Habilitation Thesis

Algebraic analysis on generalized Verma modules and differential invariants in parabolic geometries

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1 Introduction

The subject of the present thesis concerns the existence, construction and properties of branching rules and associated singular vectors for generalized Verma modules or dually, differential invariants of homogeneous manifolds equipped with a geometrical structure encapsulated by geometrical and representation theoretical properties of semisimple Lie algebras and their parabolic subalgebras. The main focus is to uncover and explore the intimate relation between several branches of mathematics including local differential geometry on homogeneous manifolds and its submanifolds with a parabolic structure, in particular conformally invariant linear and bilinear differential operators, and homomorphisms and coproducts between induced modules in representation theory of semisimple Lie algebras and groups.

In a series of pioneering papers, see [3], [4], [5], R. Baston introduced a number of general methods to study invariant differential operators on conformal manifolds, and related class of parabolic geometries, which he called "almost hermitian symmetric (AHS)" structures. In particular, he suggested that certain complexes of natural differential operators, dual to generalized Bernstein-Gelfand-Gelfand (BGG) resolutions of generalized Verma modules, could be extended from the homogeneous context (generalized flag manifolds) to manifolds modeled on these spaces. He provided a construction of such a BGG sequence (no longer a complex in general) for AHS structures, and introduced a class of induced modules, now called semiholonomic Verma modules.

Baston's work fits into the program of parabolic invariant theory initiated by Fefferman and Graham, [26]. Several authors completed these ideas and hence provided a theory of invariant operators in all parabolic geometries, including conformal geometry, projective geometry, quaternionic geometry, projective contact geometry, CR geometry and quaternionic CR geometry. In [23], Eastwood and Slovak began the study of semiholonomic generalized Verma modules and classified the generalized Verma module homomorphisms lifting to the semiholonomic modules in the conformal case. The suggestion by R. Baston of the construction of BGG sequences has been clarified in [14], and in the process, generalized to all parabolic geometries. Hence we now know that all standard homomorphisms of parabolic Verma modules induce differential operators also in the curved setting, providing us with a huge supply of invariant linear differential operators. The construction of BGG sequences has been simplified in [11] by introducing the so called A_∞-structure, which organizes these sequences of linear differential operators together with bilinear and more generally multilinear differential pairings induced by cup product on the Lie algebra cohomology into one package.

This development turned out to find its place in a wider framework of questions, characterized by mutual interaction of geometrical and analytical techniques approaching various incarnations of fundamental geometrical objects. A representative example is Q-curvature, which is an important object in conformal Riemannian geometry, see [37] and references therein. In conformal Riemannian geometry, one associates to a hypersurface or boundary $i : \Sigma \hookrightarrow M$

of a Riemannian manifold (M, g) a 1-parameter family of natural conformally covariant differential operators mapping densities on M to densities on Σ. Then the Q-curvature resp. the Graham-Jenne-Mason-Sparling (GJMS) operators of $(\Sigma, i^{\star}(g))$ appear as the linear resp. constant coefficients of the family parameter expansion of these differential operators, and their transformation properties are a direct consequence of invariance of these natural differential operators.

There are two prominent examples of curved extensions of families of invariant differential operators - the families coming from invariant calculus of tractors and the residue families related to asymptotic of eigenfunctions of ambient hyperbolic Laplace operator and scattering operator. On the other hand, Q-curvature [9] is an invariant produced by analytic techniques like variational problems for spectral functional determinants and analytic half torsion.

The setting of residue families is based on canonical extension of the metric on Σ to a metric on M, arising in the case of conformally compact Einstein metrics or Fefferman-Graham construction of the ambient metric in the case when Σ is a boundary of M. The Poincare-Einstein metric associates to a conformal class of metrics on Σ a diffeomorphism class of conformally compact Einstein metrics on M with a prescribed conformal class as its conformal infinity. In the case of invariant metric on homogeneous space, the related mathematical concept is Helgason's theory of Poisson transformation in harmonic analysis on symmetric spaces.

An inspiration for mathematically rigorous understanding of this circle of ideas is based on concepts of theoretical physics and string theory called holographic principle in quantum gravity and AdS/CFT correspondence. Another, a number theoretical, inspiration comes from analytical continuation of intertwining operators for discrete cocompact Kleinian groups Γ, whose residues contain automorphic forms supported on the limit set for Γ.

In the last couple of years, we developed in [43], [44], [55] a systematic approach to treat two intimately related problems. The first problem has its origin in basic questions of representation theory, namely the branching rules for generalized Verma modules. The answer to this question is expressed in terms of qualitative properties of the decomposition with respect to an embedded (non-)compatible semisimple Lie subalgebras of a given simple Lie algebra, and includes a far reaching generalization of Kazhdan-Lusztig polynomials. The second problem is a quantitative part of abstract (qualitative) conclusions, describing the construction of singular vectors generating submodules in an initial generalized Verma module. In the series of cited articles we applied the distribution Fourier transform to convert the quest for singular vectors to a problem of solving the system of special partial differential equations subordinate to a symmetry in question. For example, in the case of conformal parabolic subalgebra of the simple Lie algebra $so(n+1, 1, \mathbb{R})$ with compatible subalgebra $so(n, 1, \mathbb{R})$, the problem for singular vectors is transformed into the special differential equation for the Gegenbauer polynomials, or the case of semisimple Lie algebra $sl(2, \mathbb{R}) \times sl(2, \mathbb{R})$ with Borel subalgebra and the compatible diagonal subalgebra $sl(2, \mathbb{R})$ this procedure leads to the Jacobi polynomials.

The equivalence between homomorphisms of generalized Verma modules and invariant differential operators acting on smooth (holomorphic) sections of homogeneous vector bundles over homogeneous manifold yields the explicit form of differential invariants.

Another part of the project almost not touched in the present thesis is the construction of lifts to homomorphisms in the category of semiholonomic generalized Verma modules. Generalizing the pattern of the arguments given in [23], one can prove the existence of curved extensions of many differential invariants, cf. [43], [44] for some examples of this construction. The curved translation procedure and invariant calculus of tractors then allow, at least in principle, to construct the curved operators explicitly.

Let us briefly describe the structure and results in the thesis based on the work of the author, together with general scheme developed in collaboration with co-authors in [43], [44] and overviewed in Section 3, Section 4, Section 5.

The Section 2 offers a recollection of the standard material about Lie theory with emphasis on the structure of parabolic subalgebras of simple Lie algebras, universal enveloping Lie algebras of simple Lie algebras, intertwining differential operators and the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$.

Then we pass to a panorama of representative problems and examples, each of them having its own range of applications.

The Section 6 and Section 7 deal with the simplest example when either $\mathfrak{g}' = \mathfrak{g}$ or $\mathfrak{g}, \mathfrak{g}'$ are two compatible consecutive orthogonal Lie algebras with (compatible) conformal subalgebras, and the inducing representations are either characters or the spinor modules twisted by characters. In this way we recover in the case of conformal parabolic subalgebras of simple orthogonal Lie algebras and generalized Verma modules induced from characters the classification given in [21], and for two consecutive compatible orthogonal Lie algebras $\mathfrak{g}, \mathfrak{g}'$ with conformal parabolic subalgebras the results in [37]. It is worth to emphasize that we construct both the standard and non-standard homomorphisms of generalized Verma modules. Another (almost) complete classification is given in the case of the conformal geometry, when the inducing representation is spinor module twisted by family of characters. In this example the singular vectors correspond to powers of the Dirac operator.

The Section 8 is devoted to the description of the structure of composition series in the case $\mathfrak{g}, \mathfrak{g}'$ are two compatible consecutive orthogonal Lie algebras with (compatible) conformal subalgebras. Particularly interesting is the appearance of projective modules in the BGG category $\mathcal{O}^{\mathfrak{p}}$ for special values of the inducing character λ. A consequence of the detailed analysis is a proof of conjectures formulated by A. Juhl, [37].

The Section 9 contains a detailed analysis of the case when $\mathfrak{g}, \mathfrak{g}'$ are two compatible consecutive orthogonal Lie algebras with (compatible) conformal subalgebras, and as the induced representation is taken the fundamental vector representation of the Levi factor corresponding to 1-forms. Consequently, there are two series of families of operators with target either $0-$ or 1-forms on the conformal submanifold. The curved version of these family operators are currently used in the construction of higher Q-curvature on forms and Branson-Gover operators.

The next example in Section 10 corresponds to the diagonal branching problem for orthogonal Lie algebras and its parabolic subalgebras of conformal type, when the singular vectors induce bilinear differential operators generalizing Rankin-Cohen brackets, for the first time studied in the context of the ring structure on automorphic forms on $SL(2,\mathbb{R})$. The full classification of singular vectors and their dual differential operators is achieved for the inducing 1-dimensional representations.

In Section 11 we treat again the cases of orthogonal Lie algebras and its conformal parabolic subalgebras, but this time we take the inducing representation to be infinite dimensional representation given by generalized Verma module of scalar type twisted by character of the Levi subalgebra. In this case our results give the flat version of so called universal splitting operator, when the inducing representation has non-trivial composition structure with respect to the conformal parabolic subalgebra. For special values of the inducing twisting character, passing to the finite dimensional quotient yields the splitting operators on symmetric powers of the standard tractor representation. Rewriting the valuation as a formal power series in terms of an infinite product, one can write down its curved generalization in terms of the curved Casimir operator.

In Section 12 we demonstrate another flexibility of our approach, namely the possibility of the branching problem and construction of singular vectors for non-compatible couple of Lie algebras and their (non-compatible) parabolic subalgebras. In particular, we consider the case of orthogonal Lie algebra $so(7,\mathbb{C})$ with its conformal parabolic subalgebra, and the Lie algebra Lie G_2 with its unique 3-graded parabolic subalgebra. The construction of invariant differential operators in this specific case is closely related to the question of Lie G_2 distributions of type $(2,3,5)$ on manifolds with conformal structure in dimension 5. In a recent article, [55], we extended the framework of the scheme presented in the thesis to a much broader setting. In particular, on some technical assumptions, we deal in a uniform way with non-compatible parabolic subalgebras fitting into the class of Fernando-Kac algebras.

The final Appendix contains summary and properties of Gegenbauer polynomials, characterized as polynomial solutions of second order ordinary Gegenbauer differential equation.

2 Introduction of basic objects in Lie and representation theory

The material presented in this section may be found in any advanced textbook on Lie and representation theory, e.g. [38], [27]. It is an introductory material aiming to orient the interested reader when reading the main text of the thesis.

2.1 (Semiholonomic) Universal enveloping algebra

The universal enveloping complex algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is defined as a quotient of the tensor algebra $T(\mathfrak{g})$ by two-sided ideal generated by elements $X \otimes Y - Y \otimes X - [X,Y]$ for all $X, Y \in \mathfrak{g}$. The non-homogeneous ideal induces an increasing filtration $\{\mathcal{U}^i(\mathfrak{g})\}_{i\in\mathbb{N}_0}$ on $\mathcal{U}(\mathfrak{g})$:

$$\{0\} \subset \mathcal{U}^1(\mathfrak{g}) \subset \mathcal{U}^2(\mathfrak{g}) \subset \mathcal{U}^3(\mathfrak{g}) \subset \dots$$

and according to Poincare-Birkhoff-Witt theorem, the canonical inclusion $\mathfrak{g} \hookrightarrow \mathcal{U}(\mathfrak{g})$ extends to the vector space isomorphism

$$\oplus_{i=0}^{l} S^i(\mathfrak{g}) \xrightarrow{\sim} \mathcal{U}^l(\mathfrak{g}),\ l \in \mathbb{N}_0,$$

where $S^\star(\mathfrak{g})$ denotes the polynomial algebra on $\mathfrak{g}^\star$. If $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is a semisimple Lie algebra with two simple summands $\mathfrak{g}_1, \mathfrak{g}_2$, we have

$$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_1) \otimes \mathcal{U}(\mathfrak{g}_2).$$

In a basis $\{e_j\}_{j=1,\dots,dim(\mathfrak{g})}$ of $\mathfrak{g}$ the elements $e_{i_1} \cdot \dots \cdot e_{i_m}$, $m \leq l$, linearly span $\mathcal{U}^l(\mathfrak{g})$. The adjoint representation

$$ad_X : \mathfrak{g} \to \mathfrak{g},\ Y \mapsto [X,Y],\ X, Y \in \mathfrak{g},$$

extends to a $\mathfrak{g}$-module structure on $\mathcal{U}(\mathfrak{g})$, completely reducible for $\mathfrak{g}$ a reductive Lie algebra. The subalgebra of elements $X \in Z(\mathcal{U}(\mathfrak{g})) \subset \mathcal{U}(\mathfrak{g})$ such that $[X,Y] = 0$ for all $Y \in \mathfrak{g}$ is called the center of $\mathcal{U}(\mathfrak{g})$.

Let $\mathbb{V}$ be a complex representation of reductive Lie algebra $\mathfrak{g}$, given by Lie algebra homomorphism $\phi : \mathfrak{g} \to End(\mathbb{V})$. It follows from the universal property of $\mathcal{U}(\mathfrak{g})$ that there is a unique lift of ϕ to a homomorphism of associative algebras $\tilde{\phi} : \mathcal{U}(\mathfrak{g}) \to End(\mathbb{V})$. In the case $\mathbb{V}$ is irreducible, the elements from $Z(\mathcal{U}(\mathfrak{g}))$ act by scalars and the underlying homomorphism

$$\chi_{(\phi,\mathbb{V})} : Z(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$$

is called central (infinitesimal) character. Let $(\phi_1, \mathbb{V}_1), (\phi_2, \mathbb{V}_2)$ be two irreducible representations of $\mathfrak{g}$ and $T : \mathbb{V}_1 \to \mathbb{V}_2$ a $\mathcal{U}(\mathfrak{g})$-homomorphism (an intertwining linear map.) Then we have for all $v_1 \in \mathbb{V}_1, z \in Z(\mathcal{U}(\mathfrak{g}))$

$$\chi_{(\phi_2,\mathbb{V}_2)}(z)T(v_1) = zT(v_1) = Tz(v_1) = T\chi_{(\phi_1,\mathbb{V}_1)}(z)(v_1) = \chi_{(\phi_1,\mathbb{V}_1)}(z)T(v_1),$$

so that either $T = 0$ or $\chi_{(\phi_1,\mathbb{V}_1)} = \chi_{(\phi_2,\mathbb{V}_2)}$.

The geometric meaning of $\mathcal{U}(\mathfrak{g})$ is given by the tensor algebra generated by (left or right) invariant vector fields on a connected Lie group G with Lie algebra $\mathfrak{g}$, i.e. by the enveloping algebra of (left or right) invariant differential operators acting on functions on G.

Given a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a finite dimensional irreducible $\mathfrak{p}$-module $(\sigma, \mathbb{V}_\sigma)$, the generalized Verma module induced from $(\sigma, \mathbb{V}_\sigma)$ is defined by

$$\begin{aligned} M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{V}_\sigma) = M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\sigma) &= (\mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}_\sigma)/ < X \otimes v - 1 \otimes \sigma(X)v,\ X \in \mathfrak{p},\ v \in \mathbb{V}_\sigma > \\ &= \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}_\sigma\,, \end{aligned} \tag{2.1}$$

where $< >$ denotes a left $\mathcal{U}(\mathfrak{g})$-ideal generated by indicated elements. The (finite dimensional) irreducible $\mathfrak{p}$-representation $(\sigma, \mathbb{V}_\sigma)$ is embedded into $M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\sigma)$ via

$$\mathbb{V}_\sigma \to (1 \otimes_{\mathbb{C}} \mathbb{V}_\sigma) \hookrightarrow \mathcal{U}(\mathfrak{p}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}_\sigma. \tag{2.2}$$

As a consequence of the Poincare-Birkhoff-Witt theorem we have the vector space isomorphism

$$M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\sigma) \simeq \mathcal{U}(\mathfrak{n}_-) \otimes \mathbb{V}_\sigma, \tag{2.3}$$

where $\mathfrak{n}_-$ is the opposite nilradical of $\mathfrak{n}$, $\mathfrak{g} \simeq \mathfrak{n}_- \oplus \mathfrak{p}$. In the case when $\mathfrak{n}_-$ is abelian nilradical, $M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\sigma)$ is isomorphic to the space of $\mathbb{V}_\sigma$-valued polynomials on $\mathfrak{n}$:

$$\mathcal{U}(\mathfrak{n}_-) \otimes \mathbb{V}_\sigma \simeq S^*(\mathfrak{n}_-) \otimes \mathbb{V}_\sigma \simeq Pol(\mathfrak{n}) \otimes \mathbb{V}_\sigma. \tag{2.4}$$

Let us consider again two irreducible finite dimensional $\mathfrak{p}$-modules $(\sigma_1, \mathbb{V}_{\sigma_1})$, $(\sigma_2, \mathbb{V}_{\sigma_2})$ and a $\mathcal{U}(\mathfrak{g})$-homomorphism

$$T : M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\sigma_1}) \to M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\sigma_2}). \tag{2.5}$$

The semiholonomic universal enveloping algebra $\mathcal{U}_s(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ is defined as a quotient of the tensor algebra $T(\mathfrak{g})$ by a two-sided ideal generated by elements $X \otimes Y - Y \otimes X - [X, Y]$ for all $X \in \mathfrak{g}$, $Y \in \mathfrak{p}$. In other words, it is allowed to commute two elements only in the case when at least one of them is in $\mathfrak{p}$, i.e. the relations $< X \otimes Y - Y \otimes X - [X, Y] >$ do not hold for $X, Y \in \mathfrak{n}_-$. Given a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a finite dimensional irreducible $\mathfrak{p}$-module $(\sigma, \mathbb{V}_\sigma)$, the semiholonomic generalized Verma module induced from $(\sigma, \mathbb{V}_\sigma)$ is defined by

$$\begin{aligned} M_s(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\sigma) &= (\mathcal{U}_s(\mathfrak{g}) \otimes \mathbb{V}_\sigma)/ < X \otimes v - 1 \otimes \sigma(X)v,\ X \in \mathfrak{p},\ v \in \mathbb{V}_\sigma > \\ &= \mathcal{U}_s(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} \mathbb{V}_\sigma\,, \end{aligned} \tag{2.6}$$

where $< >$ denotes a left $\mathcal{U}_s(\mathfrak{g})$-ideal.

There is a canonical projection $\pi : \mathcal{U}_s(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ inducing a surjective homomorphism

$$M_s(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\sigma) \to M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\sigma) \tag{2.7}$$

for all irreducible $\mathfrak{p}$-modules $(\sigma, \mathbb{V}_\sigma)$.

2.2 Parabolic subalgebras of simple Lie algebras

Let us consider a simple Lie algebra $\mathfrak{g}$ with fixed Cartan subalgebra $\mathfrak{h}$ and a set of simple positive roots $\triangle_s^+$. Each subset $\Sigma \subset \triangle_s^+$ determines a standard parabolic subalgebra $\mathfrak{p}_\Sigma \leq \mathfrak{g}$, characterized as a linear span of root spaces whose roots have in their expansion into simple roots positive coefficients by elements from Σ, i.e.

$$\mathfrak{p}_\Sigma = \mathfrak{h} \oplus (\oplus_{\alpha \in <-\triangle_s^+ \setminus \Sigma>} \mathfrak{g}_\alpha) \oplus (\oplus_{\alpha \in <\triangle_s^+>} \mathfrak{g}_\alpha), \tag{2.8}$$

where $<>$ denotes linear combinations with positive coefficients. The sum can be conveniently rewritten by dividing all roots in the previous sum on the part $(\triangle_s^+ \setminus \Sigma) \cup (-\triangle_s^+ \setminus \Sigma)$ generating together with the Cartan subalgebra $\mathfrak{h}$ the reductive Lie subalgebra $\mathfrak{l}_\Sigma$, and its vector complement generating the nilradical $\mathfrak{n}_\Sigma$:

$$\mathfrak{p}_\Sigma \simeq \mathfrak{l}_\Sigma \oplus \mathfrak{n}_\Sigma, \, \mathfrak{g} \simeq \mathfrak{n}_{\Sigma-} \oplus \mathfrak{p}_\Sigma. \tag{2.9}$$

The parabolic subalgebra $\mathfrak{p}_\Sigma$ contains the standard Borel subalgebra, $\mathfrak{l}_\Sigma$ is called the Levi factor of $\mathfrak{p}_\Sigma$ and $\mathfrak{n}_{\Sigma-}$ the opposite nilradical to $\mathfrak{n}_\Sigma$. This decomposition of $\mathfrak{g}$ is called the Iwasawa-Langlands decomposition. In what follows, we usually omit the subscript Σ from the notation when a parabolic subalgebra is fixed. Notice that any parabolic subalgebra is conjugate to a standard parabolic subalgebra.

For a given root α of $\mathfrak{g}$, we define its $(\mathfrak{p}_\Sigma, \Sigma)$-height as the sum of coefficients by simple roots from Σ. The $(\mathfrak{p}_\Sigma, \Sigma)$-height yields the structure of $|k|$-graded Lie algebra on $\mathfrak{g} \simeq \oplus_{i=-k}^k \mathfrak{g}_i$, where $\mathfrak{g}_i$ is the linear span of root spaces associated to roots with $(\mathfrak{p}_\Sigma, \Sigma)$-height i. All graded subspaces are $\mathfrak{g}_0$-modules, in particular $\mathfrak{g}_0 = \mathfrak{l}$ is the Levi subalgebra and $\mathfrak{g}_1$ is generated by simple roots in Σ, such that

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \, i, j = -k, \dots, 0, \dots, k. \tag{2.10}$$

Conversely, the structure of $|k|$-graded simple Lie algebra on $\mathfrak{g} \simeq \oplus_{i=-k}^k \mathfrak{g}_i$ implies

$$\mathfrak{p}_\Sigma = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k, \, \mathfrak{n}_\Sigma = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_k, \, \mathfrak{n}_{\Sigma-} = \mathfrak{g}_{-1} \oplus \dots \oplus \mathfrak{g}_{-k}. \tag{2.11}$$

There is an unique element $E_0 \in \mathfrak{g}_0$ encapsulating the graded structure on $\mathfrak{g}$, i.e.

$$ad_{E_0}(X) = [E_0, X] = iX, \, X \in \mathfrak{g}_i, \, i = -k, \dots, k. \tag{2.12}$$

For all $i = -k, \dots, k-1$, $[\mathfrak{g}_{-1}, \mathfrak{g}_{i+1}] = \mathfrak{g}_i$ and if no simple factor of $\mathfrak{g}$ is contained in $\mathfrak{g}_0$, $\mathfrak{g}_{-1}$ acts on $\mathfrak{g}_1$ with surjective image $\mathfrak{g}_0$. If an element $X \in \mathfrak{g}_i$, $i = 1, \dots, k$ fulfills $[X, Y] = 0$ for all $Y \in \mathfrak{g}_{-1}$, then $X = 0$ (the result can be extended to $i = 0$ on the condition of non-existence of a simple factor contained in $\mathfrak{g}_0$.) The Killing-Cartan non-degenerate symmetric bilinear form on $\mathfrak{g}$ restricts to the isomorphism of $\mathfrak{g}_0$-modules

$$\mathfrak{g}_{-i} \xrightarrow{\sim} (\mathfrak{g}_i)^\star, \, i = 1, \dots, k. \tag{2.13}$$

As for the representation theory of parabolic subalgebras, each irreducible $\mathfrak{p}$-module is an irreducible $\mathfrak{g}_0$-module equipped with the trivial action of $\mathfrak{p}_+$. Because $\mathfrak{g}_0$ is reductive Lie algebra, a finite dimensional irreducible $\mathfrak{g}_0$-module is determined by the highest weight of $\mathfrak{g}_0^s = [\mathfrak{g}_0, \mathfrak{g}_0]$ (the semisimple part of $\mathfrak{g}_0$) and the character of $Z(\mathfrak{g}_0)$, the center of $\mathfrak{g}_0$. Most of the $\mathfrak{p}$-modules are reducible but indecomposable, i.e. carry a finite (or infinite) length $\mathfrak{p}$-filtration:

$$\mathbb{V} = \mathbb{V}_0 \supset \mathbb{V}_1 \supset \mathbb{V}_2 \cdots \supset \mathbb{V}_l \supset \{0\},\, l \in \mathbb{N}_0. \tag{2.14}$$

The subspaces $\bigcup_{i=j}^{l} V_i$, $j = 1, \ldots, l$ are $\mathfrak{p}$-submodules and the quotient of $\mathbb{V}$ by these submodules are $\mathfrak{p}$-quotient modules of $\mathbb{V}$. In particular, the quotients $\mathbb{V}_i/\mathbb{V}_{i+1}$, $i = 0, \ldots, l$ are direct sums of irreducible $\mathfrak{p}$-modules. Associated graded $\mathfrak{g}_0$-modules are distinguished by the eigenvalues of the grading element E_0.

2.3 Weyl groups and Hasse posets of simple Lie algebras and their parabolic subalgebras

Let us consider a simple Lie algebra $\mathfrak{g}$, its Cartan subalgebra $\mathfrak{h}^\star \subset \mathfrak{g}$ and a root lattice $\triangle \subset \mathfrak{h}^\star$. The Cartan-Killing form restricts to a positive definite symmetric bilinear form $< -, - >$ on $\mathfrak{h}^\star$. The hyperplane perpendicular to a root $\alpha \in \triangle$ is called the wall W_α. We denote r_α the reflection generated by W_α. The Weyl group W of $\mathfrak{g}$ is the finite group generated by reflections $r_\alpha, \alpha \in \triangle_s^+$. The length function

$$l : W \to \mathbb{N}_0,\, w \mapsto l(w)$$

is defined as a minimal integer, such that $w \in W$ can be expressed as a product of $l(w)$ simple reflections. Such an expression is generally not unique and is called a reduced expression of w. The walls $W_\alpha, \alpha \in \Delta$, stratify $\mathfrak{h}^\star$ into polyhedral cones called Weyl chambers, faithfully permuted by the action of W. Any chamber determines the set of simple roots and conversely, the dominant chamber is obtained by a choice of simple roots. Let $\lambda \in \mathfrak{h}^\star$. Then

$$r_\alpha(\lambda) = \lambda - < \lambda, \alpha^\vee > \alpha, \tag{2.15}$$

where $\alpha^\vee = \frac{2\alpha}{<\alpha,\alpha>}$ is the coroot of α.

The Weyl group W of $\mathfrak{g}$ admits the structure of partially ordered set (directed graph or Hasse graph): the vertices are elements of W and for $w_1, w_2 \in W$ there is a directed edge $w_1 \to w_2$ if $l(w_2) = l(w_1) + 1$ and $w_2 = r_\alpha w_1$ for some $\alpha \in W$. The structure of partial ordering is also called Bruhat order, and it reflects the topology of complete flag manifold G/B in terms of the stratification by affine cells (B-orbits) of increasing dimension.

Let us denote by $\rho = \frac{1}{2}\sum_{\alpha \in \Delta^+} \alpha$ the half-sum of positive roots (equal to the sum of fundamental weights.) The shifted action of W on $\mathfrak{h}^\star$ is given by

$$w \cdot \lambda = w(\lambda + \rho) - \rho,\; w \in W,\, \lambda \in \mathfrak{h}^\star \tag{2.16}$$

and conventionally appears in the formulation of the Lie algebra cohomology via Bott-Borel-Weyl theorem and Bernstein-Gelfand-Gelfand resolutions on G/B. A weight $\lambda \in \mathfrak{h}^\star$ is called singular if $\lambda + \rho$ has a nontrivial stabilizer in W (or, equivalently, lies on some wall of a chamber.) Another characterization is given by the existence of a coroot $\alpha_i^\vee$ and $w \in W$ such that $< w(\lambda + \rho), \alpha_i^\vee >= 0$. In the opposite case is λ called non-singular or regular.

In order to construct Hasse graph for standard parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ as a subgraph of the Hasse graph for Borel subalgebra $\mathfrak{b}$ discussed above, we use the characterization given by Kostant, [46]. It identifies the Hasse poset $W^\mathfrak{p}$ with the set of minimal length right coset representatives of the Weyl group $W_\mathfrak{l}$ of the reductive Levi factor $\mathfrak{l}$ in W: any $w \in W$ admits a unique decomposition $w = w_\mathfrak{p} w^\mathfrak{p}$ with $w_\mathfrak{p} \in W_\mathfrak{l}$ and $w^\mathfrak{p} \in W^\mathfrak{p}$. Moreover, $l(w) = l(w_\mathfrak{p}) + l(w^\mathfrak{p})$ and $w^\mathfrak{p}$ is the only element of the coset with the minimal length.

The Hasse diagram of a parabolic subalgebra reflects the topological structure of the generalized flag manifold G/P and the Bernstein-Gelfand-Gelfand resolutions on G/P, and controls the topology of the fiber for direct image operation between homogeneous vector bundles on generalized flag manifolds.

2.4 Invariant differential operators on homogeneous spaces and homomorphisms of generalized Verma modules

We shall first consider the case of homogeneous or Cartan model of flat geometry of type (G, P, ω) and recall the bijective correspondence between G-invariant differential operators

$$D : C^\infty(G, \mathbb{V}_{\sigma_1})^P \to C^\infty(G, \mathbb{V}_{\sigma_2})^P \tag{2.17}$$

acting between smooth sections of homogeneous vector bundles $V_{\sigma_1}, V_{\sigma_2}$ induced from irreducible P-modules $\mathbb{V}_{\sigma_1}, \mathbb{V}_{\sigma_2}$, and $\mathcal{U}(\mathfrak{g})$-module homomorphisms of generalized Verma modules

$$\tilde{D} : M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}^\star_{\sigma_2}) \to M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}^\star_{\sigma_1}). \tag{2.18}$$

The G-invariant differential operator D induces a linear map

$$\begin{aligned} \varphi : \mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^\star_{\sigma_2} &\to \mathcal{U}(\mathfrak{g}) \otimes \mathbb{V}^\star_{\sigma_1}, \\ X \otimes v_2^\star &\mapsto XU \otimes v_1^\star, \end{aligned} \tag{2.19}$$

where

$$U \in \mathcal{U}(\mathfrak{g}),\ v_1^\star \in \mathbb{V}^\star_{\sigma_1},\ v_2^\star \in \mathbb{V}^\star_{\sigma_2}$$

are characterized by

$$< D(f)(e), v_2^\star >=< U(u)(e), v_1^\star >,\ u \in C^\infty(G, \mathbb{V}_{\sigma_1})^P \tag{2.20}$$

for $U \in \mathcal{U}(\mathfrak{g})$ acting from the right on $C^\infty(G)$ and $e \in G$ the unit element.

Lemma 2.1 *The φ induces linear map*

$$
\begin{aligned}
&< X \otimes v_2^\star - 1 \otimes \sigma_2^\star(X) v_2^\star | \, X \in \mathfrak{p}, \, v_2^\star \in \mathbb{V}_{\sigma_2}^\star > \longrightarrow \\
&< X \otimes v_1^\star - 1 \otimes \sigma_1^\star(X) v_1^\star | \, X \in \mathfrak{p}, \, v_1^\star \in \mathbb{V}_{\sigma_1}^\star >
\end{aligned} \tag{2.21}
$$

of left $\mathcal{U}(\mathfrak{g})$-ideals, and consequently a homomorphism of generalized Verma modules

$$
\tilde{D} : \, M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\sigma_2}^\star) \to M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\sigma_1}^\star). \tag{2.22}
$$

Proof: First of all notice that $\tilde{D}(1 \otimes v_2^\star) = U \otimes v_1^\star$ implies

$$
\tilde{D}(1 \otimes \sigma_2^\star(X) v_2^\star) = ad(X) U \otimes v_1^\star + U \otimes \sigma_1^\star(X) v_1^\star.
$$

In fact, we have by left G-invariance of D

$$
\begin{aligned}
&< D(f)(e), \sigma_2^\star(X) v_2^\star > = < -\sigma_2(X) D(f)(e), v_2^\star > \\
&= < -(L_X)_\star D(f)(e), v_2^\star > = < -D((L_X)_\star f)(e), v_2^\star > \\
&= < -U((L_X)_\star f)(e), v_1^\star > = < (ad(X)U) f(e), v_1^\star > \\
&+ < (Uf)(e), \sigma_1^\star(X) v_1^\star >,
\end{aligned} \tag{2.23}
$$

where $(L_X)_\star$ denotes the infinitesimal left translation by $X \in \mathfrak{p}$. Hence

$$
\begin{aligned}
&\tilde{D}(X \otimes v_2^\star) - \tilde{D}(1 \otimes \sigma_2^\star(X) v_2^\star) \\
&= XU \otimes v_1^\star - (ad(X) U \otimes v_1^\star + U \otimes \sigma_1^\star(X) v_1^\star) \\
&= U(X \otimes v_1^\star - 1 \otimes \sigma_1^\star(X) v_1^\star).
\end{aligned} \tag{2.24}
$$

In the other direction, we reconstruct D from $\tilde{D}$ by

$$
u(g) \mapsto (g \mapsto (v_2^\star \mapsto \tilde{D}(1 \otimes v_2^\star)(u)(g))), \, g \in G, \tag{2.25}
$$

where $\tilde{D}(1 \otimes v_2^\star)(u)(g) = (U \otimes v_1^\star)(u)(g)$ acts on $C^\infty(G, \mathbb{V}_{\sigma_1})^P$ by

$$
< U(u)(g), v_1^\star >, \, U \otimes v_1^\star \in \mathcal{U}(\mathfrak{g}) \otimes V_{\sigma_1}^\star .
$$

This completes the proof of the Lemma.

□

2.5 Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^\mathfrak{p}$

Many questions raised in the thesis are naturally formulated and dissolved in the framework of the Bernstein-Gefand-Gelfand parabolic category $\mathcal{O}^\mathfrak{p}$.

Definition 2.2 *The BGG parabolic category $\mathcal{O}^\mathfrak{p}$ is the full subcategory of $\mathcal{U}(\mathfrak{g})$-modules, whose objects M satisfy the following conditions:*

1. *M is a finitely generated $\mathcal{U}(\mathfrak{g})$-module.*

2. *M is $\mathfrak{l}$-semisimple $\mathcal{U}(\mathfrak{g})$-module, i.e. $M \simeq \oplus_\mu M_\mu$ is a direct sum of finite dimensional simple $\mathfrak{l}$-modules.*

3. *M is locally $\mathfrak{n}$-finite ($\mathfrak{n}$ is the nilradical for $\mathfrak{p}$), i.e. for each $v \in M$, the subspace $\mathcal{U}(\mathfrak{n}) \cdot v \subset M$ is finite dimensional.*

The morphisms between two objects M, N in the BGG category $\mathcal{O}^\mathfrak{p}$ are denoted $Hom_{\mathcal{O}^\mathfrak{p}}(M, N)$.

Obviously, $\mathcal{O}^\mathfrak{p}$ contains all finite dimensional $\mathfrak{g}$-modules. There are several equivalent characterizations of objects M of the BGG parabolic category $\mathcal{O}^\mathfrak{p}$. For example, $\mathfrak{n}$-finiteness in the last item of the previous Definition is equivalent to $dim(M_\mu) = dim(M_{w\mu})$ for all $w \in W_\mathfrak{l}$ (the Weyl group of the Levi factor $\mathfrak{l}$), or the stability of the set $Weights_{\mathfrak{h}^\star}(M)$ under $W_\mathfrak{l}$.

There is a useful duality in the category $\mathcal{O}^\mathfrak{p}$, defined by

$$\begin{aligned} M &\rightarrow M^\vee, \\ \oplus_\mu M_\mu &\mapsto \oplus_\mu (M_\mu)^\star \end{aligned} \tag{2.26}$$

with $\mathfrak{g}$-module structure on $f \in M_\mu^\star$, $f \mapsto f \circ \iota$ and $\iota : \mathfrak{g} \to \mathfrak{g}$ the involution acting as $\iota : \mathfrak{g}_\alpha \mapsto \mathfrak{g}_{-\alpha}$ for all $\alpha \in \triangle$ and $\iota : \mathfrak{h} \to \mathfrak{h}$ the identity map. The BGG category $\mathcal{O}^\mathfrak{p}$ is closed under the duality $^\vee$, under direct sum, submodules, quotients and extensions in $\mathcal{O}^\mathfrak{p}$ as well as the tensor product with finite dimensional $\mathcal{U}(\mathfrak{g})$-modules. If $M \in \mathcal{O}^\mathfrak{p}$ decomposes according to the infinitesimal character as $M = \oplus_\chi M_\chi$ with $M_\chi \in \mathcal{O}^\mathfrak{p}_\chi$, then each $M_\chi \in \mathcal{O}^\mathfrak{p}$. The simple modules $L(\lambda)$ given by quotients of generalized Verma modules by their maximal submodules are in $\mathcal{O}^\mathfrak{p}$.

The basic building blocks for all objects in the category $\mathcal{O}^\mathfrak{p}$ are highest weight modules, i.e. the $\mathcal{U}(\mathfrak{g})$-modules $M \simeq \mathcal{U}(\mathfrak{g}) \cdot v$ generated by one vector $v \in M$. Despite the fact that an extension class $M \in \mathcal{O}^\mathfrak{p}$ need not be semisimple, there exists on M an increasing finite (standard) filtration $0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M$ by $\mathcal{U}(\mathfrak{g})$-submodules such that the quotient M_{i+1}/M_i is a generalized Verma module.

A central problem significant for geometric applications of Bernstein-Gelfand-Gelfand resolutions is the determination of character formulas for the simple modules $L(\lambda) \in \mathcal{O}^\mathfrak{p}$, $\lambda \in \Lambda(\mathfrak{l})$, in terms of generalized Verma modules (standard modules):

$$ch(L(\lambda)) = \sum_{w \in W^\mathfrak{p}} (-1)^{l(w)} ch(M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{w \cdot \lambda})). \tag{2.27}$$

A reformulation of this problem involves two bases of classes of symbols represented by simple modules, $[L(\lambda)]$, and generalized Verma modules, $[M(\lambda)]$, in the Grothendieck group $K(\mathcal{O}^\mathfrak{p})$. The transition matrix between the two bases involves integral unipotent matrix, which captures the information on Kazhdan-Lusztig polynomials.

The effective use of methods of homological algebra in the category $\mathcal{O}^\mathfrak{p}$ relies on the notion of projective (injective) objects. A central result leading to the BGG reciprocity states that the category $\mathcal{O}^{\mathfrak{p}}$ has enough projectives (injectives.) This means that for each $M \in \mathcal{O}^{\mathfrak{p}}$ there exists a projective object $P \in \mathcal{O}^{\mathfrak{p}}$ and an epimorphism $P \to M$. Recall that an object P in an abelian category is called projective if the left exact functor $Hom(P, -)$ is also right exact. This property can be equivalently characterized by universality, i.e. given an epimorphism $\pi : M \to N$ and a morphism $\varphi : P \to N$, there is a lift $\psi : P \to M$ fulfilling $\pi \circ \psi = \varphi$. Dually, an object $I \in \mathcal{O}^{\mathfrak{p}}$ is injective if the right exact functor $Hom(-, I)$ is also left exact. The duality $^\vee$ in $\mathcal{O}^{\mathfrak{p}}$ implies the equivalence between enough projectives and enough injectives objects in $\mathcal{O}^{\mathfrak{p}}$. In particular, for a simple highest weight module $L(\lambda)$ the epimorphism $P \to L(\lambda)$ quotients due to the projectivity of P as $P \to M(\lambda) \to L(\lambda)$ (here $M(\lambda)$ is the generalized Verma module whose simple quotient by its maximal submodule is $L(\lambda)$), and so P is the projective cover of $M(\lambda)$. Recall that each $M \in \mathcal{O}^{\mathfrak{p}}$ has a unique projective cover, i.e. there is a projective object $P \in \mathcal{O}^{\mathfrak{p}}$ unique up to an isomorphism such that the surjective homomorphism $P \to M$ is essential meaning that no proper submodule of P is mapped onto M.

The basic homological properties of the parabolic category $\mathcal{O}^{\mathfrak{p}}$ are captured by BGG reciprocity: if $\lambda_1, \lambda_2 \in \Lambda^+(\mathfrak{l})$, then

$$[P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1}) : M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_2})] = [M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_2}) : L(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})], \tag{2.28}$$

where $L(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1}) = L(\lambda_1)$ is the simple $\mathcal{U}(\mathfrak{g})$-module (the quotient of $M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})$ by its maximal submodule), and $P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})$ the projective cover of $L(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})$. Here we use the standard notation

$$[P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1}) : M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_2})] = dim\, Hom_{\mathcal{O}^{\mathfrak{p}}}(P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1}), M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_2})^\vee), \tag{2.29}$$

and $[M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_2}) : L(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})]$ denotes the multiplicity of the isomorphism class of $L(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})$ in the composition series of $M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_2})$. In fact, for each module $M \in \mathcal{O}^{\mathfrak{p}}$ holds

$$dim\, Hom_{\mathcal{O}^{\mathfrak{p}}}(P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1}), M^\vee) = [M : L(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})], \tag{2.30}$$

and each projective module $P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1}) \in \mathcal{O}^{\mathfrak{p}}$ has standard filtration for which

$$[P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1}) : M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})] = 1. \tag{2.31}$$

Moreover, $\lambda_2 > \lambda_1$ for all other subquotients $M(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_2})$ of $P(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_{\lambda_1})$.

3 Distribution Fourier transform method and analytic computation of singular vectors

A very basic aspect of the Lie theory is the connection between differential equations and their possible Lie groups of symmetries, partly motivated by aim to understand the spaces of solutions. Another point is the link to differential geometry, regarded as the study of geometric quantities invariant under the action of a Lie group of symmetries. In the present section we give a new systematic approach termed "F-method"to find large classes of systems of partial differential operators, attached to geometric structures, [43], [44]. These structures belong to the class that has been much studied in recent years, namely the so-called parabolic geometries. In particular, we present panorama of representative problems coming from conformal geometry, where the natural invariants are studied via techniques and ideas inspired by holographic correspondence.

On the algebraic level, i.e. in the dual language of homomorphisms between generalized Verma modules, the whole construction is connected to a very natural question in representation theory, namely the branching laws. It seems that no attempts at a systematic approach to branching laws for Verma modules has been made, and our results might be of independent interest from this point of view. Now restricting a generalized Verma module to a reductive subalgebra often leads to wild problems, see [41], but it is possible to find families of examples with good behavior, which are at the same time particularly important for our geometric purposes. Also, many of the operators we construct have already appeared in physics, for example (powers of) the wave operator and Dirac operator. Even here is our approach new and in many cases simplifies classification problems enormously.

Since in parabolic geometry a large amount of natural differential operators have been found already, it is worth pointing out, that the ones found here are exactly the ones that are the hardest to find by the previous methods (essentially coming from the BGG resolution). Here we work primarily in the model case situation, where the manifold is a real flag manifold, but it is pointed out in [44] that (as seen for example in the case of conformal geometry) it is both possible and interesting to extend to the "curved case" of manifolds equipped with the corresponding parabolic geometry.

The results we are going to present are inspired by geometrical considerations. In particular, they correspond to differential invariants (of higher order in general) in the case of models for parabolic geometries. These operators act on sections of homogeneous bundles over homogeneous models G/P for parabolic geometries (i.e., on representations induced from a parabolic subgroup P of a semisimple Lie group G) and they have values in similar space of section of homogeneous bundles over different homogeneous models G'/P' with (a reductive) $G' \subset G, P' = G' \cap P$. In effect, what happens is that first sections are differentiated and then restricted to a submanifold, and this combined operation is covariant for the group G'.

Our language chosen for presenting these results is algebraic, relying at certain stage on analytic techniques, however. We translate geometrical problems into a problem of branching of generalized Verma modules for (the Lie algebra of) G, induced from P, under their restriction to (the Lie algebra of) G'. It is shown that many features of such branching problems are given by general character formulas expressed in terms of the Grothendieck group of the parabolic Bernstein-Gelfand-Gelfand category $\mathcal{O}^{\mathfrak{p}'}$ (see Theorem 5.5.) This describes the result of the branching in 'generic' situations. In order to get more precise information, we develop a procedure for describing the branching for all values of the parameters. It is based on a complete description of the structure of the space of singular vectors (i.e., the highest weight vectors with respect to a parabolic subalgebra of a reductive subalgebra under consideration). The structure of the space of singular vectors as a module under the action of the Levi factor of the parabolic subgroup can be used to describe the resulting branching completely, including special ('non-generic') values of parameter(s) for the inducing representation of the generalized Verma modules.

The structure of the space of singular vectors could in principle be computed by combinatorial considerations (see [37] in the case of conformal densities.) However, such a computation could be worked out only in a quite limited number of cases due to its computational complexity.

Our new method of constructing analogues of Juhl's families of conformally equivariant differential operators is based on the "Fourier transform" of generalized Verma modules, and call it the *F-method*. In contrast to the existing algebraic techniques to find singular vectors (e.g. by recurrence relations in generalized Verma modules), this new method translates the computation of the singular vectors to a question of finding all polynomial solutions of differential equations. In many cases this procedure leads to an ordinary differential equation (due to symmetry involved), which can then often be identified with some classical differential equation for certain special functions (special polynomials). Hence this new method offers a uniform and effective tool to find explicitly singular vectors in many different cases.

The F-method itself is described in detail in Section 4. The key idea of the F-method is as indicated earlier to transform an algebraic branching problem for generalized Verma modules to an analytic problem, namely, solving a system of differential equations. While the existing methods based on combinatorial computations of recurrence relations did not explain the origin of special functions (e.g., the Gegenbauer polynomials) in formulas for singular vectors, our new method is completely different and explains their appearance in a natural way.

In Section 5, we discuss a class of branching problems for modules in the category $\mathcal{O}^{\mathfrak{p}}$ having a discrete decomposability property. Moreover, one of our guiding principles is to focus on multiplicity-free cases which was studied systematically in [40, 41]. Thus we expect a priori that the F-method should work nicely in such cases. Branching rules are formulated in terms of the Grothendieck ring of the BGG category $\mathcal{O}^{\mathfrak{p}}$, and they give a sufficient information on the branching in 'generic' cases.

The rest of the paper contains applications of the F-method for descriptions of the space of all singular vectors in particular cases of conformal geometry. It contains a complete description of the branching for the case of generalized Verma modules induced from one-dimensional P-modules in Section 6 (including the discussion of exceptional values of the inducing parameter), and from the spinor P-module in Section 7. The Section 8 treats the case of Juhl's operators for P-modules given by the fundamental vector representation twisted by characters. Section 9 is devoted to the fundamental vector representation as an inducing P-module, while in Section 10 we study the diagonal branching problem termed Rankin-Cohen brackets for P-modules given by tensor product of characters. In Section 11 we apply the technique to irreducible infinite-dimensional highest weight representation of P, while Section 12 treats the pair of non-compatible Lie algebras Lie $G_2 \stackrel{i}{\hookrightarrow} so(7)$.

As we already emphasized, our original motivation for the study of branching rules for generalized Verma modules came from differential geometry and, in fact, also from number theory. When A. Juhl introduced various families of invariant differential operators acting between manifolds of different dimensions, he was inspired by previous studies of automorphic distributions. From the point of view of differential geometry, there is a substantial relation of the curved version of the Juhl family and recently introduced notions of Q-curvature and conformally invariant powers of the Laplace operator. In [44], the curved version of the Juhl family is constructed using ideas of semi-holonomic Verma modules in the scalar case.

To summarize, we present new results concerning the relationship between several important topics in representation theory and differential parabolic geometry, namely branching laws for generalized Verma modules and the construction of covariant differential operators on manifolds, including in particular operators with natural restrictions to submanifolds. The methods can be extended further to a larger class of examples, including higher rank cases and parabolic subgroups with non-Abelian nilradical (e.g. the Heisenberg algebras).

We mostly use the following notation: $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{N}_+ = \{1, 2, \cdots\}$.

4 F-method

A general approach to a study of branching problems in parabolic category $\mathcal{O}^{\mathfrak{p}}$ and a number of basic results can be found in [40, 41]. In particular, the branching of generalized Verma modules for the case of a symmetric pair $(\mathfrak{g}, \mathfrak{g}')$ was discussed in [41] and the branching was described in a quite general situation for generic case. In [55], these results were generalized to the case of branching of generalized Verma modules for non-compatible parabolic subalgebras. However, for particular values of inducing parameters, more subtle phenomena can appear and the branching problem is much harder to understand. Here we address two different (but closely related) problems.

Problem A.

Find branching rules for all values of inducing parameters.

Problem B.
Find the structure of all irreducible $\mathfrak{g}'$-submodules in a generalized Verma module for $\mathfrak{g}$, or equivalently, to describe the structure of all singular vectors.

The second problem is a refinement of the first one, and is usually much harder. The answer to both problems depends on the ability to compute explicitly the form of all singular vectors.

This section explains the general idea of the F-method, a new approach to finding singular vectors by using distribution Fourier transform and invariant theory. The main advantage of the method is that an algebraic problem is converted to a question to find solutions of a set of partial, or ordinary, differential equations. An explicit answer to both problems in various situations (related to questions in differential geometry), is the content of the rest of thesis.

4.1 The big cell in a flag manifold

Let G be a connected real reductive Lie group with Lie algebra $\mathfrak{g}(\mathbb{R})$. Let $x \in \mathfrak{g}(\mathbb{R})$ be a hyperbolic element. This means that $\mathrm{ad}(x)$ is diagonalizable and its eigenvalues are all real. Then we have a Gelfand–Naimark decomposition

$$\mathfrak{g}(\mathbb{R}) = \mathfrak{n}_-(\mathbb{R}) + \mathfrak{l}(\mathbb{R}) + \mathfrak{n}_+(\mathbb{R}),$$

according to the negative, zero, and positive eigenvalues of $\mathrm{ad}(x)$. The subalgebra $\mathfrak{p}(\mathbb{R}) := \mathfrak{l}(\mathbb{R}) + \mathfrak{n}_+(\mathbb{R})$ is a parabolic subalgebra of $\mathfrak{g}(\mathbb{R})$, and its normalizer P in G is a parabolic subgroup of G. Subgroups $N_\pm \subset G$ are defined by $N_\pm = \exp \mathfrak{n}_\pm(\mathbb{R})$.

The fibration $p : G \to G/P$ is a principal fiber bundle with the group P over a compact manifold G/P. The manifold $M := p(N_- P)$ is an open dense subset of G/P, sometimes referred to as the big Schubert cell of G/P. Let $o := e \cdot P \in G/P$. The exponential map

$$\phi : \mathfrak{n}_- \to M, \quad \phi(X) := \exp(X) \cdot o \in G/P$$

gives the canonical identification of the vector space $\mathfrak{n}_-$ with M.

