrm{Hom}_R(eR/eJ, (1-e)R/(1-e)J) = 0 = \mathrm{Hom}_R((1-e)R/(1-e)J, eR/eJ).$$

Hence either $\mathrm{Hom}_R(eR, (1-e)R) \neq 0$ and so $(1-e)R$ contains a subfactor isomorphic to e_iR/e_iJ or $\mathrm{Hom}_R((1-e)R, eR) \neq 0$ and so eR contains a subfactor isomorphic to e_jR/e_jJ for a suitable $j \notin I$, since R is indecomposable. Now applying Lemma 3.11, either for $M = (1-e)R$ in the first case or for $M = eR$ in the second case, we see that R is not finitely mod-retractable. Hence $e_iR/e_iJ(R) \cong e_jR/e_jJ(R)$ for all i, j. Now it is well known that $R \cong \mathrm{End}_R(e_1R^n) \cong M_n(e_1Re_1)$ where e_1Re_1 is a local right and left perfect ring.

(3)⇒(1) follows by [4, Corollary 3 and Theorem 8] and Proposition 3.12. □

Now, we can formulate the following easy structural consequence of Theorems 3.3 and 3.14.

Corollary 3.15. *Let R be a left perfect ring. Then R is right mod-retractable if and only if $R \cong \prod_{i \leq k} M_{n_i}(R_i)$ for a system of a local rings R_i, $i \leq k$, which are both left and right perfect.*

Nonsingular rings

Recall that, a module M_R is said to be *singular* (respectively, *nonsingular*) if $Z(M_R) = M_R$ (respectively, $Z(M_R) = 0$), where $Z(M_R) = \{m \in M : ann_R^r(m) \trianglelefteq R\}$. If $Z(R_R) = 0$ then R is called *a right nonsingular* ring.

Lemma 3.16. *Let R be a right mod-retractable ring and M a non-singular module. Then*

(1) *for every nonzero $m \in M$ there exists a nonzero injective module $E \subseteq mR$,*
(2) *$J(M) = 0$.*

Proof. (1) Fix a nonzero $m \in M$. Since $E(M)$ is a retractable module and $mR \subseteq E(M)$, there exists a nonzero homomorphism $\varphi : E(M) \to mR$. As M is non-singular, $\mathrm{Ker}\varphi$ is not essential in $E(M)$, hence applying the same technique as in [2, Lemma 3.3] we can find $y \in E(M)$ such that $x = \varphi(y) \neq 0$, $yR \cap \mathrm{Ker}\varphi = 0$ and so $yR \cong \varphi(yR)$. This implies that $E(yR) \cap \mathrm{Ker}\varphi = 0$ where $E(yR)$ can be expressed as a direct summand of $E(M)$. As $E(yR) \cong \varphi(E(yR))$, the module $\varphi(E(yR))$ is injective.

(2) Let $m \in M \setminus \{0\}$. By (1) there exists a nonzero injective submodule E of mR, hence there is $N \subseteq M$ such that $M = E \oplus N$, which implies that $J(M) \subseteq J(E) \oplus N$. Since R is right max by Theorem 3.3, $J(E) \neq E$, thus $E \subseteq mR \not\subseteq J(E) \oplus N$. We have proved that $m \notin J(M)$ for arbitrary nonzero m, hence $J(M) = 0$. □

Corollary 3.17. *If R is a right non-singular right mod-retractable ring, then $J(R) = 0$ and for every nonzero $r \in R$, there exists a nonzero idempotent $e \in rR$ such that eR is an injective module.*

Theorem 3.18. *Every right noetherian right mod-retractable right non-singular ring is semisimple.*

Proof. Let R be a right mod-retractable right non-singular ring. First note that $J(R) = 0$ by Corollary 3.17. Assume that R is not semisimple and define two sequences of right ideals $\{I_n\}$ and $\{J_n\}$ such that $I_{n+1} \oplus J_{n+1} = J_n$, $I_n \neq 0$ and J_n is not semisimple for each $n \geq 0$.

Take a non-trivial idempotent $e \in R$ which exists by Lemma 3.17. Then either eR or $(1-e)R$ is not semisimple. If eR is not semisimple put $I_0 = (1-e)R$ and $J_0 = eR$ otherwise $I_0 = eR$ and $J_0 = (1-e)R$.

Since J_n is not semisimple, by Corollary 3.17, there exist an idempotent $f \in J_n$ such that $0 \neq fR \neq J_n$ and fR is injective. Thus $J_n = fR \oplus G$ for a suitable submodule G and we put $J_{n+1} = fR$ and $I_{n+1} = G$ if fR is not semisimple and $J_{n+1} = G$ and $I_{n+1} = fR$ otherwise.