4.2 A G-invariant pairing

Given a complex finite dimensional P-module V, we consider the unnormalized induced representation π of G on the space $\mathrm{Ind}_P^G(V)$ of smooth sections for the homogeneous vector bundle $\mathcal{V} := G \times_P V \to G/P$. We can identify this space with

$$C^\infty(G,V)^P := \{f \in C^\infty(G,V) : f(g\,p) = p^{-1} \cdot f(g),\, g \in G, p \in P\}.$$

Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}$ of $\mathfrak{g}(\mathbb{R})$. Let $V^\vee$ denote the contragredient representation. Then $V^\vee$ extends to a representation of the whole enveloping algebra $\mathcal{U}(\mathfrak{p})$. The generalized Verma module $M_\mathfrak{p}^\mathfrak{g}(V^\vee)$ is defined by

$$M_\mathfrak{p}^\mathfrak{g}(V^\vee) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} V^\vee.$$

It is a well-known fact that there is an equivariant pairing between $\mathrm{Ind}_P^G(V)$ and the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$. A detailed discussion of it and some basic facts recalled below may be found in [10]. We present here a version better adapted to our needs (it is a simple adaptation of arguments in [53, Theorem 8.3]).

Fact 4.1 *Let G be a connected semisimple Lie group with complexified Lie algebra $\mathfrak{g}$, and P a parabolic subgroup of G with complexified Lie algebra $\mathfrak{p}$. Suppose further that V is a finite dimensional P-module and $V^\vee$ its dual. Then there is a $(\mathfrak{g}, P)$-invariant natural pairing between $\mathrm{Ind}_P^G(V)$ and $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$.*

Moreover, the space of G-equivariant differential operators from $\mathrm{Ind}_P^G(V)$ to $\mathrm{Ind}_P^G(V')$ is isomorphic with the space of $(\mathfrak{g}, P)$-homomorphisms from $M_{\mathfrak{p}}^{\mathfrak{g}}((V')^\vee)$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ for two representations V and V' of P.

In order to construct explicitly equivariant differential operators it suffices to find singular vectors by Fact 4.1 (see also Theorem 4.4 below). The F-method transfers the latter problem into the following steps:

$$\text{generalized Verma module} \rightsquigarrow \text{distributions on } G/P \text{ supported at the origin}$$
$$\rightsquigarrow \text{polynomials on } \mathfrak{n}_+$$

Then singular vectors in generalized Verma modules are transferred to polynomial solutions of certain differential equations on $\mathfrak{n}_+$.

To fix the notation for actual computation by the F-method we briefly review the pairing stated in the theorem in the next subsections. The tools that we are using here go back to Kostant [46], and more details on various parts of the construction can be found in [45].

4.3 The induced modules

Let V be an irreducible complex finite dimensional P-module, and let us consider the corresponding induced representation π of G on $\mathrm{Ind}_P^G(V) \simeq C^\infty(G, V)^P$. The representation of G on $\mathrm{Ind}_P^G(V)$ will be denoted by π and the infinitesimal representation $d\pi$ of its complexified Lie algebra $\mathfrak{g}$ will be considered in the non-compact picture. We can restrict $d\pi$ to a representation of $\mathfrak{g}(\mathbb{R})$ on functions on the big cell $\mathfrak{n}_-(\mathbb{R}) \simeq M \subset G/P$ with values in V.

More precisely, we identify the space of equivariant smooth maps $C^\infty(N_-\, P, V)^P$ and the space $C^\infty(N_-, V)$ resp. $C^\infty(\mathfrak{n}_-(\mathbb{R}), V)$ as follows. A function $f \in C^\infty(N_-, V)$ corresponds to $\tilde{f} \in C^\infty(N_-\, P, V)^P$ defined by $\tilde{f}(n_-\, p) = p^{-1} \cdot f(n_-)$, $n_- \in N_-$, $p \in P$. We identify the space $C^\infty(N_-, V)$ with $C^\infty(\mathfrak{n}_-(\mathbb{R}), V)$ via the exponential map $\exp : \mathfrak{n}_-(\mathbb{R}) \to N_-$, and consider the latter as a representation of $\mathfrak{g}(\mathbb{R})$ by the action of $d\pi$.

We need to know an explicit form of the action $d\pi(Z)$ for $Z \in \mathfrak{n}_+(\mathbb{R})$ on $C^\infty(\mathfrak{n}_-(\mathbb{R}), V)$. The action is realized by vector fields on $\mathfrak{n}_-(\mathbb{R})$ with coefficients in $\mathrm{Pol}(\mathfrak{n}_-) \otimes \mathrm{End}\, V$, where $\mathrm{Pol}(\mathfrak{n}_-)$ denotes the space of all polynomials on $\mathfrak{n}_-$ ([46]).

An actual computation of the representation $d\pi(Z)$ can be carried out by the usual scheme: for a given $Z \in \mathfrak{n}_+(\mathbb{R})$, we consider the one-parameter subgroup $n(t) = \exp(tZ) \in N_+$ and rewrite the product $n(t)^{-1}x$, for $x \in N_-$ and for small t as

$$n(t)^{-1}x = \tilde{x}(t)p(t),\ \tilde{x}(t) \in N_-,\ p(t) \in P.$$

Then, for $f \in C^\infty(\mathfrak{n}_-(\mathbb{R}), V)$, we have

$$[d\pi(Z)f](x) = \frac{d}{dt}\Big|_{t=0}(p(t))^{-1} \cdot f(\tilde{x}(t)). \tag{4.1}$$

4.4 Generalized Verma modules

To describe the pairing in Fact 4.1, it is important to realize the generalized Verma modules as the space of distributions supported at the origin.

Recall that $V^\vee$ denotes the module contragredient to the irreducible complex finite dimensional $\mathfrak{p}$-module V. It extends to the representation of the whole enveloping algebra $\mathcal{U}(\mathfrak{p})$.

By the Poincaré–Birkhoff–Witt theorem, the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ is isomorphic to $\mathcal{U}(\mathfrak{n}_-) \otimes V^\vee$ as an $\mathfrak{l}$-module. It will be useful to realize the space $\mathcal{U}(\mathfrak{n}_-)$ as a suitable subspace of the Weyl algebra $\mathrm{Diff}(\mathfrak{n}_-)$ of differential operators with polynomial coefficients on $\mathfrak{n}_-$. For this, let $\mathrm{Diff}_{N_-}(\mathfrak{n}_-)$ be a subspace of $\mathrm{Diff}(\mathfrak{n}_-)$ consisting of all holomorphic differential operators which are invariant under the left action of N_-.

We define for each $Y \in \mathfrak{n}_-(\mathbb{R})$ the left N_--invariant vector field D_Y on $\mathfrak{n}_-(\mathbb{R})$ by

$$[D_Y \cdot f](xo) = \frac{d}{dt}\Big|_{t=0} f(x\exp(tY)o) \quad \text{for } x \in N_-,\ f \in C^\infty(\mathfrak{n}_-, \mathbb{C}).$$

The correspondence $Y \mapsto D_Y$ extends to a ring isomorphism $\mathcal{U}(\mathfrak{n}_-) \xrightarrow{\sim} \mathrm{Diff}_{N_-}(\mathfrak{n}_-)$. In the special case when $\mathfrak{n}_-$ is commutative, the space $\mathrm{Diff}_{N_-}(\mathfrak{n}_-)$ is nothing but the space of holomorphic differential operators on $\mathfrak{n}_-$ with constant coefficients.

4.5 Duality

Recall from Fact 4.1 that there exists a $(\mathfrak{g}, P)$-pairing between induced modules $\mathrm{Ind}_P^G(V)$ and generalized Verma modules $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$. Now we are going to describe this natural pairing. Let $\mathcal{D}'(G/P) \otimes V^\vee$ be the space of all distributions on G/P with values in $V^\vee$. The evaluation defines a canonical pairing between $\mathrm{Ind}_P^G(V)$ and $\mathcal{D}'(G/P) \otimes V^\vee$, which restricts to the pairing

$$\mathrm{Ind}_P^G(V) \times \mathcal{D}'_{[o]}(G/P) \otimes V^\vee \to \mathbb{C}, \tag{4.2}$$

where $\mathcal{D}'(G/P)_{[o]} \otimes V^\vee$ denotes the space of distributions with support in the base point $o \in G/P$. As shown in [10], the space $\mathcal{D}'_{[o]}(G/P) \otimes V^\vee$ can be identified, as a $\mathfrak{g}(\mathbb{R})$-module, with the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$. We shall now describe the isomorphism in more details.

We are considering the space of distributions with support in o, hence we can restrict to the non compact picture as well and to consider the subspace $\mathcal{D}'_{[0]} \subset \mathcal{D}'(\mathfrak{n}_-(\mathbb{R}))$ of all distributions on $\mathfrak{n}_-$ with support at 0. There is a natural N_--invariant identification of $\mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R}))$ with $\mathcal{D}'_{[0]}$ given by the map $Y \in \mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \mapsto Y\delta_0 \in \mathcal{D}'_{[0]}$, where δ_0 is the Dirac distribution at the point $0 \in \mathfrak{n}_-(\mathbb{R})$ (see [46]). More generally, we can identify the space $\mathcal{D}'_{[0]} \otimes V^\vee$ with $\mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$, which is isomorphic (as a vector space) to the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$.

The pairing (4.2) now translates to the pairing between $C^\infty(\mathfrak{n}_-(\mathbb{R}), V)$ and $\mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$, given by the evaluation

$$\langle Y, f\rangle = Y\delta_0(f),\ Y \in \mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee,\ f \in C^\infty(\mathfrak{n}_-(\mathbb{R}), V).$$

It can be checked ([10, 46]) that it gives a $\mathfrak{g}(\mathbb{R})$ -invariant pairing. It will be used to compute the dual action $d\pi^\vee(Z), Z \in \mathfrak{n}_+$ on the generalized Verma module.

The representation $d\pi$ of $\mathfrak{g}(\mathbb{R})$ on $C^\infty(\mathfrak{n}_-(\mathbb{R}), V)$ induces the contragredient representation $d\pi^\vee$ of $\mathfrak{g}(\mathbb{R})$ on $\mathcal{D}'_{[0]} \otimes V^\vee \simeq \mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$ by the formula

$$\langle d\pi^\vee(X)T, f\rangle = -\langle T, d\pi(X)f\rangle, \qquad (4.3)$$
$$\text{for} \quad T \in \mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee, f \in C^\infty(\mathfrak{n}_-(\mathbb{R}), V).$$

Then $d\pi^\vee(X)$ is given by first order differential operators with values in $\mathrm{End}(V^\vee)$. It can be extended to a representation, denoted by the same letter $d\pi^\vee$, of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ on $\mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$.

4.6 The Fourier transform

Let us consider now the case when the Lie algebra $\mathfrak{n}_-(\mathbb{R})$ is commutative. In this case, the operators $d\pi^\vee(X)$, $X \in \mathfrak{g}(\mathbb{R})$ are realized as differential operators on $\mathfrak{n}_-(\mathbb{R})$ with constant coefficients in $\mathrm{End}(V^\vee)$.

Using the Fourier transform on the space $\mathfrak{n}_-(\mathbb{R})$, the generalized Verma module $\mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$ can be identified with the space $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ and the action $d\pi^\vee$ of $\mathfrak{g}(\mathbb{R})$ on $\mathrm{Diff}_{N_-}(\mathfrak{n}_-(\mathbb{R})) \otimes V^\vee$ is translated to the action $d\tilde{\pi}$ of $\mathfrak{g}(\mathbb{R})$ on $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$, realized again by differential operators with values in $\mathrm{End}(V^\vee)$, possibly of higher order. The explicit form of $d\tilde{\pi}(X)$ is easy to compute by the Fourier transform from the explicit form of $d\pi^\vee$.

4.7 Singular vectors

As explained in Fact 4.1, there is a one-to-one correspondence relating intertwining differential operators between principal series representations and (algebraic) homomorphisms between generalized Verma modules. The latter homomorphisms are characterized by the image of the highest weight vectors with respect to the parabolic subalgebra in consideration, which are sometimes referred to as *singular vectors*, i.e., those vectors annihilated by the nilradical $\mathfrak{n}_+$.

We shall consider a more general setting. Suppose that, together with the couple $P \subset G$ used above, we take another couple $P' \subset G'$, such that $G' \subset G$ is a reductive subgroup of G and $P' = P \cap G'$ is a parabolic subgroup of G'. We shall see that this occurs if $\mathfrak{p}$ is $\mathfrak{g}'$-compatible in the sense of Definition 5.3. In this case, $\mathfrak{n}'_+ := \mathfrak{n}_+ \cap \mathfrak{g}'$ is the nilradical of $\mathfrak{p}'$, and we set $L' = L \cap G'$ for the corresponding Levi subgroup in G'. We are interested in the problem to describe explicitly branching of a generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ under the restriction from $\mathfrak{g}$ to $\mathfrak{g}'$.

Definition 4.2 *Let V be any irreducible finite dimensional $\mathfrak{p}$-module. Let us define the L'-module*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+} := \{v \in M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) : d\pi^\vee(Z)v = 0 \text{ for any } Z \in \mathfrak{n}'_+\}. \tag{4.4}$$

For $G = G'$, the set $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ is of finite dimension. Note that for $G \neq G'$, the set $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ is infinite dimensional but it is still completely reducible as an $\mathfrak{l}'$-module. Let us decompose $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ into irreducible modules of L' and take $W^\vee$ to be one of its irreducible submodules. Then we get an injective $\mathfrak{g}'$-homomorphism from $M_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee)$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$. Dually, in the language of differential operators, we get an invariant differential operator from $\mathrm{Ind}_P^G(V)$ to $\mathrm{Ind}_{P'}^{G'}(W)$ by Theorem 4.4 below. So the knowledge of all irreducible summands of $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ gives the knowledge of all possible targets for equivariant differential operators on $\mathrm{Ind}_P^G(V)$.

In the F-method, we then realize the space $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ in the space of polynomials on $\mathfrak{n}_+$ with values in $V^\vee$ with action $d\tilde{\pi}$. It can be done efficiently using the Fourier transform as follows:

Definition 4.3 *We define*

$$\begin{aligned} \mathrm{Sol} \equiv &\mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee) && (4.5)\\ := &\{f \in \mathrm{Pol}[\mathfrak{n}_+] \otimes V^\vee : d\tilde{\pi}(Z)f = 0 \text{ for any } Z \in \mathfrak{n}'_+\}. && (4.6) \end{aligned}$$

The inverse Fourier transform gives an L'-isomorphism

$$\varphi : \mathrm{Sol}(\mathfrak{g}, \mathfrak{g}'; V^\vee) \xrightarrow{\sim} M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}. \tag{4.7}$$

An explicit form of the action $d\tilde{\pi}(Z)$ leads to a (system of) differential equation for elements in Sol and it makes it possible to describe its structure completely in many particular cases of interest. We shall see in Sections 6 and 7 the full structure of the set Sol as an L'-module, and give the complete classification of $\mathfrak{g}'$-homomorphisms from $M_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee)$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$.

The transition from $M_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)^{\mathfrak{n}'_+}$ to Sol is the key point of the F-method. It makes it possible to transform algebraic problem of computation of singular vectors in Verma modules into analytic problem of solving differential equations. The F-method is often more efficient than other existing algebraic methods in finding singular vectors. Furthermore, the F-method clarifies why the combinatorial formula appearing in the coefficients of intertwining differential operators in the example of Juhl [37, Chapter 5] are related to those of the Gegenbauer polynomials. It also reduces substantially the amount of computation needed and gives a complete description of the set of singular vectors; finally it offers a systematic and effective tool for the investigation of singular vectors in many cases. It will be illustrated below with a series of different examples.

4.8 Equivariant differential operators to submanifolds

The above scheme for computing elements in Sol will be implemented below in various special cases, for different groups, different parabolics, and different induced representations. For applications in geometry, it is useful to extend the Fact 4.1 to the more general situation.

Theorem 4.4 *The set of all G'-intertwining differential operators from $\mathrm{Ind}_P^G(V)$ to $\mathrm{Ind}_{P'}^{G'}(V')$ is in one-to-one correspondence with the space of all $(\mathfrak{g}', P')$-homomorphisms from $M_{\mathfrak{p}'}^{\mathfrak{g}'}(V'^{\vee})$ to $M_{\mathfrak{p}}^{\mathfrak{g}}(V^{\vee})$.*

See [43], [44], [45] for the proof of Theorem 4.4 (and some further generalizations) and for the precise meaning of "differential operators" between different base spaces with morphisms. This correspondence will be used in the next sections.

5 Discretely decomposable branching laws

In this section we fix the notation for the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$, and summarize the algebraic framework on discretely decomposable restrictions and multiplicity-free theorems in branching laws established in [39, 40, 41]. These algebraic results are a guiding principle in the current article in finding appropriate settings in parabolic geometry, and then in obtaining explicit formulas of invariant differential operators.

5.1 Category $\mathcal{O}$ and $\mathcal{O}^{\mathfrak{p}}$

We begin with a quick review of the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ (see [35] for an introduction to this area).

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{j}$ a Cartan subalgebra. We write $\Delta \equiv \Delta(\mathfrak{g}, \mathfrak{j})$ for the root system, $\mathfrak{g}_\alpha$ $(\alpha \in \Delta)$ for the root space, and $\alpha^\vee$ for the coroot, and $W \equiv W(\mathfrak{g})$ for the Weyl group for the root system $\Delta(\mathfrak{g}, \mathfrak{j})$. We fix a positive system Δ^+, write $\rho \equiv \rho(\mathfrak{g})$ for half the sum of positive roots, and define a Borel subalgebra $\mathfrak{b} = \mathfrak{j} + \mathfrak{n}$ with nilradical $\mathfrak{n} := \oplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. The Bernstein–Gelfand–Gelfand category $\mathcal{O}$ (BGG category for short) is defined to be the full subcategory of $\mathfrak{g}$-modules whose objects are finitely generated $\mathfrak{g}$-modules X such that X are $\mathfrak{j}$-semisimple and locally $\mathfrak{n}$-finite [7].

Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{b}$, and $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ its Levi decomposition with $\mathfrak{j} \subset \mathfrak{l}$. We set $\Delta^+(\mathfrak{l}) := \Delta^+ \cap \Delta(\mathfrak{l}, \mathfrak{j})$, and define

$$\mathfrak{n}_-(\mathfrak{l}) := \bigoplus_{\alpha \in \Delta^+(\mathfrak{l})} \mathfrak{g}_{-\alpha}.$$

The parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ is the full subcategory of $\mathcal{O}$ whose objects X are locally $\mathfrak{n}_-(\mathfrak{l})$-finite. We note that $\mathcal{O}^{\mathfrak{b}} = \mathcal{O}$ by definition.

The set of λ for which $\lambda|_{\mathfrak{j} \cap [\mathfrak{l}, \mathfrak{l}]}$ is dominant integral is denoted by

$$\Lambda^+(\mathfrak{l}) := \{\lambda \in \mathfrak{j}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{N} \text{ for all } \alpha \in \Delta^+(\mathfrak{l})\}.$$

We write F_λ for the finite dimensional simple $\mathfrak{l}$-module with highest weight λ, inflate F_λ to a $\mathfrak{p}$-module via the projection $\mathfrak{p} \to \mathfrak{p}/\mathfrak{n}_+ \simeq \mathfrak{l}$, and define the generalized Verma module by

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \equiv M_{\mathfrak{p}}^{\mathfrak{g}}(F_\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} F_\lambda. \tag{5.1}$$

Then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) \in \mathcal{O}^{\mathfrak{p}}$, and any simple object in $\mathcal{O}^{\mathfrak{p}}$ is the quotient of some $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$. We say $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is of *scalar type* if F_λ is one-dimensional, or equivalently, if $\langle \lambda, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta(\mathfrak{l})$.

If $\lambda \in \Lambda^+(\mathfrak{l})$ satisfies

$$\langle \lambda + \rho, \beta^\vee \rangle \notin \mathbb{N}_+ \text{ for all } \beta \in \Delta^+ \setminus \Delta(\mathfrak{l}), \tag{5.2}$$

then $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ is simple, see [17].

Let $\mathfrak{Z}(\mathfrak{g})$ be the center of the enveloping algebra $U(\mathfrak{g})$, and we parameterize maximal ideals of $\mathfrak{Z}(\mathfrak{g})$ by the Harish-Chandra isomorphism:

$$\mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathfrak{Z}(\mathfrak{g}), \mathbb{C}) \simeq \mathfrak{j}^*/W, \quad \chi_\lambda \leftrightarrow \lambda.$$

Then the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ has a $\mathfrak{Z}(\mathfrak{g})$-infinitesimal character $\lambda + \rho \in \mathfrak{j}^*/W$.

We denote by $\mathcal{O}_\lambda^{\mathfrak{p}}$ the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ whose objects have generalized $\mathfrak{Z}(\mathfrak{g})$-infinitesimal characters $\lambda \in \mathfrak{j}^*/W$, namely,

$$\mathcal{O}_\lambda^{\mathfrak{p}} = \bigcup_{n=1}^{\infty} \{X \in \mathcal{O}^{\mathfrak{p}} : (z - \chi_\lambda(z))^n v = 0 \quad \text{for any } v \in X \text{ and } z \in \mathfrak{Z}(\mathfrak{g})\}.$$

Any $\mathfrak{g}$-module in $\mathcal{O}^{\mathfrak{p}}$ is a direct sum of finite $\mathfrak{g}$-modules belonging to some $\mathcal{O}_\lambda^{\mathfrak{p}}$. Let $K(\mathcal{O}_\lambda^{\mathfrak{p}})$ be the Grothendieck group of $\mathcal{O}_\lambda^{\mathfrak{p}}$, and set

$$K(\mathcal{O}^{\mathfrak{p}}) := \prod_{\lambda \in \mathfrak{j}^*/W}{}' K(\mathcal{O}_\lambda^{\mathfrak{p}}),$$

where $\prod'$ denotes the direct product for which the components are zero except for countably many constituents. Then $K(\mathcal{O}^{\mathfrak{p}})$ is a free $\mathbb{Z}$-module with basis elements $\mathrm{Ch}(X)$ in one-to-one correspondence with simple modules $X \in \mathcal{O}^{\mathfrak{p}}$. We note that $K(\mathcal{O}^{\mathfrak{p}})$ allows a formal sum of countably many $\mathrm{Ch}(X)$, and contains the Grothendieck group of $\mathcal{O}^{\mathfrak{p}}$ as a subgroup.

5.2 Discretely decomposable branching laws for $\mathcal{O}^{\mathfrak{p}}$

Retain the notation of Subsection 5.1. Let $\mathfrak{g}'$ be a reductive subalgebra of $\mathfrak{g}$. We note that rank $\mathfrak{g}'$ may be strictly smaller than rank $\mathfrak{g}$. Our subject here is to understand the $\mathfrak{g}'$-module structure of a $\mathfrak{g}$-module $X \in \mathcal{O}^{\mathfrak{p}}$, to which we simply refer as the *restriction* $X|_{\mathfrak{g}'}$. This question might look easy in the category $\mathcal{O}^{\mathfrak{p}}$ at first glance, however, the restriction $X|_{\mathfrak{g}'}$ behaves surprisingly in a various (and sometimes "wild") manner even when $(\mathfrak{g}, \mathfrak{g}')$ is a reductive symmetric pair. In particular, it may well happen that the restriction $X|_{\mathfrak{g}'}$ does not contain any simple module of $\mathfrak{g}'$ (see [41]).

For a simple $\mathfrak{g}$-module X, the restriction $X|_{\mathfrak{g}'}$ contains a simple $\mathfrak{g}'$-module if and only if the restriction $X|_{\mathfrak{g}'}$ is "discretely decomposable". Here we recall:

Definition 5.1 ([39, Part III]) *A $\mathfrak{g}'$-module X is discretely decomposable if there exists an increasing sequence of $\mathfrak{g}'$-modules X_j of finite length ($j \in \mathbb{N}$) such that $X = \cup_{j=0}^{\infty} X_j$.*

It then turns out that the concept of "discretely decomposable restrictions" exactly corresponds to our main interest here, namely to the construction of equivariant differential operators (including normal derivatives) in parabolic geometry.

We then ask for which triple $\mathfrak{g}' \subset \mathfrak{g} \supset \mathfrak{p}$ the restriction $X|_{\mathfrak{g}'}$ of $X \in \mathcal{O}^{\mathfrak{p}}$ is discretely decomposable as a $\mathfrak{g}'$-module. A criterion for this was established in [41] as follows: Let G be the group $\mathrm{Int}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$, $P \subset G$ the parabolic subgroup of G with Lie algebra $\mathfrak{p}$, and $G' \subset G$ a reductive subgroup with Lie algebra $\mathfrak{g}' \subset \mathfrak{g}$.

Proposition 5.2 *If $G'P$ is closed in G, then the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable for any simple $X \in \mathcal{O}^{\mathfrak{p}}$. The converse statement also holds if (G, G') is a symmetric pair.*

Proof:

See [41, Proposition 3.5 and Theorem 4.1].

□

Let us consider a simple sufficient condition for the closedness of $G'P$ in G, which will be fulfilled in all the examples discussed in the thesis. To that aim, let E be a hyperbolic element of $\mathfrak{g}$ defining a parabolic subalgebra $\mathfrak{p}(E) = \mathfrak{l}(E) + \mathfrak{n}(E)$.

Definition 5.3 ([41, Definition 3.7]) *A parabolic subalgebra $\mathfrak{p}$ is $\mathfrak{g}'$-compatible if there exists a hyperbolic element $E' \in \mathfrak{g}'$ such that $\mathfrak{p} = \mathfrak{p}(E')$.*

If $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}$ is $\mathfrak{g}'$-compatible, then $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ becomes a parabolic subalgebra of $\mathfrak{g}'$ with the following Levi decomposition:

$$\mathfrak{p}' = \mathfrak{l}' + \mathfrak{n}' := (\mathfrak{l} \cap \mathfrak{g}') + (\mathfrak{n} \cap \mathfrak{g}'),$$

and $P' := P \cap G'$ becomes a parabolic subgroup of G'. Hence, $G'/P \cap G'$ becomes automatically a closed submanifold of G/P, or equivalently, $G'P$ is closed in G. Here is a direct consequence of Proposition 5.2:

Proposition 5.4 ([41, Proposition 3.8]) *If* $\mathfrak{p}$ *is* $\mathfrak{g}'$*-compatible, then the restriction* $X|_{\mathfrak{g}'}$ *is discretely decomposable for any* $X \in \mathcal{O}^{\mathfrak{p}}$*.*

Let $\mathfrak{p}$ be a $\mathfrak{g}'$-compatible parabolic subalgebra, and keep the above notation. We denote by F'_μ a finite dimensional simple $\mathfrak{l}'$-module with highest weight $\mu \in \Lambda^+(\mathfrak{l}')$. The $\mathfrak{l}'$-module structure on the opposite nilradical $\mathfrak{n}_-$ descends to $\mathfrak{n}_-/(\mathfrak{n}_- \cap \mathfrak{g}')$, and consequently extends to the symmetric tensor algebra $S(\mathfrak{n}_-/(\mathfrak{n}_- \cap \mathfrak{g}'))$. We set

$$m(\lambda, \mu) := \dim \operatorname{Hom}_{\mathfrak{l}'}(F'_\mu, F_\lambda|_{\mathfrak{l}'} \otimes S(\mathfrak{n}_-/(\mathfrak{n}_- \cap \mathfrak{g}'))).$$

The following identity is a key step to find branching laws (in a generic case) for the restriction $X|_{\mathfrak{g}'}$ for $X \in \mathcal{O}^{\mathfrak{p}}$:

Theorem 5.5 ([41, Proposition 5.2]) *Suppose that* $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ *is a* $\mathfrak{g}'$*-compatible parabolic subalgebra of* $\mathfrak{g}$*, and* $\lambda \in \Lambda^+(\mathfrak{l})$*. Then*

1) $m(\lambda, \mu) < \infty$ *for all* $\mu \in \Lambda^+(\mathfrak{l}')$*.*

2) *We have the following identity in* $K(\mathcal{O}^{\mathfrak{p}})$*:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{\mu \in \Lambda^+(\mathfrak{l}')} m(\lambda, \mu) M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mu)$$

for any generalized Verma modules $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ *and* $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mu)$ *defined respectively by* $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} F_\lambda$*,* $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mu) = \mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} F'_\mu$*.*

Finally, we highlight the multiplicity-free case, namely, when $m(\lambda, \mu) \leq 1$ and give a closed formula of branching laws. Suppose now that $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$ is a parabolic subalgebra such that the nilradical $\mathfrak{n}_+$ is abelian. We write $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{l} + \mathfrak{n}_+$ for the Gelfand–Naimark decomposition. Let τ be an involutive automorphism of $\mathfrak{g}$ such that $\tau\mathfrak{p} = \mathfrak{p}$. We take a Cartan subalgebra $\mathfrak{j}$ of $\mathfrak{l}$ such that $\mathfrak{j}^\tau$ is a maximal abelian subspace of $\mathfrak{l}^\tau$. Here, for a subspace V in $\mathfrak{g}$, we write $V^{\pm\tau} := \{v \in V : \tau v = \pm v\}$ for the ± 1 eigenspaces of τ. Let $\Delta(\mathfrak{n}_-^{-\tau}, \mathfrak{j}^\tau)$ be the set of weights of $\mathfrak{n}_-^{-\tau}$ with respect to $\mathfrak{j}^\tau$. The roots α and β are said to be *strongly orthogonal* if neither $\alpha + \beta$ nor $\alpha - \beta$ is a root. We take a maximal set of strongly orthogonal roots $\{\nu_1, \cdots, \nu_k\}$ in $\Delta(\mathfrak{n}_-^{-\tau}, \mathfrak{j}^\tau)$ inductively as follows:

1) ν_1 is the highest root of $\Delta(\mathfrak{n}_-^{-\tau}, \mathfrak{j}^\tau)$.

2) ν_{j+1} is the highest root among the elements in $\Delta(\mathfrak{n}_-^{-\tau}, \mathfrak{j}^\tau)$ that are strongly orthogonal to $\nu_1, \cdots, \nu_j$ $(1 \leq j \leq l - 1)$.

Then we recall from [40, 41] the multiplicity-free branching law:

Theorem 5.6 *Suppose that $\mathfrak{p}$, τ, and λ are as above. Then the generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ decomposes into a multiplicity-free direct sum of generalized Verma modules of $\mathfrak{g}^\tau$ in $K(\mathcal{O}^{\mathfrak{p}})$:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)|_{\mathfrak{g}^\tau} \simeq \bigoplus_{\substack{a_1\geq\cdots\geq a_l\geq 0\\ a_1,\cdots,a_l\in\mathbb{N}}} M_{\mathfrak{p}^\tau}^{\mathfrak{g}^\tau}(\lambda|_{\mathfrak{j}^\tau}+\sum_{j=1}^{l} a_j\nu_j). \tag{5.3}$$

It is still a direct sum in the parabolic BGG category $\mathcal{O}^{\mathfrak{p}}$ if the following criterion is satisfied:
$\langle\lambda|_{\mathfrak{j}^\tau}+\rho(\mathfrak{g}^\tau)+\sum_{j=1}^{l} a_j\nu_j, \beta^\vee\rangle \in \mathbb{N}_+$ *for all* $\beta\in\Delta(\mathfrak{n}_+^\tau, \mathfrak{j}^\tau)$,
$\lambda|_{\mathfrak{j}^\tau}+\rho(\mathfrak{g}^\tau)+\sum_{j=1}^{l} a_j\nu_j$ *are all distinct in* $(\mathfrak{j}^\tau)^*/W(\mathfrak{g}^\tau)$.

Proof:
The formula (5.3) was proved in [40, Theorem 8.3] (in the framework of holomorphic discrete series representations) and in [41, Theorem 5.2] (in the framework of generalized Verma modules) under the assumption that λ is sufficiently negative. The latter proof shows in fact that the identity (5.3) holds for all λ in $K(\mathcal{O}^{\mathfrak{p}})$. Since two modules with different infinitesimal characters do not have extension, the last statement follows.

□

Remark 5.7 *In the case $\mathfrak{l}=\mathfrak{g}^\tau=\mathfrak{p}^\tau$, each summand of the right-hand side is finite dimensional.*

The pair $(\mathfrak{g}\oplus\mathfrak{g}, \mathrm{diag}(\mathfrak{g}))$ is a special case of a symmetric pair, and we can apply Theorem 5.6. For the convenience to the reader, we write the branching laws in this special case as below. Suppose $\mathfrak{p}_1$, $\mathfrak{p}_2$ are two standard parabolic subalgebras of $\mathfrak{g}$. Then the generalized Verma module $M_{\mathfrak{p}_1\oplus\mathfrak{p}_2}^{\mathfrak{g}\oplus\mathfrak{g}}(\lambda,\mu)$ of the direct product Lie algebra $\mathfrak{g}\oplus\mathfrak{g}$ is naturally isomorphic to the outer tensor product:

$$M_{\mathfrak{p}_1\oplus\mathfrak{p}_2}^{\mathfrak{g}\oplus\mathfrak{g}}(\lambda,\mu) \simeq M_{\mathfrak{p}_1}^{\mathfrak{g}}(\lambda)\boxtimes M_{\mathfrak{p}_2}^{\mathfrak{g}}(\mu).$$

The restriction to $\mathrm{diag}(\mathfrak{g})\simeq\mathfrak{g}$ is nothing but the tensor product representation:

$$M_{\mathfrak{p}_1\oplus\mathfrak{p}_2}^{\mathfrak{g}\oplus\mathfrak{g}}(\lambda,\mu)|_{\mathrm{diag}(\mathfrak{g})} \simeq M_{\mathfrak{p}_1}^{\mathfrak{g}}(\lambda)\otimes M_{\mathfrak{p}_2}^{\mathfrak{g}}(\mu). \tag{5.4}$$

The subalgebra $\mathrm{diag}(\mathfrak{g})$ is compatible with $(\mathfrak{g}\oplus\mathfrak{g}, \mathfrak{p}_1\oplus\mathfrak{p}_2)$ in the sense of Definition 5.3, and hence (5.4) decomposes discretely by Proposition 5.2. For simplicity, suppose now that $\mathfrak{p}_1=\mathfrak{p}_2=\mathfrak{p}$, where $\mathfrak{p}=\mathfrak{l}+\mathfrak{n}_+$ is a parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{n}_+$ abelian .

Let $\{\mu_1,\cdots,\mu_k\}$ be a maximal set of strongly orthogonal roots in $\Delta(\mathfrak{n}_-,\mathfrak{j})$, which are taken inductively as

1) μ_1 is the highest root among $\Delta(\mathfrak{n}_-,\mathfrak{j})$,

2) μ_{j+1} is the highest root among the elements in $\Delta(\mathfrak{n}_-,\mathfrak{j})$ that are strongly orthogonal to $\{\mu_1,\cdots,\mu_j\}$.

The following theorem is a special case of Theorem 5.6.

Theorem 5.8 ([40, Theorem 8.4]) *Let $\lambda_1, \lambda_2 \in \Lambda^+(\mathfrak{l})$. Then the tensor product representation of two generalized Verma modules of scalar type decomposes into a multiplicity-free sum of generalized Verma modules in $K(\mathcal{O}^{\mathfrak{p}})$:*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_1) \otimes M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_2) \simeq \bigoplus_{\substack{a_1 \geq \cdots \geq a_k > 0 \\ a_1, \cdots, a_k \in \mathbb{N}}} M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_1 + \lambda_2 + \sum_{j=1}^{k} a_j \mu_j). \tag{5.5}$$

The right-hand side is a direct sum in $\mathcal{O}^{\mathfrak{p}}$ if the following criterion is satisfied:

- $\langle \lambda_1 + \lambda_2 + \rho + \sum_{j=1}^{k} a_j \mu_j, \beta^\vee \rangle \notin \mathbb{N}_+$ *for* $\beta \in \Delta(\mathfrak{n}_+, \mathfrak{j})$,
- *the infinitesimal characters* $\lambda_1 + \lambda_2 + \rho + \sum_{j=1}^{k} a_j \mu_j \in \mathfrak{j}^*/W$ *are all distinct.*

6 Conformal geometry with arbitrary signature

The present section deals with a generalization of the Juhl families of invariant differential operators acting between sections of line bundles over two conformal manifolds (of different dimensions), of which the original form was constructed with completely different motivation by combinatorial techniques in [37]. In particular, we describe the basic family of the Juhl family of differential operators, extended to the case of arbitrary signature. They are acting between sections of line bundles. The case of spinor bundles over conformal manifolds and differential operators acting between their sections will be treated in the next section.

6.1 Notation

Let $p \geq 1, q \geq 2, n = p + q - 2$, and suppose that $\epsilon_i = 1$ for $i = 1, \ldots, p-1$ resp. $\epsilon_i = -1$ for $i = p, \ldots, p+q-2$. Let us consider the quadratic form $2x_0\, x_{n+1} + \sum_{i=1}^n \epsilon_i x_i^2$ and its invariance Lie group $G = SO_o(p,q)$ (the connected component containing the identity).

The group G preserves the null cone $\mathcal{N} = \mathcal{N}_{p,q} \subset \mathbb{R}^{p,q}$, and the parabolic subgroup $P \subset G$ is defined as the isotropy subgroup of the line in the null cone $\mathcal{N}$ generated by $e_0 = (1, 0, \ldots, 0)$. The homogeneous space G/P is the projective null cone $\mathbb{P}\mathcal{N}$ with its conformal structure. Let $P = LN = MAN$ be the Langlands decomposition. The group M can be identified with $SO(p-1, q-1)$, it preserves the coordinates x_0, x_{n+1} up to a sign, and the spaces $\mathfrak{n}_\pm$ are its fundamental vector, resp. dual, representations. The group M preserves the quadratic form $\sum_{i=1}^n \epsilon_i x_i^2$. Denote by $\mathbb{J}$ the matrix of this quadratic form with elements ϵ_i on the diagonal.

Elements in G (resp. in $\mathfrak{g}$) can be written as block matrices with respect to the direct sum decomposition

$$\mathbb{R}^{n+2} = \mathbb{R}(x_0) \oplus \mathbb{R}^n(x_1, \ldots, x_n) \oplus \mathbb{R}(x_{n+1}). \tag{6.1}$$

Elements in P are given by block triangular matrices

$$p = \begin{pmatrix} \epsilon(m)a & \star & \star \\ 0 & m & \star \\ 0 & 0 & \epsilon(m)a^{-1} \end{pmatrix} \in P$$

with $a \in \mathbb{R}_+$, $(m, \epsilon) \in M = SO(p-1, q-1)$. Here $\epsilon(m) = \pm 1$ according to whether $m \in M$ belongs to the connected component of the identity or not.

Using notation

$$\mathfrak{n}_+ \equiv \{Z : Z = (z_1, \ldots, z_n)\} \simeq \mathbb{R}^n, \mathfrak{n}_- \equiv \{X : {}^tX = (x_1, \ldots, x_n)\} \simeq \mathbb{R}^n,$$

we have a standard basis of $\mathfrak{n}_+$ given by $\{E_j\}$, $j = 1, \ldots, n$. We realize elements $n \in N^+, x \in N^-$ as

$$n = \exp Z = \begin{pmatrix} 1 & Z & -\frac{|Z|^2}{2} \\ 0 & \mathrm{Id} & -\mathbb{J}^t Z \\ 0 & 0 & 1 \end{pmatrix}, x = \exp X = \begin{pmatrix} 1 & 0 & 0 \\ X & \mathrm{Id} & 0 \\ -\frac{|X|^2}{2} & -{}^tX\mathbb{J} & 1 \end{pmatrix}, \tag{6.2}$$

where $|X|^2 = {}^tX\mathbb{J}X$ and $|Z|^2 = Z\mathbb{J}^tZ$.

6.2 The representations $d\pi_\lambda$ and $d\tilde{\pi}_\lambda$.

We are going now to apply the F-method explained in Section 4 to the conformal case. The first goal is to describe the action of elements in $\mathfrak{n}_+$ in terms of differential operators acting on the "Fourier image" of the Verma module. This can be systematically deduced from the explicit form of the (easily described) action of the induced representation in the non-compact picture. In particular, we shall find singular vectors in $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ by using the F-method. Later on, we shall use them to obtain equivariant differential operators from $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ to $\mathrm{Ind}_{P'}^{G'}(\mathbb{C}_{\lambda+K})$ by switching λ to $-\lambda$.

Let us consider the complex representation π_λ, $\lambda \in \mathbb{C}$, of G, induced from the character $p \mapsto a^\lambda$, $p \in P$, acting on the one dimensional representation space $\mathbb{C}_\lambda \simeq \mathbb{C}$. In other words, π_λ acts by left regular representation on $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$.

We define a family of differential operators by

$$Q_j(\lambda) = -\frac{1}{2}\epsilon_j|X|^2\partial_j + x_j(-\lambda + \sum_k x_k\partial_k),\ j = 1,\ldots,n,$$

$$P_j(\lambda) = i\left(\frac{1}{2}\epsilon_j\xi_j\Box + (\lambda - E)\partial_{\xi_j}\right),\ j = 1,\ldots,n, \tag{6.3}$$

where

$$\Box = \partial_{\xi_1}^2 + \cdots + \partial_{\xi_{p-1}}^2 - \partial_{\xi_p}^2 - \cdots - \partial_{\xi_{p+q-2}}^2$$

is the Laplace operator of signature $(p-1, q-1)$ and $E = \sum_k \xi_k\partial_{\xi_k}$ is the Euler homogeneity operator. The operators $P_j(\lambda)$ $(1 \le j \le n)$ were constructed in [42, Chapter 1], and are referred to as "fundamental differential operators".

Lemma 6.1 *The elements $E_j \in \mathfrak{n}_+$ are acting on $C^\infty(\mathfrak{n}_-, \mathbb{C}_\lambda)$ by*

$$d\pi_\lambda(E_j)(g \otimes v) = Q_j(\lambda)(g) \otimes v \quad \text{for } g \in C^\infty(\mathfrak{n}_-), v \in \mathbb{C}_\lambda. \tag{6.4}$$

The action of $d\tilde{\pi}_\lambda$ on $\mathrm{Pol}[\xi_1,\ldots,\xi_n] \otimes \mathbb{C}_\lambda^\vee$ is given by

$$d\tilde{\pi}_\lambda(E_j)(f \otimes v) = P_j(\lambda)(f) \otimes v \quad \text{for } f \in \mathrm{Pol}[\xi_1,\ldots,\xi_n], v \in \mathbb{C}_\lambda^\vee. \tag{6.5}$$

Proof:

Given elements $n \in N^+, x \in N^-$, as in 6.2, we define $a = 1 - Z\cdot X + \frac{|Z|^2|X|^2}{4}$. Then the action of $d\pi_\lambda(Z)$ on $C^\infty(\mathfrak{n}_-, \mathbb{C}_\lambda)$ is computed from the formula

$$n^{-1}\cdot x = \begin{pmatrix} a & -Z + \frac{1}{2}|Z|^{2t}X\mathbb{J} & -\frac{1}{2}|Z|^2 \\ X - \frac{1}{2}|X|^2\mathbb{J}^tZ & \mathrm{Id} - \mathbb{J}^tZ \otimes {}^tX\mathbb{J} & \mathbb{J}^tZ \\ -\frac{1}{2}|X|^2 & -{}^tX\mathbb{J} & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ a^{-1}(X - \frac{1}{2}|X|^2 \mathbb{J}^t Z) & \mathrm{Id} & 0 \\ -\frac{1}{2}a^{-1}|X|^2 & a^{-1}({}^tX\mathbb{J} + \frac{1}{2}|X|^2 Z) & 1 \end{pmatrix} \cdot$$

$$\cdot \begin{pmatrix} a & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1 & \star & \star \\ 0 & \mathrm{Id} & \star \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \tilde{x} \cdot p$$

where $\tilde{x} \in N^-$ and $p \in P$. Here the elements $\tilde{x}, a$ and m are computed up to first order in Z by

$$a \sim 1 - Z \cdot X, \quad m \sim \mathrm{Id} - \mathbb{J}^t Z \otimes {}^tX\mathbb{J} + X \otimes Z; \tag{6.6}$$

$$\tilde{x} = \exp \tilde{X}, \ \tilde{X} \sim (1 + Z \cdot X) \left(X - \frac{|X|^2 \mathbb{J}^t Z}{2} \right). \tag{6.7}$$

Taking $Z(t) = tE_j$ and applying the first derivative $\frac{d}{dt}|_{t=0}$ yields the formula (6.4) for the action $d\pi_\lambda(E_j)$.

The action of $d\tilde{\pi}_\lambda(E_j)$ is computed in two steps. The first step is to compute the dual action $d\pi^\vee$ (resulting in the multiplication of each operator by sign of its order) reversing the order in the composition of operators and adding sign changes depending on the order of the operator. The second step is to apply the (distributional) Fourier transform

$$x_j \mapsto -i\partial_{\xi_j}, \, \partial_{x_j} \mapsto -i\xi_j$$

preserving the order of operators in the composition.

□

6.3 The case $(G, G') = (SO_o(p,q), SO_o(p, q-1))$.

Let us consider the group $G' = SO_o(p, q-1)$ embedded into $G = SO_o(p,q)$ as the subgroup leaving the coordinate x_n invariant and the compatible parabolic subgroup $P' = P \cap G'$ in G' in the sense of Definition 5.3.

We recall $n = p + q - 2$. The nilpotent radical

$$\mathfrak{n}'_- \simeq \{{}^tX : X = (x_1, \dots, x_{n-1})\} \simeq \mathbb{R}^{n-1},$$

has codimension one in $\mathfrak{n}_-$. We endow $\mathfrak{n}_- \simeq \mathbb{R}^{p+q-2}$ with the standard flat quadratic form $(p-1, q-1)$, denoted by $\mathbb{R}^{p-1,q-1}$. The subspace $\mathfrak{n}_- \simeq \mathbb{R}^{n-1}$ has signature $(p-1, q-2)$.

According to the recipe of the F-method, we begin by finding the L'-module structure on the space Sol (see (4.5) for the definition) in this case.

6.3.1 The space of singular vectors

Recall the isomorphism between the space of singular vectors $M_{\mathfrak{p}}^{\mathfrak{g}}(V^{\vee})^{\mathfrak{n}'_+}$ and the space Sol. We are now going to compute the form of the set Sol, hence we describe in such a way completely the set of singular vectors.

If $C_\ell^{(\alpha)}(x)$ is the Gegenbauer polynomial, then $x^{-\ell}C_\ell^{(\alpha)}(x)$ is an even function. Hence we can write it as $x^{-\ell}C_\ell^{(\alpha)}(x) = \widetilde{C}_\ell^{\alpha}(x^2)$. More details on Jacobi and Gegenbauer polynomials can be found in the Appendix.

Theorem 6.2 *Let v_λ be a non-zero vector in the one-dimensional vector space $\mathbb{C}_\lambda$, $\lambda \in \mathbb{C}$.*

(1) For every $\lambda \in \mathbb{C}$, the L'-module Sol *corresponding to the Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ contains a direct sum of one-dimensional invariant subspaces generated by vectors $w_K = F_K \otimes v_\lambda, K \in \mathbb{N}$, with*

$$F_K(\xi) = \xi_n^K P_K(t), t = \frac{\sum_{i=1}^{n-1} \epsilon_n \epsilon_i \xi_i^2}{\xi_n^2},$$

where $P_K(t) = \widetilde{C}_K^{\alpha}(-t^{-1})$ with $\alpha = -\lambda - \frac{n-1}{2}$. Then F_K is a homogeneous polynomial of order K.

(2) If $\lambda \in \mathbb{N}$, then the space Sol *contains in addition a direct sum $\oplus_{j=1}^{\lambda+1} H'_j$, where H'_j are isomorphic to the space $\mathcal{H}^j(\mathbb{R}^{p-1,q-2})$ of homogeneous polynomial solutions of $\Box' = \sum_{j=1}^{n-1} \epsilon_j \partial_j^2$ of degree j. Let us denote by φ general isomorphism from the set* Sol *to the set $M_{\mathfrak{p}}^{\mathfrak{g}}(V^{\vee})^{\mathfrak{n}'_+}$. For each $j = 1, \ldots, \lambda+1$, the image $\varphi(H'_j)$ is included in the $\mathfrak{g}'$-Verma module generated by the vector $\varphi(w_K), K = j-1$.*

The list above is a complete description of the space Sol.

Proof:

We apply the F-method as follows. By (6.3), the equation $d\tilde{\pi}(Z)f = 0$ for $Z \in \mathfrak{n}_+$ amounts to a system of differential equations

$$P_j(\lambda)f = 0 \;\; j = 1, \ldots, n-1. \tag{6.8}$$

The operators $P_j(\lambda)$ respect homogeneity, hence we can restrict to the space of polynomials of a fixed homogeneity. When acting on a homogeneous polynomial $f \in \mathrm{Pol}[\xi_1, \ldots, \xi_n]$, we have

$$(\epsilon_i \xi_i P_j(\lambda) - \epsilon_j \xi_j P_i(\lambda))f = 0 \Longleftrightarrow (E - \lambda - 1)(\epsilon_j \xi_j \partial_{\xi_i} - \epsilon_i \xi_i \partial_{\xi_j})f = 0 \tag{6.9}$$

for all $i, j = 1, \ldots, n-1$. Hence if $\lambda \neq \deg f - 1$, any element in the kernel Sol of the action $d\tilde{\pi}_\lambda$ is a polynomial invariant under $SO_o(p-1, q-2)$.

1. $SO_o(p-1, q-2)$-invariant solutions.

This is always the case when $\lambda \notin \mathbb{N}$. Classical invariant theory says that any polynomial invariant with respect to $SO_o(p-1, q-2)$ in homogeneity K can be written in the form

$$f(\xi_1, \ldots, \xi_n) = \xi_n^K \, P(t),$$

where $t = \frac{\epsilon_n |\xi'|^2}{\xi_n^2}$, $|\xi'|^2 = \sum_{i=1}^{n-1} \epsilon_i \xi_i^2$ and P is a polynomial of degree N, (depending of parity of K, either $K = 2N$, or $K = 2N+1$).

Let us set (for simplicity) $\eta = \xi_n$. Hence we look for a solution in the form $v = \eta^K P(t),\ t = \frac{\epsilon_n |\xi'|^2}{\eta^2}$. We get immediately

$$\partial_j v = \eta^{K-2} 2\epsilon_n \epsilon_j \xi_j \, P',\ \partial_j^2 v = \eta^{K-2}[4P'' \frac{\xi_j^2}{\eta^2} + \epsilon_n \epsilon_j 2P'],\ j = 1, \ldots, n-1,$$

$$\Box' v = \epsilon_n \eta^{K-2}[4P'' t + 2(n-1)\, P'],\ \partial_\eta v = \eta^{K-1}[K\,P - 2\,P' t],$$
$$\epsilon_n \partial_\eta^2 v = \epsilon_n \eta^{K-2}[K(K-1)\,P + (-4K+6)\,P' t + 4\,P'' t^2],$$
$$\Box v = \epsilon_n \eta^{K-2}[4t(1+t)\,P'' + [2(n-1) + t(-4K+6)]\,P' + K(K-1)\,P]$$
$$(\lambda - E)\partial_j v = \epsilon_n \epsilon_j \xi_j \eta^{K-2}[2\lambda - 2K + 2]P'.$$

Collecting terms together, we get the common factor $\frac{1}{2}\epsilon_n \epsilon_j \xi_j \eta^{K-2}$ multiplied by the term

$$4t(1+t)P'' + [(2n + 4\lambda - 4K + 2) + t(-4K+6)]P' + K(K-1)P. \quad (6.10)$$

Hence the resulting system of equations (for $j = 1, \ldots, n-1$) reduces to one second order ordinary differential equation.