Now we can see that $\bigoplus_{n<\omega} I_n$ is an infinitely generated right ideal. We have proved that a right mod-retractable right non-singular ring which is not semisimple is not right noetherian. □

Proposition 3.19. *Let R be a semiartinian mod-retractable ring and I be an ideal. Then*

1. *$J(R/I)$ is T-nilpotent,*
2. *$(R/I)/J(R/I)$ is non-singular,*
3. *every nonzero ideal of R/I contains a nonzero idempotent $e \in R/I$ such that $e(R/I)/eJ(R/I)$ is an injective $(R/I)/J(R/I)$-module.*

Proof. (1) It follows by Corollary 3.4

(2) Without lost of generality, we may assume that $J(R) = 0$ and $x\mathrm{Soc}(R) = 0$ for a nonzero $x \in R$. Note that $xR \cap \mathrm{Soc}(R) \neq 0$ and take $0 \neq y \in xR \cap \mathrm{Soc}(R)$. Then $yRyR = 0$, hence $yR \in J(R)$, a contradiction.

(3) Since $R/J(R)$ is nonsingular by [15, Lemma 7.8], we may apply Corollary 3.17. Thus there exists an idempotent in $R/J(R)$ which can be lifted to an idempotent $e \in R$ such that $eR/eJ(R)$ is injective $R/J(R)$-module. □

Noetherian rings

The following lemma is analogue to [2, Proposition 3.16].

Lemma 3.20. *Let R be a right noetherian ring. If R is right mod-retractable, then it is right artinian and left perfect.*

Proof. Since $R/N(R)$ contains no nilpotent ideal, $R/N(R)$ is right non-singular by [24, Lemma II.2.5]. Note that $J(R) \subseteq N(R)$ in general and $R/N(R)$ is semisimple by Theorem 3.18, which implies that $J(R) = N(R)$. Finally, since $J(R)$ is nilpotent by [24, Lemma XV.1.4] we get that R is right artinian by Hopkins-Levitzki Theorem and R is left perfect by [1, Theorem 28.4]. □

Theorem 3.21. *Let R be a right noetherian ring. Then R is right mod-retractable if and only if $R \cong \prod_{i\leq k} M_{n_i}(R_i)$ for a system of local right artinian rings R_i, $i \leq k$.*

Proof. ($\Rightarrow$) It follows from Lemma 3.20 and Corollary 3.15.

($\Leftarrow$) Since $R \cong \prod_{i\leq k} M_{n_i}(R_i)$ is left and right perfect, the proof is clear from Corollary 3.15 □

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I. ON MODULES AND RINGS WITH THE RESTRICTED MINIMUM CONDITION

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ABSTRACT. A module M satisfies the restricted minimum condition if M/N is Artinian for every essential submodule N of M and R is called a right RM-ring whenever R_R satisfies the restricted minimum condition as a right module. Several structural necessary conditions for particular classes of RM-rings are presented in the paper. Furthermore, a commutative ring R is proved to be an RM-ring if and only if $R/\operatorname{Soc}(R)$ is Noetherian and every singular module is semiartinian.

1. INTRODUCTION

Given a module M over a ring R, recall that N is an *essential* submodule of M if there is no non-zero submodule K of M such that $K \cap N = 0$ and we say that M satisfies *restricted minimum condition* (RMC) if for every essential submodule N of M, the factor module M/N is Artinian. It is easy to see that the class of all modules satisfying RMC is closed under submodules, factors and finite direct sums. A ring R is called a right *RM-ring* if R_R satisfies RMC as a right module. An integral domain R satisfying the restricted minimum condition is called an RM-domain, i.e. R/I is Artinian for all non-zero ideals I of R (see [4]). Recall that a Noetherian domain has Krull dimension 1 if and only if it is an RM-domain [5, Theorem 1].

The purpose of the present paper is to continue in study of works [3],[4], [5], [10] and [14], in which basic structure theory of RM-rings and RM-domains was introduced, and the recent paper [1], which describes some properties of classes of torsion modules over RM-domains known and widely studied for corresponding classes of abelian groups. As the method of [1] appears to be fruitful, this paper is focused on study of structure of modules satisfying RMC, in particular singular ones. For a module M with the essential socle, we show that M satisfies RMC if and only if $M/\operatorname{Soc}(M)$ is Artinian. It is also proved, among other results, that if M is singular, then M is semiartinian for a module M over a right RM-ring R.

This tools allow us to obtain ring theoretical results for non-commutative as well as commutative rings. Namely, if R is a right RM-ring and $\operatorname{Soc}(R) = 0$, we prove that R is a nonsingular ring of finite Goldie dimension. As a consequence, in Section 2, we obtain characterizations for various classes of right RM-rings via some well

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known and important rings (semiartinian, (Von Neumann) regular, semilocal, max, perfect) plus some (socle finiteness) conditions:

In the case when R is a semilocal right RM-ring and $\operatorname{Soc}(R) = 0$, we show that R is Noetherian if and only if $J(R)$ is finitely generated if and only if the socle length of $E(R/J(R))$ is at most ω. If R is a right max right RM-ring, we prove that $R/\operatorname{Soc}(R)$ is right Noetherian. Section 3 is focused on commutative rings R, it is shown that such a ring R satisfies RMC if and only if $R/\operatorname{Soc}(R)$ is Noetherian and every singular module is semiartinian.

Throughout this paper, rings are associative with unity and modules are unital right R-modules, where R denotes such a ring and M denotes such a module. We write $J(R), J(M), \operatorname{Soc}(R), \operatorname{Soc}(M)$ for the Jacobson radical of the ring R, for the radical of the module M, the socle of R and the socle of M, respectively. We also write $N \trianglelefteq M$ and $E(M)$ to indicate that N is an essential submodule of M and the injective hull of M, respectively.

2. The structure of general right RM-rings

First state one elementary observation about submodules of modules satisfying RMC and then recall a useful folklore observation (see [11, Lemma 3.6]):

Lemma 2.1. *Let K and N be submodules of M such that $K \trianglelefteq N$. If M satisfies RMC, then N/K is Artinian.*

Proof. If we chose a submodules A for which $N \cap A = 0$ and $N \oplus A \trianglelefteq M$, then $K \oplus A \trianglelefteq M$. Hence $M/(K \oplus A)$ and $(N \oplus A)/(K \oplus A) \cong N/K$ are Artinian modules. □

Let us recall that *Goldie dimension* of a module M is defined as the supremum of the set

$$\{n \in \mathbb{N} \,|\, \exists N_1 \dots, N_n \text{ submodules of } M:\ \oplus_{i=1}^{n} N_i \subseteq M\}.$$

Lemma 2.2. *If a module M satisfies RMC, then Goldie dimension of $M/\operatorname{Soc}(M)$ is finite.*

Proof. Put $S_0 := \operatorname{Soc}(M)$ and fix a submodule S_1 of the module M such that $S_0 \subseteq S_1$ and $S_1/S_0 = \operatorname{Soc}(M/S_0)$. By Zorn's Lemma, we may choose a maximal set of elements $m_i \in M$ satisfying the condition $S_1 \cap \bigoplus_{i\in I} m_i R = 0$. It is easy to see that $S_1 \oplus \bigoplus_{i\in I} m_i R \trianglelefteq M$. Since $\bigoplus_{i\in I} m_i R \cap S_0 = 0$, every module $m_i R$ has zero socle. Hence $m_i R$ is not simple and any maximal submodule of $m_i R$ is essential in $m_i R$. For every $i \in I$, let N_i be a fix maximal submodule in $m_i R$. As $\bigoplus_{i\in I} N_i \trianglelefteq \bigoplus_{i\in I} m_i R$, the module $L = S_0 \oplus \bigoplus_{i\in I} N_i$ is essential in M. Applying RMC, we get that M/L is an Artinian module containing an isomorphic copy of $(S_1/S_0) \oplus \bigoplus_{i\in I} m_i R/N_i$ which implies that I is finite and S_1/S_0 is a finitely generated semisimple module. By [12, Proposition 6.5], we obtain that the uniform dimension of $M/\operatorname{Soc}(M)$ is finite. □

Following [7, Section 7.2] recall that the zero module is the only module of Krull dimension -1 and the class $\mathcal{M}_\alpha$ of modules with Krull dimension α is inductively defined as the class of all modules M satisfying two conditions:

(i) $M \notin \bigcup_{\beta<\alpha} \mathcal{M}_\beta$,
(ii) for every decreasing chain $M_0 \supseteq M_1 \supseteq \dots$ of submodules of M there exists n such that $M_i/M_{i+1} \in \bigcup_{\beta<\alpha} \mathcal{M}_\beta$.

It is easy to obtain information about Krull dimension of modules satisfying RMC.

Proposition 2.3. *If an module M satisfies RMC, then Krull dimension of $M/\operatorname{Soc}(M)$ is at most one.*

Proof. Note that a module has zero Krull dimension if it is an Artinian module. Suppose that $N_0 \supseteq N_1 \supseteq \dots$ is a sequence of submodules of $M/\operatorname{Soc}(M)$. As $M/\operatorname{Soc}(M)$ has a finite Goldie dimension by Lemma 2.2, there exists n such that for each $i \geq n$ either $N_i = 0$ or $N_{i+1} \trianglelefteq N_i$. Since N_{i+1}/N_i is Artinian by Lemma 2.1, we obtain that $M/\operatorname{Soc}(M)$ has a Krull dimension at most 1. □

Obviously, the class of all right RM-rings is closed under taking factors and finite products. To show that it is not the case of taking extensions we recall the notion of a *semiartinian* module, i.e. a module M such that every non-zero factor of M contains a non-zero socle. A ring R is called *right semiartinian* if R_R is a right semiartinian module. Note that every non-zero right module over a right semiartinian ring is semiartinian (for basic results cf [9]).