We can now relate the equation (6.10) and to the differential equation for Gegenbauer polynomials $C_\ell^{(\alpha)}(x)$. It has the form

$$(1-x^2){C_\ell^{(\alpha)}}''(x) - (2\alpha+1)x{C_\ell^{(\alpha)}}'(x) + n(n+2\alpha)C_\ell^{(\alpha)}(x) = 0, \quad (6.11)$$

where $n \in \mathbb{N}$ and $\alpha \in \mathbb{C}$.

The Gegenbauer equation (6.11) for $C = C_\ell^{(\alpha)}$ can be transformed by the substitution $t = -\frac{1}{x^2}, C(x) = x^\ell h(t)$ (for ℓ suitable) to the form

$$4t(1+t)h'' + [-(4\ell + 4\alpha - 4) + t(-4\ell+6)]h' + \ell(\ell-1)\,h = 0. \quad (6.12)$$

Comparing with equation (6.10), we see that they coincide for $\alpha = -\lambda - \frac{n-1}{2}$, and either $\ell = K$ or $\ell = K-1$. Because F is a polynomial, it is equal to the Gegenbauer polynomial of the corresponding degree, up to a multiple.

2. $\lambda \in \mathbb{N}$.

   If λ is a non-negative integer, the term $(\lambda - E)\partial_{\xi_j} f$ vanishes for homogeneous polynomials f of degree $k = \lambda + 1$. It means that vectors f need not be invariant under the action of $SO_o(p-1, q-2)$. But in this case, the system of equations reduces to the single equation $\Box F = 0$. Its solutions space is the space $\mathcal{H}^k(\mathbb{R}^{p-1,q-1})$ of 'harmonic' polynomials of degree $k = \lambda + 1$.

   For $j = 1, \ldots, k$, let M_j be the $\mathfrak{g}'$-modules generated by the singular vectors $\varphi(w_j)$. Comparing their (L')-highest weights, it is immediately clear that their infinitesimal characters are mutually different, hence their sum is direct.

   It is easy to see that the space

   $$\tilde{\mathcal{H}}^k := \{\varphi(F \otimes v_\lambda) : F \in \mathcal{H}^k(\mathbb{R}^{p-1,q-1})\}$$

   is included in the sum $\oplus_{j=0}^k M_j$. By the classical branching rules, the space $\mathcal{H}^k(\mathbb{R}^{p-1,q-1})$ decomposes into the direct sum of M'-modules:

   $$\mathcal{H}^k(\mathbb{R}^{p-1,q-1}) \simeq H'_k \oplus H'_{k-1} \oplus \ldots \oplus H'_1 \oplus H'_0,$$

   with $H'_j \simeq \mathcal{H}^j(\mathbb{R}^{p-1,q-2}), j = 0, \ldots, k$. At the same time the space

   $$\tilde{\mathcal{H}}^k = H'_k \oplus H'_{k-1} \oplus \ldots \oplus H'_1 \oplus H'_0 \tag{6.13}$$

   is the decomposition as L'-modules. Here we use the Langlands decomposition for the compatible parabolic subgroup $P' = L'N' = M'A'N'$ with A' being the same as A. This decomposition is analogous to the one for P.

   Let us now consider one fixed summand $H'_j;\ j = 1, \ldots, k$. Using its highest weight (resp. highest weights for $\mathfrak{l}'$ of its irreducible components), it is immediately seen that the infinitesimal character of the $\mathfrak{g}'$-module generated by H'_j is different from infinitesimal characters of all $M_i, i \neq j-1$. Hence it must be included in M_{j-1}.

□

The first part of the above proposition gives a complete description of the set of singular vectors invariant with respect to $SO_o(p-1, q-2)$. Such invariant singular vectors were found by A. Juhl (in positive signature, i.e. $p = 1$ case) in [37, Chapter 5] by a long combinatorial computation using recurrence relations. We would like to emphasize that our approach is very different and conceptual. Notice that the Juhl's computations depend on the parity of M.

An important point is that the method above gives a complete description of the set of singular vectors and its structure as L'-module. In the second part of the above proposition, we describe also all L'-submodules of higher dimensions. This is sufficient not only for understanding of the branching but also for complete description of composition series of individual summands.

Indeed, the higher dimensional submodules in the set Sol are responsible for reducibility of the corresponding summands in the branching. Note that higher dimensional components of singular vectors are much more difficult to detect by algebraic methods.

As stated in Theorem 4.1, the homomorphisms between generalized Verma modules calculated above induce equivariant differential operators acting on local sections of induced bundles on the corresponding flag manifolds. We shall describe these differential operators using the non-compact picture of the induced representation. The restriction from G to $N_-\,P$ induces the non-compact model of the induced representation by the map

$$\beta : \mathrm{Ind}_P^G(\mathbb{C}_\lambda) \to C^\infty(N_-, \mathbb{C}) \simeq C^\infty(\mathbb{R}^{p-1,q-1}).$$

Using this identification, we get the following explicit form of the induced invariant differential operator.

Theorem 6.3 *Let $\lambda \in \mathbb{C}$.*

1. *The singular vectors $w_{2N} \in M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ from Theorem 6.2 (1) induce (in the non-compact picture) the family $D_{2N}(\lambda)$ of even order G'-equivariant differential operators of the form*

$$D_{2N}(\lambda) : C^\infty(\mathbb{R}^{p-1,q-1}) \to C^\infty(\mathbb{R}^{p-1,q-2})$$

$$D_{2N}(\lambda) = \sum_{j=0}^{N} a_j(-\lambda)(\Box')^j(\frac{\partial}{\partial x_n})^{2N-2j},$$

where $\Box'$ denotes the (ultra)- wave operator on $\mathbb{R}^{p-1,q-2}$ and $D_{2N}(\lambda)$ is normalized in such a way that the coefficient a_N is equal to 1. Here the parameter for the action on the left resp. the right hand side is λ resp. $\lambda + 2N$.

Explicitly,

$$D_{2N}(\lambda) d\pi_\lambda^G(X) = d\pi_{\lambda+2N}^{G'}(X) D_{2N}(\lambda), \tag{6.14}$$

where $X \in \mathfrak{g}'$.

The coefficients a_j are explicitly given by

$$a_j(\lambda) = \frac{N!}{j!(2N-2j)!}(-2)^{N-j} \prod_{k=j}^{N-1}(2\lambda - 4N + 2k + n + 1). \tag{6.15}$$

Similarly, the singular vectors $w_{2N+1} \in M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ from Theorem 6.2 (1) induce the family $D_{2N+1}(\lambda)$ of odd order equivariant differential operators of the form

$$D_{2N+1}(\lambda) = \sum_{j=0}^{N} b_j(-\lambda)(\Box')^j(\frac{\partial}{\partial x_n})^{2N-2j+1},$$

where the operator $D_{2N}(\lambda)$ is normalized in such a way that the coefficient b_N is equal to 1. Here the parameter for the action on the left resp. the right hand side is λ resp. $\lambda + 2N + 1$. The coefficients b_j are explicitly given by

$$b_j(\lambda) = \frac{N!}{j!(2N-2j+1)!}(-2)^{N-j}\prod_{k=j}^{N-1}(2\lambda - 4N + 2k + n - 1), \quad (6.16)$$

normalized to $b_N(\lambda) = 1$.

2. *If λ is a non-negative integer, then the components $H'_j, j = 1, \ldots, \lambda+1$ of dimension larger than one in the set of singular vectors induce equivariant differential operators D_j mapping (in non-compact picture) functions to the $j-th$ power of the gradient projected to the trace free part of it.*

Remark 6.4 *Denote the representation induced from $p \mapsto a^\lambda$ as above $\pi_{\lambda,+}$ and denote the representation induced from $p \mapsto \epsilon a^\lambda$ by $\pi_{\lambda,-}$. They give rise to the same action of the Lie algebra, but on the level of induced representations we have the following intertwining relation*

$$D_K(\lambda)\pi^G_{\lambda,\epsilon_1}(g') = \pi^{G'}_{\lambda+K,\epsilon_2}(g')D_M(\lambda), \quad (6.17)$$

where $g' \in G'$ and $\epsilon_1 \cdot \epsilon_2 = (-)^M$.

The first part of the theorem follows immediately from Proposition 6.2(1) and from the fact that an element $X \in \mathfrak{n}_-$ acts on functions from $C^\infty(\mathfrak{n}_-)$ by the derivative in the direction X. The results generalize those obtained in [37, Section 5.2] for Euclidean signature (i.e. $p = 1$ case). Our proof based on the F-method is completely different from [37], and is significantly shorter even in the case $p = 1$.

The second part follows immediately from Proposition 6.2(2) and the well-known classification of conformally invariant operators on the sphere $S^{n-1} \simeq G'/P'$ (see, e.g., [53]).

Note that the operators D_j in Remark 6.4 are the first BGG operators in BGG sequences corresponding to the G'-module given by the symmetric powers of the defining representation of G', see [15].

6.3.2 The branching rules for Verma modules - generic case

The branching rules in generic cases are obtained using general results of Section 5 and they do not require knowledge of the explicit form of singular vectors.

Theorem 6.5 *For $\lambda \in \mathbb{C} \setminus \{\frac{1}{2}(k-n) : k = 2, 3, 4, \cdots\}$, the Verma module $M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)$ decomposes as a direct sum of generalized Verma modules of $\mathfrak{g}'$:*

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{N\in\mathbb{N}} M^{\mathfrak{g}'}_{\mathfrak{p}'}(\lambda - N). \quad (6.18)$$

We normalize the parameter space $\mathfrak{j}^* \simeq \mathbb{C}$ as $\rho = \frac{n}{2}$, and then $\rho' = \frac{n-1}{2}$.

Proof:

Apply Theorem 5.5 to the special case where

$$(\mathfrak{g}, \mathfrak{g}', \mathfrak{p}/\mathfrak{n}_+) \simeq (\mathfrak{so}(n+2, \mathbb{C}), \mathfrak{so}(n+1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}) + \mathbb{C}).$$

Then $l = 1$ and $\nu_1 = -1$ via the identification $\mathfrak{j}^* \simeq \mathbb{C}$. Hence we get (6.18) from Theorem 5.5 as the identity in the Grothendieck group. The criterion in Theorem 5.5 amounts to

- $\lambda - N + \frac{1}{2}(n-1) \notin \frac{1}{2}\mathbb{N}$,
- infinitesimal characters $\lambda - N + \frac{1}{2}(n-1)$ $(N \in \mathbb{N})$ are all distinct in $\mathfrak{j}^*/W \simeq \mathbb{C}/\mathbb{Z}_2$.

This is equivalent to $2\lambda + n \neq 2, 3, 4, \cdots$, whence the last statement.

□

6.3.3 The branching rules - exceptional cases

For exceptional values of the parameter, the structure of branching is more complicated. To understand it fully, we need an explicit form of singular vectors. The description of the branching rules for all values of exceptional parameters was first obtained in the case of Juhl's operator in [54].

Let us first notice that the whole Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ decomposes for all λ into an even and odd part. We have mapped it by Fourier transform to $\mathrm{Pol}(\xi_1, \ldots, \xi_n) \otimes \mathbb{C}_\lambda$, which decomposes as

$$\left(\bigoplus_{k=0}^{\infty} \mathrm{Pol}(\xi_1, \ldots, \xi_{n-1})\xi_n^{2k}\right) \otimes \mathbb{C}_\lambda \oplus \left(\bigoplus_{k=0}^{\infty} \mathrm{Pol}(\xi_1, \ldots, \xi_{n-1})\xi_n^{2k+1}\right) \otimes \mathbb{C}_\lambda.$$

Since we have realized $\mathfrak{g}' = \mathfrak{so}(n+1, \mathbb{C})$ in $\mathfrak{g} = \mathfrak{so}(n+2, \mathbb{C})$ in the first $(n+1)$-coordinates, it is easy to check that both summands are $\mathfrak{g}'$-submodules.

Let us discuss the even part (the odd one has a similar behavior). In Theorem 6.2, we have computed an explicit formula for the singular vector w_{2N}. Denote by V_{2N} the $\mathfrak{g}'$-Verma module it generates. In the generic case, individual summands in the branching are realized inside the Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda)$ by submodules V_{2N}.

In cases where submodules V_{2N} and $V_{2N'}$ have the same infinitesimal character, we know (due to knowledge of the explicit form of the singular vectors) that one of them is submodule of the other. In this cases, it is necessary to replace their sum in the branching rules by their (non-trivial) extension. We shall illustrate it in a number of examples.

We set

$$\lambda_j := \frac{1}{2}(-n+1+j). \tag{6.19}$$

Example 6.6 *The case* $\lambda_1 = \frac{-n+2}{2}$. *In this case, all infinitesimal characters of summand generated by singular vectors are mutually different up to those corresponding to* w_0 *and* w_1*, which coincide. But due to the fact that the whole* $\mathfrak{g}$*-module splits into a direct sum of even and odd parts, the whole branching is again the same as in the generic case.*

Example 6.7 *The case* $\lambda_2 = \frac{-n+3}{2}$. *In this case, infinitesimal characters of summand generated by singular vectors* w_0 *and* w_2 *coincide and are different from all others (which are mutually different, too). Denote* $\Box' = \sum_{i=1}^{n-1} \xi_i^2$. *The top two singular vectors -* w_0 $(N = 0)$ *and* w_2 $(N = 2)$ *are determined by polynomials* $P_0 = 1$ *and* $P_2(\lambda) = -(2\lambda+n-3)\xi_n^2+\Box'$*, hence* $P_2(\lambda)|_{\lambda=\frac{-n+3}{2}} = \Box'$.

Thus for $\lambda_2 = \frac{-n+3}{2}$*, the* $\mathfrak{g}'$*-generalized Verma module generated by* w_0 *contains a unique nontrivial submodule, generated by* w_2*. On the other hand, the direct sum* M_{02} *of the* $\mathcal{U}(\mathfrak{n}'_-)$ *span of* $w_0 = v_\lambda$ *and the* $\mathcal{U}(\mathfrak{n}'_-)$ *span of the vector* $\xi_n^2 \otimes v_\lambda$ *is invariant under the action of* $\mathfrak{g}'$*, and it is a (non-split) extension*

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_2) \to M_{02} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_2 - 2) \to 0. \tag{6.20}$$

All the other infinitesimal characters are different (and mutually different), hence the branching rule is now given by

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_2) \simeq M_{02} \oplus \bigoplus_{N\in\mathbb{N}, N\neq 0,2} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_2 - N).$$

Example 6.8 *The case* $\lambda_3 = \frac{-n+4}{2}$. *In this case, infinitesimal characters of summand generated by singular vectors* w_0, w_3 *respectively* w_1, w_2 *coincide, and both characters are different from each other, and different from all others (which are also mutually different). But again due to the fact that the whole* $\mathfrak{g}$*-module splits into a direct sum of even and odd parts, the whole branching is again the same as in generic case.*

Example 6.9 *Let* $\lambda_4 = \frac{-n+5}{2}$. *Computation shows that in this case, the vectors* w_0 $(N = 0)$ *and* w_4 $(N = 4)$ *given by*

$$P_4(\lambda) = \Box'^2 - (2\lambda + n - 5)\Box'\xi_n^2 + (2\lambda + n - 7)(2\lambda + n - 5)\xi_n^4$$

correspond to the same infinitesimal character. Another couple with the same infinitesimal characters (but different from the previous couple) are modules generated by w_1 *and* w_3.

The explicit form of w_4 *shows that it belongs to the* $\mathfrak{g}'$*-module generated by* w_4*, and it is again possible to show that there is in the* $\mathfrak{g}$*-Verma module a non-trivial extension of* $\mathfrak{g}'$*-Verma modules generated by* w_0 *and* w_4*. We denote it* M_{04}*. It can be defined as the module generated by* v_λ, $\xi_n^2 \otimes v_\lambda$ *and* $\xi_n^4 \otimes v_\lambda$*, quotioned by the submodule generated by* w_2*. The resulting extension takes the form*

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4) \to M_{04} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - 4) \to 0. \tag{6.21}$$

Similarly, there is a non-trivial extension

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - 1) \to M_{13} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - 3) \to 0 \tag{6.22}$$

of modules generated by w_1 *and* w_3*, denoted by* M_{13}*, and the branching rule is*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_4) \simeq M_{04} \oplus M_{13} \oplus \bigoplus_{N \in \mathbb{N}, N \neq 0,1,3,4} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_4 - N).$$

The general case looks similar. Whenever there is a couple of singular vectors corresponding to the same infinitesimal character, they do not generate direct summands in the branching sum but their sums are replaced by nontrivial extensions.

Theorem 6.10 *Recall* (6.19).

1. *For* $k = 1, 2, \ldots$*, the branching is the same as in the generic case*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k+1}) \simeq \bigoplus_{N \in \mathbb{N}} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k+1} - N). \tag{6.23}$$

2. *For* $k = 1, 2, \ldots$*, there exist for each* $j = 0, \ldots, k-1$ *modules* $M_{j,2k-j} \subset M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k})$*, realizing nontrivial extensions*

$$0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - j) \to M_{j,2k-j} \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - (2k-j)) \to 0 \tag{6.24}$$

such that

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k}) \simeq \bigoplus_{j=0}^{k-1} M_{j,2k-j} \oplus M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - k) \oplus \bigoplus_{N=2k+1}^{\infty} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - N).$$

The proof of the theorem is based on the following simple statement applied to our situation.

Lemma 6.11 *Suppose* N *is a* $\mathfrak{g}$*-module in the category* $\mathcal{O}$*, and* V_1*,* V_2 *are submodules satisfying the following three conditions:*

1) $\mathrm{Ch}(V_1) + \mathrm{Ch}(V_2) = \mathrm{Ch}(N)$*,*

2) V_1 *is irreducible,*

3) $\mathrm{Hom}_{\mathfrak{g}}(V_1, V_2) \simeq \mathrm{Hom}_{\mathfrak{g}}(V_1, N) \simeq \mathbb{C}$.

Then there exists a non-split short exact sequence of $\mathfrak{g}$*-modules:*

$$0 \to V_2 \to N \to V_1 \to 0. \tag{6.25}$$

Proof:

Since V_1 is irreducible, V_1 is isomorphic to the quotient N/V_2 by the first condition. Then the third condition implies that (6.25) does not split.

□

Proof of Theorem 6.10: By assumption, the sum

$$\bigoplus_{N=2k+1}^{\infty} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - N)$$

is direct and gives $\mathfrak{g}'$-submodule in $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_{2k})$. The quotient N' is a module in $\mathcal{O}_{\mathfrak{p}'}^{\mathfrak{g}'}$ and can be written as a direct sum

$$N' = \bigoplus_j N'_{\chi_j}, N'_{\chi_j} \in (\mathcal{O}_{\mathfrak{p}'}^{\mathfrak{g}'})_{\chi_j},$$

where $\chi_j, j = 0, 1, \ldots, k-1$ are infinitesimal characters of modules generated by w_j. We know by Theorem 5.5 that

$$\mathrm{Ch}(N'_{\chi_j}) = \mathrm{Ch}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - j)) + \mathrm{Ch}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda_{2k} - (2k - j))).$$

The explicit form of singular vectors computed in Proposition 6.2 shows that whenever the infinitesimal character is the same for two modules $M_I, M_J, I < J$, there is an inclusion $M_I \subset M_J$. Moreover, the knowledge of all singular vectors implies that M_I is irreducible. The theorem follows by application of Lemma 6.11.

□

6.3.4 Factorization identities

Let us return back to the Example 6.6. For $\lambda_0 = \frac{-n+3}{2}$, the action of $\mathfrak{g}'$ on the top two singular vectors w_0 and w_2 generate Verma modules V_0, resp. V_2 in $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_0)$. The second one is a submodule of the first one. The corresponding inclusion is a $\mathfrak{g}'$-homomorphisms ψ, whose dual differential operator is the conformally invariant Yamabe operator. If we denote by ϕ_0, resp. ϕ_2, inclusions of V_0, resp. V_2, into $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_0)$, we get the relation

$$\phi_2 = \phi_0 \circ \psi.$$

As another example, let us consider the weight $\lambda_0 = -\frac{n}{2} + \frac{5}{2}$. Then the module V_1 generated by the singular vector w_1 and the module V_3 generated by the singular vector w_3 have the same infinitesimal character. There exists a $\mathfrak{g}'$-homomorphism ψ from V_3 to V_1. The homomorphism ϕ_3 from V_3 to $M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda_0)$ can be factorized as $\phi_1 \circ \phi$.

Hence for some particular discrete subset of values for λ, there is a possibility to factorize an element in $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda'), M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$ as a composition of an element in the space $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda'), M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda''))$ and an element in $\mathrm{Hom}_{\mathfrak{g}'}(M_{\mathfrak{p}'}^{\mathfrak{g}'}(\lambda''), M_{\mathfrak{p}}^{\mathfrak{g}}(\lambda))$.

There is also a complementary possibility to factorize an element in $\mathrm{Hom}_{\mathfrak{g}'}(M^{\mathfrak{g}'}_{\mathfrak{p}'}(\lambda'), M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda))$ as a composition of an element in $\mathrm{Hom}_{\mathfrak{g}'}(M^{\mathfrak{g}'}_{\mathfrak{p}'}(\lambda'), M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda''))$ and in $\mathrm{Hom}_{\mathfrak{g}}(M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda''), M^{\mathfrak{g}}_{\mathfrak{p}}(\lambda))$.

The fact that such a behavior can happen only for discrete values of λ is consequence of classification of homomorphisms of $\mathfrak{g}$-generalized Verma modules. These properties were discovered and used effectively for curved generalizations by A. Juhl (see [37, Chapter 6]) under the name factorization identities. It is not a special feature of this particular example with $G = SO_o(1, n+1)$, but it is a more general phenomenon. It holds not only in the Juhl case (the scalar case), but it can be proved also in the spinor-valued case.

In the dual language of differential operators the factorization is described as follows: The first example above is expressing the Juhl operator D_2 as the composition of the operator D_0 and the Laplace operator. The second example shows that the operator D_3 is given by the composition of D_1 and the Laplace operator.

6.4 The case $G = G' = SO_o(p, q)$.

An interesting special case of the previous procedure gives a new independent construction of all differential intertwining operators for $G = SO_o(p, q)$-modules induced from densities.

The computation of the set Sol of singular vectors for the case $G = G'$ runs in the same way as above. We have to find homogeneous solutions of the system of PDE's

$$P_j(\lambda)F(\xi_1, \dots, \xi_n) = \mathcal{F}(d\pi_\lambda(E_j))(\mathcal{F}u(\xi_1, \dots, \xi_n)) = 0 \tag{6.26}$$

for all basis elements $E_j \in \mathfrak{n}_+$, $0 \le j \le n = p + q - 2$. By the same argument as in the proof of Theorem 6.2, either polynomial F should be invariant under the action of the group $SO_o(p-1, q-1)$, or it should be in the kernel of the Laplace operator $\Box$.

In the first case, classical invariant theory implies that the ring of $SO_o(p-1, q-1)$-invariants in $\mathrm{Pol}_{\mathbb{C}}(\mathbb{R}^{p-1,q-1})$ is generated by

$$Q(\xi) := \xi_1^2 + \cdots + \xi_{p-1}^2 - \xi_p^2 - \cdots - \xi_{p+q-2}^2$$

and the Fourier system is (recall we are in the setting $G = G'$)

$$(\frac{1}{2}\epsilon_j \xi_j \Box + (\lambda - E)\partial_{\xi_j}) f(Q(\xi)) = 0, \tag{6.27}$$

for $j = 1, \dots, n$. In homogeneity $2m$, we get that

$$\xi^{2m} = (\xi_1^2 + \cdots + \xi_{p-1}^2 - \xi_p^2 - \cdots - \xi_{p+q-2}^2)^m$$

is a solution of the previous system if and only if $\lambda = -\frac{n}{2} + m$.

In the second case, there are again higher dimensional L-modules of singular vectors given by the kernel of $\Box$ in appropriate homogeneity, as above.

It gives a complete description of the set Sol of singular vectors. Let $\mathbb{C}_\lambda$ is a one-dimensional P-module and 1_λ its highest weight vector. If F is a homomorphism from a generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(V)$ to a $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$, then the image of $1_\lambda \otimes V$ by F should form an irreducible submodule Sol. In such a way, we have got an alternative short proof of the well known classification of all homomorphisms from a generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$.

In the dual language of differential operators acting in the non-compact picture, we get powers $\Box^m$ of the Laplace operator acting from densities of the weight $-\frac{n}{2}+m$ to densities of the weight $-\frac{n}{2}-m$ and the series D_k of the first BGG operators (equations for conformal Killing tensors) given by the projection to the trace-free part of the multiple gradient $\nabla_{(a} \dots \nabla_{b)_0}\sigma$ (number of indices being k).

7 Dirac operators and $Spin_o(p,q)$

7.1 Notation

In the present section we use the same conventions as in the previous sections. Let $p \geq 1, q \geq 2, n = p+q-2, n = n'+1$, suppose that we have the quadratic form as in the Section 6.1, and let us consider the Clifford algebra $\mathcal{C}_{p,q}$ generated by an orthogonal basis $e_0, \ldots, e_{p+q-1}$ with the relations $e_i^2 = -\epsilon_i$ for $i = 1, \ldots, p+q-2$ and $e_0 e_{p+q-1} + e_{p+q-1} e_0 = 1$, and its subalgebra $\mathcal{C}_{p-1,q-1}$ generated by $e_1, \ldots, e_n, n = p+q-2$. We realize $Spin(p,q)$ in $\mathcal{C}_{p,q}$ and define $\tilde{G}$ to be the identity component of $Spin_o(p,q)$. Denote the canonical projection $p : Spin_o(p,q) \to SO_o(p,q)$. Hence $\tilde{G}$ is acting on $\mathbb{R}^{p,q}$ preserving the null cone and its projective version S^n as discussed above for $G = SO_o(p,q)$. We shall keep the notation from Section 6.1. The subgroup $\tilde{P} \subset \tilde{G}$ is defined as the stabilizer of the chosen null line generated by the vector $(1, 0, \ldots, 0)$, and $p(\tilde{P}) = P$. The Levi factor is denoted by $\tilde{L}$, and the corresponding $\tilde{M} = Spin(p-1, q-1)$.

The corresponding Lie algebras $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{p}}$ are isomorphic via p_* to those for $SO_o(p,q)$ case. We take the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ so that $\mathfrak{h} \subset \mathfrak{l}$, hence the weights of our P-modules will be in $\mathfrak{h}^*$. We denote by $\mathbb{S}^{\pm}$ the irreducible representations for $\tilde{M}$, isomorphic for $p+q$ odd and mutually non-isomorphic for $p+q$ even. The module structure is trivially extended on the N_+-part to a $\tilde{P}$-module. We often abuse the notation and write $\mathbb{S}$ for $\mathbb{S}^{\pm}$. The generators $(\epsilon_i \epsilon_j E_{ij} - E_{ji})$ for $i < j$ of $so(p-1, q-1)$ act on the spinor representations as elements $-\frac{1}{2}\epsilon_i(\, e_i\, e_j + \delta_{ij})$ of Clifford algebra. Here $E_{ij},\ i,j = 1, \ldots, n$ are elementary matrices (having 1 at the (i,j)-position) and $X_{ij} = \epsilon_i \epsilon_j E_{ij} - E_{ji}$.

7.2 Representations $d\pi_\lambda$ and $d\tilde{\pi}_\lambda$.

In this case the inducing $\tilde{P}$-module will be the $\tilde{M}$-module $\mathbb{S}_\lambda$, $\lambda \in \mathbb{C}$ with the highest weight $(\lambda + \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. Hence $\mathbb{S}_\lambda$ is isomorphic to $\mathbb{C}_\lambda \otimes \mathbb{S}$, with $\mathbb{C}_\lambda$ as in Section 6.2. The induced representation is now $\mathrm{Ind}_{\tilde{P}}^{\tilde{G}}\, \mathbb{S}_\lambda$, denoted $\pi_{S,\lambda}$ - for simplicity just π_λ. An element $F \in C^\infty(\tilde{G}, \mathbb{S}_\lambda)^{\tilde{P}}$ satisfies

$$F(g\,\tilde{m}an) = a^{-\lambda} \rho(\widetilde{m, \epsilon(m)})^{-1} F(g),\ g \in \tilde{G},\ \tilde{p} = \tilde{m}an \in \tilde{P}$$

with

$$\Pi(\tilde{m}an) = \begin{pmatrix} \epsilon(m)a & \star & \star \\ 0 & m & \star \\ 0 & 0 & \epsilon(m)a^{-1} \end{pmatrix},\ (m, \epsilon(m)) \in SO(p-1, q-1),\ a > 0, \epsilon = \pm 1,$$

where $\Pi : Spin(p-1, q-1) \to SO(p-1, q-1)$ is the twofold covering. In particular, $\Pi(\tilde{m}) = m$.

To calculate the action of $d\pi(Z), Z \in \mathfrak{n}_+$, on the induced representation, we can use the previous calculation in the scalar case. We already observed that

$$U \sim \mathbb{J}^t Z \otimes {}^t X \mathbb{J} - X \otimes Z = \sum_{i,j=1}^{n} z_i x_j (\epsilon_i \epsilon_j E_{ij} - E_{ji}), \tag{7.1}$$

where $m = \exp(U)$. Hence $d\rho(U)$ is realized as multiplication by the element $-\frac{1}{2}(\underline{z}\,\underline{x}+\sum_{i=1}^{n} \epsilon_i x_i z_i)$ in the corresponding Clifford algebra, where $\underline{x} = \sum_1^n x_i e_i$, and $\underline{z} = \sum_1^n \epsilon_i z_i e_i$. Summarizing information obtained so far, we get the following claim.

Lemma 7.1 *The standard basis elements $E_j, j = 1, \ldots, n$ for $\mathfrak{n}_+$ are acting on $C^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}_\lambda^\pm)$ $(C^\infty(\tilde{G}, \mathbb{S}_\lambda)^P)$ by*

$$d\pi_\lambda(E_j) = -\frac{1}{2}\epsilon_j |X|^2 \partial_j + x_j(-\lambda + \sum_k x_k \partial_k + \frac{1}{2}) + \frac{1}{2}(\epsilon_j e_j \underline{x}), \;\; E_j \in \mathfrak{n}_+. \tag{7.2}$$

The dual action composed with the Fourier transform is

$$(d\tilde{\pi}_\lambda(E_j)) = i\left(\frac{1}{2}\epsilon_j \xi_j \Box + (\lambda - E + \frac{1}{2})\partial_{\xi_j} - \frac{1}{2}(\epsilon_j e_j D)\right), \tag{7.3}$$

where $D := \sum_{k=1}^n e_k \partial_k$ is the Dirac operator and E is the Euler homogeneity operator.

7.3 The space Sol of singular vectors

In the spinor case, we restrict Sol to vectors invariant with respect to the action of $\mathfrak{m}$ on the space of polynomials in n variables $\xi_1, \ldots, \xi_n$ with values in $\mathrm{End}(\mathbb{S}_\lambda)$. The latter space is a subspace of the complexified Clifford algebra $\mathcal{C}^c_{p,q}$ and the classical invariant theory shows that the algebra of invariants is generated by $\underline{\xi}$ and $\underline{\eta}$, where

$$\underline{\xi} = \sum_{j=1}^{n-1} \epsilon_j\, e_j\, \xi_j\,,\; \underline{\eta} = \epsilon_n\, e_n \eta,\;\; \eta = \xi_n,$$

with $\underline{\xi}^2 = -|\xi'|^2 = -\sum_{j=1}^{n-1} \epsilon_j \xi_j^2$ and $\underline{\eta}^2 = -\eta^2$. Denote also $t = \frac{\underline{\xi}^2}{\eta^2}$.

Moreover, the equations for Sol are homogeneous, hence we can consider only invariant polynomials of a given homogeneity K. Their structure depends on the parity of K:

$$F_{2N}(t) = \eta^{2N} P(t) + \eta^{2N-2} Q(t) \underline{\xi}\underline{\eta}\,, \tag{7.4}$$

where P and Q are a polynomials in a real variable t, P is of order N and Q is of order $N-1$ and similarly,

$$F_{2N+1}(t) = \eta^{2N}[P(t)\underline{\xi} + Q(t)\underline{\eta}]\,, \tag{7.5}$$

where both P and Q are polynomials of order N. The vectors in Sol are annihilated by the operators

$$P_j = \epsilon_j \xi_j \Box - (2E - 2\lambda - 1)\partial_j - \epsilon_j e_j\, D,\; j = 1, \ldots, n-1,$$

where

$$\Box = \Box' + \partial_\eta^2, \Box' = \sum_{j=1}^{n-1} \epsilon_j \partial_{\xi_j}^2, D = D' + e_n \partial_\eta,$$

$$D' = \sum_{j=1}^{n-1} e_j \partial_{\xi_j}, \; E = \sum_{j=1}^{n-1} \xi_j \partial_{\xi_j} + \eta \partial_\eta. \tag{7.6}$$

This leads to a system of ordinary differential equations for polynomials P and Q. We shall first treat the case of even homogeneity $K = 2N$.

Lemma 7.2 *Let v_λ be the highest weight vector of $\mathbb{S}_\lambda$. The vector $v = [\eta^{2N} P(t) + \eta^{2N-2} Q(t)\underline{\xi\eta}] \cdot v_\lambda$ is a vector in Sol if and only if the following system of differential equations is satisfied:*

$$\begin{aligned}
&4\, P'' t(1+t) + P'[t(-8N+6) + (-8N+4\lambda+2n+4)] \\
&+2N(2N-1)\, P = 0, && (7.7)\\
&4\, Q'' t(1+t) + Q'[t(-8N+10) + (-8N+4\lambda+2n+8)] \\
&+(2N-1)(2N-2)\, Q = 0, && (7.8)\\
&-2\, N\, P + 2\, t\, P' + (2\lambda + n - 4N + 2)\, Q + 2\, Q' t = 0, && (7.9)\\
&-(2N-1)\, Q + 2\, Q' t - 2\, P' = 0. && (7.10)
\end{aligned}$$

The first two equations are the consequence of the last two.

Proof:
Denote $v_1 = \eta^{2N} P(t)$ and $v_2 = \eta^{2N-2} Q(t)\underline{\xi\eta}$. Then

$$\partial_j v_1 = 2\epsilon_n \epsilon_j \xi_j \eta^{2N-2} P'(t),$$

$$\partial_j^2 v_1 = 2\epsilon_n \epsilon_j \eta^{2N-2} P'(t) + 4\eta^{2N-4} \xi_j^2 P''(t),$$

$$\Box' v_1 = \epsilon_n \eta^{2N-2}[4\, t\, P'' + (2n-2)\, P'],$$

$$\partial_\eta v_1 = \eta^{2N-1}[2N\, P - 2\, t\, P'],$$

$$\partial_\eta^2 v_1 = \eta^{2N-2}[2N(2N-1)\, P + (-8N+6)\, t\, P' + 4\, t^2\, P''],$$

$$e_j D' v_1 = 2\epsilon_n\, e_j \eta^{2N-2}\, P' \underline{\xi},$$

$$e_j e_n \partial_\eta v_1 = \epsilon_n\, e_j \eta^{2N-2}[2N\, P - 2\, t\, P']\underline{\eta}.$$

Similarly,

$$\partial_j v_2 = \epsilon_n \epsilon_j \eta^{2N-4} 2Q' \xi_j \underline{\xi\eta} + \eta^{2N-2} \epsilon_j e_j Q(t) \underline{\eta},$$

$$\Box'\, v_2 = \eta^{2N-4} \epsilon_n [4\, t\, Q'' + (2n+2)\, Q'] \underline{\xi\eta},$$

$$\partial_\eta v_2 = \eta^{2N-3}[(2N-1)\, Q - 2\, t\, Q'] \underline{\xi\eta},$$

$$\partial_\eta^2 v_2 = \eta^{2N-4}[(2N-1)(2N-2)\, Q + t\,(-8N+10)\, Q' + 4\, t^2\, Q''] \underline{\xi\eta},$$

$$-e_j D' v_2 = -e_j \eta^{2N-2}[2\,t\,Q' + n'\,Q]\underline{\eta},$$
$$-e_j e_n \partial_\eta v_2 = e_j \eta^{2N-2}[(2N-1)\,Q - 2\,t\,Q']\underline{\xi}.$$

Collecting all terms by the four different expressions

$$\epsilon_j \epsilon_n \xi_j \eta^{2N-2}, \;\; \epsilon_j \epsilon_n \xi_j \eta^{2N-4}\underline{\xi\eta}, \;\; \epsilon_j\, e_j\, \epsilon_n \eta^{2N-2}\underline{\xi}, \;\; \epsilon_j\, e_j\, \epsilon_n \eta^{2N-2}\underline{\eta},$$

yields the four equations in the Lemma. It can be easily checked that a suitable linear combination of the last two equations in (7.2) yields the first two equations. Hence all equations are compatible.

□

We know from the previous analysis that the first two solutions are given by suitable Gegenbauer polynomials. The last two equations then determine the coefficients of the linear combination of P and Q.

Notice that the polynomial $P(t)$ already appeared in the previous sections in the case of generalized Verma modules of scalar type, while the second polynomial $Q(t)$ is a new one. Here we abuse the notation slightly for the polynomials: In terms of $P_M(t)$ defined in Theorem 6.2, we have $P_N(t) = \tilde{P}_{2N}(t)$ and $Q_{N-1}(t) = \tilde{P}_{2N-1}(t)$. Note also a change in the parameter, which is now $\alpha = -\lambda - \frac{n}{2}$ as opposed to the previous $\alpha = -\lambda - \frac{n-1}{2}$.

Lemma 7.3 *Let $N \in \mathbb{N}$. Denote*

$$\tilde{P}_N(t) = (-t)^N C^{(\alpha)}_{2N}(\frac{i}{\sqrt{t}}),\; \alpha = -\lambda - \frac{n}{2} \tag{7.11}$$

$$\tilde{Q}_{N-1}(t) = i(-t)^N \frac{1}{\sqrt{t}} C^{(\alpha)}_{2N-1}(\frac{i}{\sqrt{t}}),\; \alpha = -\lambda - \frac{n}{2} \tag{7.12}$$

the solutions of (7.7), resp.,(7.8).

Then the polynomial P_N has degree N and polynomial Q_{N-1} has degree $N-1$ and

$$v_{2N} = [\eta^{2N} P_N(t) + \eta^{2N-1} Q_{N-1}(t)\underline{\xi\eta}] \cdot v_\lambda,\; N \in \mathbb{N} \tag{7.13}$$

is a vector in Sol of homogeneity $2N$. The vectors v_{2N}, $N \in \mathbb{N}$ form a complete set of singular vectors of even homogeneity invariant with respect to the action of the Lie algebra $\mathfrak{m}' = so(p-1, q-2)$.

Proof:

It follows from the basic formula for the derivative of Gegenbauer polynomials (13.5) that

$$P'_N(t) = (-1)^N t^{N-1}(N C^{(\alpha)}_{2N}(\frac{i}{\sqrt{t}}) - i\frac{1}{\sqrt{t}}\alpha C^{(\alpha+1)}_{2N-1}(\frac{i}{\sqrt{t}})),$$
$$Q'_{N-1}(t) = i(-1)^N t^{N-1}((N - \frac{1}{2})\frac{1}{\sqrt{t}} C^{(\alpha)}_{2N-1}(\frac{i}{\sqrt{t}}) - i\frac{1}{t}\alpha C^{(\alpha+1)}_{2N-2}(\frac{i}{\sqrt{t}})).$$

The singular vector has a form

$$v_N = A\eta^{2N}P_N(t) + B\eta^{2N-1}Q_{N-1}(t)\underline{\xi\eta},\ N\in\mathbb{N},\ A,B\in\mathbb{C}. \qquad (7.14)$$

and the substitution into the equation (7.10) leads to

$$-A[2P'] - B[(2N-1)Q - 2tQ'] \sim -A2NC_{2N}^{(\alpha)}(\frac{i}{\sqrt{t}})$$
$$+Ai\frac{1}{\sqrt{t}}2\alpha C_{2N-1}^{(\alpha+1)}(\frac{i}{\sqrt{t}}) - B2\alpha C_{2N-2}^{(\alpha+1)}(\frac{i}{\sqrt{t}}).$$

Comparing the right hand side with another identity for Gegenbauer polynomials

$$mC_m^{(\nu)}(z) - 2\nu zC_{m-1}^{(\nu+1)}(z) + 2\nu C_{m-2}^{(\nu+1)}(z) = 0,$$

gives the relations $m = 2N, \nu = \alpha, z = \frac{i}{\sqrt{t}}$ and $A = 1, B = 1$. The equation (7.9) can be treated analogously.

□

Let $\lambda\in\mathbb{C}$, and $N\in\mathbb{N}$. The case of odd homogeneity is contained in the next two Lemmas, whose proof is left to the reader.

Lemma 7.4 *Let v_λ be the highest weight vector of $\mathbb{S}_\lambda$. The vector $v_{2N+1} = [\eta^{2N+1}(P(t)\underline{\xi}+Q(t)\underline{\eta})]\cdot v_\lambda$ is a singular vector if and only if the following system of differential equations is satisfied:*

$$\begin{aligned}
&4\,P''t(1+t) + P'[t(-8N+6) + (-8N+4\lambda+2n+4)]\\
&+2N(2N-1)\,P = 0, && (7.15)\\
&4\,Q''t(1+t) + Q'[t(-8N+2) + (-8N+4\lambda+2n)]\\
&+(2N+1)(2N)\,Q = 0, && (7.16)\\
&(2N+1)\,Q + 2\,t\,P' + (2\lambda+n-4N)\,P - 2\,Q't = 0, && (7.17)\\
&2N\,P - 2\,P't - 2\,Q' = 0. && (7.18)
\end{aligned}$$

The first two equations follow from the last two.

Lemma 7.5 *Let $N\in\mathbb{N}$ and $\lambda\in\mathbb{C}$. Recall the special polynomials introduced earlier in Lemma 7.3. Then the polynomials P_N and Q_N have degree N and*

$$v_{2N+1} = [\eta^{2N}(P_N(t)\underline{\xi} + Q_N(t)\underline{\eta})]\cdot v_\lambda,\ N\in\mathbb{N} \qquad (7.19)$$

is a vector in Sol of homogeneity $2N+1$. The vectors v_{2N+1}, $N\in\mathbb{N}$ form a complete set of vectors in Sol of odd homogeneity invariant with respect to the action of the algebra $\mathfrak{m}'$.

As in Section 4, the homomorphisms of generalized Verma modules defined by the singular vectors described above induce equivariant differential operators acting on local sections of induced bundles on the corresponding flag manifolds. We shall again describe these differential operators using the non-compact picture of the induced representations and we get, as a corollary of the previous lemmas, the following theorem.

Theorem 7.6 *Let $\Box'$ denote the Laplace operator on $\mathbb{R}^{p-1,q-2}$, let $D' = \sum_{i=1}^{n-1} e_i\partial_i$ resp. $\underline{\partial}_n = e_n\partial_n$ denote the Dirac operator in $\mathbb{R}^{p-1,q-2}$ resp. $\mathbb{R}^{0,1}$.*

The singular vectors v_{2N} of even homogeneity given by (7.13) induce in the non-compact picture the family $D_{2N}(\lambda)$ of even order equivariant differential operators of the form

$$D_{2N}(\lambda) : C^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}^\pm_\lambda) \to C^\infty(\mathbb{R}^{p-1,q-2}, \mathbb{S}^\pm_{\lambda+2N}),$$

$$D_{2N}(\lambda) = \sum_{j=0}^{N} \tilde{a}_j(-\lambda-1)(\Box')^j(\partial_n)^{2N-2j} + \sum_{j=0}^{N} \tilde{b}_j(-\lambda-1)(\Box')^j(\partial_n)^{2N-2j} D'\underline{\partial}_n.$$

The intertwining relation is

$$D_{2N}(\lambda) d\pi^G_{S,\lambda}(X) = d\pi^{G'}_{S,\lambda+2N}(X) D_{2N}(\lambda), \tag{7.20}$$

where $X \in \mathfrak{g}'$.

Similarly, the family $D_{2N+1}(\lambda)$ of odd order equivariant differential operators

$$D_{2N+1}(\lambda) : C^\infty(\mathbb{R}^{p-1,q-1}, \mathbb{S}^\pm_\lambda) \to C^\infty(\mathbb{R}^{p-1,q-2}, \mathbb{S}^\mp_{\lambda+2N+1}),$$

induced by the singular vectors v_{2N+1} has the form

$$D_{2N+1}(\lambda) = \sum_{j=0}^{N} \tilde{\tilde{a}}_j(-\lambda-1)(\Box')^j(\partial_n)^{2N-2j} D' + \sum_{j=0}^{N} \tilde{\tilde{b}}_j(-\lambda-1)(\Box')^j(\partial_n)^{2N-2j}\underline{\partial}_n.$$

Based on the conventions (6.15), (6.16), the coefficients $\tilde{a}_j, \tilde{b}_j, \tilde{\tilde{a}}_j, \tilde{\tilde{b}}_j$ are in both cases explicitly given by

$$\tilde{a}_j(\lambda) = a_j(\lambda + \frac{1}{2}), \tilde{b}_j(\lambda) = b_j(\lambda + \frac{1}{2}), \tag{7.21}$$

$$\tilde{\tilde{a}}_j(\lambda) = a_j(\lambda + \frac{1}{2}), \tilde{\tilde{b}}_j(\lambda) = b_j(\lambda + \frac{1}{2}). \tag{7.22}$$

In the article [44], we shall prove the existence of lifts of homomorphisms corresponding to singular vectors in generalized Verma modules induced from spinor representations to homomorphisms of semiholonomic generalized Verma module covering them. According to the philosophy of parabolic geometries, [15], we get curved versions of our equivariant differential operators acting on sections of spinor bundles on manifolds with conformal structure.

8 Juhl's conjectures

8.1 Introduction and Motivation

The question we are going to answer in the present section has its original motivation in geometry. Let (M^n, g) be a Riemannian manifold, $i_\Sigma : \Sigma^{n-1} \hookrightarrow M^n$ embedded codimension one (i.e. $(n-1)$-dimensional) submanifold and $i^\star_\Sigma(g)$ the induced metric on Σ^{n-1}. One of the basic problems in geometrical analysis on Riemannian or conformal manifolds defined by these data is the existence, uniqueness and properties of natural differential (scalar) operators

$$D_N(M^n, \Sigma^{n-1}, g, \lambda) : C^\infty(M^n) \to C^\infty(\Sigma^{n-1})$$

of order $N \in \mathbb{N}$ and depending polynomially on $\lambda \in \mathbb{C}$, which are conformally invariant in the sense that

$$e^{-(\lambda-N)i^*_\Sigma(\varphi)} D_N(M^n, \Sigma^{n-1}, e^{2\varphi}g, \lambda)e^{\lambda\varphi} = D_N(M^n, \Sigma^{n-1}, g, \lambda)$$

for each $\varphi \in C^\infty(M^n)$.

It is difficult to handle this problem for a general metric g, but the situation simplifies considerably in the case of a homogeneous flat domain realized on an open orbit of the partial flag manifold. The case of our interest in this article corresponds to $M^n = S^n$ resp. $M^n = \mathbb{R}^n$ and $\Sigma^{n-1} = S^{n-1}$ resp. $\Sigma^{n-1} = \mathbb{R}^{n-1}$ in the compact resp. non-compact models of induced representations of (conformal) Lie algebra $\mathfrak{g}(n+1, 1) = so(n+1, 1)$, and its signature generalizations.

For any simple Lie algebra $\mathfrak{g}$ and its parabolic subalgebra $\mathfrak{p}$ we have the Langlands decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}_+$ and the Iwasawa decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_+$. Here $\mathfrak{l}$ denotes the Levi factor of $\mathfrak{p}$, $\mathfrak{n}_+$ its nilradical and $\mathfrak{n}_-$ the opposite nilradical. In this section we focus again on the maximal parabolic subalgebra of orthogonal Lie algebra with abelian nilradical given by omitting the first simple root.

There is a well-known equivalence between invariant differential operators acting on induced representations and homomorphisms of generalized Verma modules, realized by the pairing

$$Ind^G_P(\mathbb{V}_\lambda) \times M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{V}^\star_\lambda) \to \mathbb{C} \tag{8.1}$$

for any finite dimensional irreducible $\mathfrak{p}$-module $\mathbb{V}_\lambda$ and its dual $\mathbb{V}^\star_\lambda$. This allows to turn the former motivating problem into the question of $\mathfrak{g}(p, q+1) = so(p, q+1)$-homomorphisms of generalized Verma modules

$$M^{\mathfrak{g}(p,q+1)}_{\mathfrak{p}(p,q+1)}(\mathbb{V}_{\lambda_1}) \to M^{\mathfrak{g}(p+1,q+1)}_{\mathfrak{p}(p+1,q+1)}(\mathbb{V}_{\lambda_2}), \tag{8.2}$$

where $\mathbb{V}_{\lambda_i}, i = 1, 2$ denote finite dimensional irreducible inducing representations of $\mathfrak{p}(p, q+1)$ resp. $\mathfrak{p}(p+1, q+1)$.