Let M be a semiartinian module. It is well known (see e.g. [8, 13]) that every semiartinian module contains an increasing (so called socle) chain of submodules $(S_\alpha \mid \alpha \geq 0)$ satisfying $S_0 = 0$, $S_{\alpha+1}/S_\alpha = \operatorname{Soc}(M/S_\alpha)$ for each ordinal α and $S_\alpha = \bigcup_{\beta<\alpha} S_\beta$ if α is a limit ordinal. Furthermore recall that the first ordinal σ such that $S_\sigma = M$ is said to be *the socle length* of M.

Since every semiartinian ring contains essential socle we obtain an easy observation:

Lemma 2.4. *Let R be a right semiartinian ring. Then R is a right RM-ring if and only if $R/\operatorname{Soc}(R)$ is Artinian.*

The following example shows that the class of modules satisfying RMC is not closed under extensions.

Example 2.5. Let R be a right semiartinian ring of socle length 3 and such that $R/\operatorname{Soc}(R)$ is non-Artinian, hence R is not a right RM-ring by Lemma 2.4. Then $R_0 = R/\operatorname{Soc}(R)$ is a right RM-ring by Lemma 2.4 because $R_0/\operatorname{Soc}(R_0)$ is semisimple. Clearly $\operatorname{Soc}(R)$ satisfies RMC as well. Hence the short exact sequence

$$0 \to \operatorname{Soc}(R) \to R \to R/\operatorname{Soc}(R) \to 0$$

shows that the class of all modules satisfying RMC is not closed under extensions.

In particular, using constructions of [6], we can fix a field F and take R_1 as an F-subalgebra of F-algebra F^ω of all countable sequences over F generated by ideal of ultimately zero sequences $F^{(\omega)}$ where ω denotes the first infinite ordinal, this F-subalgebra contains exactly ultimately constant sequences. Then R_2 is defined as an F-subalgebra of a natural F-algebra R_1^ω generated by $R_1^{(\omega)}$. It is easy to see that R_2 is a right semiartinian ring of socle length 3 and $R_2/\operatorname{Soc}(R_2)$ is non-Artinian.

Let us recall another well-known observation.

Lemma 2.6. *Let M be an Artinian R-module. If $J(N) \neq N$ for every nonzero submodule N of M, then M is Noetherian.*

Proof. Assume that M is not Noetherian. Then it contains a semiartinian submodule of infinite socle length. As M is Artinian, there is a minimal submodule N of infinite socle length. Thus N contains no maximal submodule, i.e. $J(N) = N$. □

Now we are able to clarify structure of RM-rings, which is similar (and in some sense dual) to structure of semiartinian rings.

Theorem 2.7. *Let R be a right RM-ring, $S(R)$ the greatest right semiartinian ideal of R and put $A := R/\operatorname{Soc}(R)$ and $S(A) := S(R)/\operatorname{Soc}(R)$.*

(i) *$\bigcap_{n<\omega} J(A)^n$ is nilpotent,*
(ii) *$S(A) \cap J(A)$ is nilpotent,*
(iii) *$S(A)/(S(A) \cap J(A))$ is Noetherian.*

Proof. First note that $S(A)$ is the greatest right semiartinian ideal of $S(R)/\operatorname{Soc}(R)$.

(i) Since Krull dimension of A is equal to 0 or 1 by Proposition 2.3, we may directly apply [7, Theorem 7.26], which proved that $\bigcap_n J(A)^n$ is a nilpotent.

(ii) Put $K := S(A)\cap(\bigcap_n J(A)^n)$ and $I := S(A)\cap J(A)$. Note that K is a nilpotent by (i). Since $S(A)$ is Artinian by Lemma 2.1, we obtain that I is Artinian. Moreover, $I^n \subseteq J(A)^n$, and so $\bigcap_n I^n \subseteq K$. Since I Artinian, there exists n for which $I^n \subseteq K$, which finishes the proof.

(iii) Note that $S(A)$ and so $M = S(A)/(S(A)\cap J(A))$ is Artinian and $J(M) = 0$. Hence $J(N) = 0$ for each submodule N of M. By applying Lemma 2.6 we get the conclusion. □

Corollary 2.8. *If $\operatorname{Soc}(R_R) = 0$ and $J(R)^2 = J(R)$ for a ring R, then R is not a right RM-ring.*

Recall that the *singular submodule* $Z(M)$ of a module M is defined by

$$Z(M) = \{m \in M : mI = 0 \text{ for some essential right ideal } I \text{ of } R\}.$$

The module M is called *singular* if $M = Z(M)$, and *nonsingular* if $Z(M) = 0$. A ring R is *(Von Neumann) regular* if for every $x \in R$ there exists $y \in R$ such that $x = xyx$. Clearly, every regular ring is non-singular (for more properties cf. [15]). Before we describe structure of singular modules over RM-rings, let us observe that the structure of regular RM-rings appears to be very lucid:

Proposition 2.9. *The following conditions are equivalent for a regular ring R.*

(i) *R is a right RM-ring,*
(ii) *$R/\operatorname{Soc}(R)$ is Artinian,*
(iii) *R is semiartinian of socle length 2.*

Proof. (i)⇒(ii) By Lemma 2.2, $R/\operatorname{Soc}(R)$ is of finite Goldie dimension. Since $R/\operatorname{Soc}(R)$ is a regular ring which cannot contain an infinite set of orthogonal set idempotents, we obtain that $R/\operatorname{Soc}(R)$ is Artinian.

(ii)⇒ (iii) It is obvious because an Artinian regular ring is semisimple.

(iii)⇒(i) It follows from Lemma 2.4. □

Lemma 2.10. *Let R be a right RM-ring. Then $Z(M)$ is semiartinian for each right R-module M.*

Proof. Let $m \in Z(M)$ and put $r(m) = \{a \in A | ma = 0\}$. Then $r(m)$ is an essential right ideal of R, hence $mR \cong R/r(m)$ is Artinian and so semiartinian. □

Theorem 2.11. *Let R be a right RM-ring and M be a right R-module.*

(i) *If M is singular, then M is semiartinian.*
(ii) *$E(M)/M$ is semiartinian.*
(iii) *If M is semiartinian, then $E(M)$ is semiartinian. In particular, $E(S)$ is semiartinian for every simple module S.*

Proof. Assume that M is singular. By Lemma 2.10, $Z(M) = M$ is semiartinian, hence (i) holds. Since $E(M)/M$ is a singular module by [12, Examples 7.6(3)] and the class of semiartinian modules is closed under taking essential extensions, (ii) and (iii) hold. □

Since for a ring R with no simple submodule we obtain $Z(R) = 0$ by Lemma 2.10, we can formulate the following consequence of Lemma 2.2:

Corollary 2.12. *If $\operatorname{Soc}(R) = 0$ for a right RM-ring R, then R is a nonsingular ring of finite Goldie dimension.*

Recall that a ring R is called *semilocal* if $R/J(R)$ is semisimple Artinian.

Lemma 2.13. *If R is a semilocal ring, then $J(R) + \operatorname{Soc}(R) \trianglelefteq R$.*

Proof. Let $J(R) + \operatorname{Soc}(R)$ is not essential in R. Then there exists a nonzero right ideal $I \subseteq R$ such that $I \cap (J(R) + \operatorname{Soc}(R)) = 0$. Since $\operatorname{Soc}(I) = \operatorname{Soc}(R) \cap I = 0$ and $R/J(R)$ contains an ideal which is isomorphic to I, we obtain that $\operatorname{Soc}(R/J(R)) \neq R/J(R)$. Hence R is not semilocal, a contradiction. □

The following example shows that the converse of Lemma 2.13 is not true.

Example 2.14. Suppose that R is a local commutative domain with the maximal ideal J. It is easy to see that J^ω is the Jacobson radical of the ring R^ω and it is essential in R^ω, however R^ω is not semilocal.

Recall that $J(R/J(R)) = \{0 + J(R)\}$ for an arbitrary ring R.

Proposition 2.15. *Assume that R is a right RM-ring.*

(i) *If $\operatorname{Soc}(R) = 0$, then $J(R) \trianglelefteq R$ if and only if R is semilocal.*

(ii) *If R is a semilocal ring, then $J(R)/\operatorname{Soc}(J(R))$ is finitely generated as a two-sided ideal.*

Proof. (i) Since $J(R) \trianglelefteq R_R$ and R_R satisfies right RMC, we obtain that $R/J(R)$ is an Artinian ring. On the other hand, $J(R/J(R)) = \{0+J(R)\}$ implies that $R/J(R)$ is semisimple, and hence R is semilocal. The converse follows from Lemma 2.13.