Let us denote the standard inclusion

$$i : \mathfrak{g}(n, 1) \hookrightarrow \mathfrak{g}(n+1, 1),$$

characterized by the fact that the highest weight vector Y_n of $\mathfrak{l}$-module $\mathfrak{n}$ is preserved by $i(\mathfrak{l}')$, the image of the Levi factor of $\mathfrak{g}(n,1)$. Recently in [37], A. Juhl constructed a collection of elements

$$D_N(\lambda) \in Hom_{\mathcal{U}(\mathfrak{g}(n,1))}(M^{\mathfrak{g}(n,1)}_{\mathfrak{p}(n,1)}(\mathbb{C}_{\lambda-N}), M^{\mathfrak{g}(n+1,1)}_{\mathfrak{p}(n+1,1)}(\mathbb{C}_\lambda))$$

numbered by $N \in \mathbb{N}$ and polynomially dependent on the character $\lambda \in \mathbb{C}$ of $\mathfrak{p}(n,1)$-module $\mathbb{C}_{\lambda-N}$, such that $D_N(\lambda) \in \mathcal{U}(\mathfrak{n}_-(n+1,1))$ is induced by

$$\begin{aligned} \mathcal{U}(\mathfrak{g}(n,1)) \otimes \mathbb{C}_{\lambda-N} &\to \mathcal{U}(\mathfrak{g}(n+1,1)) \otimes \mathbb{C}_\lambda, \\ V \otimes 1 &\mapsto i(V)D_N(\lambda). \end{aligned} \tag{8.3}$$

Then he formulated the following conjectures (see [37] for the case $\mathfrak{g}(n,2)$; the cases of remaining signatures have, according to A. Juhl, analogous formulation):

Conjecture 8.1 *The set of families* $\{D_N(\lambda)\}_{N\in\mathbb{N}}$ *generates the space*

$$M^{\mathfrak{g}(p,q+1)}_{\mathfrak{p}(p,q+1)}(\mathbb{C}_{\lambda-N}) \to M^{\mathfrak{g}(p+1,q+1)}_{\mathfrak{p}(p+1,q+1)}(\mathbb{C}_\lambda), \ N \in \mathbb{N}$$

of all homomorphisms $\mathcal{U}(\mathfrak{g}(p,q+1))$*-modules.*

This construction was then subsequently considered also in the Lorentzian signature, [1].

The main aim of the present section is to prove these conjectures for a generic value of the inducing character λ in the case of any signature. The tool used to complete this task is based on the analysis of character formulas for corresponding parabolic subalgebras. The second aim is a direct analysis of the space of homomorphisms or, equivalently, the space of singular vectors, for certain discrete subset of the values of inducing character. The result is that in these special cases some of the $\mathfrak{g}(p,q+1)$-generalized Verma modules which decompose a given $\mathfrak{g}(p+1,q+1)$-generalized Verma module form non-trivial extensions, i.e. they represent projective objects (nontrivial extension classes) in the BGG parabolic category $\mathcal{O}^{\mathfrak{p}}$ of the pair $\mathfrak{g}(p,q+1), \mathfrak{p}(p,q+1)$. We give the complete list of projective modules appearing in the branching problem, thus completing the decomposition as a task in the BGG parabolic category $\mathcal{O}^{\mathfrak{p}}$ rather then the set of Verma modules alone, see e.g. [35], [7].

Notice that in the special case corresponding to the domain of unitarity D_U for generalized Verma modules $M^{\mathfrak{g}(p+1,q+1)}_{\mathfrak{p}(p+1,q+1)}(\mathbb{C}_\lambda)$ of scalar type, this decomposition was treated by geometrical techniques and orbit methods in [39]. In the recent article [41], the author develops branching problem for generalized Verma modules even for non-standard embeddings of smaller Lie algebra in the bigger one. However, the decomposition is defined only if the induction parameters belong to a certain region in the dual of Cartan subalgebra.

8.2 $so(p,q+1)$-homomorphisms and $so(p,q+1)$-singular vectors for $so(p+1,q+1)$-generalized Verma modules induced from character

In this subsection we construct discrete family of 1-dimensional continuous families of $\mathcal{U}(so(p,q+1))$-homomorphisms between generalized Verma $so(p,q+1)$- resp. $so(p+1,q+1)$-modules induced from character. This also amounts to the construction of $so(p,q+1)$-singular vectors in the target generalized $so(p+1,q+1)$-Verma module.

For the Lie algebra $so(p+1,q+1)$ of signature $(p+1,q+1)$ $(p+q=n)$, let J be the diagonal matrix with the number of $(p+1)$ 1's and $(q+1)$ -1's. The set of matrices $(i,j=1,\dots,n)$

$$M_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_i^T \otimes e_j - Je_j^T \otimes e_i J & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; H_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$Y_i^- = \sqrt{2} \begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & -Je_i^T \\ 0 & 0 & 0 \end{pmatrix}, \; Y_i^+ = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ e_i^T & 0 & 0 \\ 0 & -e_i J & 0 \end{pmatrix} \tag{8.4}$$

gives the matrix realization of Iwasawa decomposition of $so(p+1,q+1)$. Here $\{e_i\}_i$ is the basis of $\mathbb{R}^{p,q}$. The following commutation relations for $so(p+1,q+1)$ will be useful:

$$\begin{aligned} &[Y_i^+, Y_j^-] = 2(\delta_{ij} H_0 + M_{ij}), \\ &[H_0, Y_i^\pm] = \pm Y_i^\pm, \\ &[M_{ij}, Y_k^\pm] = \delta_{jk} Y_i^\pm - \delta_{ik} Y_j^\pm. \end{aligned} \tag{8.5}$$

It is now elementary to extend the results in [37] (for the signature $(n+1,1)$) resp. [1] (for the signature $(n,2)$) to any signature. The following identities will be helpful:

$$\begin{aligned} &[Y_1^+, (Y_1^-)^2] = -2Y_1^- + 4Y_1^- H_0, \\ &[Y_1^+, (Y_i^-)^2] = 2Y_1^- + 4Y_1^- M_{1i}, \; i = 2, \dots, n. \end{aligned}$$

The next Lemma is a key step to construct continuous families of homomorphisms of generalized Verma modules. Its proof differs from [37] and is based on suitable inductive procedure. We use the obvious shorthand notation for Lie subalgebras appearing in the Iwasawa decomposition of $so(p+1,q+1)$, e.g. $\mathfrak{m}_n = \mathfrak{m}(so(p+1,q+1)) = so(p,q)$ and $\mathfrak{n}_{n-} = \mathfrak{n}_-(so(p+1,q+1)) \simeq \mathbb{R}^{p,q}$, etc. As we shall see all the results are independent of signature and depend on $n=p+q$ only. The subscript by $\Box^-$ denotes the underlying dimension.

Lemma 8.2 *For any signature $(p+1,q+1)$ $(p+q=n)$ and $j \in \mathbb{N}$, we have*

$$\begin{aligned} &[Y_1^+, (\Box_{n-1}^-)^j] - 2j(p+q-1-2j) Y_1^- (\Box_{n-1}^-)^{j-1} - 4j Y_1^- (\Box_{n-1}^-)^{j-1} H_0 \\ &\in \mathcal{U}(\mathfrak{n}_{n-}) \mathfrak{m}_n. \end{aligned} \tag{8.6}$$

Proof:

The proof goes by induction on j. Let us recall the conventional notation $\Box^-_{n-1} := \sum_{j=1}^{n-1}(Y^-_j)^2$. The case $j=1$ amounts to

$$\begin{aligned}
&[Y_1^+, \sum_{j=1}^{p}(Y_j^-)^2 + \sum_{j=p+1}^{n-1}(Y_j^-)^2] = (-2Y_1^- + 4Y_1^- H_0) + \\
&(2Y_1^- + 4Y_2^- M_{1,2}) + \cdots + (2Y_1^- + 4Y_p^- M_{1,p}) + \\
&(2Y_1^- + 4Y_{p+1}^- M_{1,p+1}) + \cdots + (2Y_1^- + 4Y_{n-1}^- M_{1,n-1}) = \\
&2(p+q-3)Y_1^- + 4Y_1^- H_0 \ mod\, \mathcal{U}(\mathfrak{n}_{n\,-})\mathfrak{m}_n
\end{aligned} \tag{8.7}$$

and the claim is proved.

Let us now assume that the claim is true for $j \in \mathbb{N}$, i.e.

$$\begin{aligned}
&[Y_1^+, (\Box^-_{n-1})^j] = 2j(p+q-1-2j)Y_1^-(\Box^-_{n-1})^{j-1} + \\
&4jY_1^-(\Box^-_{n-1})^{j-1}H_0 \ mod\, \mathcal{U}(\mathfrak{n}_{n\,-})\mathfrak{m}_n.
\end{aligned} \tag{8.8}$$

Then

$$\begin{aligned}
&[Y_1^+, (\Box^-_{n-1})^{j+1}] = \Box^-_{n-1}[Y_1^+, (\Box^-_{n-1})^j] + [Y_1^+, \Box^-_{n-1}](\Box^-_{n-1})^j = \\
&\Box^-_{n-1}(2j(p+q-1-2j)Y_1^-(\Box^-_{n-1})^{j-1} + 4jY_1^-(\Box^-_{n-1})^{j-1}H_0) + \\
&(2(p+q-3)Y_1^- + 4Y_1^- H_0)(\Box^-_{n-1})^j \\
&= 2(j+1)(p+q-1-2(j+1))Y_1^-(\Box^-_{n-1})^j + \\
&4(j+1)Y_1^-(\Box^-_{n-1})^j H_0 \ mod\, \mathcal{U}(\mathfrak{n}_{n\,-})\mathfrak{m}_n
\end{aligned} \tag{8.9}$$

and the claim follows.

$\Box$

A direct consequence of the previous Lemma yields the explicit form of homomorphisms or, when evaluated, singular vectors in the target generalized Verma module. The Theorem is divided into two parts according to the homogeneity of the homomorphism.

Theorem 8.3 *1. (Families of even order) Let $(so(p+1,q+1), so(p,q+1))$ $(p+q=n)$ be the couple of orthogonal Lie algebras. For any $p,q,N \in \mathbb{N}$, $(p+q) \geq 3$ and $\lambda \in \mathbb{C}$, the element*

$$D_{2N}(\lambda) = \sum_{j=0}^{N} a_j(\lambda)(\Box^-_{n-1})^{j-1}(Y_n^-)^{2N-2j} \in \mathcal{U}(\mathfrak{n}_{n+1\,-}) \tag{8.10}$$

satisfies

$$\begin{aligned}
&[Y_i^+, \sum_{j=0}^{N} a_j(\lambda)(\Box^-_{n-1})^{j-1}(Y_n^-)^{2N-2j}] \\
&\in \mathcal{U}(\mathfrak{n}_{n+1\,-})(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H_0 - \lambda))
\end{aligned} \tag{8.11}$$

for $i = 1, \ldots, n-1$ iff the coefficients $\{a_j\}_{j=0}^N$ fulfill the recursive relations

$$(N-j+1)(2N-2j+1)a_{j-1}$$
$$+j(p+q-1+2\lambda-4N+2j)a_j = 0, \qquad (8.12)$$

$j = 1, \ldots, N$. In effect, the left multiplication by this element induces $\mathcal{U}(so(p,q+1))$-homomorphism

$$M^{\mathfrak{g}(p,q+1)}_{\mathfrak{p}(p,q+1)}(\mathbb{C}_{\lambda-N}) \to M^{\mathfrak{g}(p+1,q+1)}_{\mathfrak{p}(p+1,q+1)}(\mathbb{C}_\lambda) \qquad (8.13)$$

2. *(Families of odd order) For any $p, q, N \in \mathbb{N}$, $p+q = n$ ($(p+q) \geq 3$) and $\lambda \in \mathbb{C}$ the element*

$$D_{2N+1}(\lambda) = \sum_{j=0}^{N} b_j(\lambda)(\Box^-_{n-1})^{j-1}(Y^-_n)^{2N-2j+1} \in \mathcal{U}(\mathfrak{n}_{n+1\,-}) \qquad (8.14)$$

satisfies

$$[Y^+_i, \sum_{j=0}^{N} a_j(\lambda)(\Box^-_{n-1})^{j-1}(Y^-_n)^{2N-2j+1}] \qquad (8.15)$$
$$\in \mathcal{U}(\mathfrak{n}_{n+1\,-})(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H_0-\lambda)), i = 1, \ldots, n-1 \qquad (8.16)$$

iff the coefficients $\{b_j\}_{j=0}^N$ fulfill the recursive relations

$$(N-j+1)(2N-2j+3)b_{j-1} \qquad (8.17)$$
$$+j(p+q-3+2\lambda-4N+2j)b_j = 0, j = 1, \ldots, N.$$

As we shall prove in the next section, this set of singular vectors (enumerated by $N \in \mathbb{N}$) is complete and sufficient to decompose a given generalized Verma module with respect to a rank one less orthogonal Lie subalgebra.

8.3 The composition series for branching problem of generalized Verma modules

In the previous subsection we produced a collection of $so(p,q+1)$-homomorphism from $\mathfrak{g}' = so(p,q+1)$-generalized Verma modules to a fixed $\mathfrak{g} = so(p+1,q+1)$-generalized Verma module (regarded as $so(p,q+1)$-module via standard embedding $so(p,q+1) \hookrightarrow so(p+1,q+1)$) or, when evaluated, the collection of $so(p,q+1)$-singular vectors in the $so(p+1,q+1)$-generalized Verma module. The remaining question is whether the construction in the previous section produced complete (exhausting) family of singular vectors.

One way to analyze this question is based on character identities for the restriction of generalized Verma modules with respect to a reductive subalgebra

$\mathfrak{g}'$ for which the parabolic subalgebra $\mathfrak{p}' := \mathfrak{g}' \cap \mathfrak{p}$ is standard ($\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+, \mathfrak{p}' = \mathfrak{l}' + \mathfrak{n}'_+$), see e.g. [41]. Let $\mathbb{V}_\lambda$ be a finite dimensional $\mathfrak{l}$-module with highest weight $\lambda \in \Lambda^+(\mathfrak{l})$ and likewise $\mathbb{V}_{\lambda'}$ be a finite dimensional representation of $\mathfrak{l}'$, $\lambda' \in \Lambda^+(\mathfrak{l}')$. Given a vector space V we denote $S(V) = \oplus_{l=0}^{\infty} S_l(V)$ the symmetric tensor algebra on V. Let us extend the adjoint action of $\mathfrak{l}'$ on $\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}')$ to $S(\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}'))$. We set

$$m(\lambda', \lambda) = Hom_{\mathfrak{l}'}(\mathbb{V}_{\lambda'}, \mathbb{V}_\lambda|_{\mathfrak{l}'} \otimes S(\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}'))). \tag{8.18}$$

Theorem 8.4 *([41], Theorem 3.9) Suppose $\mathfrak{p}$ is $\mathfrak{g}'$-compatible standard parabolic subalgebra of $\mathfrak{g}$, $\lambda \in \Lambda^+(\mathfrak{l})$. Then*

1. *$m(\lambda', \lambda) < \infty$ for all $\lambda' \in \Lambda^+(\mathfrak{l}')$.*
2. *In the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$ there is $\mathfrak{g}'$-isomorphism*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{\lambda' \in \Lambda^+(\mathfrak{l}')} m(\lambda', \lambda) M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda'}). \tag{8.19}$$

A consequence of this Theorem is that in the case of multiplicity free $\mathfrak{l}'$-module $\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}')$) and for generic character λ the decomposition of generalized Verma module $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ (induced from character λ) with respect to $\mathfrak{g}'$ is multiplicity free. Moreover, for any value of the character λ the following relation holds true in the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$:

Corollary 8.5 *For $\mathfrak{g} \equiv so(p+1, q+1), \mathfrak{g}' \equiv so(p, q+1)$ with standard maximal parabolic subalgebras $\mathfrak{p}, \mathfrak{p}'$ given by omitting the first simple root, we have $\mathfrak{n}_+ \simeq \mathbb{R}^{p,q}, \mathfrak{n}_+ \cap \mathfrak{g}' \simeq \mathbb{R}^{p-1,q}$ and $\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}') \simeq \mathbb{R}$ transforms as the character of the Levi subalgebra of $\mathfrak{g}'$. Then $m(\lambda, \lambda') = 1$ if and only if $\lambda' = \lambda - j, j \in \mathbb{N}$ and $m(\lambda, \lambda') = 0$ otherwise. In the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$ holds*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda) \simeq \bigoplus_{j \in \mathbb{N}} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda - j}). \tag{8.20}$$

8.4 Branching rules for the generic value λ of inducing character of generalized Verma module

In this subsection we prove that the previous observation on the relation in the Grothendieck group corresponds, in case of a generic inducing character, to the actual branching rule for the couple $(\mathfrak{g}, \mathfrak{g}') \equiv (so(p+1, q+1), so(p, q+1))$ and a generalized Verma $\mathcal{U}(\mathfrak{g})$-module induced from the character $\lambda \in \mathbb{C}$ of $\mathfrak{p} \subset \mathfrak{g}$.

First of all we note that for any λ there is always a direct sum decomposition of $\mathfrak{g}'$-module $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ into the even and odd part,

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda) = \left(\sum_{k=0}^{\infty} \langle w_{2k} \rangle \right) \oplus \left(\sum_{k=0}^{\infty} \langle w_{2k+1} \rangle \right) = U^{even} \oplus U^{odd},$$

according to the homogeneity of an element in the polynomial algebra $\mathcal{U}(\mathfrak{n}_-)$. Here w_l denotes the singular vector in $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ corresponding to the image of the homomorphism $D_l(\lambda)$, $l \in \mathbb{N}$, and $< w_l >$ denotes its $\mathcal{U}(\mathfrak{g}')$-span. The spaces $\langle w_j \rangle = \mathcal{U}(\mathfrak{g}')w_j$ are invariant under the action of $\mathfrak{n}'_-$, while under the action of $\mathfrak{n}'$ the spaces $\langle w_j \rangle$ are mapped to the sum $\sum_{\ell \in \mathbb{N}} \langle w_{j-2\ell} \rangle$. Hence

$$W_{2k} = \sum_{\ell=0}^{k} \langle w_{2l} \rangle,\ k \in \mathbb{N}$$

form a $\mathfrak{g}'$-filtration of U^{even} by invariant subspaces. Analogous result is true for the odd part and consequently the spaces U^{even} and U^{odd} have invariant filtrations under the $\mathfrak{g}'$-action.

We shall now discuss decomposition problem of U^{even}, U^{odd} for $\lambda \neq k - \frac{n}{2}$. The values of $\lambda \in \mathbb{C}, \lambda \neq k - \frac{n}{2}$ are henceforth termed generic.

Theorem 8.6 *Let $\lambda \in \mathbb{C}, \lambda \neq k - \frac{n}{2}$ for $k \in \mathbb{N}$, i.e. let λ be generic. Then*

$$U^{even} = \bigoplus_{j=0}^{\infty} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2j}),$$

$$U^{odd} = \bigoplus_{i=0}^{\infty} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2i-1}), \tag{8.21}$$

gives the direct sum decomposition of the left hand side into irreducible submodules under the restriction from $\mathfrak{g}$ to $\mathfrak{g}'$. The embedding of $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2j}) \hookrightarrow M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ resp. $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2i-1}) \hookrightarrow M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ is induced by the singular vector $w_{2j} \in M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ resp. $w_{2i+1} \in M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$, $i, j \in \mathbb{N}$.

Proof:

The singular vector w_l generates a cyclic $\mathfrak{g}'$-submodule $\langle w_l \rangle$ in $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_{\lambda-1})$ with the highest weight $\lambda_l = (\lambda - l|0, \dots, 0)$. The vectors have mutually different infinitesimal characters, because the difference of the quadratic Casimir for w_j resp. w_i is

$$|\lambda_j + \delta|^2 - |\lambda_i + \delta|^2 = (i - j)(2\lambda + n - (i + j)),$$

which is nonzero by assumptions of Theorem. Here δ denotes the half of the sum of simple roots of $\mathfrak{g}'$. This conclusion implies direct sum decomposition in Corollary 8.5 and the result follows.

□

8.5 Branching rules for the non-generic value λ of inducing character of generalized Verma module

The remaining task is the analysis of the composition series for non-generic values $\lambda \in \mathbb{C}$ of induced character in a given block of $\mathcal{O}^{\mathfrak{p}'}$, characterized as a locus given by special values of quadratic Casimir operator. Recall that these values correspond to the known classification of homomorphisms of generalized Verma $so(p,q+1)$-modules induced from character, [21]. As we shall see, parabolic $\mathcal{O}^{\mathfrak{p}'}$ category naturally appears in our decomposition problem.

Because the weights used to induce generalized Verma modules are characters of reductive Levi factor of $\mathfrak{g}$, we are basically left with $sl(2)$-theory (generated by the first simple root of $\mathfrak{g}$). It is then natural to remind as a motivation the structure of $\mathcal{O}^{\mathfrak{b}}$ category for $sl(2)$ and then return back to our former problem.

Example 8.7 *Throughout this example we use the notation*

$$M_\lambda = M_{\mathfrak{b}}^{sl(2,\mathbb{C})}(\mathbb{C}_\lambda), \mathcal{L}_\lambda = \mathcal{L}_\lambda(sl(2,\mathbb{C}),\mathfrak{b}), \mathcal{P}_\lambda = \mathcal{P}_\lambda(sl(2,\mathbb{C}),\mathfrak{b}).$$

As for $\mathfrak{g} = sl(2,\mathbb{C})$, the dual of Cartan subalgebra $\mathfrak{h}^\star$ is isomorphic to $\mathbb{C}$. The non-integral weights are linked by action of the Weyl group to no comparable weights (in the standard ordering), and so the only interesting subcategories (blocks) $\mathcal{O}_\lambda$ of the Borel category $\mathcal{O}^{\mathfrak{b}}$ are given by $\lambda \in \mathbb{Z}$. Let us consider the orbit of the Weyl group for $\lambda, \mu := w \cdot \lambda = -\lambda - 2$, $\lambda \in \mathbb{N}$. There is no lower weight associated to μ, consequently $\mathcal{L}_\mu = M_\mu$ ($dim(\mathcal{L}_\mu) = \infty$). For $\lambda \in \mathbb{N}$ we get $dim(\mathcal{L}_\lambda) < \infty$ and there is a short exact sequence

$$0 \to \mathcal{L}_\mu \to M_\lambda \to \mathcal{L}_\lambda \to 0. \tag{8.22}$$

For the dominant integral weight λ we have $\mathcal{P}_\lambda = M_\lambda$. Its dual in $\mathcal{O}^{\mathfrak{b}}$-category $\mathcal{Q}(\lambda) := \tilde{\mathcal{P}}_\lambda$ is the injective module whose socle is $\mathcal{L}_\lambda$ and its head is $\mathcal{L}_\mu$. The top quotient of $\mathcal{P}_\mu$ is $\mathcal{L}_\mu = M_\mu$, and it follows from the BGG reciprocity $[\mathcal{P}_\mu : M_\lambda] = [M_\lambda : \mathcal{L}_\mu]$, $\lambda, \mu \in \mathfrak{h}^\star$ that there is a non-split short exact sequence

$$0 \to M_\lambda \to \mathcal{P}_\mu \to M_\mu \to 0. \tag{8.23}$$

The dual of projective module $\tilde{\mathcal{P}}_\mu$ is $\mathcal{Q}_\mu \simeq \mathcal{P}_\mu$. The (quadratic) Casimir operator $z \in \mathcal{U}(\mathfrak{g})$ acts by scalar $\lambda^2 + 2\lambda$ on both M_λ and M_μ. The element $z - (\lambda^2 + 2\lambda)$ is nonzero when acting on $\mathcal{P}_\mu$, but $(z - (\lambda^2 + 2\lambda))^2$ is trivial on $\mathcal{P}_\mu$.

In conclusion, there are five isomorphism classes of indecomposable objects in $\mathcal{O}^{\mathfrak{b}}_\lambda$:

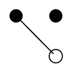
$$\mathcal{L}_\lambda, \mathcal{L}_\mu = M_\mu, M_\lambda = \mathcal{P}_\lambda, \mathcal{Q}_\lambda = \tilde{M}_\lambda, \mathcal{P}_\mu = \mathcal{Q}_\mu.$$

We shall now analyze explicitly the first few cases when the nontrivial composition series emerges. We focus on the nontrivial part of the decomposition, which means that the vector complement in the decomposition consists of generalized Verma modules with mutually different infinitesimal characters, hence direct summands in the decomposition. We discuss the even case only, the discussion of odd case goes along the same lines. Recall the convention $\Box' := \Box^-_{n-1}$ ($n = p + q$.)

The first non-trivial case corresponds to the value λ for which $2\lambda + n - 3 = 0$, $N = 1$ and $D_2(\lambda) = (2\lambda + n - 3)Y_n^{-2} + \Box'$. Hence for this value of λ the homomorphism reduces to $\Box'$ and so with respect to the homogeneity of Y_n^-, the first row $M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_\lambda)$ given by $\mathcal{U}(\mathfrak{g}')$-span of highest weight vector of $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_\lambda)$ contains the nontrivial submodule (its singular vector is generated by the image of $\Box'$). Taken together with the third row $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_{\lambda-2})$ form the nontrivial (non split) extension class

$$0 \to M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_\lambda) \to \mathcal{P}_{\lambda-2}(\mathfrak{g}',\mathfrak{p}') \to M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_{\lambda-2}) \to 0, \tag{8.24}$$

where $\mathcal{P}_{\lambda-2}(\mathfrak{g}',\mathfrak{p}')$ is an object in the block of the parabolic category $\mathcal{O}^{\mathfrak{p}'}$. The picture representing such a situation is



where (anti)diagonals represent the singular vectors and the degeneration of particular singular vector for the previously mentioned value of λ is pictured in such a way that the missing monomials (in $Y_n^{-2}, \Box'$) correspond to white circles and the nontrivial present monomials to black circles. The first resp. the third rows correspond to $\mathcal{U}(\mathfrak{g}')$-span of v_λ resp. $Y_n^{-2} v_\lambda$, where v_λ is the highest weight vector of $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_\lambda)$.

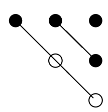
The next case is related to the appearance of a non-trivial composition series, whose source is the fourth order operator ($N = 2$)

$$D_4(\lambda) = (2\lambda + n - 7)(2\lambda + n - 5)Y_n^{-4} + (2\lambda + n - 5)Y_n^{-2}\Box' + (\Box')^2.$$

The computation of infinitesimal character implies that this happens for λ fulfilling $2\lambda + n - 5 = 0$. For such λ, the generalized Verma $\mathcal{U}(\mathfrak{g}')$-modules $M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_\lambda), M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_{\lambda-4})$ form nontrivial extension

$$0 \to M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_\lambda) \to \mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}') \to M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_{\lambda-4}) \to 0, \tag{8.25}$$

realizing an object $\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$. The generalized Verma module $M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_\lambda)$ has a nontrivial composition structure in itself - its nontrivial submodule $M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_{\lambda-4}) \subset M^{\mathfrak{g}'}_{\mathfrak{p}'}(\mathbb{C}_\lambda)$ is generated by the image of $\Box'^2$. The picture in which the first and the third row represent $\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ and the second row has a different infinitesimal character is drawn on the following picture:



The last explicit case we mention corresponds to λ fulfilling $2\lambda + n - 7 = 0$. The sixth order operator (here $N = 3$) generating the family of singular vector is

$$D_6(\lambda) = (2\lambda+n-11)(2\lambda+n-9)(2\lambda+n-7)Y_n^{-6} +(2\lambda+n-9)(2\lambda+n-7)Y_n^{-4}\Box' +(2\lambda+n-7)Y_n^{-2}(\Box')^2+(\Box')^3. \quad (8.26)$$

In this case we observe the emergence of two objects in the parabolic category $\mathcal{O}^{\mathfrak{p}'}$. The first comes from the nontrivial extension

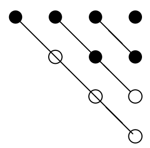
$$0\to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_\lambda)\to \mathcal{P}_{\lambda-6}(\mathfrak{g}',\mathfrak{p}')\to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-6})\to 0, \quad (8.27)$$

while the second from

$$0\to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2})\to \mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')\to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-4})\to 0. \quad (8.28)$$

Note that $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_\lambda)$ has nontrivial filtered structure - its submodule is a generalized Verma module generated by the image of $(\Box')^3$. Similarly, $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2})$ has nontrivial composition series - its maximal generalized Verma submodule $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-4})$ is generated by the image of $\Box'$. The modules $\mathcal{P}_{\lambda-6}(\mathfrak{g}',\mathfrak{p}'), \mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ have different infinitesimal character.

The picture in which the first and the fourth resp. the second and the third row represent $\mathcal{P}_{\lambda-6}(\mathfrak{g}',\mathfrak{p}')$ resp. $\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ is



Theorem 8.8 *Let $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ be a family of generalized Verma $\mathcal{U}(\mathfrak{g})$-modules induced from character λ, where $\mathfrak{g}=so(p+1,q+1)$ $(p+q=n)$ and $\mathfrak{p}\subset\mathfrak{g}$ its standard maximal parabolic subalgebra given by omitting the first simple root. Let $\mathfrak{g}'=so(p,q+1)$ be the reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{p}'=\mathfrak{g}'\cap\mathfrak{p}$.*

As an $\mathcal{U}(\mathfrak{g}')$-module, $M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda)$ has a contribution to the non-trivial composition structure from both the even and odd homogeneity homomorphisms:

1. *The case of even homogeneity homomorphisms corresponds to $\lambda\in\mathbb{C}$ fulfilling $2\lambda+n=2N+1$, $N\in\mathbb{N}_+$. In the decomposition there are $[\frac{N+1}{2}]$ modules*

$$\mathcal{P}_{\lambda-2N}(\mathfrak{g}',\mathfrak{p}'),\mathcal{P}_{\lambda-2N+2}(\mathfrak{g}',\mathfrak{p}'),\dots,\mathcal{P}_{\lambda-2N+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}').$$

These modules appear as nontrivial extensions in short exact sequences

$$
\begin{aligned}
&0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2[\frac{N-1}{2}]}) \to \mathcal{P}_{\lambda-2N+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}') \\
&\quad \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N+2[\frac{N-1}{2}]}) \to 0, \\
&\dots \\
&0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2j}) \to \mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}') \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N+2j}) \to 0, \\
&\dots \\
&0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda}) \to \mathcal{P}_{\lambda-2N}(\mathfrak{g}',\mathfrak{p}') \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N}) \to 0,
\end{aligned} \tag{8.29}
$$

where $j = 0, 1, \dots, [\frac{N-1}{2}]$*. The* j*-th module* $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2j})$, $j = 0, 1, \dots, [\frac{N-1}{2}]$ *has nontrivial composition series - its maximal submodule* $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N+2j})$ *is generated by the image of* ${\Box'}^{N-2j} := {\Box_{n-1}}^{N-2j}$ *and the quotient* $\mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')/M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2j})$ *is simple module. The module* $\mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')$ *is realized in the generalized Verma* $so(p+1,q+1)$*-module by* $\mathcal{U}(\mathfrak{g}')$*-span of singular vectors* w_{2j}, w_{2N-2j}*.*

Let us introduce the finite set $S := \{\lambda-2j, \lambda-2N+2j | j = 0, 1, \dots, [\frac{N-1}{2}]\}$*, so* $S' := \{\{\lambda - 2\mathbb{N}\} \setminus S | N \in \mathbb{N}\}$ *is infinite. Then we have the branching rule*

$$
M_{\mathfrak{p}}^{\mathfrak{g}^{even}}(\mathbb{C}_\lambda) \simeq \bigoplus_{j=0,1,\dots,[\frac{N-1}{2}]} \mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}') \bigoplus_{\lambda' \in S'} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda'}). \tag{8.30}
$$

2. *The case of odd homogeneity homomorphisms corresponds to* $\lambda \in \mathbb{C}$ *fulfilling* $2(\lambda-1)+n = 2N+1$, $N \in \mathbb{N}_+$*. In the decomposition there are* $[\frac{N+1}{2}]$ *modules*

$$
\mathcal{P}_{\lambda-2N-1}(\mathfrak{g}',\mathfrak{p}'), \mathcal{P}_{\lambda-2N+1}(\mathfrak{g}',\mathfrak{p}'), \dots, \mathcal{P}_{\lambda-2N-1+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}').
$$

These modules appear as nontrivial extensions in short exact sequences

$$
\begin{aligned}
&0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-1-2[\frac{N-1}{2}]}) \to \mathcal{P}_{\lambda-2N-1+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}') \\
&\quad \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N-1+2[\frac{N-1}{2}]}) \to 0, \\
&\dots \\
&0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-1-2j}) \to \mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}',\mathfrak{p}') \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N-1+2j}) \to 0, \\
&\dots \\
&0 \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-1}) \to \mathcal{P}_{\lambda-2N-1}(\mathfrak{g}',\mathfrak{p}') \to M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N-1}) \to 0,
\end{aligned} \tag{8.31}
$$

where $j = 0, 1, \dots, [\frac{N-1}{2}]$*. The* j*-th module* $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2j-1})$, $j = 0, 1, \dots, [\frac{N-1}{2}]$ *has nontrivial composition series - its maximal submodule* $M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-2N-1+2j})$ *is generated by the image of* $Y_n^- {\Box'}^{N-2j}$ *and the quotient* $\mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}',\mathfrak{p}')/M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda-1-2j})$

is simple module. The module $\mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}',\mathfrak{p}')$ is realized in the generalized Verma $so(p+1,q+1)$-module by $\mathcal{U}(\mathfrak{g}')$-span of singular vectors $w_{2j+1}, w_{2N+1-2j}$.

Let us introduce the finite set $\tilde{S} := \{\lambda-1-2j, \lambda-2N-1+2j | j = 0,1,\dots,[\frac{N-1}{2}]\}$, so $\tilde{S}' := \{\{\lambda-2\mathbb{N}-1\} \setminus \tilde{S} | N \in \mathbb{N}\}$ is infinite. Then we have the branching rule

$$M_{\mathfrak{p}}^{\mathfrak{g}^{odd}}(\mathbb{C}_\lambda) \simeq \bigoplus_{j=0,1,\dots,[\frac{N-1}{2}]} \mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}',\mathfrak{p}') \bigoplus_{\lambda'\in\tilde{S}'} M_{\mathfrak{p}'}^{\mathfrak{g}'}(\mathbb{C}_{\lambda'}). \quad (8.32)$$

Finally, we have direct sum decomposition of $\mathfrak{g}'$-modules (which is even true for any λ):

$$M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda) \simeq M_{\mathfrak{p}}^{\mathfrak{g}^{even}}(\mathbb{C}_\lambda) \bigoplus M_{\mathfrak{p}}^{\mathfrak{g}^{odd}}(\mathbb{C}_\lambda). \quad (8.33)$$

Proof:

The general case follows the scheme indicated in the discussion of the structure of singular vectors preceded this Theorem. Corollary 8.5 implies that the elements constructed in Theorem 8.3 cover all singular vectors. Moreover, for non-generic value of the inducing character λ (determined in Theorem 8.6) there is finite number of couples of singular vectors with equal infinitesimal character, as follows again from Theorem 8.6. These couples are enumerated in Equation 8.29 for even homogeneity resp. Equation 8.31 for odd homogeneity case. The non-triviality of each extension class is an elementary direct check applied to the singular vector (evaluated at the corresponding non-generic value λ) based on Equation 8.5, Equation 8.6.

□

The techniques used in the present section do not allow further analysis of constructed extension classes of generalized Verma modules. In [8], there are certain partial results describing (non-recursive) scheme to compute Kazhdan-Lusztig polynomials associated to Hermitian symmetric spaces. In particular, Kazhdan-Lusztig polynomials are determined in the case of regular block of zero weight in Proposition 5.1., p. 288, [8], with the following result (in the even dimensional orthogonal case): $P_{w_i,w_j}(u)$ is trivial for incomparable w_i, w_j, $P_{w_i,w_j}(u) = 1+u^{j-n-1}$ for $n+2 \leq j \leq 2n-1$, $1 \leq i \leq 2n-j$ and $P_{w_i,w_j}(u) = 1$ otherwise. In the odd orthogonal case, the structure of Kazhdan-Lusztig polynomials is even simpler. As the extension classes $Ext^{\star}_{\mathcal{U}(\mathfrak{g})}(M_{w_i}, \mathcal{L}_{w_j})$ are the coefficients of Kazhdan-Lusztig polynomials, they are at most one dimensional. In a basic example, taking into account the relationship between extension classes and Lie algebra cohomology classes of commutative nilradical ([35]), one can directly compare the extension class produced in the branching rule with its geometrical realization based on Lie algebra cohomology method, see [5]. However, in many cases are our results realized in singular infinitesimal character for which the structure of Kazhdan-Lusztig polynomials are to our best knowledge not available.

Another remark, closely related to the remarks in the last paragraph, is that in many cases the extension classes appearing in the main Theorem are projective objects of parabolic BGG category $\mathcal{O}^{\mathfrak{p}'}$.

9 Branching problem for couple of (codimension one) conformal geometries and inducing fundamental vector representation

We shall start with qualitative analysis of character identities for the restriction of generalized Verma modules with respect to a reductive subalgebra $\mathfrak{g}'$ for the standard compatible parabolic subalgebra $\mathfrak{p}' := \mathfrak{g}' \cap \mathfrak{p}$ ($\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+, \mathfrak{p}' = \mathfrak{l}' + \mathfrak{n}'_+$), see e.g. [43]. Let V_λ be a finite dimensional $\mathfrak{l}$-module with highest weight $\lambda \in \Lambda^+(\mathfrak{l})$ and likewise $V_{\lambda'}$ be a finite dimensional representation of $\mathfrak{l}'$, $\lambda' \in \Lambda^+(\mathfrak{l}')$. Given a vector space V we denote $S(V) = \oplus_{l=0}^{\infty} S_l(V)$ the symmetric tensor algebra on V. Let us extend the adjoint action of $\mathfrak{l}'$ on $\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}')$ to $S(\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}'))$. We set

$$m(\lambda', \lambda) = Hom_{\mathfrak{l}'}(V_{\lambda'}, V_\lambda|_{\mathfrak{l}'} \otimes S(\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}'))). \tag{9.1}$$

Theorem 9.1 *([43], Theorem 3.9) Suppose $\mathfrak{p}$ is $\mathfrak{g}'$-compatible standard parabolic subalgebra of $\mathfrak{g}$, $\lambda \in \Lambda^+(\mathfrak{l})$. Then*

1. *$m(\lambda', \lambda) < \infty$ for all $\lambda' \in \Lambda^+(\mathfrak{l}')$.*
2. *In the Grothendieck group of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}'}$ there is $\mathfrak{g}'$-isomorphism*

$$M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)|_{\mathfrak{g}'} \simeq \bigoplus_{\lambda' \in \Lambda^+(\mathfrak{l}')} m(\lambda', \lambda) M_{\mathfrak{p}}^{\mathfrak{g}}(V_{\lambda'}).$$

In particular, we focus on and consequently classify $\mathfrak{g}' = so(p, q+1)$-homomorphisms between $\mathfrak{g}' = so(p, q+1)$-generalized Verma modules and a fixed $\mathfrak{g} = so(p+1, q+1)$-generalized Verma module (regarded as $so(p, q+1)$-module via standard embedding $\mathfrak{g}' = so(p, q+1) \hookrightarrow \mathfrak{g} = so(p+1, q+1)$) induced from fundamental form representations or, when evaluated, the collection of $\mathfrak{g}' = so(p, q+1)$-singular vectors in the $\mathfrak{g} = so(p+1, q+1)$-generalized Verma module. Inductively, one can construct the singular vectors in the case of the difference in the ranks of $\mathfrak{g}, \mathfrak{g}'$ higher than one.

The remaining question is whether the construction based on the F-method produces complete (exhausting) family of singular vectors. It follows from the isomorphism of $\mathfrak{m}'$-modules

$$\Lambda^p(\mathbb{C}^n) \simeq \Lambda^p(\mathbb{C}^{n-1}) \oplus \Lambda^{p-1}(\mathbb{C}^{n-1})$$

and the fact $dim_{\mathbb{C}}(\mathfrak{n}_+/(\mathfrak{n}_+ \cap \mathfrak{g}')) = 1$ that the singular vectors constructed in the article form complete set and realize branching problem in the Grothendieck group $K(\mathcal{O}^{\mathfrak{p}'})$ of the BGG parabolic category $\mathcal{O}^{\mathfrak{p}'}$. For the generic value of the inducing character λ, the branching is a direct sum decomposition of multiplicity free $(\mathfrak{g}', P')$-modules.

As for the inducing representation given by fundamental vector representation, we use the notation $e_i, i = 1, \ldots, n$ for its basis. We shall work directly in the Fourier dual space, and retain the notation used in previous sections.

9.1 Branching problem for couple of conformal geometries and inducing fundamental vector representation twisted by character - qualitative analysis and the construction of singular vectors

Example 9.2 *The singular vector of homogeneity one corresponding to a* $(\mathfrak{g}', P')$*-homomorphism*

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} \mathbb{C}_{\lambda-2} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda)$$

is of the form

$$v_1(\lambda) = A\xi_n \otimes e_n + B\sum_{i=1}^{n-1} \xi_i \otimes e_i \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda)$$

for some $A, B \in \mathbb{C}$ *to be determined. The condition*

$$v_1(\lambda) \in Ker(P_j(\lambda) = \frac{1}{2}\xi_j \Box_\xi + (\lambda - E_\xi)\partial_j - \sum_{k=1}^{n} \partial_k(e_k \otimes e_j^\star - e_j \otimes e_k^\star)),$$

$j = 1, \ldots, n-1$*, with the notation* $\partial_k = \frac{\partial}{\partial \xi_k}$ *understood is equivalent to*

$$A + B(\lambda + n - 2) = 0,$$

which yields the family of linear singular vectors

$$v_1(\lambda) = -(\lambda + n - 2)\xi_n \otimes e_n + \sum_{i=1}^{n-1} \xi_i \otimes e_i. \tag{9.2}$$

Example 9.3 *The singular vector of homogeneity two corresponding to a* $(\mathfrak{g}', P')$*-homomorphism*

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} \mathbb{C}_{\lambda-3} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda)$$

is of the form

$$v_2(\lambda) = A\sum_{i=1}^{n-1} \xi_i^2 \otimes e_n + B\xi_n^2 \otimes e_n + C\sum_{i=1}^{n-1} \xi_n\xi_i \otimes e_i = \xi_n^2(At + B) \otimes e_n$$
$$+\xi_n C\sum_{i=1}^{n-1} \xi_i \otimes e_i \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda) \tag{9.3}$$

for some $A, B, C \in \mathbb{C}$ *to be determined and* $P(t) = At + B, Q(t) = C$ *(*$deg(P) = 1, deg(Q) = 0$*). The condition*

$$v_2(\lambda) \in Ker(P_j(\lambda) = \frac{1}{2}\xi_j \Box_\xi + (\lambda - E_\xi)\partial_j - \sum_{k=1}^{n} \partial_k(e_k \otimes e_j^\star - e_j \otimes e_k^\star)),$$

$j=1,\dots,n-1$, *is equivalent to*

$$\xi_j\otimes e_n(A(n-1)+A2(\lambda-1)+B-C)+\xi_n\otimes e_j(2B+C(\lambda-1)-C(2-n))=0,$$

$j=1,\dots,n-1$, *or to the system of linear equations*

$$(2\lambda+n-3)A+B-C=0,$$
$$2B+(\lambda+n-3)C=0.$$

Its solution yields the family of quadratic singular vectors

$$v_2(\lambda)=(\lambda+n-1)\sum_{i=1}^{n-1}\xi_i^2\otimes e_n-(\lambda+n-3)(2\lambda+n-3)\xi_n^2\otimes e_n$$
$$+2(2\lambda+n-3)\sum_{i=1}^{n-1}\xi_n\xi_i\otimes e_i. \qquad (9.4)$$

Example 9.4 *The singular vector of homogeneity three corresponding to a $(\mathfrak{g}',P')$-homomorphism*

$$\mathcal{U}(\mathfrak{g}')\otimes_{\mathcal{U}(\mathfrak{p}')}\mathbb{C}_{\lambda-4}\to\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})}(\Lambda^1(\mathbb{C}^n)\otimes\mathbb{C}_\lambda)$$

is of the form

$$v_3(\lambda)=A\xi_n\sum_{i=1}^{n-1}\xi_i^2\otimes e_n+B\xi_n^3\otimes e_n+$$
$$C\sum_{k=1}^{n-1}\xi_k^2\sum_{i=1}^{n-1}\xi_i\otimes e_i+D\xi_n^2\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$=\quad \xi_n^3(At+B)\otimes e_n+\xi_n^2(Ct+D)\sum_{i=1}^{n-1}\xi_i\otimes e_i\in\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})}(\Lambda^1(\mathbb{C}^n)\otimes\mathbb{C}_\lambda)$$

for some $A,B,C,D\in\mathbb{C}$ to be determined and $P(t)=At+B, Q(t)=Ct+D$ ($deg(P)=1, deg(Q)=1$). The condition

$$v_3(\lambda)\in Ker(P_j(\lambda)=\frac{1}{2}\xi_j\Box_\xi+(\lambda-E_\xi)\partial_j-\sum_{k=1}^{n}\partial_k(e_k\otimes e_j^\star-e_j\otimes e_k^\star)),\ j=1,\dots,n-1$$

is equivalent to

$$\xi_l\xi_n\otimes e_n(A(n-1)+3B+A2(\lambda-2)-2D)+ \qquad (9.5)$$
$$\xi_l\sum_{i=1}^{n-1}\xi_i\otimes e_i(C(n+1)+D+2C(\lambda-2)-2C)+$$
$$\sum_{i=1}^{n-1}\xi_i^2\otimes e_l(C(\lambda-2)-(-nC-A))+$$
$$\xi_n^2\otimes e_l(D(\lambda-2)-((2-n)D-3B))=0 \qquad (9.6)$$

for all $j, l = 1, \dots, n-1$, or to the system of linear equations

$$(\lambda+n-2)C + A = 0,$$
$$3B + (\lambda+n-4)D = 0,$$
$$D + (2\lambda+n-5)C = 0,$$
$$(2\lambda+n-5)A + 3B - 2D = 0.$$

The determinant of this linear system is trivial and its solution yields the family of cubic homogeneity singular vectors

$$\begin{aligned} v_3(\lambda) &= -(\lambda+n-2)\xi_n \sum_{i=1}^{n-1} \xi_i^2 \otimes e_n + \frac{1}{3}(2\lambda+n-5)(\lambda+n-4)\xi_n^3 \otimes e_n \\ &\quad + \sum_{k=1}^{n-1} \xi_k^2 \sum_{i=1}^{n-1} \xi_i \otimes e_i - (2\lambda+n-5)\xi_n^2 \sum_{i=1}^{n-1} \xi_i \otimes e_i. \end{aligned} \tag{9.7}$$

9.2 Families of odd homogeneity homomorphisms from bulk 1-forms to boundary 0-forms

It follows from the $\mathfrak{m}'$-module structure on $Hom_{\mathbb{C}}(\Lambda^0(\mathbb{C}^{n-1}), \Lambda^1(\mathbb{C}^n))$ and the structure of the algebra of $\mathfrak{m}'$-invariants $Pol(\sum_{i=1}^{n-1} \xi_i^2, \xi_n) \subset Pol(\xi_1, \dots, \xi_{n-1}, \xi_n)$ that a singular vector of odd homogeneity $(2N+1)$, $N \in \mathbb{N}$, corresponding to a $(\mathfrak{g}', P')$-homomorphism

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} \mathbb{C}_{\lambda-(2N+1)-1} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda)$$

is of the form

$$v_{2N+1}(\lambda) = \xi_n^{2N}[P(t)\xi_n \otimes e_n + Q(t) \sum_{i=1}^{n-1} \xi_i \otimes e_i],\ j = 1, \dots, n-1 \tag{9.8}$$

for some polynomials $P(t), Q(t)$ of homogeneity N to be determined. Based on the change of variables

$$\partial_j = \frac{2\xi_j}{\xi_n^2}\frac{\partial}{\partial t}\ (j = 1, \dots, n-1),\ \partial_n = -\frac{2}{\xi_n} t \frac{\partial}{\partial t},$$

the action of

$$P_j(\lambda) = \frac{1}{2}\xi_j \Box_\xi + (\lambda - E_\xi)\partial_j - \sum_{k=1}^{n} \partial_k (e_k \otimes e_j^\star - e_j \otimes e_k^\star)$$

on $\xi_n^{2N} P(t)\xi_n \otimes e_n$ is

$$\begin{aligned} &P_j(\lambda)(\xi_n^{2N} P(t)\xi_n \otimes e_n) = \xi_n^{2N-1}[(2tP'' + (n-1)P')(\xi_j \otimes e_n) \\ &+(N(2N+1)P + (-4N+1)tP' + 2t^2P'')(\xi_j \otimes e_n) \\ &+2(\lambda - 2N)P'(\xi_j \otimes e_n) + ((2N+1)P - 2tP')(\xi_n \otimes e_j)] \end{aligned}$$

and on $\xi_n^{2N}Q(t)\sum_{i=1}^{n-1}\xi_i\otimes e_i$ is

$$
\begin{aligned}
&P_j(\lambda)(\xi_n^{2N}Q(t)\sum_{i=1}^{n-1}\xi_i\otimes e_i)=\xi_j\xi_n^{2N-2}(2tQ''+(n-1)Q'+2Q')(\sum_{i=1}^{n-1}\xi_i\otimes e_i)\\
&+\xi_j\xi_n^{2N-2}(N(2N-1)Q+(-4N+3)tQ'+2t^2Q'')(\sum_{i=1}^{n-1}\xi_i\otimes e_i)\\
&+(\lambda-2N)(\xi_n^{2N-2}\xi_j 2Q'(\sum_{i=1}^{n-1}\xi_i\otimes e_i)+\xi_n^{2N-1}Q\xi_n\otimes e_j)\\
&-2\xi_j\xi_n^{2N-2}Q'\sum_{i=1}^{n-1}\xi_i\otimes e_i-\xi_n^{2N-1}Q\xi_n\otimes e_j-\xi_n^{2N-1}(2NQ-2tQ')\xi_j\otimes e_n\\
&+\xi_n^{2N-1}2tQ'\xi_n\otimes e_j+\xi_n^{2N-1}(n-1)Q\xi_n\otimes e_j.
\end{aligned}
$$

The system of differential equations

$$P_j(\lambda)(v_{2N+1}(\lambda))=0,\ j=1,\dots,n-1$$

is equivalent to the system of three ordinary differential equations for P,Q representing the coefficients of $\xi_j\otimes e_n,\xi_n\otimes e_j$ and $\xi_j\sum_{i=1}^{n-1}\xi_i\otimes e_i$, respectively:

$$
\begin{aligned}
&(2t^2+2t)P''+[(2\lambda+n-4N-1)+(-4N+1)t]P'+N(2N+1)P\\
&-2NQ+2tQ'=0,\\
&-2tQ'-(n+\lambda-2N-2)Q+2tP'-(2N+1)P=0,\\
&(2t^2+2t)Q''+[(n+2\lambda-4N-1)+(-4N+3)t]Q'+N(2N-1)Q=0.
\end{aligned}
\tag{9.9}
$$

In what follows we prove that there is a unique solution of the system (9.9). As a consequence of the substitution of the second equation of (9.9) into the first one, the (unique) solution of (9.9) solves the rank two system ($deg(P)=deg(Q)=N$):

$$
\begin{pmatrix}
\begin{matrix}(2t^2+2t)\partial_t^2+(2\lambda+n-4N-1)\partial_t+\\(-4N+3)t\partial_t+(N-1)(2N+1)\end{matrix} & -(n+\lambda-2)\\
0 & \begin{matrix}(2t^2+2t)\partial_t^2+(n+2\lambda-4N-1)\partial_t+\\(-4N+3)t\partial_t+N(2N-1)\end{matrix}
\end{pmatrix}
\begin{pmatrix}P(t)\\Q(t)\end{pmatrix}=0.
$$

Lemma 9.5 *The first equation in (9.9) can be rewritten, modulo the second and the third one, as*

$$(n+2\lambda-2N-2)P'-N(n+\lambda-1)Q+(-\lambda+2N+1)Q'+(n+\lambda-1)tQ'=0.\tag{9.10}$$

Proof:

The application of the first derivative ∂_t to the second equation (9.9) and consequent multiplication by t yields its differential consequences,

$$
\begin{aligned}
&2P' + 2tP' - (2N+1)P' = 2Q' + 2tQ'' + (n+\lambda-2N-2)Q', \\
&2tP' + 2t^2P' - (2N+1)tP' = 2tQ' + 2t^2Q'' + (n+\lambda-2N-2)tQ'.
\end{aligned}
$$

Summing up both of them together and subtracting the first and the third equation in (9.9) cancels out the terms $(2t^2+2t)P'', (2t^2+2t)Q''$ and gives

$$
\begin{aligned}
&(2\lambda + n - 2N - 2)P' - 2NtP' + N(2N+1)P + (-\lambda + 2N + 1)Q' \\
&+(n+\lambda+2N-1)tQ' - N(2N+1)Q = 0. \qquad (9.11)
\end{aligned}
$$

The second and the third terms in the last equation combine together and enter the second equation in (9.9), which finally results in

$$
(n+2\lambda-2N-2)P' - N(n+\lambda-1)Q + (-\lambda+2N+1)Q' + (n+\lambda-1)tQ' = 0.
$$

□

The consequence of the Lemma is that the first equation in (9.9) is a differential consequence of the second and the third ones. In what follows we prove the existence and uniqueness of the solution of the second and the third equation in (9.9), polynomial in λ, t.