(ii) We note that there exists a finitely generated right ideal $F \subseteq J(R)$ such that $F + (\operatorname{Soc}(R) \cap J(R)) \trianglelefteq J(R)$ since $J(R)/(\operatorname{Soc}(R) \cap J(R))$ has a finite Goldie dimension by Lemma 2.2. Thus $RF + \operatorname{Soc}(R)$ is a two-sided ideal which is essential in R as a right ideal by Lemma 2.13. By the hypothesis, $R/(RF+\operatorname{Soc}(R))$ is a right Artinan ring. As $J(R)+\operatorname{Soc}(R)/(RF+\operatorname{Soc}(R))$ is finitely generated as a right ideal and

$$\begin{aligned}(J(R) + \operatorname{Soc}(R))/(RF + \operatorname{Soc}(R)) &\cong J(R)/(J(R) \cap (RF + \operatorname{Soc}(R)))\\ &= J(R)/(RF + (J(R) \cap \operatorname{Soc}(R)))\\ &= J(R)/(RF + \operatorname{Soc}(J(R))),\end{aligned}$$

the ideal $J(R)/\operatorname{Soc}(J(R))$ is finitely generated. □

Note that every Artinian module is semiartinian and recall that ω denotes the first infinite ordinal.

Lemma 2.16. *Let M be an Artinian R-module. Then the following are equivalent:*

(i) *The socle length of M is greater then ω,*

(ii) *M contains a cyclic submodule with infinitely generated Jacobson radical,*

(iii) *M contains a cyclic submodule which is not noetherian.*

Proof. (i)⇒(ii) Let M be an Artinian module of nonlimit infinite socle length and fix $x \in M$ such that xR has the socle length $\omega + 1$. Denote the α-th member of the socle sequence of xR by S_α. Since xR is Artinian, we obtain that $J(xR)$ is the intersection of finitely many maximal submodules, which implies that $xR/J(xR)$ is semisimple. As xR/S_ω is semisimple as well, we have $J(xR) \subseteq S_\omega$. Hence the socle length of $J(xR)$ is at most ω. Assume that $J(xR)$ is finitely generated. Then the socle length of $J(xR)$ is non-limit, hence finite. This implies that xR has a finite socle length, a contradiction, i.e. $J(xR)$ is infinitely generated.

(ii)⇒(iii) Clear.

(iii)⇒(i) As a cyclic non-Noetherian Artinian module is of infinite non-limit socle length it have to be greater than ω. □

The next result characterizes semilocal right RM-rings further:

Theorem 2.17. *If R is a semilocal right RM-ring and $\operatorname{Soc}(R) = 0$. Then the following conditions are equivalent:*

(i) *R is right Noetherian,*
(ii) *$J(R)$ is finitely generated as a right ideal,*
(iii) *the socle length of $E(R/J(R))$ is at most ω.*

Proof. (i)$\Rightarrow$(ii) Obvious.

(ii)$\Rightarrow$(iii) First, note that every cyclic submodule of $E(R/J(R))$ is Artinian by Theorem 2.11. Suppose that the socle length of $E(R/J(R))$ is greater than ω. Hence it contains an Artinian submodule of the socle length greater than ω. By Lemma 2.16, there exists a cyclic module xR with infinitely generated Jacobson radical. Fix right ideals I_1 and I_2 such that $xR \cong R/I_1$, $I_1 \subseteq I_2$ and $I_2/I_1 = J(R/I_1)$. It is easy to see that I_2 is infinitely generated and $J(R) \subseteq I_2$. Since $I_2/J(R)$ is a right ideal of the semisimple ring $R/J(R)$, we obtain that $I_2/J(R)$ is finitely generated, and hence $J(R)$ is an infinitely generated right ideal.

(iii)$\Rightarrow$(i) Let I be a right ideal. We show that I is finitely generated. By Lemma 2.2, there exists finitely generated right ideals F and G such that $F \trianglelefteq I$, $I \cap G = 0$ and $F + G \trianglelefteq R$. First we note that $R/(F+G)$ is an Artinian module with a submodule isomorphic to I/F and it is also easy to see that $R/(F+G)$ is isomorphic to a submodule of $\bigoplus_{i\leq n} E(S_i)$ for some simple modules $S_1, \dots S_n$. Since each $E(S_i)$ is isomorphic to some submodule of $E(R/J(R))$, we obtain that the socle length of $\bigoplus_{i\leq n} E(S_i)$ and so of $R/(F+G)$ is at most ω. As $R/(F+G)$ is a cyclic module, it is an Artinian module of finite socle length, which implies that $R/(F+G)$ is also a Noetherian module. Therefore I/F and so I are finitely generated modules. □

Recall that a ring R is called *right max* if every non-zero right module has a maximal proper submodule.

Theorem 2.18. *If R is a right max right RM-ring, then $R/\operatorname{Soc}(R)$ is right Noetherian.*

Proof. Let I be a right ideal of $R/\operatorname{Soc}(R)$. It is enough to show that I is finitely generated. If we apply Lemma 2.2 to I, we see that there exists a finitely generated right ideal F such that $F \trianglelefteq I$ and I/F is Artinian. Since R is a right max ring, every nonzero submodule of I/F contains a maximal submodule, and so I/F is Noetherian. By Lemma 2.6, it is finitely generated. Thus I is finitely generated as well. □

As right perfect rings are right max, we get

Corollary 2.19. *If R is a right perfect right RM-ring, then $R/\operatorname{Soc}(R)$ is right Noetherian.*

The following example shows that a perfect right RM-ring needs not be a (right) Noetherian ring.

Example 2.20. Let F be a commutative field and V be a vector space over F. Consider the trivial extension $R = F \times V$. Then R is a local ring, hence it is perfect.

The proper ideals of R are the $0 \times W$, where W is an F-subspace of V. Hence the only essential ideals of R are R and the maximal ideal $0 \times V$. Then R_R satisfies right RMC. We note that if V is infinite dimensional, then R is not Noetherian.

Moreover, since every left perfect ring is right Artinian, we can formulate the following consequence of Lemma 2.4.

Corollary 2.21. *If R is a left perfect right RM-ring, then $R/\operatorname{Soc}(R)$ is right Artinian.*

3. Characterization of commutative RM-rings

The aim of this section is to generalize results of the work [1] for the class of all RM-rings.

First, we recall the terminology that we need. Let P be a maximal ideal of a domain R. For every R-module M, the symbol $M_{[P]}$ denotes the sum of all finite length submodules U of M such that all composition factors of U are isomorphic to R/P. A module M is *self-small*, if the functor $\operatorname{Hom}(M, -)$ commutes with all direct powers of M. Recall that M is not self-small if and only if there exists a chain $M_1 \subseteq M_2 \subseteq ... \subseteq M$ of submodules such that $\bigcup_n M_n = M$ and $\operatorname{Hom}(M/M_n, M) \neq 0$ for each n. Denote by $\operatorname{Max}(M)$ a set of all maximal submodules of M.

For readers convenience let us formulate results of [1] in one criterion:

Theorem 3.1. [1, Theorem 6, Lemma 3(2), Theorem 9] *The following conditions are equivalent for a commutative domain R:*

(i) *R is an RM-domain,*
(ii) *$M = \oplus_{P \in \operatorname{Max}(R)} M_{[P]}$ for all torsion modules M,*
(iii) *R is Noetherian and every non-zero (cyclic) torsion R-module has an essential socle,*
(iv) *R is Noetherian and every self-small torsion module is finitely generated.*

First, make an easy observation.

Lemma 3.2. *Every cyclic Artinian module is Noetherian over each commutative ring.*

The following example shows that the assumption of commutativity in Lemma 3.2 is not superfluous.

Example 3.3. Let F be a field and $I = \mathbb{N} \cup \{\omega\}$ be a countable set (I consists of all natural numbers plus a further index ω). The ring R is the ring of non-commutative polynomials with coefficients in F and in the non-commutative indeterminates x_i, $i \in I$. The cyclic module will be a vector space V over F of countable dimension, with basis v_i, $i \in I$, over the field F.

We must say how R acts on V. For every $n \in \mathbb{N}$, set $x_n v_i = v_n$ if $i \geq n$ and $i \in N$, $x_n v_i = 0$ if $i < n$ and $i \in N$, $x_n v_\omega = v_n$. Moreover, set $x_\omega v_i = 0$ for every

$i \in N$, and $x_\omega v_\omega = v_\omega$. Thus we obtain a left R-module ${}_RV$. Now ${}_RV$ is cyclic generated by v_ω (because $x_n v_\omega = v_n$).

The R-submodules of ${}_RV$ are (each one is contained in the following one):

$$Rv_0 \subset Rv_1 \subset Rv_2 \subset \cdots \subset \bigcup_{i \in \mathbb{N}} Rv_i \subset Rv_\omega = V.$$

Thus the lattice of R-submodules of ${}_RV$ is isomorphic to $\mathbb{N} \cup \{\omega\}$, that is, is order-isomorphic to the cardinal $\omega + 1$. Thus the cyclic R-module ${}_RR$ is Artinian but not Noetherian.

The following observation generalizes [1, Lemma 3(2)].