Lemma 9.6 *Let us denote the coefficients of the polynomials P, Q by p_j and q_j, respectively, i.e. $P(t) = \sum_{i=0}^{N} p_j t^j$ and $Q(t) = \sum_{i=0}^{N} q_j t^j$. Given the collection $\{q_j\}_j$, the second equation in (9.9) yields a unique collection $\{p_j\}$ fulfilling*

$$
p_j = \frac{n+\lambda+2j-2N-2}{2j-2N-1} q_j, \; j = 0, \dots, N. \qquad (9.12)
$$

There exists a unique polynomial (both in λ and t) solution of the second and the third equation in (9.9). In particular, the third equation in (9.9) gives the unique solution Q with required properties (Gegenbauer type polynomial) and the coefficients of the polynomial P are (uniquely) determined from the last display, equivalent to the second equation in (9.9). The first equation in (9.9) is a differential consequence of the second and the third equation.

Proof:

The uniqueness of the solution with required properties of the system (9.9) results from the following observations. The polynomial solution (of homogeneity N) of the third equation in (9.9) is the Gegenbauer type polynomial $Q(t) = \sum_{i=0}^{N} q_j t^j$, whose coefficients fulfill

$$
(N-j+1)(2N-2j+1)q_{j-1} + j(n+2\lambda-4N+2j-3)q_j = 0, \; j = 1, \dots N.
$$

The relation (9.12) easily follows from the second equation in (9.9) - the coefficient by monomial t^j yields

$$
-2jq_j - (n+\lambda-2N-2)q_j + 2jp_j - (2N+1)p_j = 0
$$

and the result follows.

Finally, as we have already discussed in the previous Lemma, the first equation in (9.9) is equivalent to (9.10) and we show that it is a differential consequence of the second and the third equation in (9.9). The polynomial equality in (9.10) reduced to the monomial t^j is

$$\begin{aligned}&(j+1)(2\lambda+n-2N-2)\frac{n+\lambda+2j-2N}{2j-2N+1}q_{j+1}-N(n+\lambda-1)q_j\\&+(j+1)(-\lambda+2N+1)q_{j+1}+j(n+\lambda-1)q_j=0.\end{aligned} \tag{9.13}$$

Using the identity

$$\begin{aligned}&(n+\lambda+2j-2N)(2\lambda+n-2N-2)+(2j-2N+1)(-\lambda+2N+1)\\&=(n+\lambda-1)(n+2\lambda-4N+2j-1),\end{aligned} \tag{9.14}$$

the previous equation reduces to

$$(j-N)(2j-2N+1)q_j+(j+1)(n+2\lambda-4N+2j-1)q_{j+1}=0,\ j=1,\dots N$$

and the shift in j, $j\to j-1$, yields the required claim by recursive property of the coefficients of Gegenbauer polynomial. The proof is complete.

□

9.3 Families of even homogeneity homomorphisms from bulk 1-forms to boundary 0-forms

As in the case of odd homogeneity, a singular vector of even homogeneity $2N$ corresponding to a $(\mathfrak{g}', P')$-homomorphism

$$\mathcal{U}(\mathfrak{g}')\otimes_{\mathcal{U}(\mathfrak{p}')}\mathbb{C}_{\lambda-2N-1}\to\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})}(\Lambda^1(\mathbb{C}^n)\otimes\mathbb{C}_\lambda)$$

is of the form

$$\begin{aligned}v_{2N}(\lambda)&=\xi_n^{2N-1}[P(t)\xi_n\otimes e_n+Q(t)\sum_{i=1}^{n-1}\xi_i\otimes e_i]\\&=\xi_n^{2N}P(t)\otimes e_n+\xi_n^{2N-1}Q(t)\sum_{i=1}^{n-1}\xi_i\otimes e_i\end{aligned} \tag{9.15}$$

for $j=1,\dots,n-1$ and polynomials $P(t),Q(t)$ of homogeneity N and $N-1$ in $t=\frac{\sum_{i=1}^{n-1}\xi_i^2}{\xi_n^2}$ to be determined. The system of differential equations

$$P_j(\lambda)(v_{2N}(\lambda))=0,\ j=1,\dots,n-1$$

is equivalent to the system of three ordinary differential equations for P, Q representing coefficients of $\xi_j \otimes e_n, \xi_n \otimes e_j$ and $\xi_j \sum_{i=1}^{n-1} \xi_i \otimes e_i$, respectively:

$$
\begin{aligned}
&(2t^2+2t)P'' + [(2\lambda+n-4N+1)+(-4N+3)t]P' + N(2N-1)P \\
&-(2N-1)Q + 2tQ' = 0, \\
&-2tQ' - (n+\lambda-2N-1)Q + 2tP' - 2NP = 0, \\
&(2t^2+2t)Q'' + [(n+2\lambda-4N+1)+(-4N+5)t]Q' + (N-1)(2N-1)Q = 0.
\end{aligned} \tag{9.16}
$$

As a consequence the solution of this system solves $(deg(P) = N, deg(Q) = N-1)$:

$$
\begin{pmatrix}
\begin{matrix}(2t^2+2t)\partial_t^2 + (2\lambda+n-4N+1)\partial_t + \\ (-4N+5)t\partial_t + (2N-3)N\end{matrix} & -(n+\lambda-2) \\
0 & \begin{matrix}(2t^2+2t)\partial_t^2 + (n+2\lambda-4N+1)\partial_t + \\ (-4N+5)t\partial_t + (N-1)(2N-1)\end{matrix}
\end{pmatrix}
\begin{pmatrix} P(t) \\ Q(t) \end{pmatrix} = 0. \tag{9.17}
$$

Lemma 9.7 *There exists a unique polynomial solution of the system (9.16). In particular, the third equation in the system (9.16) gives the recursive relation for the coefficients* $\{q_j\}_j$*,*

$$
\begin{aligned}
&(2j-2N+1)(j-N+1)q_j \\
&+(j+1)(n+2\lambda-4N+2j+1)q_{j+1} = 0,
\end{aligned} \tag{9.18}
$$

$j = 0, \dots, N-1$*, and the second equation in the system (9.16) gives the collection* $\{p_j\}_j$*:*

$$
p_j = \frac{(n+\lambda-2N+2j-1)}{2j-2N} q_j, \; j = 0, \dots, N-1 \tag{9.19}
$$

The first equation of the system (9.16) is a differential consequence of the second and the third equations.

Proof:

The monomial expansion of the third equation amounts to

$$
\begin{aligned}
&[2j(j-1) + (-4N+5)j + (N-1)(2N-1)]q_j + [2j(j+1) \\
&+(n+2\lambda-4N+1)(j+1)]q_{j+1} = 0
\end{aligned} \tag{9.20}
$$

and the first result follows from an elementary factorization.

The result for $\{p_j\}_j$ is an immediate consequence of monomial expansion of the second equation.

As in the odd homogeneity case it is elementary to verify that the first equation is a differential consequence of the second and the third equations, and the proof is complete.

□

9.4 Even and odd order family homomorphisms from bulk 1-forms to boundary 1-forms

It follows from the $\mathfrak{m}'$-module structure of $Hom_{\mathbb{C}}(\Lambda^1(\mathbb{C}^{n-1}), \Lambda^1(\mathbb{C}^n))$ and the structure of the algebra of $\mathfrak{m}'$-invariants $Pol(\sum_{i=1}^{n-1} \xi_i^2, \xi_n) \subset Pol(\xi_1, \dots, \xi_{n-1}, \xi_n)$ that a singular vector of homogeneity N corresponding to a $(\mathfrak{g}', P')$-homomorphism

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^1(\mathbb{C}^{n-1}) \otimes \mathbb{C}_{\lambda-N}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda)$$

is of the form

$$\begin{aligned} v_N(\lambda) \quad = & \sum_{a,b \in \mathbb{N} | a+2b+2=N} A_{a,b} (\xi_n)^a (\sum_{i=1}^{n-1} \xi_k \xi_k)^b \xi_j \sum_{i=1}^{n-1} \xi_i \otimes e_i \\ & + \sum_{a,b \in \mathbb{N} | a+2b=N} B_{a,b} (\xi_n)^a (\sum_{i=1}^{n-1} \xi_k \xi_k)^b \otimes e_j \\ & + \sum_{a,b \in \mathbb{N} | a+2b+1=N} C_{a,b} (\xi_n)^a (\sum_{i=1}^{n-1} \xi_k \xi_k)^b \xi_j \otimes e_n . \end{aligned}$$

for some $A_{a,b}, B_{a,b}, C_{a,b} \in \mathbb{C}$ to be determined.

Let us list a few examles of lowest homogeneity ($A, B, C, D, E \in \mathbb{C}$):

1. Homomorphism of homogeneity 0 is induced by singular vector

$$A \otimes e_j$$

2. Homomorphism of homogeneity 1 is induced by singular vector

$$A\xi_n \otimes e_j + B\xi_j \otimes e_n ,$$

3. Homomorphism of homogeneity 2 is induced by singular vector

$$A \sum_{i=1}^{n-1} \xi_i \xi_j \otimes e_i + B\xi_n^2 \otimes e_j + C\xi_n \xi_j \otimes e_n + D \sum_{i=1}^{n-1} \xi_i \xi_i \otimes e_j ,$$

4. Homomorphism of homogeneity 3 is induced by singular vector

$$A\xi_n \sum_{i=1}^{n-1} \xi_i \xi_j \otimes e_i + B\xi_n^3 \otimes e_j + C\xi_n \sum_{i=1}^{n-1} \xi_i^2 \otimes e_j + D\xi_j \sum_{i=1}^{n-1} \xi_i^2 \otimes e_n + E\xi_n^2 \xi_j \otimes e_n .$$

In the next part we convert both the even and odd homogeneity case to a system of three ordinary differential equations encapsulated by upper triangular matrix valued operator with Gegenbauer type differential equation on the diagonal.

Example 9.8 *The singular vector of homogeneity two corresponding to a* $(\mathfrak{g}', P')$-*homomorphism*

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^1(\mathbb{C}^{n-1}) \otimes \mathbb{C}_{\lambda-2}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda)$$

is of the form

$$v_2(\lambda) = A\sum_{i=1}^{n-1} \xi_i\xi_j \otimes e_i + B\xi_n^2 \otimes e_j + C\xi_n\xi_j \otimes e_n + D\sum_{i=1}^{n-1} \xi_i\xi_i \otimes e_j \in \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda)$$

for some $A, B, C, D \in \mathbb{C}$ *to be determined and* $P(t) = A, Q(t) = Dt + B, R(t) = C$ $(deg(P) = 0, deg(Q) = 1, deg(R) = 0)$*. The condition*

$$v_2(\lambda) \in Ker(P_j(\lambda) = \frac{1}{2}\xi_j \Box_\xi + (\lambda - E_\xi)\partial_j - \sum_{k=1}^{n} \partial_k(e_k \otimes e_j^\star - e_j \otimes e_k^\star)),\ j = 1, \ldots, n-1$$

is equivalent to

$$\begin{aligned} &Q + ((2\lambda + n - 3) - t)Q' = 0, \\ &-2Q + 2tQ' + (\lambda - 1)R = 0, \\ &(\lambda + n - 2)P + R + 2Q' = 0 \end{aligned}$$

or to the system of linear equations

$$\begin{aligned} &B + (2\lambda + n - 3)D = 0, \\ &-2B + (\lambda - 1)C = 0, \\ &(\lambda + n - 2)A + C + 2D = 0. \end{aligned}$$

Its solution yields the family of quadratic singular vectors

$$\begin{aligned} v_2(\lambda) = &\sum_{i=1}^{n-1} \xi_i\xi_j \otimes e_i - \frac{1}{2}(\lambda - 1)(n + 2\lambda - 3)\xi_n^2 \otimes e_j \\ &-(n + 2\lambda - 3)\xi_n\xi_j \otimes e_n + \frac{1}{2}(\lambda - 1)\sum_{i=1}^{n-1} \xi_i\xi_i \otimes e_j. \end{aligned} \tag{9.21}$$

9.5 Families of even homogeneity homomorphisms from bulk 1-forms to boundary 1-forms

We construct singular vectors of homogeneity $2N$ corresponding to $(\mathfrak{g}', P')$-homomorphisms

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^1(\mathbb{C}^{n-1}) \otimes \mathbb{C}_{\lambda-2N}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda).$$

Based on the change of variables

$$\partial_l = \frac{2\xi_l}{\xi_n^2}\frac{\partial}{\partial t}\ (l = 1, \ldots, n-1),\ \partial_n = -\frac{2}{\xi_n}t\frac{\partial}{\partial t},$$

the action of

$$P_l(\lambda) = \frac{1}{2}\xi_l \Box_\xi + (\lambda - E_\xi)\partial_l - \sum_{k=1}^{n} \partial_k (e_k \otimes e_l^\star - e_l \otimes e_k^\star)$$

is given on particular summands as follows:

1.
$$v_{2N}^1(\lambda) = \xi_n^{2(N-1)} P(t) \xi_j \sum_{i=1}^{n-1} \xi_i \otimes e_i,$$

of ξ-homogeneity $2N$, where $P(t)$ is of t-homogeneity $(N-1)$ with coefficients $A_{a,b}$.

$$\frac{1}{2}\xi_l \Box_\xi v_{2N}^1(\lambda) = \xi_n^{2(N-1)-2}[(N-1)(2N-3)P + ((n+3) + (-4N+7)t)P'$$
$$+(2t+2t^2)P'']\xi_l\xi_j \sum_{i=1}^{n-1} \xi_i \otimes e_i + \xi_n^{2(N-1)} P \xi_l \otimes e_j,$$
$$(\lambda - E_\xi)\partial_l v_{2N}^1(\lambda) = (\lambda - (2N-1))[\xi_n^{2(N-1)-2} 2\xi_l \xi_j P' \sum_{i=1}^{n-1} \xi_i \otimes e_i$$
$$+\xi_n^{2(N-1)} P \delta_{jl} \sum_{i=1}^{n-1} \xi_i \otimes e_i + \xi_n^{2(N-1)} P \xi_j \otimes e_l],$$
$$\sum_{k=1}^{n} \partial_k (e_k \otimes e_j^\star - e_j \otimes e_k^\star) v_{2N}^1(\lambda) = \xi_n^{2(N-1)-2} 2\xi_j \xi_l P' \sum_{i=1}^{n-1} \xi_i \otimes e_i$$
$$+\xi_n^{2(N-1)-1}(2(N-1)P - 2tP')\xi_j\xi_l \otimes e_n + \xi_n^{2(N-1)} P \xi_l \otimes e_j$$
$$+\xi_n^{2(N-1)}(-2tP' - (n-1)P)\xi_j \otimes e_l,$$

and so

$$P_l(\lambda)(v_{2N}^1(\lambda)) = \xi_n^{2(N-1)-2}[(N-1)(2N-3)P + ((2\lambda + n - 4N + 3)$$
$$+(-4N+7)t)P' + (2t^2+2t)P'']\xi_l\xi_j \sum_{i=1}^{n-1} \xi_i \otimes e_i$$
$$+\xi_n^{2(N-1)}(\lambda - (2N-1))P\delta_{jl} \sum_{i=1}^{n-1} \xi_i \otimes e_i$$
$$+\xi_n^{2(N-1)-1}(-2(N-1)P + 2tP')\xi_j\xi_l \otimes e_n$$
$$+\xi_n^{2(N-1)}((\lambda + n - 2N)P + 2tP')\xi_j \otimes e_l \qquad (9.22)$$

for $l = 1, \dots, n-1$.

2.
$$v_{2N}^2(\lambda) = \xi_n^{2N} Q(t) \otimes e_j,$$

of ξ-homogeneity $2N$, where $Q(t)$ is of t-homogeneity N with coefficients $B_{a,b}$.

$$(\frac{1}{2}\xi_l\Box_\xi + (\lambda - E_\xi)\partial_l)v^2_{2N}(\lambda) = \xi_n^{2(N-1)}[N(2N-1)Q$$
$$+((n+2\lambda-4N+1)+(-4N+3)t)Q' + (2t+2t^2)Q'']\xi_l \otimes e_j,$$
$$\sum_{k=1}^{n}\partial_k(e_k\otimes e_j^\star - e_j\otimes e_k^\star)v^2_{2N}(\lambda) = \xi_n^{2(N-1)}2Q'\delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$+\xi_n^{2N-1}(2NQ-2tQ')\delta_{jl}\otimes e_n - \xi_n^{2(N-1)}2Q'\xi_j\otimes e_l,$$

and so

$$P_l(\lambda)(v^2_{2N}(\lambda)) = \xi_n^{2(N-1)}[N(2N-1)Q + ((2\lambda+n-4N+1)$$
$$+(-4N+3)t)Q' + (2t^2+2t)Q'']\xi_l\otimes e_j - \xi_n^{2(N-1)}2Q'\delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$-\xi_n^{2N-1}(2NQ-2tQ')\delta_{jl}\otimes e_n + \xi_n^{2(N-1)}2Q'\xi_j\otimes e_l$$

for $l = 1, \ldots, n-1$.

3.
$$v^3_{2N}(\lambda) = \xi_n^{2N-1}R(t)\xi_j\otimes e_n,$$

of ξ-homogeneity $2N$, where $R(t)$ is of t-homogeneity $(N-1)$ with coefficients $C_{a,b}$.

$$\frac{1}{2}\xi_l\Box_\xi v^3_{2N}(\lambda) = \xi_n^{2N-3}[(N-1)(2N-1)R + ((n+1)+(-4N+5)t)R'$$
$$+(2t+2t^2)R'']\xi_l\xi_j\otimes e_n,$$
$$(\lambda - E_\xi)\partial_l v^3_{2N}(\lambda) = (\lambda-(2N-1))[\xi_n^{2N-3}2R'\xi_l\xi_j\otimes e_n + \xi_n^{2N-1}R\delta_{jl}\otimes e_n],$$
$$\sum_{k=1}^{n}\partial_k(e_k\otimes e_j^\star - e_j\otimes e_k^\star)v^3_{2N}(\lambda) = -\xi_n^{2N-2}((2N-1)R-2tR')\xi_j\otimes e_l,$$

and so

$$P_l(\lambda)(v^3_{2N}(\lambda)) = \xi_n^{2N-3}[(N-1)(2N-1)R + ((2\lambda+n-4N+3)$$
$$+(-4N+5)t)R' + (2t^2+2t)R'']\xi_l\xi_j\otimes e_n + \xi_n^{2N-1}(\lambda-(2N-1))R\delta_{jl}\otimes e_n$$
$$+\xi_n^{2N-2}((2N-1)R-2tR')\xi_j\otimes e_l$$

for $l = 1, \ldots, n-1$.

Taken together, we have

$$
\begin{aligned}
&P_l(\lambda)(v^1_{2N}(\lambda)+v^2_{2N}(\lambda)+v^3_{2N}(\lambda)) = \xi_n^{2N-4}[(N-1)(2N-3)P \\
&+((2\lambda+n-4N+3)+(-4N+7)t)P'+(2t^2+2t)P'']\xi_l\xi_l\sum_{i=1}^{n-1}\xi_i\otimes e_i \\
&+\xi_n^{2(N-1)}[(\lambda-(2N-1))P-2Q']\delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i \\
&+\xi_n^{2N-3}[(N-1)(2N-1)R+((2\lambda+n-4N+3)+(-4N+5)t)R' \\
&+(2t^2+2t)R''-2(N-1)P+2tP']\xi_l\xi_j\otimes e_n+\xi_n^{2N-2}[(\lambda+n-2N)P \\
&+2tP'+(2N-1)R-2tR'+2Q']\xi_j\otimes e_l+\xi_n^{2N-2}[N(2N-1)Q \\
&+((2\lambda+n-4N+1)+(-4N+3)t)Q'+(2t^2+2t)Q'']\xi_l\otimes e_j \\
&+\xi_n^{2N-1}[-2NQ+2tQ'+(\lambda-(2N-1))R]\delta_{jl}\otimes e_n,
\end{aligned}
$$

and so the system of differential equations

$$P_l(\lambda)(v_{2N}(\lambda))=0,\ l=1,\dots,n-1$$

is equivalent to the system of six ordinary differential equations for P,Q,R representing coefficients of $\xi_l\xi_j\sum_{i=1}^{n-1}\xi_i\otimes e_i, \delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i, \xi_l\xi_j\otimes e_n, \xi_j\otimes e_l, \xi_l\otimes e_j$ and $\delta_{jl}\otimes e_n$:

$$
\begin{aligned}
&(N-1)(2N-3)P+((2\lambda+n-4N+3)+(-4N+7)t)P'+(2t^2+2t)P''=0,\\
&N(2N-1)Q+((2\lambda+n-4N+1)+(-4N+3)t)Q'+(2t^2+2t)Q''=0,\\
&(N-1)(2N-1)R+((2\lambda+n-4N+3)+(-4N+5)t)R'+(2t^2+2t)R''\\
&-2(N-1)P+2tP'=0,\\
&-2NQ+2tQ'+(\lambda-(2N-1))R=0,\\
&(\lambda-(2N-1))P-2Q'=0,\\
&(\lambda+n-2N)P+2tP'+(2N-1)R-2tR'+2Q'=0.
\end{aligned}
\tag{9.23}
$$

The solution of this system solves the extension class of a rank three system corresponding to $(deg(P)=N-1, deg(Q)=N, deg(R)=N-1)$:

$$
\begin{pmatrix}
\lambda-(2N-1) & 0 & -2N+2t\partial_t\\
0 & \lambda-(2N-1) & -2\partial_t\\
0 & 0 & O_1
\end{pmatrix}
\begin{pmatrix} R(t)\\ P(t)\\ Q(t)\end{pmatrix}=0,
\tag{9.24}
$$

where

$$O_1=N(2N-1)+((2\lambda+n-4N+1)+(-4N+3)t)\partial_t+(2t^2+2t)\partial_t^2. \tag{9.25}$$

In what follows we discuss the existence and uniqueness of a solution of (9.23).

Lemma 9.9 *The polynomial solutions* $P(t) = \sum_{j=0}^{N-1} P_j t^j, Q(t) = \sum_{j=0}^{N} Q_j t^j$ *of the first and the second differential equation of the system (9.23) are Gegenbauer polynomials, i.e. the coefficients* $\{P_j\}_j, \{Q_j\}_j$ *fulfill the recursive relations*

$$\begin{aligned} &P_{j-1}(N-j)(2N-2j-1) + P_j(2\lambda + n - 4N + 2j + 1)j = 0, \\ &Q_{j-1}(N-j+1)(2N-2j+1) + Q_j(2\lambda + n - 4N + 2j - 1)j = 0. \end{aligned} \tag{9.26}$$

Proof:

A direct comparison of the coefficients by monomials t^{j-1}, $j = 1, \dots, N+1$ after substitution of $P(t), Q(t)$ into the first and the second equation of the system (9.23) gives the recursive relations

$$\begin{aligned} &P_{j-1}[(N-1)(2N-3) + (-4N+7)(j-1) + 2(j-1)(j-2)] \\ &+P_j[j(2\lambda + n - 4N + 3) + 2j(j-1)] = 0, \\ &Q_{j-1}[N(2N-1) + (-4N+3)(j-1) + 2(j-1)(j-2)] \\ &+Q_j[j(2\lambda + n - 4N + 1) + 2j(j-1)] = 0 \end{aligned} \tag{9.27}$$

and some elementary manipulation yields the required result.

□

Lemma 9.10 *The fifth equation in the system (9.23) gives the mutual normalization of two collections of coefficients* $\{P_j\}_j, \{Q_j\}_j$*:*

$$Q_{j+1} = \frac{(\lambda - (2N-1))}{2(j+1)} P_j, \; j = 0, \dots, N. \tag{9.28}$$

Proof:

It follows from the two formulas in previous Lemma 9.9

$$\begin{aligned} &P_{j-1}(N-j)(2N-2j-1) + P_j(2\lambda + n - 4N + 2j + 1)j = 0, \\ &Q_j(N-j)(2N-2j-1) + Q_{j+1}(2\lambda + n - 4N + 2j + 1)(j+1) = 0 \end{aligned} \tag{9.29}$$

and so

$$\frac{P_{j-1}}{jP_j} = \frac{Q_j}{(j+1)Q_{j+1}}$$

implies the proportionality relation $P_j \sim (j+1)Q_{j+1}$, $j = 0, \dots, N$. The fifth equation in the system (9.23) makes the proportionality explicit and the result follows.

□

Lemma 9.11 *The fourth equation in the system (9.23) yields a unique solution for the polynomial* $R(t) = \sum_{j=1}^{N-1} R_j t^j$ *in the form*

$$Q_j = \frac{(\lambda - (2N-1))}{(2N-2j)} R_j \; j = 0, \dots, N-1 \tag{9.30}$$

or, equivalently, as

$$R_j = \frac{(N-j)}{j} P_{j-1},\ j = 1, \dots, N. \tag{9.31}$$

Proof:
The proof is a straightforward consequence of the fourth equation in the system (9.23).

□

Lemma 9.12 *The third and the sixth equations in the system (9.23) are differential consequences of the remaining equations.*

Proof:
Add the fifth and the sixth equations in the system (9.23):

$$(\lambda + n - 2N)P + 2tP' + (2N-1)R - 2tR' + (\lambda - (2N-1))P = 0.$$

Substituting the result of the previous Lemma and comparing the contribution by monomials yields

$$j(2\lambda + n - 4N + 2j + 1)P_j + (2N - 2j - 1)(N - j)P_{j-1} = 0,\ j = 0, \dots, N,$$

which is the defining recursive property of the collection $\{P_j\}_j$. This completes the claim that the sixth equation is an algebraic consequence of the remaining (except the third) equations in the system (9.23).

As for the sixth equation, the proof is more complicated. Let us differentiate by ∂_t and then multiply by t the equality

$$(\lambda + n - 2N)P + 2tP' + (2N-1)R - 2tR' + (\lambda - (2N-1))P = 0$$

to get

$$\begin{aligned}&(2\lambda + n - 4N + 3)P' + 2tP'' + (2N-3)R' - 2tR'' = 0,\\ &(2\lambda + n - 4N + 3)tP' + 2t^2P'' + (2N-3)tR' - 2t^2R'' = 0.\end{aligned}$$

Their sum is

$$\begin{aligned}&(2\lambda + n - 4N + 3)P' + (2\lambda + n - 4N + 3)tP' + (2t + 2t^2)P''\\ &+(2N-3)R' + (2N-3)tR' - (2t + 2t^2)R'' = 0\end{aligned}$$

and adding the first and the third equation gives

$$\begin{aligned}&-(N-1)(2N-1)P + (2\lambda + n - 2)tP' + (N-1)(2N-1)R\\ &+(2\lambda + n - 2N)R' + j(-2N+2)tR' = 0.\end{aligned} \tag{9.32}$$

Substituting for the polynomials P, R and using (9.31), this coverts after some manipulation into

$$(n + 2\lambda - 4N + 2j + 1)P_j + (2N - 2j - 1)R_j = 0.$$

The last substitution of

$$Q_{j+1} = \frac{(\lambda - (2N-1))}{2(j+1)} P_j,\ R_j = \frac{2N-2j}{(\lambda - (2N-1))} Q_j \tag{9.33}$$

leads to

$$(j+1)(n+2\lambda-4N+2j+1)Q_{j+1} + (2N-2j-1)(N-j)Q_j = 0,$$

which are the defining recursive relations for the coefficients of $Q(t) = \sum_{j=0}^{N} Q_j t^j$. This completes the proof that the third equation is a differential consequence of the remaining ones.

□

9.6 Families of odd homogeneity homomorphisms from bulk 1-forms to boundary 1-forms

We construct singular vectors of homogeneity $2N+1$ corresponding to $(\mathfrak{g}', P')$-homomorphisms

$$\mathcal{U}(\mathfrak{g}') \otimes_{\mathcal{U}(\mathfrak{p}')} (\Lambda^1(\mathbb{C}^{n-1}) \otimes \mathbb{C}_{\lambda-(2N+1)}) \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p})} (\Lambda^1(\mathbb{C}^n) \otimes \mathbb{C}_\lambda).$$

The operator

$$P_l(\lambda) = \frac{1}{2}\xi_l \Box_\xi + (\lambda - E_\xi)\partial_l - \sum_{k=1}^{n} \partial_k (e_k \otimes e_l^\star - e_l \otimes e_k^\star)$$

acts on particular summands of odd homogeneity $2N+1$ as follows:

1.
$$v^1_{2N+1}(\lambda) = \xi_n^{2N-1} P(t) \xi_j \sum_{i=1}^{n-1} \xi_i \otimes e_i,$$

of ξ-homogeneity $2N+1$, where $P(t)$ is of t-homogeneity $(N-1)$ with coefficients $A_{a,b}$.

$$\frac{1}{2}\xi_l \Box_\xi v^1_{2N+1}(\lambda) = \xi_n^{2N-3}[(N-1)(2N-1)P + ((n+3) + (-4N+5)t)P'$$
$$+(2t+2t^2)P'']\xi_l\xi_j \sum_{i=1}^{n-1} \xi_i \otimes e_i + \xi_n^{2N-1} P \xi_l \otimes e_j,$$

$$(\lambda - E_\xi)\partial_l v^1_{2N+1}(\lambda) = (\lambda - 2N)[\xi_n^{2N-3} 2\xi_l\xi_j P' \sum_{i=1}^{n-1} \xi_i \otimes e_i$$
$$+\xi_n^{2N-1} P \delta_{jl} \sum_{i=1}^{n-1} \xi_i \otimes e_i + \xi_n^{2N-1} P \xi_j \otimes e_l],$$

$$\sum_{k=1}^{n} \partial_k (e_k \otimes e_j^\star - e_j \otimes e_k^\star) v^1_{2N+1}(\lambda) = \xi_n^{2N-3} 2\xi_j\xi_l P' \sum_{i=1}^{n-1} \xi_i \otimes e_i$$
$$+\xi_n^{2N-2}((2N-1)P - 2tP')\xi_j\xi_l \otimes e_n + \xi_n^{2N-1} P \xi_l \otimes e_j$$
$$+\xi_n^{2N-1}(-2tP' - (n-1)P)\xi_j \otimes e_l$$

and so

$$P_l(\lambda)(v^1_{2N+1}(\lambda)) = \xi_n^{2N-3}[(N-1)(2N-1)P + ((2\lambda+n-4N+1)$$
$$+(-4N+5)t)P' + (2t^2+2t)P'']\xi_l\xi_j\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$+\xi_n^{2N-1}(\lambda-2N)P\delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i + \xi_n^{2N-2}(-(2N-1)P+2tP')\xi_j\xi_l\otimes e_n$$
$$+\xi_n^{2N-1}((\lambda+n-2N-1)P+2tP')\xi_j\otimes e_l \tag{9.34}$$

for $l = 1, \dots, n-1$.

2.

$$v^2_{2N+1}(\lambda) = \xi_n^{2N+1}Q(t)\otimes e_j,$$

of ξ-homogeneity $2N+1$, where $Q(t)$ is of t-homogeneity N with coefficients $B_{a,b}$.

$$(\frac{1}{2}\xi_l\Box_\xi + (\lambda - E_\xi)\partial_l)v^2_{2N+1}(\lambda) = \xi_n^{2N-1}[N(2N+1)Q$$
$$+((n+2\lambda-4N-1)+(-4N+1)t)Q' + (2t+2t^2)Q'']\xi_l\otimes e_j,$$
$$\sum_{k=1}^{n}\partial_k(e_k\otimes e_j^\star - e_j\otimes e_k^\star)v^2_{2N+1}(\lambda) = \xi_n^{2N-1}2Q'\delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$+\xi_n^{2N}((2N+1)Q-2tQ')\delta_{jl}\otimes e_n - \xi_n^{2N-1}2Q'\xi_j\otimes e_l$$

and so

$$P_l(\lambda)(v^2_{2N+1}(\lambda)) = \xi_n^{2N-1}[N(2N+1)Q + ((2\lambda+n-4N-1)$$
$$+(-4N+1)t)Q' + (2t^2+2t)Q'']\xi_l\otimes e_j - \xi_n^{2N-1}2Q'\delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$-\xi_n^{2N}((2N+1)Q-2tQ')\delta_{jl}\otimes e_n + \xi_n^{2N-1}2Q'\xi_j\otimes e_l$$

for $l = 1, \dots, n-1$.

3.

$$v^3_{2N+1}(\lambda) = \xi_n^{2N}R(t)\xi_j\otimes e_n,$$

of ξ-homogeneity $2N+1$, where $R(t)$ is of t-homogeneity N with coefficients $C_{a,b}$.

$$\frac{1}{2}\xi_l\Box_\xi v^3_{2N+1}(\lambda) = \xi_n^{2N-2}[N(2N-1)R + ((n+1)+(-4N+3)t)R'$$
$$+(2t+2t^2)R'']\xi_l\xi_j\otimes e_n,$$
$$(\lambda-E_\xi)\partial_l v^3_{2N+1}(\lambda) = (\lambda-2N)[\xi_n^{2N-2}2R'\xi_l\xi_j\otimes e_n + \xi_n^{2N}R\delta_{jl}\otimes e_n],$$
$$\sum_{k=1}^{n}\partial_k(e_k\otimes e_j^\star - e_j\otimes e_k^\star)v^3_{2N+1}(\lambda) = -\xi_n^{2N-1}(2NR-2tR')\xi_j\otimes e_l$$

and so

$$P_l(\lambda)(v^3_{2N+1}(\lambda)) = \xi_n^{2N-2}[N(2N-1)R + ((2\lambda+n-4N+1)$$
$$+(-4N+3)t)R' + (2t^2+2t)R'']\xi_l\xi_j \otimes e_n + \xi_n^{2N}(\lambda-2N)R\delta_{jl}\otimes e_n$$
$$+\xi_n^{2N-1}(2NR-2tR')\xi_j\otimes e_l$$

for $l = 1, \ldots, n-1$.

Taken together, we have

$$P_l(\lambda)(v^1_{2N+1}(\lambda) + v^2_{2N+1}(\lambda) + v^3_{2N+1}(\lambda)) = \xi_n^{2N-3}[(N-1)(2N-1)P$$
$$+((2\lambda+n-4N+1)+(-4N+5)t)P' + (2t^2+2t)P'']\xi_l\xi_l\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$+\xi_n^{2N-1}[(\lambda-2N)P-2Q']\delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i$$
$$+\xi_n^{2N-2}[N(2N-1)R + ((2\lambda+n-4N+1)+(-4N+3)t)R'$$
$$+(2t^2+2t)R'' - (2N-1)P + 2tP']\xi_l\xi_j\otimes e_n + \xi_n^{2N-1}[(\lambda+n-2N-1)P$$
$$+2tP' + 2NR - 2tR' + 2Q']\xi_j\otimes e_l + \xi_n^{2N-1}[N(2N+1)Q$$
$$+((2\lambda+n-4N-1)+(-4N+1)t)Q' + (2t^2+2t)Q'']\xi_l\otimes e_j$$
$$+\xi_n^{2N}[-(2N+1)Q + 2tQ' + (\lambda-2N)R]\delta_{jl}\otimes e_n,$$

and so the system of differential equations

$$P_l(\lambda)(v_{2N+1}(\lambda)) = 0,\ l = 1, \ldots, n-1$$

is equivalent to the system of six ordinary differential equations for P, Q, R representing coefficients of $\xi_l\xi_j\sum_{i=1}^{n-1}\xi_i\otimes e_i, \delta_{jl}\sum_{i=1}^{n-1}\xi_i\otimes e_i, \xi_l\xi_j\otimes e_n, \xi_j\otimes e_l, \xi_l\otimes e_j$ and $\delta_{jl}\otimes e_n$:

$$(N-1)(2N-1)P + ((2\lambda+n-4N+1)+(-4N+5)t)P' + (2t^2+2t)P'' = 0,$$
$$N(2N+1)Q + ((2\lambda+n-4N-1)+(-4N+1)t)Q' + (2t^2+2t)Q'' = 0,$$
$$N(2N-1)R + ((2\lambda+n-4N+1)+(-4N+3)t)R' + (2t^2+2t)R''$$
$$-(2N-1)P + 2tP' = 0,$$
$$-(2N+1)Q + 2tQ' + (\lambda-2N)R = 0,$$
$$(\lambda-2N)P - 2Q' = 0,$$
$$(\lambda+n-2N-1)P + 2tP' + 2NR - 2tR' + 2Q' = 0. \tag{9.35}$$

The solution of this system solves the extension class of a rank three system corresponding to $(deg(P) = N-1, deg(Q) = N, deg(R) = N)$:

$$\begin{pmatrix} \lambda-2N & 0 & -(2N+1)+2t\partial_t \\ 0 & \lambda-2N & -2\partial_t \\ 0 & 0 & O_1 \end{pmatrix}\begin{pmatrix} R(t) \\ P(t) \\ Q(t) \end{pmatrix} = 0, \tag{9.36}$$

where

$$O_1 = N(2N+1) + ((2\lambda + n - 4N - 1) + (-4N+1)t)\partial_t + (2t^2 + 2t)\partial_t^2. \tag{9.37}$$

In what follows we discuss the existence and uniqueness of solutions of the system (9.23).

Lemma 9.13 *The polynomial solutions $P(t) = \sum_{j=0}^{N-1} P_j t^j, Q(t) = \sum_{j=0}^{N} Q_j t^j$ of the first and the second differential equation of the system (9.23) are Gegenbauer polynomials, i.e. the coefficients $\{P_j\}_j, \{Q_j\}_j$ fulfill the recursive relations*

$$\begin{aligned} &P_{j-1}(N-j)(2N-2j+1) + P_j(2\lambda + n - 4N + 2j - 1)j = 0, \\ &Q_{j-1}(N-j+1)(2N-2j+3) + Q_j(2\lambda + n - 4N + 2j - 3)j = 0. \end{aligned} \tag{9.38}$$

Proof:
A direct comparison of the coefficients by monomials t^{j-1}, $j = 1, \dots, N+1$ after the substitution of $P(t), Q(t)$ into the first and the second equation of the system (9.23) gives the recursive relations

$$\begin{aligned} &P_{j-1}[(N-1)(2N-1) + (-4N+5)(j-1) + 2(j-1)(j-2)] \\ &+P_j[j(2\lambda + n - 4N + 1) + 2j(j-1)] = 0, \\ &Q_{j-1}[N(2N+1) + (-4N+1)(j-1) + 2(j-1)(j-2)] \\ &+Q_j[j(2\lambda + n - 4N - 1) + 2j(j-1)] = 0 \end{aligned}$$

and an elementary manipulation yields the required result.

□

Lemma 9.14 *The fifth equation in the system (9.23) gives the mutual normalization of two collections of coefficients $\{P_j\}_j, \{Q_j\}_j$:*

$$Q_{j+1} = \frac{(\lambda - 2N)}{2(j+1)} P_j, \; j = 0, \dots, N. \tag{9.39}$$

Proof:
It follows from the two formulas in Lemma 9.9

$$\begin{aligned} &P_{j-1}(N-j)(2N-2j+1) + P_j(2\lambda + n - 4N + 2j - 1)j = 0, \\ &Q_j(N-j)(2N-2j+1) + Q_{j+1}(2\lambda + n - 4N + 2j - 1)(j+1) = 0 \end{aligned} \tag{9.40}$$

and so

$$\frac{P_{j-1}}{jP_j} = \frac{Q_j}{(j+1)Q_{j+1}}$$

implies the proportionality relation $P_j \sim (j+1)Q_{j+1}$, $j = 0, \dots, N$. The fifth equation in the system (9.23) makes the proportionality explicit and the result follows.

□

Lemma 9.15 *The fourth equation in the system (9.23) yields a unique solution for the polynomial* $R(t) = \sum_{j=1}^{N} R_j t^j$ *in the form*

$$Q_j = \frac{(\lambda - 2N)}{(2N - 2j + 1)} R_j, \; j = 0, \ldots, N \tag{9.41}$$

or, equivalently, as

$$R_j = \frac{(2N - 2j + 1)}{2j} P_{j-1}, \; j = 1, \ldots, N. \tag{9.42}$$

Proof:
The proof is a straightforward consequence of the fourth equation in the system (9.23).

□

Lemma 9.16 *The third and the sixth equations in the system (9.23) are differential consequences of the remaining equations.*

Proof:
Add the fifth and the sixth equations in the system (9.23):

$$(\lambda + n - 2N - 1)P + 2tP' + 2NR - 2tR' + (\lambda - 2N)P = 0.$$

Substituting the result of the previous Lemma and comparing the contributions by monomials yields

$$j(2\lambda + n - 4N + 2j - 1)P_j + (2N - 2j + 1)(N - j)P_{j-1} = 0, \; j = 0, \ldots, N,$$

which is the defining recursive property of the collection $\{P_j\}_j$. This completes the claim that the sixth equation is an algebraic consequence of the remaining (except the third) equations in the system (9.23).

As for the sixth equation, the proof is more complicated. Let us differentiate by ∂_t and then multiply by t the equality

$$(\lambda + n - 2N - 1)P + 2tP' + 2NR - 2tR' + (\lambda - 2N)P = 0$$

to get

$$\begin{aligned} &(2\lambda + n - 4N + 1)P' + 2tP'' + (2N - 2)R' - 2tR'' = 0, \\ &(2\lambda + n - 4N + 1)tP' + 2t^2P'' + (2N - 2)tR' - 2t^2R'' = 0. \end{aligned}$$

Their sum is

$$\begin{aligned} &(2\lambda + n - 4N + 1)P' + (2\lambda + n - 4N + 1)tP' + (2t + 2t^2)P'' \\ &+(2N - 2)R' + (2N - 2)tR' - (2t + 2t^2)R'' = 0 \end{aligned}$$

and adding the first and the third equation gives

$$-N(2N-1)P+(2\lambda+n-2)tP'+N(2N-1)R+(2\lambda+n-2N-1)R'+j(-2N+1)tR' = 0.$$

Substituting for the polynomials P, R and using (9.31), this coverts after some manipulation into

$$(n + 2\lambda - 4N + 2j - 1)P_j + (2N - 2j)R_j = 0.$$

The last substitution of

$$Q_{j+1} = \frac{(\lambda - 2N)}{2(j+1)}P_j,\ R_j = \frac{2N - 2j + 1}{(\lambda - 2N)}Q_j \tag{9.43}$$

leads to

$$(j + 1)(n + 2\lambda - 4N + 2j - 1)Q_{j+1} + (2N - 2j + 1)(N - j)Q_j = 0, \tag{9.44}$$

which are the defining recursive relations for the coefficients of $Q(t) = \sum_{j=0}^{N} Q_j t^j$. This completes the proof that the third equation is a differential consequence of the remaining ones.

□

9.7 The realization of singular vectors by invariant calculus of tractors in parabolic geometries

In the present subsection we comment on the relationship between the F-method and the invariant calculus of tractors for parabolic geometries, see [13] and the references therein. We remark that both conformal geometries (i.e., the conformal geometry and its codimension one conformal subgeometry) are examples of parabolic geometries. The tractor approach relies on the iteration of a basic invariant differential operator together with branching rules for finite dimensional (tractor) representations, while the F-method is based on analytic tool of solving the system of differential equations in the Fourier dual picture.

The normal vector field $N^a \in \mathcal{E}^a$ ($N_a N^a = 1$) to an embedded (conformal or Riemannian) submanifold $i : \Sigma \hookrightarrow M$ allows to decompose $\omega_a \in \mathcal{E}_a$ as

$$\omega_a \xrightarrow{\sim} (\omega_{a'}, \omega_n),$$

where the first component is in the image of the projector

$$\Pi_a^b = \delta_a^b - N_a N^b : \mathcal{E}_b \to \mathcal{E}_a$$

and the second component is identified with the kernel of Π_a^b. We will use the notation $i^\star$ for the composition of restriction to Σ followed by the projection Π_a^b.

The embedding of conformal manifolds $i : (\Sigma, i^\star[g]) \hookrightarrow (M, [g])$ induces reduction of tractor bundles

$$i^\star(\mathcal{E}^A) = \mathcal{E}^A|_{(\Sigma, i^\star[g])} \xrightarrow{\sim} \mathcal{E}^A_\Sigma \oplus N^A,$$

where N^A is the normal tractor bundle whose sections are denoted by

$$\begin{pmatrix} 0 \\ N^a \\ -H \end{pmatrix} \in \mathcal{E}(N^A),\ N^a \in \mathcal{E}^a(M)[1],\ H \in \mathcal{E}(M)[-1].$$

Conformally invariant projection of tractor bundle

$$\Pi_B^A = (\delta_B^A - N^A N_B) : i^\star(\mathcal{E}^A) \to Ker(N^A \lrcorner)$$

induces an isomorphism

$$\Pi_B^A : \mathcal{E}^A_\Sigma \xrightarrow{\sim} Ker(N^A \lrcorner),$$
$$\begin{pmatrix} \sigma \\ \omega_a \\ \varrho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \Pi_a^b \omega_b + H N_a \sigma \\ \varrho - \frac{1}{2} H^2 \sigma \end{pmatrix}.$$

In the previous display we denoted by H the mean curvature of the embedded submanifold. The inverse isomorphism is given by

$$V^A := \begin{pmatrix} \sigma \\ \omega_a \\ \varrho \end{pmatrix}, N_A \lrcorner V^A = 0 \mapsto \tilde{V}^A := \begin{pmatrix} \sigma \\ \Pi_a^b \omega_b \\ \varrho + \frac{1}{2} H^2 \sigma \end{pmatrix} \in \mathcal{E}^A_\Sigma. \tag{9.45}$$

The injectors of irreducible subquotients of a bundle on M are denoted X^A, Z^A_a, Y^A and for a bundle on Σ by $\tilde{X}^A, \tilde{Z}^A_a, \tilde{Y}^A$. When restricted from M on Σ, the injectors are related

$$\begin{aligned}
\tilde{Y}^A &= Y^A + Z^A_a N^a H - \frac{1}{2}H^2 X^A,\\
\tilde{Z}^A_a &= \Pi^b_a Z^A_b,\\
\tilde{X}^A &= X^A,
\end{aligned}$$

and an invariant isomorphism $\mathcal{E}^A_\Sigma \simeq Ker(N^A \lrcorner) \xrightarrow{\sim} \mathcal{E}^{A'}$ is given by

$$\begin{pmatrix} \sigma \\ \omega_a \\ \varrho \end{pmatrix} \mapsto \begin{pmatrix} \sigma \\ \Pi^b_a \omega_b \\ \varrho + H\omega_a N^a - \frac{1}{2}H^2\sigma \end{pmatrix}. \tag{9.46}$$

It is easy to verify that the operator (in dimension n)

$$\begin{aligned}
E^{bA} &: \mathcal{E}_b[w] \to \mathcal{E}^A[w-1],\\
\omega_b &\mapsto \begin{pmatrix} 0 \\ (n+w-2)\omega^a \\ -\nabla^c\omega_c \end{pmatrix}
\end{aligned}$$

is an invariant spitting operator. We define a family of first order conformally invariant operators Σ as $\tilde{D}_B \Pi^B_A E^{bA}$ acting on 1-forms on M and valued in densities on Σ. We have for $\omega_b \in \mathcal{E}_b[w]$

$$\begin{aligned}
&\tilde{D}_B \Pi^B_A E^{bA}\omega_b = \tilde{D}_B \begin{pmatrix} 0 \\ (n+w-2)\Pi^b_a\omega_b \\ -\nabla^b\omega_b + (n+w-2)H\omega_a N^a \end{pmatrix}\\
&= (n+2(w-1)-1)\tilde{\nabla}^a\Pi^b_a\omega_b + (n+2(w-1)-1)(n+w-3)(-\nabla^b\omega_b\\
&+(n+w-2)HN^b\omega_b) = (n+2w-3)[(n+w-2)\tilde{\nabla}^a\Pi^b_a\omega_b\\
&+(n+w-3)(-\nabla^b\omega_b + (n+w-2)HN^b\omega_b)] = (n+2w-3)[\delta^a\omega_a\\
&-(n+w-3)\delta_n\omega_n + (n+w-2)(n+w-3)HN^b\omega_b].
\end{aligned} \tag{9.47}$$

Let us emphasize that the symbol of this operator is, up to an overall factor $(n+2w-3)$, a linear combination of tangent codifferential of the cotangent part ω_a and normal codifferential of the normal component ω_n.

The first order operator $\tilde{D}_B \Pi^B_A E^{bA}$ as well as the zero order operator given by projection on ω_n, can be translated by fundamental tractor D-operator D^A and its dual tractor operator D_A to even and odd order operators, respectively, acting on 1-forms on M.