Theorem 3.4. *Let R be a commutative ring. Then R is an RM-ring if and only if $R/\operatorname{Soc}(R)$ is Noetherian and every singular module is semiartinian.*

Proof. (:⇒) Let R be an RM-ring. If A is the greatest semiartinian ideal in R, then R/A has zero socle and $\operatorname{Soc}(R) \trianglelefteq A$. By Lemma 2.1, $A/\operatorname{Soc}(R)$ is Artinian, and so Noetherian by Lemma 3.2. It remains to show that R/A is Noetherian. Without loss of generality, we may suppose that $\operatorname{Soc}(R) = 0$. Let I be an ideal of R. We must show that it is finitely generated. Repeating the argument of the implication $(iii) \Rightarrow (i)$ of the proof of Theorem 2.17, we can find finitely generated ideals F and G such that $F \trianglelefteq I$, $I \cap G = 0$ and $F + G \trianglelefteq R$. Hence $R/(F+G)$ is Artinian and it has a submodule which is isomorphic to I/F. Since $R/(F+G)$ is Noetherian by Lemma 3.2, we have I/F as well as I are finitely generated. The rest follows from Lemma 2.10.

(⇐:) Let $R/\operatorname{Soc}(R)$ be Noetherian and every singular module be semiartinian. Fix an ideal $I \trianglelefteq R$. Then R/I is singular and so semiartinian by Lemma 2.10. Moreover, R/I is Noetherian and semiartinian, hence it is Artinian which finishes the proof. □

In light of Theorem 3.4, we ask the following.

Problem 3.5. *Is $R/\operatorname{Soc}(R)$ Noetherian for each non-commutative right RM-ring R?*

Lemma 3.6. *R be a commutative RM-ring and M a singular module. Then $M = \oplus_{P \in \ Max(R)} M_{[P]}$.*

Proof. Assume that $M \neq \oplus_{P \in \operatorname{Max}(R)} M_{[P]}$ and fix $m \in M \setminus \oplus_{P \in \operatorname{Max}(R)} M_{[P]}$. Since M is singular, mR is Artinian and $mR \cong R/r(m) \cong \prod_{r(m) \subseteq I} A_I$ where A_I are local commutative Artinian rings with maximal ideal I. Since $A_I \subseteq M_{[I]}$ and there are only finitely many $I \in \operatorname{Max}(R)$, we get a contradiction. □

Theorem 3.7. *The following conditions are equivalent for a commutative ring R:*

(i) *R is an RM-ring,*
(ii) *$M = \oplus_{P \in Max(R)} M_{[P]}$ for all singular modules M,*

(iii) $R/\operatorname{Soc}(R)$ *is Noetherian and every self-small singular module is finitely generated.*

Proof. (i) $\Rightarrow$ (ii) It follows by Lemma 3.6.

(ii) $\Rightarrow$ (i) The proof is the same as the proof of [1, Theorem 6]. Let I be an essential ideal of R. Then R/I is a cyclic singular module, hence $R/I \cong \oplus_{P\in Max(R)} A_{[P]}$ where all $A_{[P]}$ are cyclic and only finitely many $A_{[P]}$ are non-zero. Since every cyclic module $A_{[P]}$ is a submodule of a sum of finite-length modules, it is Artinian. Thus R/I is Artinian and R is an RM-ring.

(i)$\Rightarrow$(iii) By Theorem 3.4 and Lemma 2.16, we have $R/\operatorname{Soc}(R)$ is Noetherian and every singular module is semiartinian of socle length less or equal than ω. Let M be a self-small singular module. Then $M = \oplus_{P\in Max(R)} M_{[P]}$ by Lemma 3.6, hence $M_{[P]} \neq 0$ for only finitely many $[P]$. Since $\operatorname{Hom}(M_{[P]}, M_{[Q]}) = 0$ for all $P \neq Q$, we may suppose that $M = M_{[P]}$ for a single maximal ideal P by [16, Proposition 1.6]. Denote by M_i the i-th member of the socle sequence of M. It is easy to see that $M_i = \{m \in M \mid mP^i = 0\}$. Assume that socle length of M is infinite, i.e. $M_i \neq M_{i+1}$ and $M = \bigcup_{i<\omega} M_i$. Then for each $i < \omega$, there exist $m_i \in M_{i+1} \setminus M_i$ and $p_i \in P^i$ such that $0 \neq m_i p_i \in \operatorname{Soc}(M)$. Then multiplication by p_i forms a nonzero endomorphism on M for which $M_i \subseteq \ker p_i$, a contradiction with the fact that M is self-small. We have proved there exists n such that $M_n = M$ and so M has a natural structure of a self-small module over commutative Artinian ring R/P^n. Hence M is finitely generated by [2, Proposition 2.9].

(iii)$\Rightarrow$(i) The argument is similar as in the proof of [1, Theorem 9]. If I is an essential ideal of R, then $\operatorname{Soc}(R) \subseteq I$, hence R/I is Noetherian. Moreover, every self-small module over R/I is singular as an R-module, and so it is finitely generated. Now, the conclusion follows immediately from [2, Proposition 3.17]. □

Remark 3.8. Note that Theorem 3.1 is a direct consequence of Theorems 3.4 and 3.7 since singular modules over commutative domains are exactly torsion modules.

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