The tractor D-operator

$$D^B : \mathcal{E}_a(M)[w] \longrightarrow \mathcal{E}^B_a(M)[w-1]$$

is given by

$$\omega_a \mapsto \begin{pmatrix} \star \\ (n+2w-4)(n+w-2)(w-1)\nabla^b\omega_a + (n+2w-4)(n+w-2)\nabla_a\omega^b \\ -(n+2w-4)w\delta_a^b\nabla^c\omega_c \\ \star \end{pmatrix},$$

where $\star$-symbol indicates for our purposes irrelevant components of the tractor. It follows from the normalization of N^a that $i^\star\nabla_N\omega_a = \nabla_N\omega_{a'}$, and this together with $i^\star N_a = 0$ implies that

$$i^\star N_B D^B : \mathcal{E}_a(M)[w] \to \mathcal{E}_a^B(M)[w-1] \to \mathcal{E}_{a'}(\Sigma)[w-1]$$

is the first order invariant operator from bulk 1-forms to boundary 1-forms, given in the flat case by

$$\begin{aligned} & i^\star[(n+2w-4)(n+w-2)(w-1)N_b\nabla^b\omega_a \\ & +(n+2w-4)(n+w-2)\nabla_a(N_b\omega^b) - (n+2w-4)wN_a\nabla^c\omega_c] \\ & = (n+2w-4)(n+w-2)[(w-1)N_b\nabla^b\omega_{a'} + \nabla_{a'}\omega_n]. \end{aligned} \tag{9.48}$$

In particular, the operator is the linear combination of the normal derivative of tangent part of the bulk form ω_a and tangent derivatives of its normal part. The zero order operator given by projection on the normal part of the bulk 1-form and the first order operator $i^\star N_B D^B$ can be translated by fundamental tractor D-operator D^A and its dual tractor operator D_A to even and odd order operators, respectively, acting on bulk 1-forms.

10 F-method for diagonal branching problem and Rankin-Cohen brackets for orthogonal Lie algebras

The bilinear invariant differential operators, as the simplest representatives of multilinear invariant differential operators organized in an A_∞-homotopy structure, appear in a wide range of Lie theoretic applications. For example, the classical Rankin-Cohen brackets realized by holomorphic $SL(2,\mathbb{R})$-invariant bilinear differential operators on the upper half plane $\mathbb{H}$ are devised, originally in a number theoretic context, to produce from a given pair of modular forms another modular form. They turn out to be intertwining operators producing ring structure on $SL(2,\mathbb{R})$ holomorphic discrete series representations, and can be analytically continued to the full range of inducing characters. Consequently, such operators were constructed in several specific situations of interest related to Jacobi forms, Siegel modular forms, holomorphic discrete series of causal symmetric spaces of Cayley type, etc., [28], [18], [52].

The main reason behind the underlying classification scheme for such class of operators is inspired by geometrical analysis on manifolds with, e.g., the conformal structure, and related PDE problems of geometrical origin. For $\tilde{M}$ a smooth (or, complex) manifold equipped with a filtration of its tangent bundle $T\tilde{M}$, $\mathcal{V}$ a smooth (or holomorphic) vector bundle on $\tilde{M}$ and $J^k\mathcal{V}$ the weighted jet bundle, a bilinear differential pairing between sections of the bundle $\mathcal{V}$ and sections of the bundle $\mathcal{W}$ to sections of a bundle $\mathcal{Y}$ is a sheaf homomorphism

$$B : J^k\mathcal{V} \otimes J^l\mathcal{W} \to \mathcal{Y}.$$

In the case $\tilde{M} = \tilde{G}/\tilde{P}$ is a generalized flag manifold, a pairing is called invariant if it commutes with the action of $\tilde{G}$ on sections of the homogeneous vector bundles $\mathcal{V}, \mathcal{W}, \mathcal{Y}$. Denoting $\mathbb{V}$, $\mathbb{W}$, $\mathbb{Y}$ the inducing $\tilde{P}$-representations of homogeneous vector bundles $\mathcal{V}, \mathcal{W}, \mathcal{Y}$, $\tilde{G}$-invariant differential pairings can be algebraically characterized as the space

$$Hom_{\mathcal{U}(\tilde{\mathfrak{g}})}(\mathcal{M}(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}}, \mathbb{Y}), (\mathcal{U}(\tilde{\mathfrak{g}}) \otimes \mathcal{U}(\tilde{\mathfrak{g}})) \otimes_{\mathcal{U}(\tilde{\mathfrak{p}}) \otimes \mathcal{U}(\tilde{\mathfrak{p}})} (\mathbb{V}^\vee \otimes \mathbb{W}^\vee)). \tag{10.1}$$

In other words, the former geometrical problem for finding bilinear invariant differential operators on $\tilde{G}/\tilde{P}$ acting on induced representations turns into a Lie algebraic problem of the characterization of homomorphisms of generalized Verma modules. In the geometrical context of flag manifolds and general curved manifolds with parabolic structure, a classification of first order bilinear differential operators for parabolic subalgebras with commutative nilradicals (so called AHS structures) was completed in [48]. One of the main applications of bilinear differential operators is that they act via invariant cup product as symmetries of invariant differential operators, see e.g. [20] for the case of conformally invariant Laplace operator.

Yet another approach to these questions is purely analytical and consists of meromorphic continuation of invariant distributions given by a multilinear form on the principal series representations. For example, a class of $G = SO_0(n+1,1,\mathbb{R})$ (i.e., conformally)-invariant linear and bilinear differential operators was constructed as residues of meromorphically continued invariant trilinear form on principal series representations induced from characters, see [6].

To summarize, the present section contains a general Lie theoretic classification of Rankin-Cohen-like brackets for the couple of real orthogonal Lie algebra $so(n+1,1,\mathbb{R})$ and its conformal parabolic Lie subalgebra, and their explicit - in the sense of dependence on representation theoretical parameters - construction for characters as inducing representations.

The structure of present section goes as follows. As already mentioned, we first reformulate the existence of invariant bilinear differential operators (or equivalently, Rankin-Cohen-like brackets) in terms of purely abstract Lie theoretic classification scheme for diagonal branching rules of generalized Verma modules, associated to the real orthogonal Lie algebra $so(n+1,1,\mathbb{R})$ and its conformal parabolic Lie subalgebra $\mathfrak{p}$. The reason behind the choice for this parabolic subalgebra is its fundamental property of having the commutative nilradical. Consequently, the branching problem takes value in the Grothendieck group $K(\mathcal{O}^{\mathfrak{p}})$ of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$. Here the main device are character formulas and their reduction in the branching problems. The quantitative part of the problem consists of the construction of singular vectors. It is based on the procedure of rewriting the representation theoretical action in the Fourier dual picture, where the positive nilradical of $\mathfrak{p}$ is acting on symmetric algebra of the (commutative) opposite nilradical. This action produces the four term functional equation for singular vectors, and its solution is technically the most difficult part with both analytic and combinatorial aspects arising from generalized hypergeometric equation. The last subsection determines the explicit formulas for bilinear conformally invariant differential operators representing these singular vectors.

10.1 Abstract characterization of diagonal branching rules applied to generalized Verma modules for $so(n+1,1,\mathbb{R})$

The present subsection contains qualitative results on the diagonal branching rules for $so(n+1,1,\mathbb{R})$ applied to generalized Verma modules.

Let $n \in \mathbb{N}$ such that $n \geq 3$. Throughout the section $\mathfrak{g}$ denotes the real Lie algebra $so(n+1,1,\mathbb{R})$ of the connected and simply connected simple Lie group $G = SO_o(n+1,1,\mathbb{R})$. Let $\mathfrak{p}$ be its maximal parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}$, in the Dynkin diagrammatic notation for parabolic subalgebras given by omitting the first simple root of $\mathfrak{g}$. The Levi factor $\mathfrak{l}$ of $\mathfrak{p}$ is isomorphic to $so(n,\mathbb{R})\times\mathbb{R}$ and the commutative nilradical $\mathfrak{n}$ (resp. the opposite nilradical $\mathfrak{n}_-$) is isomorphic to $\mathbb{R}^n$. Let $diag : (\mathfrak{g},\mathfrak{p}) \hookrightarrow (\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$ denote the diagonal embedding.

The main task of the present section concerns the branching problem for the family of scalar generalized Verma $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$-modules induced from characters of the center of $\mathfrak{l}\oplus\mathfrak{l}$, with respect to $diag(\mathfrak{g},\mathfrak{p})$. An inducing character

$\chi_{\lambda,\mu}$ on $\mathfrak{p}\oplus\mathfrak{p}$ is determined by two complex characters χ_μ,χ_λ on $\mathfrak{l}$:

$$\begin{aligned}\chi_{\lambda,\mu}\equiv(\chi_\lambda,\chi_\mu)\quad:\quad & \mathfrak{p}\oplus\mathfrak{p}\to\mathbb{C}\\ & (p_1,p_2)\mapsto\chi_\lambda(p_1)\cdot\chi_\mu(p_2),\end{aligned}\tag{10.2}$$

where the homomorphism (χ_λ,χ_μ) quotients through the semisimple subalgebra $[\mathfrak{p},\mathfrak{p}]\oplus[\mathfrak{p},\mathfrak{p}]$. The generalized Verma $(\mathfrak{g}\oplus\mathfrak{g},\mathfrak{p}\oplus\mathfrak{p})$-module induced from character (χ_λ,χ_μ) $(\lambda,\mu\in\mathbb{C})$ is

$$M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_{\lambda,\mu})=\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})}(\mathbb{C}_\lambda\otimes\mathbb{C}_\mu),\tag{10.3}$$

where $\mathbb{C}_\lambda\otimes\mathbb{C}_\mu$ is a 1-dimensional representation (χ_λ,χ_μ) of $\mathfrak{p}\oplus\mathfrak{p}$. As a vector space, $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_{\lambda,\mu})$ is isomorphic to the symmetric algebra $S^\star(\mathfrak{n}_-\oplus\mathfrak{n}_-)$, where $\mathfrak{n}_-\oplus\mathfrak{n}_-$ is the vector complement of $\mathfrak{p}\oplus\mathfrak{p}$ in $\mathfrak{g}\oplus\mathfrak{g}$.

A way to resolve this branching problem abstractly is based on character identities for the restriction of $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_{\lambda,\mu})$ to the diagonal subalgebra $diag(\mathfrak{g})$ with standard compatible parabolic subalgebra

$$diag(\mathfrak{p}):=diag(\mathfrak{g})\cap(\mathfrak{p}\oplus\mathfrak{p}),\ \mathfrak{p}=\mathfrak{l}\ltimes\mathfrak{n},\ diag(\mathfrak{p})=diag(\mathfrak{l})\ltimes diag(\mathfrak{n}).$$

For a fixed choice of positive simple roots of $\mathfrak{g}$ we denote by $\Lambda^+(\mathfrak{l}\oplus\mathfrak{l})$ the set of weights dominant for $\mathfrak{l}\oplus\mathfrak{l}$ and integral for $[\mathfrak{l},\mathfrak{l}]\oplus[\mathfrak{l},\mathfrak{l}]$. Let $\mathbb{V}_{\lambda,\mu}$ be a finite dimensional irreducible $\mathfrak{l}\oplus\mathfrak{l}$-module with highest weight $(\lambda,\mu)\in\Lambda^+(\mathfrak{l}\oplus\mathfrak{l})$, and likewise $\mathbb{V}_{\lambda'}$ be a finite dimensional representation of $diag(\mathfrak{l})$, $\lambda'\in\Lambda^+(diag(\mathfrak{l}))$. Given a vector space $\mathbb{V}$ we denote $S^\star(\mathbb{V})=\oplus_{l=0}^\infty S_l(\mathbb{V})$ the symmetric tensor algebra on $\mathbb{V}$. Let us extend the adjoint action of $diag(\mathfrak{l})$ on $(\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))$ to $S^\star\big((\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\big)$. Notice that we have an isomorphism

$$(\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\simeq\mathfrak{n}_-$$

of $diag(\mathfrak{l})$-quotient modules. We set

$$\begin{aligned}& m\big(\lambda',(\lambda,\mu)\big):=\\ & Hom_{diag(\mathfrak{l})}\Big(\mathbb{V}_{\lambda'},\mathbb{V}_{\lambda,\mu}|_{diag(\mathfrak{l})}\otimes S^\star\big((\mathfrak{n}_-\oplus\mathfrak{n}_-)/((\mathfrak{n}_-\oplus\mathfrak{n}_-)\cap diag(\mathfrak{g}))\big)\Big).\end{aligned}\tag{10.4}$$

Let us recall

Theorem 10.1 *([41], Theorem 3.9) Let $(\tilde{\mathfrak{g}},\tilde{\mathfrak{p}})$, $(\tilde{\mathfrak{g}}',\tilde{\mathfrak{p}}')$ be a compatible couple of simple Lie algebras and their parabolic subalgebras, and suppose $\tilde{\mathfrak{p}}$ is $\tilde{\mathfrak{g}}'$-compatible standard parabolic subalgebra of $\tilde{\mathfrak{g}}$, $(\lambda,\mu)\in\Lambda^+(\tilde{\mathfrak{l}}\oplus\tilde{\mathfrak{l}})$. Then*

1. *$m(\lambda',(\lambda,\mu))<\infty$ for all $\lambda'\in\Lambda^+(\tilde{\mathfrak{l}}')$.*
2. *In the Grothendieck group of Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\tilde{\mathfrak{p}}'}$ there is $\tilde{\mathfrak{g}}'$-isomorphism*

$$M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{C}_{\lambda,\mu})|_{\tilde{\mathfrak{g}}'}\simeq\bigoplus_{\lambda'\in\Lambda^+(\tilde{\mathfrak{l}}')}m(\lambda',(\lambda,\mu))M^{\tilde{\mathfrak{g}}'}_{\tilde{\mathfrak{p}}'}(\mathbb{C}_{\lambda'}).$$

Applied to the case of our interest $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{p}}) = (\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{p} \oplus \mathfrak{p})$, $(\tilde{\mathfrak{g}}', \tilde{\mathfrak{p}}') = diag(\mathfrak{g}, \mathfrak{p}) = (diag(\mathfrak{g}), diag(\mathfrak{p}))$, we see that the polynomial ring $S^\star\big((\mathfrak{n}_- \oplus \mathfrak{n}_-)/((\mathfrak{n}_- \oplus \mathfrak{n}_-) \cap diag(\mathfrak{g}))\big)$ decomposes as $diag(\mathfrak{l})$-module on irreducibles with higher multiplicities. In particular, each $diag(\mathfrak{l})$-module realized in homogeneity k polynomials also appears in polynomials of homogeneity $(k+2)$, $k \in \mathbb{N}$. As we already explained, we focus on the case of 1-dimensional inducing representations $\mathbb{V}_{\lambda,\mu} \simeq \mathbb{C}_\lambda \otimes \mathbb{C}_\mu$ as $(\mathfrak{l} \oplus \mathfrak{l})$-modules and $\mathbb{V}_{\lambda'} \simeq \mathbb{C}_\nu$ as $diag(\mathfrak{l})$-modules $(\lambda, \mu, \nu \in \mathbb{C})$. The multiplicity formula then implies that a nontrivial homomorphism in (10.4) occurs for each 1-dimensional $diag(\mathfrak{l})$-module in $S^\star\big((\mathfrak{n}_- \oplus \mathfrak{n}_-)/((\mathfrak{n}_- \oplus \mathfrak{n}_-) \cap diag(\mathfrak{g}))\big)$, and it is a result in classical invariant theory (see [29], [47]) that for each even homogeneity there is just one 1-dimensional module. Because $\mathfrak{n}_-$ is as $(diag(\mathfrak{l})/[diag(\mathfrak{l}), diag(\mathfrak{l})])$-module isomorphic to the character $\mathbb{C}_{-1}$, the following relation holds true in the Grothendieck group of $\mathcal{O}^{(\mathfrak{p})}$ with $\mathfrak{p} \simeq diag(\mathfrak{p})$:

Corollary 10.2 *For $\mathfrak{g} \oplus \mathfrak{g} = so(n+1,1,\mathbb{R}) \oplus so(n+1,1,\mathbb{R}), diag(\mathfrak{g}) \simeq so(n+1,1,\mathbb{R})$ with standard maximal parabolic subalgebras $\mathfrak{p} \oplus \mathfrak{p}, diag(\mathfrak{p})$ given by omitting the first simple root in the corresponding Dynkin diagrams, $m(\nu,(\lambda,\mu)) = 1$ if and only if $\nu = \lambda + \mu - 2j, j \in \mathbb{N}$ and $m(\nu,(\lambda,\mu)) = 0$ otherwise.*

Consequently, in the Grothendieck group of the Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$ holds

$$M_{\mathfrak{p}\oplus\mathfrak{p}}^{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{C}_{\lambda,\mu})|_{diag(\mathfrak{g})} \simeq \bigoplus_{j \in \mathbb{N}} M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_{\lambda+\mu-2j}),$$

where $\nu = \lambda + \mu - 2j$.

Although we work in one specific signature $(n+1,1)$, the results are easily extended to a real form of any signature.

10.2 The construction of singular vectors for diagonal branching rules applied to generalized Verma modules for $so(n+1,1,\mathbb{R})$

The rest of the section is devoted to the construction of singular vectors, whose abstract existence was concluded in Corollary 10.2. This can be regarded as a quantitative part of our diagonal branching problem.

10.2.1 Description of the representation

In this subsection we describe the representation of $\mathfrak{g} \oplus \mathfrak{g}$, acting upon the generalized Verma module

$$M_{\mathfrak{p}\oplus\mathfrak{p}}^{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{C}_{\lambda,\mu}) = \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{p}\oplus\mathfrak{p})} (\mathbb{C}_{\lambda,\mu}),\ \mathbb{C}_{\lambda,\mu} = \mathbb{C}_\lambda \otimes \mathbb{C}_\mu,$$

in its Fourier image, i.e. apply the framework for the F-method explained in Section 4 to

$$M_{\mathfrak{p}\oplus\mathfrak{p}}^{\mathfrak{g}\oplus\mathfrak{g}}(\mathbb{C}_{\lambda,\mu}) \simeq M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\lambda) \otimes M_{\mathfrak{p}}^{\mathfrak{g}}(\mathbb{C}_\mu). \qquad (10.5)$$

The first goal is to describe the action of elements in the nilradical $diag(\mathfrak{n})$ of $diag(\mathfrak{p})$ in terms of differential operators acting on the Fourier image of $M^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{p}\oplus\mathfrak{p}}(\mathbb{C}_{\lambda,\mu})$. This can be derived from the explicit form of the action on the induced representation realized in the non-compact picture, and it follows from (10.5) that the problem can be reduced to the question on each component of the tensor product separately. Let us consider the complex representation π_λ, $\lambda \in \mathbb{C}$, of $G = SO_o(n+1,1,\mathbb{R})$, induced from the character $p \mapsto a^\lambda$, $p \in P$, acting on the one dimensional representation space $\mathbb{C}_\lambda \simeq \mathbb{C}$. Here $a \in A = \mathbb{R}^\star$ is the abelian subgroup in the Langlands decomposition $P = MAN$, $M = SO(n)$, $N = \mathbb{R}^n$. In other words, π_λ acts by left regular representation on $\mathrm{Ind}_P^G(\mathbb{C}_\lambda)$.

Let x_j be the coordinates with respect to the standard basis on $\mathfrak{n}_-$, and ξ_j, $j = 1,\dots,n$ the coordinates on the Fourier transform of $\mathfrak{n}_-$. We consider the family of differential operators

$$Q_j(\lambda) = -\frac{1}{2}|x|^2\partial_j + x_j(-\lambda + \sum_k x_k\partial_k),\ j = 1,\dots,n, \tag{10.6}$$

$$P_j^\xi(\lambda) = i\left(\frac{1}{2}\xi_j\Box^\xi + (\lambda - \mathbb{E}^\xi)\partial_{\xi_j}\right),\ j = 1,\dots,n, \tag{10.7}$$

where $|x|^2 = x_1^2 + \cdots + x_n^2$,

$$\Box^\xi = \partial^2_{\xi_1} + \cdots + \partial^2_{\xi_n}$$

is the Laplace operator of positive signature, $\partial_j = \frac{\partial}{\partial x_j}$ and $\mathbb{E}^\xi = \sum_k \xi_k\partial_{\xi_k}$ is the Euler homogeneity operator ($i \in \mathbb{C}$ the complex unit.) The following result is a routine computation:

Lemma 10.3 *([43]) Let us denote by E_j the standard basis elements of $\mathfrak{n}$, $j = 1,\dots,n$. Then $E_j \in \mathfrak{n}$ are acting on $C^\infty(\mathfrak{n}_-,\mathbb{C}_{-\lambda})$ by*

$$d\tilde{\pi}_\lambda(E_j)(s\otimes v) = Q_j(\lambda)(s)\otimes v,\ s \in C^\infty(\mathfrak{n}_-,\mathbb{C}),\ v \in \mathbb{C}_{-\lambda}, \tag{10.8}$$

and the action of $(d\tilde{\pi})^F_\lambda$ on $\mathrm{Pol}[\xi_1,\dots,\xi_n]\otimes\mathbb{C}^\vee_\lambda$ is given by

$$(d\tilde{\pi})^F_\lambda(E_j)(f\otimes v) = P^\xi_j(\lambda)(f)\otimes v,\ f \in \mathrm{Pol}[\xi_1,\dots,\xi_n],\ v \in \mathbb{C}^\vee_{-\lambda}. \tag{10.9}$$

As for the action of remaining basis elements of $\mathfrak{g}$ in the Fourier image of the induced representation, the action of $\mathfrak{n}_-$ is given by multiplication by coordinate functions, the standard basis elements of the simple part of the Levi factor $[\mathfrak{l},\mathfrak{l}] = so(n)$ act by differential operators

$$M^\xi_{ij} = (\xi_j\partial_{\xi_i} - \xi_i\partial_{\xi_j}),\ i,j = 1,\dots,n$$

and the basis element of the Lie algebra of A acts as the homogeneity operator, $\mathbb{E}^\xi = \sum_{i=1}^n \xi_i\partial_{\xi_i}$.

In the Fourier image of the tensor product of two induced representations in the non-compact realization on $\mathfrak{n}_-\oplus\mathfrak{n}_-$ with coordinates ξ_i resp. ν_i on the first resp. second copy of $\mathfrak{n}_-$ in $\mathfrak{n}_-\oplus\mathfrak{n}_-$, the generators of the diagonal subalgebra $diag(\mathfrak{g})$ act on the representation $\mathrm{Ind}^{G\times G}_{P\times P}(\mathbb{C}_{\lambda,\mu})$ induced from (χ_λ,χ_μ) as

1. Multiplication by
$$(\xi_j \otimes 1) + (1 \otimes \nu_j),\ j = 1, \dots, n \tag{10.10}$$
for the elements of $diag(\mathfrak{n}_-)$,

2. First order differential operators with linear coefficients
$$\begin{aligned} M_{ij}^{\xi,\nu} &= (M_{ij}^{\xi} \otimes 1) + (1 \otimes M_{ij}^{\nu}) \\ &= (\xi_j \partial_{\xi_i} - \xi_i \partial_{\xi_j}) \otimes 1 + 1 \otimes (\nu_j \partial_{\nu_i} - \nu_i \partial_{\nu_j}), \end{aligned} \tag{10.11}$$
$i, j = 1, \dots, n$ for the elements of the simple subalgebra of $diag(\mathfrak{l})$ and
$$\mathbb{E}^{\xi} \otimes 1 + 1 \otimes \mathbb{E}^{\nu} = \sum_{i=1}^{n} (\xi_i \partial_{\xi_i} \otimes 1 + 1 \otimes \nu_i \partial_{\nu_i}), \tag{10.12}$$
for the generator of $diag(\mathfrak{l})/[diag(\mathfrak{l}), diag(\mathfrak{l})]$,

3. Second order differential operators with linear coefficients
$$\begin{aligned} P_j^{\xi,\nu}(\lambda,\mu) &= (P_j^{\xi}(\lambda) \otimes 1) + (1 \otimes P_j^{\nu}(\mu)) \\ &= i(\frac{1}{2}\xi_j \Box^{\xi} + (\lambda - \mathbb{E}^{\xi})\partial_{\xi_j}) \otimes 1 \\ &\quad + i1 \otimes (\frac{1}{2}\nu_j \Box^{\nu} + (\mu - \mathbb{E}^{\nu})\partial_{\nu_j}), \end{aligned} \tag{10.13}$$
$j = 1, \dots, n$ for the elements $diag(\mathfrak{n})$.

This completes the description of the first part of abstract procedure in the case of the diagonal branching problem of our interest.

10.2.2 Reduction to a hypergeometric differential equation in two variables

It follows from the previous discussion that $diag(\mathfrak{l})$-modules inducing singular vectors for the diagonal branching rules are one dimensional. This means that they are annihilated by $diag(\mathfrak{l}^s) = diag([\mathfrak{l}, \mathfrak{l}]) \simeq so(n, \mathbb{R})$, the simple part of the diagonal Levi factor $diag(\mathfrak{l}) \simeq so(n, \mathbb{R}) \times \mathbb{R}$. It follows that the singular vectors are invariants of $diag(\mathfrak{l})$ acting diagonally on the algebra of polynomials on $\mathfrak{n}_- \oplus \mathfrak{n}_-$ as a $\mathfrak{l} \oplus \mathfrak{l}$-module. The following result is a special case of the first fundamental theorem in classical invariant theory, see e.g. [29], [47].

Lemma 10.4 *Let $(V, <,>)$ be a finite dimensional real vector space with bilinear form $<,>$ and $SO(V)$ the Lie group of automorphisms of $(V, <,>)$. Then the subalgebra of $SO(V)$-invariants in the complex polynomial algebra $Pol[V \oplus V]$ ($SO(V)$ acting diagonally on $V \oplus V$) is polynomial algebra generated by $< \xi, \xi >$, $< \xi, \nu >$ and $< \nu, \nu >$. Here we use the convention that ξ is a vector in the first component V of $V \oplus V$ and ν in the second summand.*

In our case, the complex polynomial algebra is $Pol[\xi_1,\dots,\xi_n,\nu_1,\dots,\nu_n]$ and we use the notation $Pol[r,s,t]$ for the (complex) subalgebra of invariants:

$$\begin{aligned} r &:=< \xi,\nu >= \sum_{i=1}^{n}\xi_i\nu_i, \\ s &:=< \xi,\xi >= \sum_{i=1}^{n}\xi_i\xi_i, \\ t &:=< \nu,\nu >= \sum_{i=1}^{n}\nu_i\nu_i. \end{aligned} \tag{10.14}$$

The task of the present subsection is to rewrite the operators $P_j^{\xi,\nu}(\lambda,\mu)$ in the variables r,s,t, i.e. we reduce the action of $P_j^{\xi,\nu}(\lambda,\mu)$ from the polynomial ring to the ring of $diag(\mathfrak{n}_-)$-invariants on $\mathfrak{n}_-\oplus\mathfrak{n}_-$.

We compute

$$\partial_{\nu_i}r=\xi_i,\ \partial_{\xi_i}r=\nu_i, \partial_{\nu_i}s=0,\ \partial_{\xi_i}s=2\xi_i, \partial_{\nu_i}t=2\nu_i,\ \partial_{\xi_i}t=0, \tag{10.15}$$

and

$$\partial_{\xi_i}=\nu_i\partial_r+2\xi_i\partial_s, \Box^\xi=t\partial_r^2+4r\partial_r\partial_s+2n\partial_s+4s\partial_s^2,\ i=1,\dots,n. \tag{10.16}$$

Note that analogous formulas for ∂_{ν_i}, $\Box^\nu$ can be obtained from those for ξ by applying the involution

$$\xi_i\longleftrightarrow\nu_i,\ s\longleftrightarrow t,\ r\longleftrightarrow r. \tag{10.17}$$

We also have for all $i=1,\dots,n$

$$\mathbb{E}^\xi\partial_{\xi_i}=\nu_i(\mathbb{E}^r+2\mathbb{E}^s)\partial_r+\xi_i(2\mathbb{E}^r+4\mathbb{E}^s+2)\partial_s, \tag{10.18}$$

and so taking all together we arrive at the operators

$$\begin{aligned} P_i^{r,s,t}(\lambda,\mu) &= \xi_i(\frac{1}{2}t\partial_r^2+(n+2\lambda-2-2\mathbb{E}^s)\partial_s-(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r) \\ &+\nu_i(\frac{1}{2}s\partial_r^2+(n+2\mu-2-2\mathbb{E}^t)\partial_t-(\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r) \end{aligned} \tag{10.19}$$

acting on complex polynomial algebra $Pol[r,s,t]$, $i=1,\dots,n$. A consequence of the system of equations ($i=1,\dots,n$) is

$$\begin{aligned} & P_\xi^{r,s,t}(\lambda,\mu):=\sum_{i=1}^{n}\xi_iP_i^{r,s,t}(\lambda,\mu)=s(\frac{1}{2}t\partial_r^2+(n+2\lambda-2-2\mathbb{E}^s)\partial_s \\ & -(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r)+r(\frac{1}{2}s\partial_r^2+(n+2\mu-2-2\mathbb{E}^t)\partial_t-(\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r), \\ & P_\nu^{r,s,t}(\lambda,\mu):=\sum_{i=1}^{n}\nu_iP_i^{r,s,t}(\lambda,\mu)=r(\frac{1}{2}t\partial_r^2+(n+2\lambda-2-2\mathbb{E}^s)\partial_s \\ & -(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r)+t(\frac{1}{2}s\partial_r^2+(n+2\mu-2-2\mathbb{E}^t)\partial_t-(\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r). \end{aligned} \tag{10.20}$$

Notice that the second equation follows from the first one by the action of involution

$$\lambda \longleftrightarrow \mu,\ s \longleftrightarrow t,\ r \longleftrightarrow r.$$

In what follows we construct a set of homogeneous polynomial solutions of $P_\xi^{r,s,t}(\lambda,\mu)$, $P_\nu^{r,s,t}(\lambda,\mu)$ solving the system $\{P_i^{r,s,t}(\lambda,\mu)\}_i$, $i = 1,\ldots,n$. The uniqueness of the solution for the generic values of the inducing parameters implies the unique solution of the former system of PDEs.

Notice that (10.20) is the system of differential operators preserving the space of homogeneous polynomials in the variables r,s,t, i.e. $P_\xi^{r,s,t}(\lambda,\mu)$, $P_\nu^{r,s,t}(\lambda,\mu)$ commute with $\mathbb{E}^{r,s,t} := \mathbb{E}^r + \mathbb{E}^s + \mathbb{E}^t$.

Let $p = p(r,s,t)$ be a homogeneous polynomial of degree N, $deg(p) = N$, and write

$$\begin{aligned} p &= r^N p(\frac{s}{r},\frac{t}{r}) = r^N \tilde{p}(u,v),\ u := \frac{s}{r}, v := \frac{t}{r}, \\ \tilde{p}(u,v) &= \sum_{i,j|0\leq|i+j|\leq N} A_{i,j}u^i v^j, \end{aligned} \qquad (10.21)$$

where $\tilde{p}$ is the polynomial of degree N. We have

$$\partial_s = \frac{1}{r}\partial_u + \frac{1}{u}\partial_r,\ \partial_t = \frac{1}{r}\partial_v + \frac{1}{v}\partial_r$$

and so the summands in (10.20) transform as

$$\begin{aligned} &\frac{1}{2}st\partial_r^2 = \frac{1}{2}N(N-1)uv, \\ &s(n+2\lambda-2-2\mathbb{E}^s)\partial_s = (n+2\lambda-2\mathbb{E}^u-2N)(\mathbb{E}^u+N), \\ &-s(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r = -N(3N-3+2\mathbb{E}^v-\mu)u, \\ &\frac{1}{2}rs\partial_r^2 = \frac{1}{2}N(N-1)u, \\ &r(n+2\mu-2-2\mathbb{E}^t)\partial_t = (n+2\mu-2\mathbb{E}^v-2N)(\partial_v+\frac{N}{v}), \\ &-r(\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r = -N(2\mathbb{E}^u-\lambda+3N-3) \end{aligned} \qquad (10.22)$$

when acting on $r^N\tilde{p}(u,v)$. Taken all together, we get

Lemma 10.5 *The substitution (10.21) transforms the former system of PDEs (10.20) to a hypergeometric differential operator*

$$\begin{aligned} P_\xi^{u,v}(\lambda,\mu) \;=\; & \frac{1}{2}N(N-1)uv + (n+2\lambda-2\mathbb{E}^u-2N)(\mathbb{E}^u+N) \\ & -N(3N-3+2\mathbb{E}^v-\mu)u + \frac{1}{2}N(N-1)u \\ & +(n+2\mu-2\mathbb{E}^v-2N)(\partial_v+\frac{N}{v}) - N(2\mathbb{E}^u-\lambda+3N-3), \end{aligned} \qquad (10.23)$$

fulfilling

$$P_\nu^{u,v}(\lambda,\mu) = P_\xi^{v,u}(\mu,\lambda).$$

In the next subsection we find, for generic values of the inducing parameters λ, μ, a unique solution for this hypergeometric equation for a given homogeneity.

10.2.3 Solution of the hypergeometric differential equation in two variables

We start with a couple of simple examples.

Example 10.6 *Let us consider the polynomial of homogeneity one,*

$$p(r,s,t) = Ar + Bs + Ct, \;\; A, B, C \in \mathbb{C}.$$

The application of $P_i^{r,s,t}(\lambda,\mu)$ *yields*

$$P_i^{r,s,t}(\lambda,\mu)(Ar + Bs + Ct) = \xi_i(B(n+2\lambda-2) + A\mu) + \nu_i(C(n+2\mu-2) + A\lambda) \tag{10.24}$$

for all $i = 1, \dots, n$. *When* A *is normalized to be equal to* 1, *we get*

$$C = -\frac{\lambda}{n+2\mu-2}, \; B = -\frac{\mu}{n+2\lambda-2}.$$

The unique homogeneous (resp. non-homogeneous) solution of $P_i^{r,s,t}(\lambda,\mu)$ *for all* $i = 1, \dots, n$ *is then*

$$p(r,s,t) = (n+2\lambda-2)(n+2\mu-2)r - \mu(n+2\mu-2)s - \lambda(n+2\lambda-2)t. \tag{10.25}$$

Example 10.7 *Let*

$$p(r,s,t) = Ar^2 + Bs^2 + Ct^2 + Drs + Est + Frt, \;\; A, B, C, D, E, F \in \mathbb{C}$$

be a general polynomial of homogeneity two. We have

$$\begin{aligned}
&P_i^{r,s,t}(\lambda,\mu)p(r,s,t) = \xi_i[r(D(n+2\lambda-2) + A2(\mu-1)) + s(B2(n+2\lambda-4) + D\mu)\\
&+t(A + E(n+2\lambda-2) + F(\mu-2))],\\
&+\nu_i[r(F(n+2\mu-2) + A2(\lambda-1)) + s(A + E(n+2\mu-2) + D(\lambda-2))\\
&+t(C2(n+2\mu-4) + F\lambda)],
\end{aligned} \tag{10.26}$$

for all $i = 1, \dots, n$. *The equations*

$$\sum_i \xi_i P_{\xi_i}^{r,s,t}(\lambda,\mu) = 0, \; \sum_i \nu_i P_{\nu_i}^{r,s,t}(\lambda,\mu) = 0$$

are equivalent to two systems of linear equations:

$$\begin{aligned}
&D(n+2\lambda-2) + A2(\mu-1) + A + E(n+2\mu-2) + D(\lambda-2) = 0,\\
&F(n+2\mu-2) + A2(\lambda-1) = 0,\\
&B2(n+2\mu-4) + D\mu = 0,\\
&A + E(n+2\lambda-2) + F(\mu-2) = 0,\\
&C2(n+2\mu-4) + F\lambda) = 0,
\end{aligned} \tag{10.27}$$

resp.

$$
\begin{aligned}
&D(n+2\lambda-2)+A2(\mu-1)=0,\\
&B2(n+2\mu-4)+D\mu=0,\\
&A+E(n+2\lambda-2)+F(\mu-2)+F(n+2\mu-2)+A2(\lambda-1)=0,\\
&A+E(n+2\mu-2)+D(\lambda-2)=0,\\
&C2(n+2\mu-4)+F\lambda=0.
\end{aligned}
\tag{10.28}
$$

Both systems are equivalent under the involution

$$
A\longleftrightarrow A,\ E\longleftrightarrow E,\ D\longleftrightarrow F,\ B\longleftrightarrow C,\ \lambda\longleftrightarrow\mu
$$

and its unique solution invariant under this involution is

$$
\begin{aligned}
&A=1,\ F=\frac{-2(\lambda-1)}{n+2\mu-2},\ C=\frac{\lambda(\lambda-1)}{(n+2\mu-2)(n+2\mu-4)},\\
&E=2\frac{(\lambda-2)(\mu-2)-(1+\frac{n}{2})}{(n+2\mu-2)(n+2\lambda-2)},\ D=\frac{-2(\mu-1)}{(n+2\lambda-2)},\\
&B=\frac{\mu(\mu-1)}{(n+2\lambda-2)(n+2\lambda-4)}.
\end{aligned}
\tag{10.29}
$$

The vector

$$
\begin{aligned}
p(r,s,t)\ =\ &(n+2\lambda-2)(n+2\lambda-4)(n+2\mu-2)(n+2\mu-4)r^2\\
&+\mu(\mu-1)(n+2\mu-2)(n+2\mu-4)s^2\\
&+\lambda(\lambda-1)(n+2\lambda-2)(n+2\lambda-4)t^2\\
&-2(\mu-1)(n+2\lambda-4)(n+2\mu-2)(n+2\mu-4)rs\\
&+2((\lambda-2)(\mu-2)-(1+\frac{n}{2}))(n+2\lambda-4)(n+2\mu-4)st\\
&-2(\lambda-1)(n+2\lambda-2)(n+2\lambda-4)(n+2\mu-4)rt.
\end{aligned}
\tag{10.30}
$$

is then the unique solution of $P_i^{r,s,t}(\lambda,\mu)$ *of homogeneity two.*

We now return back to the situation of a general homogeneity. The dehomogenesation $(r,s,t,)\to(r,u,v)$ is governed by coordinate change

$$
u:=\frac{s}{r},\ v:=\frac{t}{r},\ r:=r,
\tag{10.31}
$$

and so

$$
\begin{aligned}
&\partial_s\to\frac{1}{r}\partial_u,\ \partial_t\to\frac{1}{r}\partial_v,\ \partial_r\to-\frac{1}{r}u\partial_u-\frac{1}{r}v\partial_v+\partial_r\,,\\
&\mathbb{E}^s\to\mathbb{E}^u,\ \mathbb{E}^t\to\mathbb{E}^v,\ \mathbb{E}^r\to-\mathbb{E}^u-\mathbb{E}^v+\mathbb{E}^r\,.
\end{aligned}
\tag{10.32}
$$

The terms in $P^{r,s,t}_{\xi}(\lambda,\mu)$ transform into

$$\begin{aligned}
\frac{1}{2}st\partial_r^2 &\to \frac{1}{2}uv(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r+1)(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r),\\
-r(\mathbb{E}^r+2\mathbb{E}^s-\lambda)\partial_r &\to (\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r)(\mathbb{E}^u-\mathbb{E}^v+\mathbb{E}^r-\lambda-1),\\
r(n+2\mu-2-2\mathbb{E}^t)\partial_t &\to (n+2\mu-2-2\mathbb{E}^v)\partial_v,\\
\frac{1}{2}rs\partial_r^2 &\to \frac{1}{2}u(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r+1)(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r),\\
-s(\mathbb{E}^r+2\mathbb{E}^t-\mu)\partial_r &\to u(\mathbb{E}^u+\mathbb{E}^v-\mathbb{E}^r)(-\mathbb{E}^u+\mathbb{E}^v+\mathbb{E}^r-\mu-1),\\
s(n+2\lambda-2-2\mathbb{E}^s)\partial_s &\to (n+2\lambda-2\mathbb{E}^u)\mathbb{E}_u, \qquad (10.33)
\end{aligned}$$

and when acting on a polynomial of homogeneity N, $p(r,s,t)=r^N\tilde{p}(u,v)$ for a polynomial $\tilde{p}(u,v)$ of degree N in u,v, $\mathbb{E}^r=N$ and we get

$$\begin{aligned}
P^{u,v}_{\xi}(\lambda,\mu) &= \frac{1}{2}uv(\mathbb{E}^u+\mathbb{E}^v-N+1)(\mathbb{E}^u+\mathbb{E}^v-N)\\
&\quad -(\mathbb{E}^u)^2+\mathbb{E}^u(n+\lambda-1)+(\mathbb{E}^v-N)(-\mathbb{E}^v+N-\lambda-1)\\
&\quad +(n+2\mu-2-2\mathbb{E}^v)\partial_v\\
&\quad +\frac{1}{2}u(\mathbb{E}^u+\mathbb{E}^v-N)(-\mathbb{E}^u+3\mathbb{E}^v+N-2\mu-1). \qquad (10.34)
\end{aligned}$$

Similarly, one gets

$$\begin{aligned}
P^{u,v}_{\nu}(\lambda,\mu) &= \frac{1}{2}uv(\mathbb{E}^u+\mathbb{E}^v-N+1)(\mathbb{E}^u+\mathbb{E}^v-N)\\
&\quad -(\mathbb{E}^v)^2+\mathbb{E}^v(n+\mu-1)+(\mathbb{E}^u-N)(-\mathbb{E}^u+N-\mu-1)\\
&\quad +(n+2\lambda-2-2\mathbb{E}^u)\partial_u\\
&\quad +\frac{1}{2}v(\mathbb{E}^v+\mathbb{E}^u-N)(-\mathbb{E}^v+3\mathbb{E}^u+N-2\lambda-1). \qquad (10.35)
\end{aligned}$$

Let us denote $A_{i,j}(\lambda,\mu)$ the coefficient by monomial u^iv^j in the polynomial $\tilde{p}(u,v)$. The assumption $A_{i,j}(\lambda,\mu)=A_{j,i}(\mu,\lambda)$, combined with the symmetry between $P^{u,v}_{\xi}(\lambda,\mu)$ and $P^{u,v}_{\nu}(\lambda,\mu)$, allows to restrict to the action of $P^{u,v}_{\xi}(\lambda,\mu)$ on a polynomial of degree N of the form

$$\tilde{p}(u,v)=\sum_{i,j|i\leq N,\, j\leq N} A_{i,j}(\lambda,\mu)u^iv^j,\ A_{i,j}(\lambda,\mu)=A_{j,i}(\mu,\lambda), \qquad (10.36)$$

thereby converting the differential equation (10.34) into the four-term functional relation

$$\begin{aligned}
&\frac{1}{2}(i+j-N-1)(i+j-N-2)A_{i-1,j-1}(\lambda,\mu)\\
&+(-i^2+i(n+\lambda-1)+(j-N)(-j+N-\lambda-1)A_{i,j}(\lambda,\mu)\\
&+(j+1)(n+2\mu-2-2j)A_{i,j+1}(\lambda,\mu)\\
&+\frac{1}{2}(i+j-N-1)(-i+3j+N-2\mu)A_{i-1,j}(\lambda,\mu)=0 \qquad (10.37)
\end{aligned}$$

for $i,j=1,\dots,N$ and $j\geq i$, which recursively computes $A_{i,j+1}(\lambda,\mu)$ in terms of $A_{i-1,j-1}(\lambda,\mu), A_{i-1,j}(\lambda,\mu)$ and $A_{i,j}(\lambda,\mu)$.

There is still one question we have not mentioned yet, concerning the normalization of $A_{i,j}(\lambda,\mu)$. A singular vector can be normalized by multiplication by common denominator resulting in the coefficients valued in $Pol[\lambda,\mu]$ rather than the field $\mathbb{C}(\lambda,\mu)$. As we shall prove in the next Theorem, a consequence of (10.37) is the uniqueness of its solution in the range $\lambda,\mu\notin\{m-\frac{n}{2}|m\in\mathbb{N}\}$. We observe that the uniqueness of solution fails for $\lambda,\mu\in\{m-\frac{n}{2}|m\in\mathbb{N}\}$, which indicates the appearance of a non-trivial composition structure in the branching problem for generalized Verma modules.

In the following Theorem we construct a set of singular vectors, which will be the representatives realizing abstract character formulas of the diagonal branching problem in Corollary 8.5.

Theorem 10.8 *Let $\lambda,\mu\in\mathbb{C}\setminus\{m-\frac{n}{2}|m\in\mathbb{N}\}$, $N\in\mathbb{N}$, and $(x)_l=x(x+1)\dots(x+l-1)$, $l\in\mathbb{N}$ be the Pochhamer symbol for $x\in\mathbb{C}$. The four-term functional equation (10.37) for the set $\{A_{i,j}(\lambda,\mu)\}_{i,j\in\{1,\dots,N\}}$ fulfilling*

$$A_{j,i}(\lambda,\mu)=A_{i,j}(\mu,\lambda),\ j\geq i,$$

has a unique nontrivial solution

$$A_{i,j}(\lambda,\mu)=$$
$$\frac{\Gamma(i+j-N)\Gamma(1-\frac{n}{2}-\mu)\Gamma(1-i+j-N+\lambda)\Gamma(\lambda+\frac{n}{2}-i)}{2^{i+j}(-)^{i+j}i!j!\,\Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)\Gamma(\lambda+\frac{n}{2})}\cdot$$
$$\sum_{k=0}^{i}(-)^k\binom{i}{k}(j-i+1+k)_{i-k}(\lambda+\frac{n}{2}-i)_{i-k}(\mu-N+1)_k(\lambda-N+1-k)_k. \tag{10.38}$$

Out of the range $\lambda,\mu\in\mathbb{C}\setminus\{m-\frac{n}{2}|m\in\mathbb{N}\}$, (10.38) is still a solution, but not necessarily unique.

Proof:

Let us first discuss the uniqueness of the solution. The knowledge of $A_{i,j}(\lambda,\mu)$ for $i+j\leq k_0$ allows to compute the coefficient $A_{i,j+1}(\lambda,\mu)$ with $i+j=k_0+1$ from the recursive functional equation, because of assumption $\lambda,\mu\notin\{m-\frac{n}{2}|m\in\mathbb{N}\}$. The symmetry condition for $A_{i,j}(\lambda,\mu)$ gives $A_{j,i}(\mu,\lambda)=A_{i,j}(\lambda,\mu)$ and the induction proceeds by passing to the computation of $A_{i,j+2}(\lambda,\mu)$. Note that all coefficients are proportional to $A_{0,0}(\lambda,\mu)$ and its choice affects their explicit form.

The proof of the explicit form for $A_{i,j}(\lambda,\mu)$ is based on the verification of the recursion functional equation (10.37). To prove that the left hand side of (10.37) is trivial is equivalent to the following check: up to a product of linear factors coming from Γ-functions, the left hand side is the sum of four polynomials in λ,μ. A simple criterion for the triviality of a polynomial of degree d we use is that it has d roots (counted with multiplicity) and the leading monomial in a corresponding variable has coefficient zero.

It is straightforward but tedious to check that the left hand side of (10.37) has, as a polynomial in λ, the roots $\lambda = k - \frac{n}{2}$ for $k = 1, \ldots, i$ and its leading coefficient is zero. Let us first consider $\lambda = i - \frac{n}{2}$, so get after substitution

$$A_{i,j}(i - \frac{n}{2}, \mu) = \frac{(-)^j (i+j-N-1)\ldots(-N)}{2^{i+j} i! j!} \cdot \frac{(j-N-\frac{n}{2})\ldots(i-N-\frac{n}{2}+1)(1-\frac{n}{2}-N)_i(\mu-N+1)_i}{(j-\frac{n}{2}-\mu)\ldots(1-\frac{n}{2}-\mu)(\lambda+\frac{n}{2}-1)\ldots(\lambda+\frac{n}{2}-i)} \quad (10.39)$$

and

$$A_{i,j+1}(i - \frac{n}{2}, \mu) = A_{i,j}(i - \frac{n}{2}, \mu) \cdot \frac{(-)(i+j-N)(j-N-\frac{n}{2}+1)}{2(j+1)(j-\frac{n}{2}-\mu+1)}. \quad (10.40)$$

Taken together, there remain just two contributions on the left hand side of (10.37) given by $A_{i,j}(i - \frac{n}{2}, \mu)$, $A_{i,j+1}(i - \frac{n}{2}, \mu)$. Up to a common rational factor, their sum is proportional to

$$i(\frac{n}{2} - 1) + (j-N)(-j+N-i+\frac{n}{2}-1) + (j+1)(n+2\mu-2-2j)(-)\frac{(i+j-N)(j-N-\frac{n}{2}+1)}{2(j+1)(j-\frac{n}{2}-\mu+1)} = 0,$$

which proves the claim. The proof of triviality of the left hand side at special values $\lambda = i-1-\frac{n}{2}, \ldots, 1-\frac{n}{2}$ is completely analogous.

Note that there are some other equally convenient choices for λ, μ allowing the triviality check for (10.37), for example based on the choice $\lambda = k+N-1$, $k = 1, \ldots, i$ or $\mu = N-k$, $k = 1, \ldots, i$.

The remaining task is to find the leading coefficient on the left hand side of (10.37) as a polynomial in λ. Because

$$(\lambda + \frac{n}{2} - i)_{i-k} \overset{\lambda\to\infty}{\sim} \lambda^{i-k},$$
$$(\lambda - N + 1 - k)_k \overset{\lambda\to\infty}{\sim} \lambda^k, \quad (10.41)$$

the polynomial is of degree $\lambda^{j-i}\frac{\lambda^i}{\lambda^i} = \lambda^{j-i}$, $j \geq i$. The leading coefficient of $A_{i,j}(\lambda, \mu)$ is

$$\lim_{\lambda\to\infty} \frac{A_{i,j}(\lambda,\mu)}{\lambda^{j-i}} = (\sum_{k=0}^{i} (-)^k \binom{i}{k} (j-i+1+k)_{i-k}(\mu-N+1)_k) \cdot \frac{(-)^{i+j}(i+j-N-1)\ldots(-N)}{2^{i+j} i! j! \; (j-\frac{n}{2}-\mu)\ldots(1-\frac{n}{2}-\mu)}. \quad (10.42)$$

There are three contributions to (10.37):

$$
\begin{aligned}
&(N-j+i)\lim_{\lambda\to\infty}\frac{A_{i,j}(\lambda,\mu)}{\lambda^{j-i}},\\
&(j+1)(n+2\mu-2-2j)\lim_{\lambda\to\infty}\frac{A_{i,j+1}(\lambda,\mu)}{\lambda^{j+1-i}},\\
&\frac{1}{2}(i+j-N-1)(-i+3j+N-2\mu)\lim_{\lambda\to\infty}\frac{A_{i-1,j}(\lambda,\mu)}{\lambda^{j+1-i}}, \qquad (10.43)
\end{aligned}
$$

whose sum is a polynomial in μ multiplied by common product of linear polynomial. In order to prove triviality of this polynomial, it suffices as in the first part of the proof to find sufficient amount of its roots and to prove the triviality of its leading coefficient. For example in the case $\mu = N-1$, we get from (10.43) that the coefficients of this polynomial are proportional to the sum

$$
(N-j+i)+\frac{(j+1)(i+j-N)}{(j-i+1)}-\frac{i(-i+3j+N-2(N-1))}{(j-i+1)},
$$

which equals to zero. The verification of the required property for $\mu = N-k$, $k = 2,\dots,i$ is completely analogous. This completes the proof.

□

This completes the description of the set $Sol(\mathfrak{g}\oplus\mathfrak{g}, diag(\mathfrak{g}), \mathbb{C}_{\lambda,\mu})$ characterizing solution space of a diagonal branching problem for $so(n+1,1,\mathbb{R})$, (4.5).

Remark 10.9 *It is an interesting observation that the four term functional equation (10.37) for $A_{i,j}(\lambda,\mu)$ can be simplified using hypergeometric functions ${}_3F_2$:*

$$
{}_3F_2(a_1,a_2,a_3;b_1,b_2;z) := \sum_{m=0}^{\infty}\frac{(a_1)_m(a_2)_m(a_3)_m}{(b_1)_m(b_2)_m}\frac{z^m}{m!},
$$

where $a_1,a_2,a_3,b_1,b_2\in\mathbb{C}$ and $(x)_m = x(x+1)\dots(x+m-1)$. In particular, it can be converted into four term functional equation

$$
\begin{aligned}
&\frac{(n+2\lambda)\Gamma(i+j-N)\Gamma(-\frac{n}{2}-\lambda)\Gamma(1-i+j-N+\lambda)\Gamma(1-\frac{n}{2}-\mu)}{2^{i+j}(-)^{i+j}\Gamma(1+i)\Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)}\cdot\\
&(i(-2j+n+2\mu){}_3F_2(1-i,N-\lambda,1-N+\mu;1-i+j,1-\frac{n}{2}-\lambda;1)+\\
&i(-1+i-j+N-\lambda)(i-3j-N+2\mu)\cdot\\
&\cdot{}_3F_2(1-i,N-\lambda,1-N+\mu;2-i+j,1-\frac{n}{2}-\lambda;1)+\\
&(i^2-i(-1+n+\lambda)+(j-N)(1+j-N+\lambda))\cdot\\
&\cdot{}_3F_2(-i,N-\lambda,1-N+\mu;1-i+j,1-\frac{n}{2}-\lambda;1)+\\
&(1+j)(i+j-N)(-1+i-j+N-\lambda)\cdot\\
&\cdot{}_3F_2(-i,N-\lambda,1-N+\mu;2-i+j,1-\frac{n}{2}-\lambda;1))=0. \qquad (10.44)
\end{aligned}
$$

This functional equation, whose knowledge would clearly simplify the formulation of the proof of the last Theorem, is advanced to be found in any standard textbook *on special function theory of several variables (see e.g., [25], [2]), and does not seem to be accessible in the literature.*

In fact, a large part of the monograph [2] is devoted to evaluations of generalized hypergeometric functions ${}_{p+1}F_p$ at $z = 1$, at least for reasonably small values of $p \in \mathbb{N}$. However, the Saalschutz's theorems of the form

$$\sum_{r=0}^{n} \frac{(\frac{1}{2}a)_r(\frac{1}{2}+\frac{1}{2}a-b)_r(-4)^r(a+2r)_{n-r}}{r!(n-r)!(1+a-b)_r} = \frac{(a)_n}{n!}{}_3F_2(\frac{1}{2}+\frac{1}{2}a-b, a+n, -n; 1+a-b, \frac{1}{2}+\frac{1}{2}a; 1) \qquad (10.45)$$

are too special and restrictive to be of direct use for our needs.

Example 10.10 *As an example, we have*

$$A_{1,j}(\lambda,\mu) = \frac{\Gamma(1+j-N)\Gamma(j-N+\lambda)\Gamma(1-\frac{n}{2}-\mu)}{2^{j+1}(-)^{j+1}\Gamma(1+j)\Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)} \cdot \cdot\frac{(j(-2+n+2\lambda)+2(N-\lambda)(1-N+\mu))}{(n+2\lambda-2)} \qquad (10.46)$$

for all $j \in \{1, \dots, N\}$.

Let us also remark that for special values $\lambda, \mu \in \{m - \frac{n}{2} | m \in \mathbb{N}\}$, the formula $A_{i,j}(\lambda, \mu)$ simplifies due to the factorization of the underlying polynomial. Our experience suggests that the factorization indicates so called factorization identity, when a homomorphism of generalized Verma modules quotients through a homomorphism of generalized Verma modules of one of its summands (in the source) or a target homomorphism of generalized Verma modules. This naturally leads to the question of full composition structure of the branching problem going beyond the formulation in terms of the Grothendieck group of Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$.

Let us mention another interesting observation. The diagonal coefficients $A_{i,i}(\lambda, \mu) = A_{i,i}(\mu, \lambda)$ are, up to a rational multiple coming from the ratio of the product of Γ-functions, symmetric with respect to $\lambda \leftrightarrow \mu$. As a consequence, these polynomials belong to the algebra of $\mathbb{Z}_2$-invariants:

$$\mathbb{C}[\lambda, \mu]^{\mathbb{Z}_2} \xrightarrow{\sim} \mathbb{C}[\lambda\mu, \lambda + \mu].$$

Lemma 10.11 *The diagonal coefficients can be written as*

$$A_{i,i}(\lambda,\mu) = \frac{\Gamma(2i-N)}{2^{2i+1}i!\,\Gamma(-N)\Gamma(1+i-\frac{n}{2}-\mu)\Gamma(1+i-\frac{n}{2}-\lambda)} \cdot (\Gamma(1+i-\frac{n}{2}-\lambda)\Gamma(1-\frac{n}{2}-\mu){}_3F_2(-i, N-\lambda, 1-N+\mu; 1, 1-\frac{n}{2}-\lambda; 1) + \Gamma(1+i-\frac{n}{2}-\mu)\Gamma(1-\frac{n}{2}-\lambda){}_3F_2(-i, 1-N+\lambda, N-\mu; 1, 1-\frac{n}{2}-\mu; 1))$$

Proof:
It follows from the definition of ${}_3F_2$ that

$$\frac{1}{\Gamma(1+i-\frac{n}{2}-\mu)\Gamma(1+i-\frac{n}{2}-\lambda)}\cdot$$
$$(\Gamma(1+i-\frac{n}{2}-\lambda)\Gamma(1-\frac{n}{2}-\mu){}_3F_2(-i,N-\lambda,1-N+\mu;1,1-\frac{n}{2}-\lambda;1)+$$
$$\Gamma(1+i-\frac{n}{2}-\mu)\Gamma(1-\frac{n}{2}-\lambda){}_3F_2(-i,1-N+\lambda,N-\mu;1,1-\frac{n}{2}-\mu;1))=$$
$$\sum_{m=0}^{i}(\frac{(-i)_m(N-\lambda)_m(1-N+\mu)_m}{(1_m)(1-\frac{n}{2}-\lambda)_m(1-\frac{n}{2}-\mu)_{i-1}}$$
$$+\frac{(-i)_m(N-\mu)_m(1-N+\lambda)_m}{(1_m)(1-\frac{n}{2}-\mu)_m(1-\frac{n}{2}-\lambda)_{i-1}})\frac{1}{m!}, \tag{10.47}$$

where the sum is finite due to the presence of $(-i)_m$ in the nominator. Using basic properties of the Pochhamer symbol, e.g. $(x)_m=(-)^m(-x+m-1)_m$, an elementary manipulation yields the result.

□

Example 10.12 *As an example, in the case of $i=1$ we have*

$$A_{1,1}(\lambda,\mu)=\frac{N(N-1)(\lambda\mu-N(\lambda+\mu)+(1-\frac{n}{2}+N(N-1)))}{(2\lambda+n-2)(2\mu+n-2)} \tag{10.48}$$

Let us summarize our results in

Theorem 10.13 *Let $\mathfrak{g}=so(n+1,1,\mathbb{R})$ be a simple Lie algebra and $\mathfrak{p}$ its conformal parabolic subalgebra with commutative nilradical. Then the diagonal branching problem for the scalar generalized Verma $\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})$-modules induced from character (λ,μ) is determined in the Grothendieck group of Bernstein-Gelfand-Gelfand parabolic category $\mathcal{O}^{\mathfrak{p}}$ by $\mathcal{U}(\mathfrak{g})$-isomorphism*

$$M^{\mathfrak{g}\oplus\mathfrak{g}}_{\mathfrak{p}\oplus\mathfrak{p}}(\mathbb{C}_{\lambda,\mu})|_{\mathfrak{g}}\simeq\bigoplus_{j=0}^{\infty}M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_{\lambda+\mu-2j}). \tag{10.49}$$

Here the summand $M^{\mathfrak{g}}_{\mathfrak{p}}(\mathbb{C}_{\lambda+\mu-2j})$ is generated by singular vector of homogeneity $2j$ and the form (10.21) with coefficients given by equation (10.38), $j\in\mathbb{N}$. In particular, the singular vectors are non-zero, unique up to a multiple, linearly independent and of expected weight (given by homogeneity), and the cardinality of the set of singular vectors is as predicted by Corollary 8.5.

The explicit formulas for the singular vectors will be given, in the dual language of bilinear differential operators, in the next section.

10.3 Application - the classification of bilinear conformal invariant differential operators on line bundles

Let M be a smooth (or complex) manifold equipped with the filtration of its tangent bundle

$$0 \subset T^1M \subset \cdots \subset T^{m_0}M = TM,$$

$\mathcal{V} \to M$ a smooth (or holomorphic) vector bundle on M and $J^k\mathcal{V} \to M$ the weighted jet bundle over M, defined by

$$J^k\mathcal{V} = \bigcup_{x \in M} J^k_x\mathcal{V},\ J^k_x\mathcal{V} \xrightarrow{\sim} \oplus_{l=1}^k Hom(\mathcal{U}_l(gr(T_xM)), \Gamma(\mathcal{V}_x)),$$

where $\mathcal{U}_l(gr(T_xM))$ is the subspace of homogeneity k-elements in the universal enveloping algebra of the associated graded $gr(T_xM)$. A bilinear differential pairing between sections of the bundle $\mathcal{V}$ and sections of the bundle $\mathcal{W}$ to sections of the bundle $\mathcal{Y}$ is a vector bundle homomorphism

$$B : J^k\mathcal{V} \otimes J^l\mathcal{W} \to \mathcal{Y}. \tag{10.50}$$

In the case $\tilde{M} = \tilde{G}/\tilde{P}$ is a generalized flag manifold, a pairing is called invariant if it commutes with the action of $\tilde{G}$ on local sections of the homogeneous vector bundles $\mathcal{V}, \mathcal{W}, \mathcal{Y}$. Denoting $\mathbb{V}$, $\mathbb{W}$, $\mathbb{Y}$ the inducing $\tilde{P}$-representations of homogeneous vector bundles $\mathcal{V}, \mathcal{W}, \mathcal{Y}$, the space of $\tilde{G}$-invariant differential pairings can be algebraically characterized as

$$\begin{aligned} &((\mathcal{U}(\tilde{\mathfrak{g}}) \otimes \mathcal{U}(\tilde{\mathfrak{g}})) \otimes_{\mathcal{U}(\tilde{\mathfrak{p}}) \otimes \mathcal{U}(\tilde{\mathfrak{p}})} Hom(\mathbb{V} \otimes \mathbb{W}, \mathbb{Y}))^{\tilde{P}} \simeq \\ &Hom_{\mathcal{U}(\tilde{\mathfrak{g}})}(M^{\tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}}}(\mathbb{Y}), M^{\tilde{\mathfrak{g}} \oplus \tilde{\mathfrak{g}}}_{\tilde{\mathfrak{p}} \oplus \tilde{\mathfrak{p}}}(\mathbb{V}^\vee \otimes \mathbb{W}^\vee)), \end{aligned} \tag{10.51}$$

where the superscript denotes the space of $\tilde{P}$-invariant elements and $\mathbb{V}^\vee, \mathbb{W}^\vee$ denote the dual representations, see e.g., [48]. In our case, we get

Theorem 10.14 *Let $G = SO_o(n+1, 1, \mathbb{R})$ and P its conformal parabolic subgroup, $\lambda, \mu \in \mathbb{C} \setminus \{m - \frac{n}{2} | m \in \mathbb{N}\}$, $N \in \mathbb{N}$. Let us denote by $\mathcal{L}_{-\lambda}$ the homogeneous line bundle on n-dimensional sphere $G/P \simeq S^n$ induced from the character $\chi_{-\lambda}$ of P. Then there exists up to a multiple a unique set of bilinear conformally invariant operators*

$$B_N : C^\infty(G/P, \mathcal{L}_{-\lambda}) \times C^\infty(G/P, \mathcal{L}_{-\mu}) \to C^\infty(G/P, \mathcal{L}_{-\lambda-\mu-2N}) \tag{10.52}$$

of the form

$$B_N = \sum_{0 \le i,j,k \le N | i+j+k=N} A_{i,j}(\lambda, \mu) \tilde{s}^i \tilde{t}^j \tilde{r}^k, \tag{10.53}$$

where

$$A_{i,j}(\lambda,\mu) = \frac{\Gamma(i+j-N)\Gamma(1-\frac{n}{2}-\mu)\Gamma(1-i+j-N+\lambda)\Gamma(\lambda+\frac{n}{2}-i)}{2^{i+j}(-)^{i+j}i!j!\,\Gamma(-N)\Gamma(1-N+\lambda)\Gamma(1+j-\frac{n}{2}-\mu)\Gamma(\lambda+\frac{n}{2})}\cdot$$
$$\sum_{k=0}^{i}(-)^k\binom{i}{k}(j-i+1+k)_{i-k}(\lambda+\frac{n}{2}-i)_{i-k}(\mu-N+1)_k(\lambda-N+1-k)_k,$$
$$A_{i,j}(\lambda,\mu) = A_{j,i}(\mu,\lambda), \tag{10.54}$$

such that

$$\tilde{s} = \sum_{i=1}^{n}\partial_{x_i}^2 = \Box_x,\ \tilde{t} = \sum_{i=1}^{n}\partial_{y_i}^2 = \Box_y,\ \tilde{r} = \sum_{i=1}^{n}\partial_{x_i}\partial_{y_i}. \tag{10.55}$$

Out of the range $\lambda, \mu \in \mathbb{C} \setminus \{m - \frac{n}{2} | m \in \mathbb{N}\}$, B_N is still an element of the previous set, but there might be additional ones indicating the emergence of a nontrivial composition structure.

Proof:

The proof is a direct consequence of Theorem 10.38 and duality (4.2), together with the application of inverse Fourier transform

$$x_j \longleftrightarrow -i\partial_{\xi_j},\ \partial_{x_j} \longleftrightarrow -i\xi_j$$

with $i \in \mathbb{C}$ the imaginary complex unit.

□

In many applications, it is perhaps more convenient to express the bilinear differential operators in terms of tangent resp. normal coordinates $t_i = \frac{1}{2}(\xi_i+\nu_i)$ resp. $n_i = \frac{1}{2}(\xi_i - \nu_i)$, $i = 1, \dots, n$ to the diagonal submanifold, where

$$r = \frac{1}{4}(< t,t > - < n,n >),$$
$$s = \frac{1}{4}(< t,t > + < n,n > +2 < t,n >),$$
$$t = \frac{1}{4}(< t,t > + < n,n > -2 < t,n >). \tag{10.56}$$

11 The branching problem and singular vectors for infinite dimensional indecomposable inducing representations

Not much is known on the classification of homomorphisms between generalized Verma modules induced from indecomposable representations, where the typical representative examples of interest are $\mathfrak{p}$-modules given by restriction of finite dimensional $\mathfrak{g}$-modules. In case of homogeneous parabolic geometry, the splitting operator is a homomorphism from a module induced by an irreducible representation of $\mathfrak{p}$ to a module induced by an indecomposable $\mathfrak{p}$-module.

Invariant differential operators acting between sections of vector bundles induced from irreducible modules are rare and they appear in discrete families. On the contrary, the splitting operators appear in families indexed by continuous parameters. In particular, splitting operators are basic building blocks in the discussion of translation functors, which play a prominent role in proper understanding of discrete families of operators between bundles induced from irreducible representations.

In the present section we give the construction of homomorphism for which the source space is the bundle induced by an irreducible finite dimensional module W of the parabolic subalgebra $\mathfrak{p}$, and the target space of the operator is the bundle induced by the generalized Verma module $M(W)$ induced from W. Hence the target space is infinite dimensional vector bundle and the splitting operator is a differential operator of infinite but locally finite order. This means that on each homogeneity level of the target is the operator in question of finite order, but there is an infinite number of homogeneity levels.

These splitting operators form meromorphic families depending in the case of orthogonal algebra and its conformal (maximal) parabolic subalgebra on one complex parameter, the value of inducing character twisting generalized Verma module $M(W)$. The analytically continued values at singularities offer an important information, e.g. the residues resp. bottom coefficients at a singularity define invariant differential operators between bundles induced by irreducible $\mathfrak{p}$-modules. Hence the splitting operators can be used to construct such operators.

It is well known that generalized Verma modules are universal objects in the BGG category $\mathcal{O}^{\mathfrak{p}}$ of highest weight $\mathfrak{g}$-modules. Each finite dimensional $\mathfrak{g}$-module can be realized as the quotient of a generalized Verma module with the same highest resp. lowest weight by its maximal submodule. Similarly, each product $W \otimes \mathbb{V}$ of an irreducible $\mathfrak{l}$-module W and an irreducible $\mathfrak{g}$-module $\mathbb{V}$ can be realized as a quotient of the generalized Verma module induced from W. Similarly, splitting operators with values in generalized Verma modules are universal among splitting operators with values in the product $W \otimes V$. The latter ones are usual splitting operators used in a variety of constructions.

Splitting operators with values in generalized Verma modules collect in one definition a sequence of splitting operators with values in different twisted tractor bundles (i.e., the bundles of type $W \otimes \mathbb{V}$ described above). The latter ones are obtained by projection from the former ones. In this sense we can say that the splitting operators with values in generalized Verma modules are universal splitting operators.

There is a very simple relation between splitting operators valued in generalized Verma modules and splitting operators with values in twisted tractor bundles constructed via Casimir operators. The latter ones are given by polynomials in Casimir operators, defined as finite products of linear factors. The former ones are given by limit of the latter ones, i.e. by infinite (but locally finite) suitably normalized products of linear factors. This reformulation leads to a curved generalization given by replacing Casimir operator by curved Casimir operator, cf. [16].

11.1 Analytic computation of universal splitting operator - the conformal case of lowest weight scalar Verma module

Following the general introduction, we determine the representation action by basis elements of $\mathfrak{n}$ on generalized (lowest weight) Verma modules induced from the $\mathfrak{p}$-module given by tensor product of a character and generalized (lowest weight) $\mathfrak{g}$-Verma module (induced from the trivial character). Up to a non-zero multiple, which will not be of any importance for us due to the fact that we are interested in the kernel of these operators, the representation action of i-th standard basis element of $\mathfrak{n}$ on the generalized (lowest weight) Verma module isomorphic to $\mathbb{C}[u_1, \dots, u_n]$ is given by multiplication by u_i. Consequently,

$$P_i(\lambda) = \frac{1}{2}\xi_i \Box^\xi + (\lambda - \mathbb{E}^\xi)\partial_{\xi_i} + u_i, \ i = 1, \dots, n. \tag{11.1}$$

The distributive Fourier transform leads to the algebra

$$\mathbb{C}[\xi_1, \dots, \xi_n, u_1, \dots, u_n] \simeq \mathbb{C}[\xi_1, \dots, \xi_n] \otimes \mathbb{C}[u_1, \dots, u_n]$$

and classical invariant theory implies that the subalgebra of $so(n)$-invariants is generated by $|\xi|^2 = \sum_i \xi_i \xi_i$, $|u|^2 = \sum_i u_i u_i$ and $< \xi, u >= \sum_i \xi_i u_i$. We impose the grading $|\xi_i| = (1, 0)$ and $|u_i| = (0, 1)$, so $||\xi|^2| = (2, 0), ||u|^2| = (0, 2)$ and $| < u, \xi > | = (1, 1)$. By definition, we need to work with graded subalgebra of $\mathbb{C}[|\xi|^2, |u|^2, < \xi, u >]$ generated by elements of homogeneity $(p, p), p \in \mathbb{N}$, and again classical invariant theory implies that this subalgebra is isomorphic to $\mathbb{C}[< u, \xi >, |\xi|^2|u|^2]$. We have $||\xi|^2|u|^2| = (2, 2)$ and use the notation

$$s :=< u, \xi >, t := |\xi|^2|u|^2$$

with

$$(\mathbb{E}^\xi + \mathbb{E}^u)s = 2s, (\mathbb{E}^\xi + \mathbb{E}^u)t = 4t.$$

The Jacobian of the map $s = s(\xi_i, u_i), t = t(\xi_i, u_i)$ has components

$$\begin{aligned} &\frac{\partial s}{\partial \xi_i} = u_i, \frac{\partial s}{\partial u_i} = \xi_i, \\ &\frac{\partial t}{\partial \xi_i} = 2\xi_i |u|^2, \frac{\partial t}{\partial u_i} = 2u_i |\xi|^2. \end{aligned} \tag{11.2}$$

The vector field ∂_{ξ_i} transforms as

$$\partial_{\xi_i} = u_i\partial_s + 2\xi|u|^2\partial_t,$$

and so the Laplace operator takes in the variables s, t the form

$$\sum_i \partial_{\xi_i}\partial_{\xi_i} = |u|^2(\partial_s^2 + 4s\partial_s\partial_t + 4t\partial_t^2 + 2n\partial_t),$$

i.e.

$$\frac{1}{2}\xi_i\Box^\xi = \frac{1}{2}\xi_i|u|^2(\partial_s^2 + 2(2\mathbb{E}^s + 2\mathbb{E}^t + n))$$

Similarly, we have

$$\mathbb{E}^\xi\partial_{\xi_i} = u_i(\mathbb{E}^s + 2\mathbb{E}^t)\partial_s + 2\xi_i|u|^2(\mathbb{E}^s + 2\mathbb{E}^t + 1)\partial_t. \tag{11.3}$$

Collecting all terms together allows to rewrite the representation action on the subalgebra $\mathbb{C}[s, t]$.

Theorem 11.1 *The basis elements of the nilradical* $\mathfrak{n}$ *act on the subalgebra* $\mathbb{C}[s, t] \subset \mathbb{C}[\xi_1, \dots, \xi_n, u_1, \dots, u_n]$ *as*

$$P_i(\lambda) = \xi_i|u|^2(\frac{1}{2}\partial_s^2 + (2\lambda - 2\mathbb{E}^t + (n-2))\partial_t + u_i(\lambda - \mathbb{E}^s - 2\mathbb{E}^t)\partial_s + 1) \tag{11.4}$$

for $i = 1, \dots, n$.

The multiplication by ξ_i resp. u_i and summing over i yields two partial differential equations $\sum_i \xi_i P_i(\lambda) = 0$ resp. $\sum_i u_i P_i(\lambda) = 0$ in the variables s, t:

$$(\frac{1}{2}t\partial_s^2 + (2\lambda - 2\mathbb{E}^s - 2\mathbb{E}^t + n)\mathbb{E}^t + (\lambda + 1 - \mathbb{E}^s)\mathbb{E}^s + s)P(s, t) = 0, \tag{11.5}$$

$$((\lambda - \frac{1}{2}\mathbb{E}^s - 2\mathbb{E}^t)\partial_s + s(2\lambda - 2\mathbb{E}^t + (n-2))\partial_t + 1)P(s, t) = 0. \tag{11.6}$$

The only relation used to derive the previous expression is

$$s\mathbb{E}^s\partial_s = {\mathbb{E}^s}^2 - \mathbb{E}^s, t\mathbb{E}^t\partial_t = {\mathbb{E}^t}^2 - \mathbb{E}^t.$$

Note that (11.5) can be classified as a degenerate hypergeometric equation in two variables s, t. It is genuinely not homogeneous, i.e. it can not be rewritten as a differential equation in the variable $z := \frac{t}{s^2}$. Note that $\frac{1}{2}t\partial_s^2 = 2z^3\partial_z^2 + 3z^2\partial_z$, $\mathbb{E}^s = -2\mathbb{E}^z$ and $\mathbb{E}^t = \mathbb{E}^z$.

Let us consider an increasing filtration $\{F^m(\mathbb{C}[s, t])\}_{m\in\mathbb{N}}$ associated to the graded polynomial ring $\mathbb{C}[s, t], gr(s) = 1, gr(t) = 2$, with $F^m(\mathbb{C}[s, t])$ given by elements of grading $\leq m$. The second operator preserves $F^m(\mathbb{C}[s, t])$, while the first does not. On the other hand, one can define decreasing filtration $\tilde{F}^m(\mathbb{C}[s, t])$ given by elements of grading less or equal to m. Clearly, the first operator preserves $\tilde{F}^m(\mathbb{C}[s, t])$, while the second does not.

The application of the first equation allows to introduce suitable induction procedure. Note that we shall work in the ring of formal power series $\mathbb{C}[[s,t]]$, which is more convenient for our purposes. The proof of the next Lemma is elementary and follows from the recursion formula.

Lemma 11.2 *Let us consider the formal power series* $P(s,t) := \sum_k A_{k,0}s^k$ *(constant in* t*), such that the first coefficient is normalized to* $A_{0,0} = 1$ *and* $k \in \mathbb{N}$*. Then there is a recursive formula*

$$A_{k,0} = -A_{k-1,0}\frac{1}{c_k^0},$$

whose solution is

$$A_{k,0} = \prod_{i=1}^{k} \frac{-1}{c_i^0} = \frac{(-1)^k}{c_1^0 c_2^0 \dots c_k^0}.$$

As a next step, we pass to polynomials (or formal power series) linear in t and solve analogous recursive relations as in the previous Lemma. Note that we can (and so we will) omit the monomials independent of t, because their coefficients were determined by previous Lemma.

Lemma 11.3 *Let us consider the formal power series* $P(s,t) := \sum_k A_{k,1}s^k t$ *(linear in* t*),* $k \in \mathbb{N}$*. Then there is a recursive formula among* $A_{k+2,0}, A_{k-1,1}$ *and* $A_{k,1}$*, whose solution is*

$$A_{k,1} = \frac{(k+1)(k+2)}{2}\frac{(-1)^{k+1}}{c_1^0 c_2^0 \dots c_{k+2}^0 c_2^1}.$$

Proof:

The equation (11.5) implies the three-term relations among aforementioned coefficients

$$A_{k+2,0}\frac{(k+2)(k+1)}{2} + A_{k-1,1} + A_{k,1}((k+2)\lambda + n - (k^2+k+2)) = 0,$$

as a coefficient by monomial $s^k t$. One can immediately verify that

$$A_{k,1} = \frac{(k+1)(k+2)}{2}\frac{(-1)^{k+1}}{c_1^0 c_2^0 \dots c_{k+2}^0 c_2^1}$$

is (a unique) solution of the recursive relation.

□

Now we verify that the solution of equation (11.5) described in previous Lemmas satisfies equation (11.6) automatically. In other words, the second differential equation is just a consequence of the first one.

Lemma 11.4 *Let* $k \in \mathbb{N}$*. The relation among* $A_{k+1,0}, A_{k,0}$ *and* $A_{k-1,1}$*, coming from equation (11.6), is the consequence of the relations coming from the two previous Lemmas (i.e., the consequence of equation (11.5).)*

Proof:
The relation by monomial s^k coming out of equation (11.6) is

$$(\lambda - \frac{k}{2})(k+1)A_{k+1,0} + A_{k,0} + (2\lambda + n - 2)A_{k-1,1} = 0.$$

The result of direct computation now follows from an easy to check relation

$$(\lambda - \frac{k}{2})(k+1) - c^0_{k+1} = \frac{k(k+1)}{2},$$

and the proof follows.

□

The next Theorem computes explicit form of the coefficient $A_{k,l}$.

Theorem 11.5 *Let $k, l \in \mathbb{N}$. Then the collection of coefficients $A_{k,l}$,*

$$A_{k,l} = \frac{(-1)^{k+l}(k+1)\dots(k+2l)}{2^l c^0_1 \dots c^0_{k+2l} c^1_2 \dots c^l_{2l}} \tag{11.7}$$

is the unique solution to the recursive relation

$$\frac{(k+1)(k+2)}{2}A_{k+2,l-1} + A_{k-1,l} + ((k+2l)\lambda - k^2 + k(1-2l) + l(n-2l))A_{k,l} = 0$$

coming out of the equation (11.5).

Proof:
The equation (11.5) contributes to the three term relation among $A_{k+2,l-1}, A_{k-1,l}, A_{k,l}$ by the monomial $s^k t^l$ just by the aforementioned relation. It follows by direct check that this relation is equivalent to the identity

$$\begin{aligned} &-(k+2l)(k+1)(k+2)c^l_{2l} - k(k+1)(k+2)c^0_{k+2l} + \\ &(k+1)(k+2)(k+2l)(l(n+2\lambda-2l) + k(\lambda - k - 2l + 1)) = 0, \end{aligned} \tag{11.8}$$

which is easy to verify with $c^l_{2l} = l(n+2\lambda-2l)$, $c^0_{k+2l} = (k+2l)(\lambda - k - 2l + 1)$. This completes the proof.

□

The remaining task is to check the consistency of this solution with equation (11.6).

Theorem 11.6 *Let $k, l \in \mathbb{N}$ and $A_{k,l}$ be the collection of constants solving equation (11.5). Then the formal power series $P(s,t) = \sum_{k,l} A_{k,l} s^k t^l$ also solves equation (11.6), i.e. equation (11.6) is the consequence of equation (11.5).*

Proof:
Equation (11.6) is equivalent to

$$(k+1)(\lambda - \frac{k}{2} - 2l)A_{k+1,l} + A_{k,l} + (l+1)(2\lambda - 2l + n - 2)A_{k-1,l+1} = 0,$$

and can be easily reduced to

$$-2(\lambda - \frac{k}{2} - 2l)(k+2l+1)c^{l+1}_{2l+2} + 2c^0_{k+2l+1}c^{l+1}_{2l+2}$$
$$+k(k+2l+1)(l+1)(2\lambda - 2l + n - 2) = 0. \quad (11.9)$$

This is elementary to verify and the claim follows.

□

11.2 Algebraic relations among eigenvalues of Casimir operator

In the present subsection we prove a vast number of algebraic relations among the eigenvalues of Casimir operator, which is useful in the construction of families of invariant differential operators, comparisons for expansions of invariants in different bases, etc. We focus on the collection of eigenvalues coming from the generalized Verma modules of scalar type.

In particular, let $c^l_{2l} = l(n+2\lambda-2l)$, $l \in \mathbb{N}$, be the family of such eigenvalues (see the next subsection for their description) and $\tilde{c}^m_{2m}$, $m \in \mathbb{N}$, be the corresponding set of eigenvalues coming from the dual Verma module by comparison with l-th symmetric trace free representation $S^l_0\mathbb{R}^{n+1,1}$ of $so(n+1,1)$. We denote by l the half of the order of constructed operator or, equivalently, the symmetric tensor power of the representation $\mathbb{R}^{n+1,1}$ used in this construction such that for a given l, $k = l, l+1, l+2, \dots$ denotes descended relation of the same homogeneity as the primary one. Recall that for the fixed l we have

$$\underline{\tilde{c}}^m_{2m} = -m(n+2\lambda-4l+2m-1), \quad (11.10)$$

but we need just the values $m = l, l+1, \dots$ (as already mentioned, the notation for such m is k.) The underline sign $\_$ by c's means that n was replaced by $n-1$.

Let us introduce the two simplest cases, $l = 1$ with its descendants and $l = 2$ with its descendants.

Lemma 11.7 *(Linear relations) For $l = 1$, there is for any $k = 1, 2, 3, \dots$ the linear relation*

$$-\frac{(2k-1)}{n+4k-4}c^0_{2k} - \frac{k(n+2k-3)}{n+4k-4}c^1_2 = \underline{\tilde{c}}^k_{2k}.$$

The first two relations appearing in order two and four are

$$-\frac{1}{n}c^0_2 - \frac{n-1}{n}c^1_2 = \underline{\tilde{c}}^1_2,$$
$$-\frac{3}{n+4}c^0_4 - \frac{2(n+1)}{n+4}c^1_2 = \underline{\tilde{c}}^2_4. \quad (11.11)$$

Lemma 11.8 *(Quadratic relations) For $l = 2$, there is for any $k = 2, 3, \dots$ the quadratic relation*

$$\begin{aligned}
&\frac{2(k-1)(2k-5)(2k-7)}{(2k-1)(n+4k-6)(n+4k-8)} c^0_{2k-1} c^0_{2k} - \\
&\frac{(k-1)(2k-7)(n+6k-13)}{(n+4k-4)(n+4k-8)} c^0_{2k} c^2_4 + \\
&\frac{k(k-1)(n+6k-11)(n+6k-13)}{2(n+4k-4)(n+4k-6)} c^1_2 c^2_4 = \underset{\sim}{\widetilde{c}}{}^{k-1}_{2(k-1)} \underset{\sim}{\widetilde{c}}{}^{k}_{2k}. \qquad (11.12)
\end{aligned}$$

The first in this series, $k = 2$, appearing in order four is

$$\frac{2}{n(n+2)} c^0_3 c^0_4 + \frac{3(n-1)}{n(n+4)} c^0_4 c^2_4 + \frac{(n-1)(n+1)}{(n+2)(n+4)} c^1_2 c^2_4 = \underset{\sim}{\widetilde{c}}{}^1_2 \underset{\sim}{\widetilde{c}}{}^2_4. \qquad (11.13)$$

Both Lemmas are easily proved by (uniquely) solvable system of linear equations coming from substitution of suitable values for λ in c's.

Let us focus on general situation $l = 1, 2, \dots$, associated to the representation $S^l_0 \mathbb{R}^{n+1,1}$.

Theorem 11.9 *For a given $l \in \mathbb{N}$ and $k = l, l+1, \dots$, we have the following (homogeneity l) algebraic relations among eigenvalues of Casimir operator:*

$$\begin{aligned}
&A_0 c^0_{2k} c^0_{2k-1} \dots c^0_{2k-l+1} + A_1 c^0_{2k} c^0_{2k-1} \dots c^0_{2k-l+2} c^l_{2l} + \\
&A_2 c^0_{2k} c^0_{2k-1} \dots c^0_{2k-l+3} c^{l-1}_{2l-2} c^l_{2l} + \dots + A_l c^1_2 c^2_4 \dots c^{l-1}_{2l-2} c^l_{2l} \\
&\qquad = \underset{\sim}{\widetilde{c}}{}^k_{2k} \underset{\sim}{\widetilde{c}}{}^{k-1}_{2(k-1)} \dots \underset{\sim}{\widetilde{c}}{}^{k-1+1}_{2(k-l+1)}, \qquad (11.14)
\end{aligned}$$

where

$$\begin{aligned}
A_j \quad = \quad & \frac{k(k-1)\dots(k-l+1)}{j! 2^j 2k(2k-1)\dots(2k-l+j+1)} \cdot \frac{(2l-2k+2j+1)(2l-2k+2j+3)}{(l-2k-\frac{n}{2}-j+1)(l-2k-\frac{n}{2}-j+2)\dots} \\
& \frac{\dots(2l-2k+2l-1)(6k-6l+n-1)(6k-6l+n+1)\dots(6k-6l+n+2j-3)}{\dots(l-2k\widehat{-\frac{n}{2}-}2j+1)\dots(l-2k-\frac{n}{2}+l-j+1)} \\
& j = 0, \dots, l. \qquad (11.15)
\end{aligned}$$

For example, we have

$$\begin{aligned}
A_0 &= \frac{k(k-1)\dots(k-l+1)}{2k(2k-1)\dots(2k-l+1)}\cdot \\
&\cdot\frac{(2l-2k+1)(2l-2k+3)\dots(2l-2k+2l-1)}{(l-2k-\frac{n}{2}+1)(l-2k-\frac{n}{2}+2)\dots(l-2k-\frac{n}{2}+l)}, \\
A_1 &= \frac{k(k-1)\dots(k-l+1)}{22k(2k-1)\dots(2k-l+2)}\cdot \\
&\cdot\frac{(2l-2k+3)(2l-2k+5)\dots(2l-2k+2l-1)(6k-6l+n-1)}{(l-2k-\frac{n}{2})(l-2k-\frac{n}{2}+1)\dots(l-2k-\frac{n}{2}+l-2)(l-2k-\frac{n}{2}+l)}, \\
A_l &= \frac{k(k-1)\dots(k-l+1)}{l!2^l}\cdot \\
&\cdot\frac{(6k-4l+n-3)(6k-4l+n-5)\dots(6k-4l+n-2l-1)}{(l-2k-\frac{n}{2}+1)(l-2k-\frac{n}{2})\dots(l-2k-\frac{n}{2}-l+3)(l-2k-\frac{n}{2}-l+2)}.
\end{aligned}$$

We will illustrate the proof of Theorem (11.9) in one explicit case and then return back to its proof in full generality. To that aim, fix $l=2$ and let us find λ such that $(l-2)(n+2\lambda-2(l-2))=c^{l-2}_{2l-4}=0$. We get $\lambda=l-\frac{n}{2}-2$, and the assumption of knowledge of A_0, A_1 yields the linear equation for A_2:

$$\begin{aligned}
&A_2(2l-2k-\frac{n}{2}-2)(2l-2k-\frac{n}{2}-1)(2l-2k-\frac{n}{2})2k(2k-1)\dots \\
&(2k-l+3)l(l-1)(l-2k-\frac{n}{2}-1)(l-2k-\frac{n}{2})\dots(l-2k-\frac{n}{2}+l-4)(-2)(-4)= \\
&k(k-1)\dots(k-l+1)(2l-2k+5)(2l-2k+7)\dots(2l-2k+2l-1)\cdot \\
&\{(4l-2k+1)(4l-2k+3)(2l-2k-\frac{n}{2}-2)(2l-2k-\frac{n}{2})(2l-2k-\frac{n}{2}-1) \\
&-(2l-2k+3)(l-2k-\frac{n}{2}-1)(6l-6k-n+1)(2l-2k-\frac{n}{2}-1)2l \\
&-(2l-2k+1)(2l-2k+3)(l-2k-\frac{n}{2}-1)(l-2k-\frac{n}{2})(2l-2k-\frac{n}{2}-2)\}.
\end{aligned} \tag{11.16}$$

The only technical point is to prove that the polynomial of degree 5

$$\begin{aligned}
&(4l-2k+1)(4l-2k+3)(2l-2k-\frac{n}{2}-2)(2l-2k-\frac{n}{2})(2l-2k-\frac{n}{2}-1) \\
&-(2l-2k+3)(l-2k-\frac{n}{2}-1)(6l-6k-n+1)(2l-2k-\frac{n}{2}-1)2l \\
&-(2l-2k+1)(2l-2k+3)(l-2k-\frac{n}{2}-1)(l-2k-\frac{n}{2})(2l-2k-\frac{n}{2}-2)
\end{aligned} \tag{11.17}$$

is divisible by linear polynomials $l, l-1, 2l-2k-\frac{n}{2}, 6k-6l+n-1, 6k-6l+n+1$ and this can be verified directly. Moreover, the comparison of coefficients of leading monomials l^5 on both sides reveals that the polynomial of degree 5 is just the product of these linear polynomials and the required formula for A_2 follows.

Proof: (of Theorem 11.9)

A close inspection reveals that the system of linear equations for $\{A_0, A_1, \dots, A_l\}$ is triangular, i.e. jth equation solves A_j in terms of already computed variables $A_0, \dots, A_{j-1}$. When combined together, we arrive at the question of factorization of degree $2j+1$ polynomial, and it is routine but tedious to verify that all divisors of this polynomial are as required.

□

11.3 Casimir eigenvalues for $\mathfrak{l}$-modules in generalized Verma modules of scalar type

Here we describe procedure of the decomposition of a symmetric tensor into its irreducible components, necessary to determine $\mathfrak{l}$-structure of generalized Verma modules. Let us restrict to the even dimension n, $n = 2m$, and write

$$X = T_{k-2} \odot g + T_{k-4} \odot g \odot g + T_{k-6} \odot g \odot g \odot g + \dots, \tag{11.18}$$

where T_l is a symmetric trace-free tensor of valence l. The task is to compute the collection of tensors $\{T_l\}_l$ from $\{tr^j(X)\}_j, j \in \mathbb{N}$, where tr^j denotes j-th power of an application of the trace operator to a given symmetric tensor. For example, for $j = 1$ we get

$$tr(X) = \frac{4(m+k-2)}{k(k-1)} T_{k-2} + \frac{8(m+k-3)}{k(k-1)} T_{k-4} \odot g + \dots. \tag{11.19}$$

The coefficients in the upper triangular matrix corresponding to linear transformation $\{T_l\}_l \to \{tr^j(X)\}_j$ for k-th symmetric tensor X,

$$\begin{aligned} &tr^0(X) = X = T_k + T_{k-2} \odot g + T_{k-4} \odot g \odot g + \dots, \\ &tr^1(X) = 0 + c^1_1(k) T_{k-2} + c^1_2(k) T_{k-4} \odot g + \dots, \\ &tr^2(X) = 0 + 0 + c^2_2(k) T_{k-4} + \dots, \\ &\dots \end{aligned} \tag{11.20}$$

form the matrix $C = AB$, where A is a diagonal matrix with entries

$$a^i_j = \frac{4^i \delta^i_j}{k(k-1)\dots(k-2i+1)}$$

and B is an upper triangular matrix with entries

$$b^i_j = (j-1)(j-2)\dots(j-i+1)(m+k-j)(m+k-j-1)\dots(m+k-i-j+2)$$

for $i \leq j$. The matrix D is defined as $D = C^{-1}$, i.e. $D = B^{-1}A^{-1}$ and its explicit form is

$$\begin{pmatrix} 1 & -\frac{k(k-1)}{4(m+k-2)} & \frac{k(k-1)(k-2)(k-3)}{4^2 2(m+k-2)(m+k-3)} & -\frac{k\dots(k-5)}{4^3 6(m+k-2)(m+k-3)(m+k-4)} & \dots \\ 0 & -\frac{k(k-1)}{4(m+k-2)} & -\frac{k(k-1)(k-2)(k-3)}{4^2 (m+k-2)(m+k-4)} & \frac{k\dots(k-5)}{4^3 2(m+k-2)(m+k-4)(m+k-5)} & \dots \\ 0 & 0 & \frac{k(k-1)(k-2)(k-3)}{4^2 2(m+k-3)(m+k-4)} & -\frac{k\dots(k-5)}{4^3 2(m+k-3)(m+k-4)(m+k-6)} & \dots \\ 0 & 0 & 0 & \frac{k\dots(k-5)}{4^3 6(m+k-4)(m+k-5)(m+k-6)} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

An interested reader can derive a formula for the general element of this matrix using modular arithmetic.

In particular, $d^j_j(k) = (c^j_j(k))^{-1}$ for all k and we have

$$\begin{aligned} &T_k = d^0_0(k)X + d^0_1(k)tr(X) \odot g + d^0_2(k)tr^2(X) \odot g \odot g + \dots, \\ &T_{k-2} = 0 + d^1_1(k)tr(X) + d^1_2(k)tr^2(X) \odot g + \dots, \\ &T_{k-4} = 0 + 0 + d^2_2(k)tr^2(X) + \dots. \\ &\dots \end{aligned} \tag{11.21}$$

In terms of highest weights written in Euclidean basis, the (k, j)-th $\mathfrak{g}_0 = so(2n)$-module in the $so(2n+1, 1)$-Verma module induced from the spinor representation (our convention is that k denotes the homogeneity and j the number of traces in this homogeneity) twisted by character z has highest weight $(z - k, k - 2j, 0, \dots, 0)$. The Kostant-Laplace operator evaluated on this $\mathfrak{g}_0$-module gives the $\mathfrak{g}_0$-Casimir eigenvalues

$$\Box_{(k,j)} \equiv c^j_k = kz + 2jm + [(k-2j)(1-k) - 2j^2]. \tag{11.22}$$

For small values of (j, k), the collection of these eigenvalues is

$$\begin{aligned} &c^0_0 = 1, \\ &c^0_1 = z, \\ &c^0_2 = 2(z-1), c^1_2 = n + 2z - 2. \end{aligned} \tag{11.23}$$

Notice that there is a clash in the notation for elements of the matrix C and Casimir eigenvalues, but we hope the attentive reader will be able to distinguished them in the mathematical text.

12 Branching problem and singular vectors for the pair Lie $G_2 \stackrel{i}{\hookrightarrow} so(7)$ and non-compatible parabolic subalgebras

Let us assume that $\mathfrak{g}$ is a complex semisimple Lie algebra, $i(\mathfrak{g}')$ is reductive in $\mathfrak{g}$ and $i(\mathfrak{b}') \subset \mathfrak{b} \subset \mathfrak{p}$, where $\mathfrak{b}'$ and $\mathfrak{b}$ are Borel subalgebras of respectively $\mathfrak{g}'$ and $\mathfrak{g}$. Let $M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)$ be the generalized Verma $\mathfrak{g}$-module induced from the irreducible finite dimensional $\mathfrak{p}$-module with highest weight λ. We define the branching problem of $M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)$ over $\mathfrak{g}'$ to be the problem of finding all $\mathfrak{b}'$-singular vectors in $M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)$, that is, the set of all vectors annihilated by image of the nilradical of $\mathfrak{b}'$ on which the image of the Cartan subalgebra of $\mathfrak{b}'$ has diagonal action.

Recall that for an arbitrary $\mathfrak{g}$-module M, the Fernando-Kac subalgebra of $\mathfrak{g}$ associated to M is the Lie subalgebra of elements that act locally finitely on every vector $v \in M$. As the Fernando-Kac subalgebra associated to $M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)$ is $\mathfrak{p}$, it follows that the Fernando-Kac subalgebra of $\mathfrak{g}'$ associated to $M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)$ equals $i^{-1}(i(\mathfrak{g}')\cap\mathfrak{p})$. Then the requirement that $\mathfrak{p}$ contains the image of a Borel subalgebra of $\mathfrak{g}'$ implies the discrete decomposability of $M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)$ over $i(\mathfrak{g}')$, see [55].

If we drop the requirement $i(\mathfrak{b}') \subset \mathfrak{p}$, it appears that there is no good understanding of the simple $\mathfrak{g}'$-modules with Fernando-Kac subalgebras of the form $i^{-1}(i(\mathfrak{g}')\cap\mathfrak{p})$. Even more, there appears to be no complete understanding of the structure of the Lie algebra $i^{-1}(i(\mathfrak{g}')\cap\mathfrak{p})$. We note that if $\mathfrak{p}$ does not contain an image of the Borel subalgebra of $\mathfrak{g}'$, we can restrict our attention to a maximal reductive in $\bar{\mathfrak{g}}$ subalgebra $\mathfrak{g}_1'$ with the property that it has a Borel subalgebra whose image is contained in $\mathfrak{p}$. If $\mathfrak{g}_1' \neq \{0\}$, the branching problem of $M_{\mathfrak{p}}^{\mathfrak{g}}(V_\lambda)$ over $\mathfrak{g}_1'$ is well-posed.

Here we apply the distribution Fourier transform to the case Lie $G_2 \stackrel{i}{\hookrightarrow} so(7)$. In particular, we fix the conformal parabolic subalgebra of $\mathfrak{p} \subset so(7)$ and parabolic subalgebra $\mathfrak{p}' \subset$ Lie G_2 not compatible with $(\mathfrak{g}, \mathfrak{p})$, $\mathfrak{p}' = i(\mathfrak{g}') \cap \mathfrak{p}$. The nilradical $\mathfrak{n}$ of $\mathfrak{p}$ is commutative, while the nilradical $\mathfrak{n}'$ of $\mathfrak{p}'$ is the 3-step nilpotent Lie algebra with dimensions of $\mathfrak{p}'$-submodules $(2, 3, 5)$. As for the $so(7)$ generalized Verma modules, we restrict to the case of 1-dimensional inducing representations of $\mathfrak{p}$.

12.1 Branching problem and (non-compatible) parabolic subalgebras for the pair Lie $G_2 \stackrel{i}{\hookrightarrow} so(7)$

In the present Section we introduce the Lie theoretic conventions for the complex Lie algebra $so(7)$, exceptional Lie algebra Lie G_2, and Levi resp. parabolic subalgebras $\mathfrak{p}$ of $so(7)$ relative to parabolic subalgebras $i(\mathfrak{p}')$ of $i(\text{Lie } G_2)$. For more detailed review, cf. [55].

We start by fixing a Chevalley-Weyl basis of the Lie algebra $so(2n+1)$. Let the defining vector space V of $so(2n+1)$ have a basis $e_1, \dots e_n, e_0, e_{-1}, \dots e_{-n}$, where the defining symmetric bilinear form B of $so(2n+1)$ is given by $B(e_i, e_j) :=$

$0, i \neq -j$, $B(e_i, e_{-i}) := 1$, $B(e_i, e_0) := 0$, $B(e_0, e_0) := 1$, or alternatively defined as an element of $S^2(V^*)$,

$$B := \sum_{i=-n}^{n} e_i^* \otimes e_{-i}^* = (e_0^*)^2 + 2\sum_{i=1}^{n} e_i^* e_{-i}^*, \tag{12.1}$$

under the identification $v^* w^* := \frac{1}{2!}\left(v^* \otimes w^* + w^* \otimes v^*\right)$.

In the basis $e_1, \dots e_n, e_0, e_{-1}, \dots e_{-n}$, the matrices of the elements of $so(2n+1)$ are of the form

$$\left(\begin{array}{c|c|c} A & \begin{matrix} v_1 \\ \vdots \\ v_n \end{matrix} & C = -C^T \\ \hline w_1 \quad \dots \quad w_n & 0 & -v_1 \quad \dots \quad -v_n \\ \hline D = -D^T & \begin{matrix} -w_1 \\ \vdots \\ -w_n \end{matrix} & -A^T \end{array}\right),$$

i.e., all matrices $\mathbf{C}$ such that $\mathbf{A}^t\mathbf{B} + \mathbf{B}\mathbf{A} = 0$. We fix $e_1^*, \dots e_n^*, e_0^*, e_{-1}^*, \dots e_{-n}^*$ to be basis of V^* dual to $e_1, \dots e_n, e_0, e_{-1}, \dots e_{-n}$. We identify elements of $End(V)$ with elements of $V \otimes V^*$. In turn, we identify elements of $End(V)$ with their matrices in the basis $e_1, \dots, e_n$, e_0, $e_{-1}, \dots, e_{-n}$.

Fix the Cartan subalgebra $\mathfrak{h}$ of $so(2n+1)$ to be the subalgebra of diagonal matrices, i.e., the subalgebra spanned by the vectors $e_i \otimes e_i^* - e_{-i} \otimes e_{-i}^*$. Then the basis vectors $e_1, \dots e_n, e_0, e_{-1}, \dots e_{-n}$ are a basis for the $\mathfrak{h}$-weight vector decomposition of V. Let the $\mathfrak{h}$-weight of $e_i, i > 0$, be ε_i. Then the $\mathfrak{h}$-weight of $e_{-i}, i > 0$ is $-\varepsilon_i$, and an $\mathfrak{h}$-weight decomposition of $so(2n+1)$ is given by the elements $g_{\varepsilon_i - \varepsilon_j} := e_i \otimes e_j^* - e_{-j} \otimes e_{-i}^*$, $g_{\pm(\varepsilon_i + \varepsilon_j)} := e_{\pm i} \otimes e_{\mp j}^* - e_{\pm j} \otimes e_{\mp i}^*$ and $g_{\pm\varepsilon_i} := \sqrt{2}\left(e_{\pm i} \otimes e_0^* - e_0 \otimes e_{\mp i}^*\right)$, where $i, j > 0$.

Define the symmetric bilinear form $\langle \bullet, \bullet \rangle_{\mathfrak{g}}$ on $\mathfrak{h}^*$ by $\langle \varepsilon_i, \varepsilon_j \rangle_{\mathfrak{g}} = 1$ if $i = j$ and zero otherwise.

The root system of $so(2n+1)$ with respect to $\mathfrak{h}$ is given by $\Delta(\mathfrak{g}) := \Delta^+(\mathfrak{g}) \cup \Delta^-(\mathfrak{g})$, where we define

$$\Delta^+(\mathfrak{g}) := \{\varepsilon_i \pm \varepsilon_j | 1 \leq i < j \leq n\} \cup \{\varepsilon_i | 1 \leq i \leq n\} \tag{12.2}$$

and $\Delta^-(\mathfrak{g}) := -\Delta^+(\mathfrak{g})$. We fix the Borel subalgebra $\mathfrak{b}$ of $so(2n+1)$ to be the subalgebra spanned by $\mathfrak{h}$ and the elements $g_\alpha, \alpha \in \Delta^+(\mathfrak{g})$. The simple positive roots corresponding to $\mathfrak{b}$ are then given by

$$\eta_1 := \varepsilon_1 - \varepsilon_2, \dots, \eta_{n-1} := \varepsilon_{n-1} - \varepsilon_n, \eta_n := \varepsilon_n \quad .$$

For the remainder of this Section we fix the odd orthogonal Lie algebra to be $so(7)$. We order the 18 roots of $so(7)$ in graded lexicographic order with respect to their simple basis coordinates. We then label the negative roots by the indices $-9, \dots, -1$ and the positive roots by the indices $1, \dots, 9$. Finally, we abbreviate the Chevalley-Weyl generator $g_\alpha \in so(7)$ by g_i, where i is the label of the corresponding root. For example, $g_{\pm 1} = g_{\pm(\varepsilon_1 - \varepsilon_2)}$, $g_{\pm 2} = g_{\pm(\varepsilon_2 - \varepsilon_3)}$, $g_{\pm 3} = g_{\pm(\varepsilon_3)}$ are the simple positive and negative generators, the element $g_{-9} = g_{-\varepsilon_1 - \varepsilon_2}$ is the Chevalley-Weyl generator corresponding to the lowest root, and so on. We furthermore set $h_1 := [g_1, g_{-1}]$, $h_2 := [g_2, g_{-2}]$, $h_3 := 1/2[g_3, g_{-3}]$.

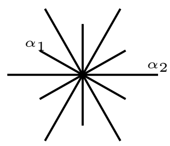
Let now $\mathfrak{g}' = \mathrm{Lie}\, G_2$. One way of defining the positive root system of Lie G_2 is by setting it to be the set of vectors

$$\Delta(\mathfrak{g}') := \{\pm(1,0), \pm(0,1), \pm(1,1), \pm(1,2), \pm(1,3), \pm(2,3)\}. \tag{12.3}$$

We set $\alpha_1 := (1,0)$ and $\alpha_2 := (0,1)$. We fix a bilinear form $\langle \bullet, \bullet \rangle_{\bar{\mathfrak{g}}}$ on $\mathfrak{h}'$, proportional to the one induced by Killing form by setting

$$\begin{pmatrix} \langle \alpha_1, \alpha_1 \rangle_{\bar{\mathfrak{g}}} & \langle \alpha_1, \alpha_2 \rangle_{\bar{\mathfrak{g}}} \\ \langle \alpha_2, \alpha_1 \rangle_{\bar{\mathfrak{g}}} & \langle \alpha_2, \alpha_2 \rangle_{\bar{\mathfrak{g}}} \end{pmatrix} := \begin{pmatrix} 2 & -3 \\ -3 & 6 \end{pmatrix}. \tag{12.4}$$

In an $\langle \bullet, \bullet \rangle_{\bar{\mathfrak{g}}}$-orthogonal basis the root system of Lie G_2 is often drawn as



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Similarly to the $so(7)$ case, we order the 12 roots of Lie G_2 in the graded lexicographic order with respect to their simple basis coordinates, and label the roots with the indices $-6, \dots, -1, 1, \dots, 6$. We fix a basis for the Lie algebra Lie G_2 by giving a set of Chevalley-Weyl generators g'_i, $i \in \{\pm 1, \cdots \pm 6\}$, and by setting $h'_1 := [g'_1, g'_{-1}]$, $h'_2 := 3[g'_2, g'_{-2}]$. Just as in the $so(7)$ case, we ask that the generator $g'_{\pm i}$ correspond to the root space labelled by $\pm i$.

All embeddings Lie $G_2 \stackrel{i}{\hookrightarrow} so(7)$ are conjugate over $\mathbb{C}$. One such embedding is given via

$$i(g'_{\pm 2}) := g_{\pm 2}, \quad i(g'_{\pm 1}) := g_{\pm 1} + g_{\pm 3} \quad .$$

As $g'_{\pm 1}, g'_{\pm 2}$ generate Lie G_2, the preceding data determines the map i and one can directly check it is a Lie algebra homomorphism. Alternatively, we can use $i(g'_{\pm 1}), i(g'_{\pm 2})$ to generate a Lie subalgebra of $so(7)$, verify that this subalgebra is indeed 14-dimensional and simple, and finally use this 14-dimensional image to compute the structure constants of Lie G_2.

We denote by $\omega_1 := \varepsilon_1$, $\omega_2 := \varepsilon_1 + \varepsilon_2$ and $\omega_3 := \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ the fundamental weights of $so(7)$ and by $\psi_1 := 2\alpha_1 + \alpha_2$, $\psi_2 := 3\alpha_1 + 2\alpha_2$ the fundamental weights of Lie G_2.

Let $\mathrm{pr} : \mathfrak{h}^* \to \mathfrak{h}'^*$ be the map naturally induced by i. Then

$$\mathrm{pr}(\underbrace{\varepsilon_1 - \varepsilon_2}_{\eta_1}) = \mathrm{pr}(\underbrace{\ \varepsilon_3\ }_{\eta_3}) = \alpha_1, \quad \mathrm{pr}(\underbrace{\varepsilon_2 - \varepsilon_3}_{\eta_2}) = \alpha_2, \tag{12.5}$$

or equivalently

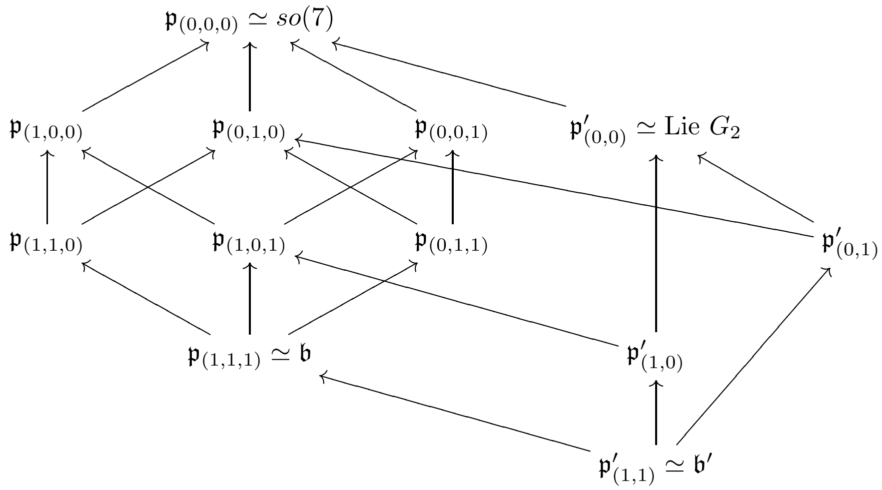
$$\mathrm{pr}(\omega_1) = \mathrm{pr}(\omega_3) = \psi_1, \quad \mathrm{pr}(\omega_2) = \psi_2.$$

Conversely, $\iota : \mathfrak{h}'^* \to \mathfrak{h}^*$ is the map

$$\iota(\alpha_2) = 3\eta_2 = 3\varepsilon_2 - 3\varepsilon_3, \quad \iota(\alpha_1) = \eta_1 + 2\eta_3 = \varepsilon_1 - \varepsilon_2 + 2\varepsilon_3. \tag{12.6}$$

We recall from [55] that the pairwise inclusions between the parabolic subalgebras of $so(7)$ and the embeddings of the parabolic subalgebras of Lie G_2 are given as follows.

Lemma 12.1 *For the pair* $G_2 \stackrel{i}{\hookrightarrow} so(7)$*, let* $\mathfrak{h}, \mathfrak{b}, \mathfrak{p}, \mathfrak{h}', \mathfrak{b}', \mathfrak{p}'$ *denote respectively Cartan, Borel and parabolic subalgebras with the assumptions that* $i(\mathfrak{h}') \subset \mathfrak{h} \subset \mathfrak{b}$*,* $i(\mathfrak{b}') \subset \mathfrak{b} \subset \mathfrak{p}$*,* $\mathfrak{b}' \subset \mathfrak{p}'$*. Then we have the following inclusion diagram for all possible values of* $\mathfrak{p}, \mathfrak{p}'$*.*



If a path of arrows exists from one node of the diagram to the other, then the corresponding parabolic subalgebras lie inside one another. If in the diagram a direct arrow exists from a parabolic subalgebra $\mathfrak{p}'$ *of* Lie G_2 *to a parabolic subalgebra* $\mathfrak{p}$ *of* $so(7)$*, then* $\mathfrak{p}' = i^{-1}(i(\mathfrak{g}') \cap \mathfrak{p})$*.*

The structure of $so(7)$ as a module over the Levi part of parabolic subalgebras of Lie G_2 is described in detail in [55, Lemma 5.2], and we will implicitly use it throughout Subsection 12.2.

Note that in the special case of conformal parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b} \supset \mathfrak{h}$ of $so(7)$ and $\mathfrak{p}'$ of i(Lie G_2) given by $\mathfrak{p}' = i(\mathfrak{g}) \cap \mathfrak{p}$ are not compatible.

12.2 Lie $G_2 \cap \mathfrak{p}'$-singular vectors in the $so(7)$-generalized Verma modules of scalar type for the conformal parabolic subalgebra

In this subsection we determine the $\mathfrak{g}' = i(\text{Lie } G_2) \cap \mathfrak{p}$-singular vectors in the family of $\mathfrak{g} = so(7)$ generalized Verma modules $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ induced from character $\chi_\lambda : \mathfrak{p} \to \mathbb{C}$ of the weight $\lambda\varepsilon_1$ (ε_1 is the first fundamental weight of $so(7)$).

In this way, the results computed in the present section are analytic counterpart realized by F-method of the algebraic results developed in [55].

Let us denote by v_λ the highest weight vector of the generalized Verma $so(7)$-module $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{V}_\lambda)$. Notice that as $i(h_{\alpha_1}) = 3h_{\varepsilon_2-\varepsilon_3}$ and $i(h_{\alpha_2}) = h_{\varepsilon_1-\varepsilon_2} + 2h_{\varepsilon_3}$, the $\mathfrak{h}'$-weight of v_λ is $\mu = \lambda(\alpha_1 + 2\alpha_2)$ because $\langle \mu, \alpha_1\rangle = 0$, $\langle \mu, \alpha_2\rangle = \lambda$.

The opposite nilradical $\mathfrak{n}_-$ associated to the parabolic subalgebra $\mathfrak{p}$ is commutative,

$$\mathcal{U}(\mathfrak{n}_-) \otimes \mathbb{V}^\vee \simeq \mathrm{Pol}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_5}\right) \otimes \mathbb{C}_\lambda \simeq \mathrm{Pol}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_5}\right)$$

and the variables $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_5}$ denote the following $so(7)$-root spaces:

$$\begin{array}{lll} \frac{\partial}{\partial x_1} := g_{-\varepsilon_1+\varepsilon_2} = g_{-1}, & \frac{\partial}{\partial x_2} := g_{-\varepsilon_1-\varepsilon_3} = g_{-8}, & \frac{\partial}{\partial x_3} := g_{-\varepsilon_1} = g_{-6}, \\ \frac{\partial}{\partial x_5} := g_{-\varepsilon_1+\varepsilon_3} = g_{-4}, & \frac{\partial}{\partial x_4} := g_{-\varepsilon_1-\varepsilon_2} = g_{-9}. & \end{array}$$

Here, we recall that $[x_i, \frac{\partial}{\partial x_j}] = -[\frac{\partial}{\partial x_j}, x_i] = \begin{cases} 0 & \text{if } i \neq j \\ -1 & \text{if } i = j \end{cases}$ is the adjoint action of the differential operator x_i on the differential operator $\frac{\partial}{\partial x_j}$.

By Lemma 12.1, the simple part of the Levi factor of $i(\tilde{\mathfrak{p}})$ is isomorphic to $sl(2)$ and its action on $\mathfrak{n}_-$ can be extended to action on $\mathcal{U}(\mathfrak{n}_-) \simeq S^\star(\mathfrak{n}_-)$. The elements $h := h_2, e := g_2, f := g_{-2}$ give the standard h, e, f-basis of $sl(2)$, i.e., $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. Then the action of h on $\mathfrak{n}_-$ is the adjoint action of $x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_4} - x_5\frac{\partial}{\partial x_5}$, the action of e is the adjoint action of $-x_5\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2}$ and the action of f is the adjoint action of $-x_1\frac{\partial}{\partial x_5} + x_2\frac{\partial}{\partial x_4}$.

We now proceed to generate all $\mathfrak{l}'$-invariant singular vectors in $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$, i.e., the singular vectors that induce $i(\mathrm{Lie}\ G_2)$-generalized Verma modules induced from character (scalar generalized Verma modules). To do that we need the following Lemma from the classical invariant theory of reductive Lie algebras.

Lemma 12.2 *Then the $sl(2)$-invariants of $S^\star(\mathfrak{n}_-)$ are an associative algebra generated by the elements $u_1 := \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_5}$ and $u_2 = \frac{\partial}{\partial x_3}$.*

Proof:

Direct computation shows that u_1, u_2 are invariants. Alternatively, as the direct sum of two two-dimensional $sl(2)$-modules gives a natural embedding $sl(2) \hookrightarrow sl(2) \times sl(2)$, we can view u_1 as the invariant element induced by the defining symmetric bilinear form of $so(4) \simeq sl(2) \times sl(2)$. Let the positive root of $sl(2)$ be η , and the multiplicity of the $sl(2)$-module with highest weight $t\frac{\eta}{2}$ in $S^l(\mathfrak{n}_-)$ be $b(l, t)$. Denoting by x, z a couple of formal variables, we have that $\sum_{l\in\mathbb{Z}_{\geq 0}, t\in\mathbb{Z}_{\geq 0}} b(l, t)(z^l x^t + z^l x^{-1-t})$ is the power series expansion of the rational function

$$(1 - x^{-2})\frac{1}{(1 - zx)^2}\frac{1}{(1 - zx^{-1})^2}\frac{1}{(1 - z)}.$$

Direct computation shows that $b(l,t)$ equals $-1/2t^2+1+1/2tl+1/2l+1/2t$ whenever $l+t$ is even and $-1/2t^2+1/2+1/2tl+1/2l$ whenever $l+t$ is odd, and l, t satisfy the inequalities $l \geq t \geq 0$. Finally, substituting with $t=0$, we get $b(l,0)=1+l/2$ for even l and $b(l,0)=1/2+l/2$. For a fixed l, this is exactly the dimension of the vector space generated by the linearly independent invariants $u_1^q u_2^r \in S^l(\mathfrak{n}_-)$ with $r+2q=l$, which completes the proof of our Lemma.

□

It follows from the definition of embedding map i that

$$\begin{aligned}
&\mathrm{ad}(i(g_1)) = -x_2\partial_4 + x_1\partial_5,\\
&\mathrm{ad}(i(g_{-1})) = -x_4\partial_2 + x_5\partial_1,\\
&\mathrm{ad}(i(h_1)) = [\mathrm{ad}(i(g_1)), \mathrm{ad}(i(g_{-1}))] = x_1\partial_{x_1} + x_2\partial_{x_2} - x_4\partial_{x_4} - x_5\partial_{x_5},\\
&\mathrm{ad}(i(h_2)) = -3x_2\partial_2 - x_3\partial_3 + x_5\partial_5 - 2x_1\partial_1,
\end{aligned}$$

and therefore

$$\mathrm{ad}(i(3h_1+2h_2)) = -x_1\partial_{x_1} - 3x_2\partial_{x_2} - 2x_3\partial_{x_3} - 3x_4\partial_{x_4} - x_5\partial_{x_5}$$

represents the central element of the Levi factor $i(\mathfrak{l}')$. It induces a grading gr on the Weyl algebra of $\mathfrak{n}_-$ in the variables

$$\{x_1, x_2, x_3, x_4, x_5, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_5}\},$$

we have

$$\begin{aligned}
&\mathrm{gr}(x_1) = -\mathrm{gr}(\partial_1) = -1,\ \mathrm{gr}(x_2) = -\mathrm{gr}(\partial_2) = -3,\\
&\mathrm{gr}(x_3) = -\mathrm{gr}(\partial_3) = -2,\ \mathrm{gr}(x_4) = -\mathrm{gr}(\partial_4) = -3,\\
&\mathrm{gr}(x_5) = -\mathrm{gr}(\partial_5) = -1 \quad ,
\end{aligned}$$

i.e. the invariants $u_1 = \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_5}$ and $u_2 = (\frac{\partial}{\partial x_3})^2$ are homogeneous with respect to the gr-grading.

Employing the distributive Fourier transform means that we pass to the polynomial ring $Pol[\xi_1,\dots,\xi_5]$, where ξ_i denote the Fourier images of $\frac{\partial}{\partial x_i}$, $i = 1,\dots,5$. In the image of the Fourier transform, the subalgebra of $\mathfrak{l}'_s = sl(2)$-invariants with respect to the Fourier dual representation is the polynomial ring $Pol[\xi_1\xi_4+\xi_2\xi_5, \xi_3]$, cf. Lemma 12.2.

Theorem 12.3 *Let v_λ be the highest weight vector of the $so(7)$-generalized Verma module $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ induced from character χ_λ, $\lambda \in \mathbb{C}$. Let $N \in \mathbb{N}$ be a positive integer and $A_i \in \mathbb{C}$, $i \in \mathbb{N}$ a collection of complex numbers such that at least one of them is non-zero.*

1. *A vector $u \cdot v_\lambda$ is $i(\mathrm{Lie}\ G_2) \cap \mathfrak{p}$-singular vector ("singular vector of scalar type") of homegenuity $2N$ if and only if $\lambda = N - 5/2$ and $u = (2u_1+u_2)^N = (2u_1+u_2)^{\lambda+5/2}$.*

2. $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ *has no* $i(\text{Lie } G_2) \cap \mathfrak{p}$*-singular vector of homogeneity* $2N+1$.

3. *A vector in* $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ *is* $so(7) \cap \mathfrak{p}$*-singular if and only if it is the* $i(\text{Lie } G_2) \cap \mathfrak{p}$*-singular vector given in point 1.*

Proof:

1. By Lemma 12.2 and Subsection 12.1 a $\mathfrak{p}'$-singular vector of homogeneity $2N$ must be of the form $u := \sum_{k=0}^{N} A_k u_1^k u_2^{N-k}$.

First we determine the action of the second simple positive root g_2 in the Fourier dual representation $d\tilde{\pi}(\text{ad}(i(g_2)))$, acting on $Pol[\xi_1, \dots, \xi_5]$.

Let n_i be non-negative integers. Then

$$
\begin{aligned}
& i(g_2) \cdot (\xi_1^{n_1} \xi_2^{n_2} \xi_3^{n_3} \xi_4^{n_4} \xi_5^{n_5} \cdot v_\lambda) = \\
& \big((-n_1^2 + n_1)\xi_1^{n_1-1} \xi_2^{n_2} \xi_3^{n_3} \xi_4^{n_4} \xi_5^{n_5} - n_2 \xi_1^{n_1} \xi_2^{n_2-1} \xi_3^{n_3+1} \xi_4^{n_4} \xi_5^{n_5} \\
& + n_1 \lambda \xi_1^{n_1-1} \xi_2^{n_2} \xi_3^{n_3} \xi_4^{n_4} \xi_5^{n_5} + (n_3^2 - n_3) \xi_1^{n_1} \xi_2^{n_2} \xi_3^{n_3-2} \xi_4^{n_4+1} \xi_5^{n_5} \\
& + 2n_3 \xi_1^{n_1} \xi_2^{n_2} \xi_3^{n_3-1} \xi_4^{n_4} \xi_5^{n_5+1} - n_1 n_5 \xi_1^{n_1-1} \xi_2^{n_2} \xi_3^{n_3} \xi_4^{n_4} \xi_5^{n_5} \\
& + n_2 n_5 \xi_1^{n_1} \xi_2^{n_2-1} \xi_3^{n_3} \xi_4^{n_4+1} \xi_5^{n_5-1} - n_1 n_2 \xi_1^{n_1-1} \xi_2^{n_2} \xi_3^{n_3} \xi_4^{n_4} \xi_5^{n_5} \\
& - n_1 n_3 \xi_1^{n_1-1} \xi_2^{n_2} \xi_3^{n_3} \xi_4^{n_4} \xi_5^{n_5}\big) \cdot v_\lambda \\
& = (-\xi_1 \partial_1^2 - \xi_3 \partial_2 + \lambda \partial_1 + \xi_4 \partial_3^2 + 2\xi_5 \partial_3 - \xi_5 \partial_1 \partial_5 + \xi_4 \partial_2 \partial_5 \\
& - \xi_2 \partial_1 \partial_2 - \xi_3 \partial_1 \partial_3) \cdot (\xi_1^{n_1} \xi_2^{n_2} \xi_3^{n_3} \xi_4^{n_4} \xi_5^{n_5}) \cdot v_\lambda, \qquad (12.7)
\end{aligned}
$$

where ∂_i is short notation for the differential operator $\frac{\partial}{\partial \xi_i}$, $i = 1, \dots, 5$. Let $P(\lambda)$ denote the differential operator on $\mathbb{C}[\xi_1, \xi_2, \xi_3, \xi_4, \xi_5]$ obtained in the following computation:

$$
\begin{aligned}
& (-\xi_1 \partial_1^2 - \xi_3 \partial_2 + \lambda \partial_1 + \xi_4 \partial_3^2 + 2\xi_5 \partial_3 \\
& -\xi_5 \partial_1 \partial_5 + \xi_4 \partial_2 \partial_5 - \xi_2 \partial_1 \partial_2 - \xi_3 \partial_1 \partial_3) \\
= \; & (-\xi_3 \partial_2 + \xi_4 \partial_3^2 + 2\xi_5 \partial_3 + (-\xi_5 \partial_1 + \xi_4 \partial_2)\partial_5 \\
& -(\xi_1 \partial_1 + \xi_2 \partial_2 + \xi_3 \partial_3 - \lambda)\partial_1) \\
= \; & (-\xi_3 \partial_2 + \xi_4 \partial_3^2 + 2\xi_5 \partial_3 + \partial_5(-\xi_5 \partial_1 + \xi_4 \partial_2) \\
& -(\xi_1 \partial_1 + \xi_2 \partial_2 + \xi_3 \partial_3 - \lambda - 1)\partial_1).
\end{aligned}
$$

We compute

$$
\begin{aligned}
\partial_1 \cdot (u_1^{b_1} u_2^{b_2}) &= b_1 \xi_4 u_1^{b_1-1} u_2^{b_2}, \\
\partial_2 \cdot (u_1^{b_1} u_2^{b_2}) &= b_1 \xi_5 u_1^{b_1-1} u_2^{b_2}, \\
(\xi_1 \partial_1 + \xi_2 \partial_2) \cdot (u_1^{b_1} u_2^{b_2}) &= b_1 u_1^{b_1} u_2^{b_2}, \\
\partial_3 \cdot (u_1^{b_1} u_2^{b_2}) &= 2b_2 \xi_3 u_1^{b_1} u_2^{b_2-1}, \\
\partial_3^2 \cdot (u_1^{b_1} u_2^{b_2}) &= 2b_2(2b_2 - 1) u_1^{b_1} u_2^{b_2-1},
\end{aligned}
$$

and so

$$\begin{aligned}
& (-\xi_3\partial_2 + \xi_4\partial_3^2 + 2\xi_5\partial_3 + \partial_5(-\xi_5\partial_1 + \xi_4\partial_2) - \\
& (\xi_1\partial_1 + \xi_2\partial_2 + \xi_3\partial_3 - \lambda - 1)\partial_1) \cdot (u_1^{b_1} u_2^{b_2}) \\
= \; & (-\xi_3\partial_2 + \xi_4\partial_3^2 + 2\xi_5\partial_3 - (\xi_1\partial_1 + \xi_2\partial_2 + \xi_3\partial_3 - \lambda - 1)\partial_1) \cdot (u_1^{b_1} u_2^{b_2}) \\
= \; & -b_1\xi_3\xi_5 u_1^{b_1-1} u_2^{b_2} + 2b_2(2b_2 - 1)\xi_4 u_1^{b_1} u_2^{b_2-1} + 4b_2\xi_5\xi_3 u_1^{b_1} u_2^{b_2-1} \\
& -(\xi_1\partial_1 + \xi_2\partial_2 + \xi_3\partial_3 - \lambda - 1) \cdot (b_1\xi_4 u_1^{b_1-1} u_2^{b_2}) \\
= \; & -b_1\xi_3\xi_5 u_1^{b_1-1} u_2^{b_2} + 2b_2(2b_2 - 1)\xi_4 u_1^{b_1} u_2^{b_2-1} + 4b_2\xi_5\xi_3 u_1^{b_1} u_2^{b_2-1} \\
& +(-b_1 + 1 + \lambda + 1 - 2b_2) b_1\xi_4 u_1^{b_1-1} u_2^{b_2} \\
= \; & 2b_2((2b_2 - 1)\xi_4 + 2\xi_5\xi_3) u_1^{b_1} u_2^{b_2-1} \\
& +b_1((-b_1 - 2b_2 + \lambda + 2)\xi_4 - \xi_3\xi_5) u_1^{b_1-1} u_2^{b_2} \quad . \qquad (12.8)
\end{aligned}$$

The operator $P(\lambda)$ is homogeneous with respect to the grading in (12.7), and its application to a homogeneous polynomial in $u_1 = u_1(\xi_1, \dots, \xi_5)$, $u_2 = u_2(\xi_1, \dots, \xi_5)$ yields

$$\begin{aligned}
& P(\lambda)(\textstyle\sum_{k=0}^N A_k u_1^k u_2^{N-k}) \\
= \; & \textstyle\sum_{k=0}^N A_k (2(N-k)((2(N-k) - 1)\xi_4 + 2\xi_5\xi_3) u_1^k u_2^{N-k-1} \\
& +k((-k - 2(N-k) + \lambda + 2)\xi_4 - \xi_3\xi_5) u_1^{k-1} u_2^{N-k}) \\
= \; & \textstyle\sum_{s=1}^{N+1} 2A_{s-1}(N - (s-1))((2(N - (s-1)) - 1)\xi_4 \\
& +2\xi_5\xi_3) u_1^{(s-1)} u_2^{N-(s-1)-1} \\
& + \textstyle\sum_{k=0}^N k A_k ((-k - 2(N-k) + \lambda + 2)\xi_4 - \xi_3\xi_5) u_1^{k-1} u_2^{N-k} \\
= \; & \textstyle\sum_{s=1}^N (2A_{s-1}(N - s + 1)((2N - 2s + 1)\xi_4 + 2\xi_5\xi_3) \\
& \qquad + sA_s((s - 2N + \lambda + 2)\xi_4 - \xi_3\xi_5)) u_1^{s-1} u_2^{N-s}).
\end{aligned}$$

The $2N$ summands of the form $\xi_4 u_1^{s-1} u_2^{N-s}$ and $\xi_3\xi_5 u_1^{s-1} u_2^{N-s}$ are linearly independent and therefore the above sum is zero if and only if

$$\begin{aligned}
& 2A_{s-1}(N - s + 1)((2N - 2s + 1)\xi_4 + 2\xi_5\xi_3) \\
& +sA_s((s - 2N + \lambda + 2)\xi_4 - \xi_3\xi_5)) \qquad (12.9)
\end{aligned}$$

equals zero for all values of s. When $s = N$, the above sum becomes

$$2A_{N-1}(\xi_4 + 2\xi_3\xi_5) + NA_N((-N + \lambda + 2)\xi_4 - \xi_3\xi_5) \quad .$$

It is a straightforward check that if A_N vanishes, then $A_{N-1}, A_{N-2}, \dots$ must also vanish; therefore we may assume $A_N \neq 0$. The vanishing of the coefficient in front of ξ_4 implies $A_{N-1} = -\frac{1}{2} N A_N (-N + \lambda + 2)$ and in turn, the vanishing of the coefficient in front of $\xi_3\xi_5$ implies $-5 + 2N - 2\lambda = 0$. Therefore

$$\lambda = N - 5/2 \quad .$$

Substituting λ back into (12.9), we get

$$\begin{aligned}
& 2A_{s-1}(N - s + 1)((2N - 2s + 1)\xi_4 + 2\xi_5\xi_3) \\
& +sA_s((-N + s - 1/2)\xi_4 - \xi_3\xi_5) = 0.
\end{aligned}$$

This implies $A_s = \frac{4(N-s+1)}{s} A_{s-1} = \cdots = 4^s \binom{N}{s} A_0$, which completes the proof of 1).

2. A $i(\text{Lie } G_2) \cap \mathfrak{p}$-singular vector is, in particular, $sl(2) \simeq i([\mathfrak{l}', \mathfrak{l}'])$-singular and by Lemma 12.2 must be of the form $u = \xi_3 \sum_{k=0}^{N} A_k u_1^k u_2^{N-k}$. The application of $2\xi_5 \partial_3$ converts $A_N(\xi_1\xi_4 + \xi_2\xi_5)^N \xi_3$ into $2A_N(\xi_1\xi_4 + \xi_2\xi_5)^N \xi_5$. Furthermore $2A_N(\xi_1\xi_4 + \xi_2\xi_5)^N \xi_5$ contains in its binomial expansion $2A_N(\xi_1\xi_4)^N \xi_5$. Direct check shows that the action of $P(\lambda)$ on $(\xi_1\xi_4 + \xi_2\xi_5)^{N-i}\xi_3^{1+2i}$ for $i > 0$ does not contain the monomial $(\xi_1\xi_4)^N \xi_5$. This implies that $A_N = 0$ and by induction, the polynomial is trivial. Consequently, there is no nontrivial odd homogeneity polynomial solving the differential equation $P(\lambda)$.

As an illustration, for $N = 0$ we have $P(\lambda)(A_0\xi_3) = 2A_0\xi_5$. This vanishes provided $A_0 = 0$, which implies the polynomial is trivial.

3. An $so(7) \cap \mathfrak{p}$-singular vector must be $i(\text{Lie } G_2) \cap \mathfrak{p}$-singular. From 1) we know that there is at most one $i(\text{Lie } G_2) \cap \mathfrak{p}$-singular vector. The simple part of $\mathfrak{l}$ is isomorphic to $so(5)$ and induces the quadratic form with matrix in the coordinates $\xi_1, \ldots, \xi_5$

$$Q = \begin{pmatrix} 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix},$$

i.e., the metric of the form

$$g(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = (d\xi_3)^2 + 2(d\xi_1 \otimes d\xi_4 + d\xi_4 \otimes d\xi_1) + 2(d\xi_2 \otimes d\xi_5 + d\xi_5 \otimes d\xi_2).$$

The Fourier transform of the $so(5)$-invariant Laplace operator associated to Q is

$$\mathcal{F}(\Box_\xi) = Q(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) = 4(\xi_1\xi_4 + \xi_2\xi_5) + \xi_3^2.$$

Relying on $\Box_\xi$ and the binomial formula for $(4(\xi_1\xi_4 + \xi_2\xi_5) + \xi_3^2)^s$, we see that the Lie $G_2 \cap \mathfrak{p}$-singular vector constructed 1) is indeed $so(7) \cap \mathfrak{p}$-singular. The proof is complete.

□

We note that an alternative proof of 3) can be given as follows. From a well known example (see e.g., [21], [43], [44]) of singular vectors in conformal geometry of dimension 5 describing conformally invariant powers of the Laplace operator, we know that for $\lambda \in \{-3/2, -1/2, 1/2, \ldots\}$ there exists one $so(7) \cap \mathfrak{p}$-singular vector in $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$. On the other hand points 1) and 2) of Theorem 12.3 present us with only one such candidate, so that candidate must be the $so(7) \cap \mathfrak{p}$-singular vector in question.

For $\lambda \in \{-3/2, -1/2, 1/2, \ldots\}$, the $\mathfrak{h}$-weight of the $so(7) \cap \mathfrak{p}$-singular vector in $M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda)$ given by Theorem 12.3 equals $(\lambda - 2N)\varepsilon_1 = (\lambda - 2(\lambda + 5/2))\varepsilon_1 = (-\lambda - 5)\varepsilon_1$. Therefore the vector from Theorem 12.3 corresponds to the homomorphism of generalized Verma modules

$$M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_\lambda) \hookrightarrow M^{so(7)}_{\mathfrak{p}_{(1,0,0)}}(\mathbb{C}_{-\lambda-5}). \tag{12.10}$$

In an analogous fashion we conclude that Theorem 12.3 gives a homomorphism of generalized Verma modules

$$M^{\text{Lie } G_2}_{\mathfrak{p}'_{(1,0)}}(\mathbb{V}_{\lambda\psi_1}) \to M^{\text{Lie } G_2}_{\mathfrak{p}'_{(1,0)}}(\mathbb{C}_{(-\lambda-5)\psi_1}). \tag{12.11}$$

We conclude this paper with the following observation from [50], a proof of which we include for completeness.

Proposition 12.4 *Suppose* $\lambda \in \{-3/2, -1/2, 1/2, \dots\}$. *Then both* (12.10) *and* (12.11) *are non-standard homomorphisms.*

Proof:

1. Let $\rho_{\mathfrak{l}}$ be the half-sum of the positive roots of $\mathfrak{l}$, i.e., $\rho_{\mathfrak{l}} := 3/2\varepsilon_2 + 1/2\varepsilon_3$, and let s_{η_3} denote the reflection with respect to the simple root $\eta_3 = \varepsilon_3$. Then

$$s_{\eta_3}(\lambda\varepsilon_1 + \rho_{\mathfrak{l}}) - ((-\lambda - 5)\varepsilon_1 + \rho_{\mathfrak{l}}) = (2\lambda + 5)\varepsilon_1 - \varepsilon_3. \tag{12.12}$$

   As $\lambda \in \{-3/2, -1/2, 1/2, \dots\}$, the expression (12.12) is a sum of positive roots of $so(7)$. Therefore by [19, Chapter 7] the non-generalized Verma module with highest weight $(-\lambda - 5)\varepsilon_1 + \rho_{\mathfrak{l}}$ lies in the non-generalized Verma module with highest weight $s_{\eta_3}(\lambda\varepsilon_1 + \rho_{\mathfrak{l}})$. Therefore by [49, Proposition 3.3] the homomorphism (12.10) is non-standard.

2. Let $\rho_{\mathfrak{l}'} = 1/2\alpha_2$. Let s_{α_2} denote the reflection with respect to the simple root α_2 (in $\mathfrak{h}'^*$). Then

$$s_{\alpha_1}(\lambda\psi_1 + \rho_{\mathfrak{l}'}) - ((-5 - \lambda)\psi_1 + \rho_{\mathfrak{l}'}) = (2\lambda + 6)\alpha_2 + (4\lambda + 16)\alpha_1$$

   is clearly a positive integral combination of positive roots of Lie G_2 and the statement follows again by [19, Chapter 7] and [49, Proposition 3.3].

□

13 Appendix: Jacobi and Gegenbauer polynomials

In the Appendix we summarize for reader's convenience a few basic conventions and properties related to Jacobi and Gegenbauer polynomials.

First, we define the analytic continuation of binomial coefficient by

$$\binom{z}{n} := \frac{\Gamma(z+1)}{\Gamma(n+1)\Gamma(z-n+1)},$$

where $\Gamma(z)$ is the Gamma function. Then $\binom{z}{n} = 0$ if $n - z \in \mathbb{N}_+$ and $z \notin -\mathbb{N}_+$.

The Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ are polynomials of degree n with two parameters α, β defined by special values of the hypergeometric function:

$$\begin{aligned} P_n^{(\alpha,\beta)}(z) =& \binom{n+\alpha}{n} {}_2F_1\left(-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-z}{2}\right) \\ =& \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \sum_{m=0}^{n} \binom{n}{m} \frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(\alpha+m+1)} \left(\frac{z-1}{2}\right)^m \\ =& \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} \left(\frac{z-1}{2}\right)^{n-j} \left(\frac{z+1}{2}\right)^j . \end{aligned}$$

The normalization is given by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n} = \frac{(\alpha+1)_n}{n!},$$

where $(\alpha+1)_n$ is the Pochhammer symbol for the rising factorial.

Jacobi polynomials satisfy the orthogonality condition

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)\, dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!} \delta_{nm} \tag{13.1}$$

for $\alpha > -1$ and $\beta > -1$.

The polynomials have various symmetries, e.g.

$$P_n^{(\alpha,\beta)}(-z) = (-1)^n P_n^{(\beta,\alpha)}(z).$$

In the special case when the four quantities n, $n+\alpha$, $n+\beta$, and $n+\alpha+\beta$ are nonnegative integers, the Jacobi polynomial can be written as

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) \quad = \quad & (n+\alpha)!(n+\beta)! \sum_s [s!(n+\alpha-s)!(\beta+s)!(n-s)!]^{-1} \\ & \left(\frac{x-1}{2}\right)^{n-s} \left(\frac{x+1}{2}\right)^s , \end{aligned} \tag{13.2}$$

where the sum on s runs over all integer values for which the arguments of the factorials are nonnegative.

The kth derivative of $P_n^{(\alpha,\beta)}(z)$ leads to

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k} P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+\beta+n+1+k)}{2^k\Gamma(\alpha+\beta+n+1)} P_{n-k}^{(\alpha+k,\beta+k)}(z). \tag{13.3}$$

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are solution of the hypergeometric differential equation

$$[(1-x^2)\frac{d^2}{dx^2} + (\beta-\alpha-(\alpha+\beta+2)x)\frac{d}{dx} + n(n+\alpha+\beta+1)]y = 0. \tag{13.4}$$

The Jacobi polynomials specialize for $\alpha = \beta$ to the Gegenbauer polynomials, which can be defined in terms of their generating function

$$\frac{1}{(1-2xt+t^2)^\alpha} = \sum_{n=0}^{\infty} C_n^{(\alpha)}(x) t^n$$

and satisfy the recurrence relation

$$C_n^\alpha(x) = \frac{1}{n}[2x(n+\alpha-1)C_{n-1}^\alpha(x) - (n+2\alpha-2)C_{n-2}^\alpha(x)]$$

with $C_0^\alpha(x) = 1, C_1^\alpha(x) = 2\alpha x$. Gegenbauer polynomials are solutions of the Gegenbauer differential equation

$$[(1-x^2)\frac{d^2}{dx^2} - (2\alpha+1)x\frac{d}{dx} + n(n+2\alpha)]y = 0.$$

When $\alpha = 1/2$, the equation reduces to the Legendre equation, and the Gegenbauer polynomials reduce to the Legendre polynomials. Again, they are given as Gaussian hypergeometric series in certain cases when the series is finite

$$C_n^{(\alpha)}(z) = \frac{(2\alpha)_n}{n!}\, {}_2F_1\left(-n, 2\alpha+n; \alpha+\frac{1}{2}; \frac{1-z}{2}\right).$$

Explicitly,

$$C_n^{(\alpha)}(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{\Gamma(n-k+\alpha)}{\Gamma(\alpha)k!(n-2k)!}(2z)^{n-2k}.$$

As special cases of the Jacobi polynomials they fulfill

$$C_n^{(\alpha)}(x) = \frac{(2\alpha)_n}{(\alpha+\frac{1}{2})_n} P_n^{(\alpha-1/2,\alpha-1/2)}(x),$$

whose consequence is the Rodrigues formula

$$C_n^{(\alpha)}(x) = \frac{(-2)^n}{n!}\frac{\Gamma(n+\alpha)\Gamma(n+2\alpha)}{\Gamma(\alpha)\Gamma(2n+2\alpha)}(1-x^2)^{-\alpha+1/2}\frac{d^n}{dx^n}\left[(1-x^2)^{n+\alpha-1/2}\right]$$

and the basic formula for derivative of Gegenbauer polynomials

$$\frac{d}{dz} C_{2N}^{(\alpha)}(z) = 2\alpha C_{2N-1}^{(\alpha+1)}(z). \tag{13.5}$$

The polynomials are orthogonal on $[-1, 1]$ with respect to the weighting function $w(z) = \left(1 - z^2\right)^{\alpha - \frac{1}{2}}$.

14 Bibliography

References

[1] P. Bäcklund, Families of Equivariant Differential Operators and Anti-de Sitter Spaces, Journal of Lie Theory 19 (2009), No. 1, 149–183.

[2] W. N. Bailey, Generalised Hypergeometric Series, Cambridge, England, Cambridge University Press, 1935.

[3] R. J. Baston, Almost Hermitian symmetric spaces I: Local twistor theory, Duke Math. J., Volume 63, Number 1 (1991), 81–112.

[4] R. J. Baston, Verma modules and differential conformal invariants, J. Differential Geom., Volume 32, Number 3 (1990), 851–898.

[5] R. J. Baston, Almost Hermitian symmetric manifolds II, Differential invariants, Duke Math. J., Volume 63, Number 1 (1991), 113–138.

[6] R. Beckmann, J. L. Clerc, Singular invariant trilinear forms and covariant (bi-)differential operators under the conformal group, Journal of Functional Analysis, Volume 262, Issue 10 (2012), 4341–4376.

[7] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, A certain category of $\mathfrak{g}$-modules, Funkcional. Anal. i Prilozen, **10** (1976), 1–8.

[8] B. Boe, Kazhdan-Lusztig polynomials for Hermitean symmetric spaces, Transactions of AMS, Vol. 309, No. 1 (1989), 279-294.

[9] T. Branson, Q-curvature and spectral invariants, Rend. Circ. Mat. di Pal., Serie II, Suppl. 75 (2005), pp. 11–55.

[10] D. Colllingwood, B. Shelton, A duality theorem for extensions of induced highest weight modules, Pacific Jour. Math., 146, 2, (1990), 227–237.

[11] D. Calderbank, T. Diemer, Differential invariants and curved Bernstein-Gelfand-Gelfand sequences, Journal fur die reine und angewandte Mathematik, (2001), no. 537, 67-103.

[12] D. Calderbank, T. Diemmer, V. Souček, Ricci-corrected derivatives and invariant differential operators, Diff. Geom. Appl., 23, (2005), 149–175.

[13] A. Cap, A. R. Gover, Tractor calculi for parabolic geometries, Trans. Amer. Math. Soc. 354 (2002), 1511–1548.

[14] A. Cap, J. Slovak, V. Soucek, Bernstein-Gelfand-Gelfand sequences, Annals of Mathematics, 154 (2001), 97–113.

[15] A. Čap, J. Slovák, Parabolic geometries, I: Background and General Theory, Mathematical Surveys and Monographs, American Mathematical Society, ISBN-13: 978-0-8218-2681-2, (2009), 628 pp.

[16] A. Cap, V. Soucek, Curved Casimir Operators and the BGG Machinery, SIGMA: Symmetry, Integrability and Geometry: Methods and Applications (2007), Volume 3, page 111, pp. 17.

[17] N. Conze-Berline and M. Duflo, Sur les représentations induites des groupes semisimples complexes, Compositio Math. 34, (1977), 307–336.

[18] Y. J. Choie, W. Eholzer, Rankin-Cohen operators for Jacobi and Siegel forms, J. Number Theory 68 (1998), 160–177.

[19] J. Dixmier, Algebres Enveloppantes, Gauthier-Villars Editeur, Paris-Bruxelles-Montreal, 1974.

[20] M. G. Eastwood, Higher symmetries of the Laplacian, Ann. Math, (2005), 161, 1645–1665.

[21] M. G. Eastwood, C. R. Graham, Invariants of conformal densities, Duke Math. J. 63, (1991), 633–671.

[22] M. G. Eastwood, J. W. Rice, Conformally invariant differential operators on Minkowski space and their curved analogues, Comm. in Math. Phys., 109, (1987), http://dx.doi.org/10.1007/BF01215221207–228.

[23] M. G. Eastwood, J. Slovák, Semi-holonomic Verma modules, Journal of Algebra, 197, (1997), 424–448.

[24] T. Enright, R. Howe, N. Wallach, A classification of unitary highest weight modules, Representation theory of reductive groups, Progress in Mathematics **40**, Birkhäuser, (1983), pp. 97–143.

[25] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, Inc.. ISBN 978-0-89874-206-0, 1953.

[26] C. Fefferman, R. Graham, Conformal invariants, in lie Cartan et les Mathmatiques d'Aujourdui, Asterisque (1985), 95-116.

[27] W. Fulton, J. Harris, Representation Theory, A First Course, Springer-Verlag, New York, 1991.

[28] A. El Gradechi, The Lie theory of the Rankin-Cohen brackets and allied bi-differential operators, Adv. Math. 207 (2006), no. 2, 484–531.

[29] R. Goodman, N. R. Wallach, Symmetry, Representations, and Invariants, Springer, ISBN 978-0-387-79851-6, 2009.

[30] C. R. Graham, Conformally invariant powers of the Laplacian, II. Nonexistence, J. London Math. Soc., 46 (1992), 566–576.

[31] C. R. Graham, R. Jehne, L. Mason, G. Sparling, Conformally invariant powers of the Laplacian, I. Existence, J. London Math. Soc., 46 (1992), 557–565.

[32] A. R. Gover, C. R. Graham, CR-invariant powers of the sub-Laplacian, J. Reine Angew. Math. 583 (2005), 1–27.

[33] R. Goodman, N. Wallach, Symmetry, Representations, and Invariants, Graduate Texts in Mathematics, 255, Springer, 2009.

[34] J.-S. Huang, Intertwining differential operators and Verma modules, Math. Ann., 297 (1993), 309–324.

[35] J. E. Humphreys, Jr., Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$, Graduate Studies in Mathematics 94, American Mathematical Society, 2008.

[36] H. P. Jakobsen, M. Vergne, Restrictions and Expansions of Holomorphic Representations, Journal of Functional Analysis 34, (1979), 29–53.

[37] A. Juhl, Families of conformally covariant differential operators, Q-curvature and holography, Progress in Math. 275, Birkhäuser, 2009, xiv+488 pp.

[38] A. Knapp, Lie Groups, Beyond an Introduction, Progress in Mathematics, Birkhauser Boston, 2002.

[39] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups and its applications, Invent. Math. 117, (1994), http://dx.doi.org/10.1007/BF01232239181–205; Part II, Ann. of Math. (2) 147 (1998), http://dx.doi.org/10.2307/120963709–729; Part III, Invent. Math. 131 (1998), http://dx.doi.org/10.1007/s002220050203229–256.

[40] T. Kobayashi, Multiplicity-free theorems of the restriction of unitary highest weight modules with respect to reductive symmetric pairs, Progress in Math. http://dx.doi.org/10.1007/978-0-8176-4646-23, 280, Birkhäuser, 2007, pp. 45–109.

[41] T. Kobayashi, Restrictions of generalized Verma modules to symmetric pairs, Transformation Groups 17 (2012), 523-546, http://dx.doi.org/10.1007/s00031-012-9180-y.

[42] T. Kobayashi, G. Mano, The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$, Memoirs of the American Mathematical Society, Vol. 212, http://dx.doi.org/10.1090/S0065-9266-2011-00592-7no.1000, (2011), vi+132 pp.

[43] T. Kobayashi, B. Ørsted, P. Somberg, V. Souček, Branching laws for Verma modules and applications in parabolic geometry. I, preprint.

[44] T. Kobayashi, B. Ørsted, P. Somberg, V. Souček, Branching laws for Verma modules and applications in parabolic geometry. II, preprint.

[45] T. Kobayashi, M. Pevzner, Rankin–Cohen operators for symmetric pairs, Archive http://arxiv.org/abs/1301.2111, preprint, 2013.

[46] B. Kostant, Verma modules and the existence of quasi-invariant differential operators, Lecture Notes in Math. 466, Springer Verlag, (1974), 101–129.

[47] Kraft H., Procesi C., Classical invariant theory. A primer, Lecture Notes, 1996, Available at http://jones.math.unibas.ch/ kraft/Papers/KP-Primer.pdf.

[48] J. Kroeske, Invariant bilinear differential pairings on parabolic geometries, thesis, University of Adelaide, (2008), arXiv:0904.3311v1.

[49] J. Lepowski, A generalization of the Bernstein–Gelfand–Gelfand resolution, J. Algebra, 49, (1977), 496–511.

[50] H. Matumoto, The homomorphisms between scalar generalized Verma modules associated to maximal parabolic subalgebras, Duke Math. J., 131, (2006), 75–118.

[51] M. Pevzner, Rankin–Cohen brackets and associativity, Letters in Mathematical Physics, 85, (2008), 195–202.

[52] G. van Dijk, M. Pevzner, Ring structures for holomorphic discrete series and Rankin–Cohen brackets, J. Lie theory, 17, (2007), 283-305.

[53] J. Slovák: Invariant operators on conformal manifolds, Research Lecture Notes, University of Vienna, 138 pages, 1992.

[54] P. Somberg, Homomorphisms of generalized Verma modules, BGG parabolic category $\mathcal{O}^{\mathfrak{p}}$ and Juhl's conjecture, Journal of Lie Theory 22 (2012) 541-555.

[55] T. Milev, P. Somberg, The branching problem for generalized Verma modules, with application to the pair $(so(7), LieG_2)$, http://xxx.lanl.gov/abs/1209.3970.