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1. INTRODUCTION

This work collects several articles of the author and his colleagues devoted to the direct sum decompositions of infinitely generated projective modules.

Before giving a summary of this thesis let us introduce the concept of a projective module. Throughout the introduction a ring means an associative ring with unit and a module over a ring R stands for a unital right R-module.

A module M over a ring R is called *free* if M is isomorphic to $R^{(\kappa)}$ (direct sum of κ copies of R) for some cardinal κ. To some extent free modules resemble vector spaces from linear algebra, for example every morphism between two free modules can be displayed as a multiplication by a column-finite matrix over R. On the other hand the structure of a free module can be very complicated. A module is called *projective* if it is isomorphic to a direct summand of a free module. In our work we study projective modules that are not finitely generated. We should stress that nonfinitely generated projective modules are definitely less important than finitely generated ones. Therefore let us say several facts about direct sum decompositions of finitely generated projectives.

Let R be a ring. We define a monoid $V(R)$ as follows. The elements of $V(R)$ are isoclasses of finitely generated projective R-modules, write $[P]$ for the class of modules isomorphic to P (of course, there is a set-theoretical problem, formally $[P]$ cannot be a set but it is easy to get rid of this problem). The binary operation on $V(R)$ is given by $[P] + [Q] := [P \oplus Q]$. Observe that the monoid $V(R)$ satisfies the following two conditions: (i) If $x, y \in V(R)$ then $x + y = 0$ implies $x = 0$ and $y = 0$ (such monoids are called reduced). (ii) There exists $u \in V(R)$ such that for every $x \in V(R)$ there exist $y \in V(R)$ and $n \in \mathbb{N}$ such that $x + y = nu$ (any element u having this property is called an order unit of $V(R)$, we can put $u = [R]$). The monoid $V(R)$ encodes the classification of finitely generated projective modules over R and also direct sum decompositions of finitely generated projective modules. A deep theorem of Bergman and Dicks [8] says that the conditions (i) and (ii) are the only restrictions we have on $V(R)$ in general. More precisely, every reduced commutative monoid with an order unit is isomorphic to $V(R)$ of some ring R. So one can see that the theory of finitely generated projective modules can be really complicated.

Sometimes it can be quite easy to describe $V(R)$ of a particular ring. For example, if R is a principal ideal domain, then $V(R) \simeq \mathbb{N}_0$. The same is true if $R = k[x_1, \dots, x_n]$ is a ring of polynomials over a field k but it is not easy at all. In fact, this problem was formulated by Serre and has been answered independently by Suslin and Quillen approximately 20 years later.

An invariant related to $V(R)$ studied in algebraic K-theory is the group $K_0(R)$. One can define it as a quotient group of $V(R)$ or alternatively first take the free abelian group $\mathbb{Z}^{(V(R))}$ and mod out the subgroup generated by $\{[P] - [P_1] - [P_2] \mid P \simeq P_1 \oplus P_2\}$ where we consider all possible direct sum decompositions of all finitely generated projective modules (see for example [25, Theorem 1.13] for details). This group $K_0(R)$ encodes the classification of finitely generated projective modules up to stable equivalence. Recall that finitely generated projective modules P, Q are called *stably equivalent* if there exists a finitely generated free module F such that $P \oplus F \simeq Q \oplus F$. It is quite common that one can calculate $K_0(R)$ while having no idea about the structure of $V(R)$. On the other hand if $V(R)$ is a cancellative monoid then $V(R)$ is a submonoid of $K_0(R)$.

We conclude this part of introduction by two appearances of projective modules in number theory and in geometry.

Let R be a Dedekind domain with a quotient field K, for example R can be the ring of algebraic integers in a number field K. A *fractional ideal* of R is a finitely generated R-submodule of K. If I, L are fractional ideals of R then $IL = \{\sum_{t=1}^{n} i_t l_t \mid i_1, \ldots, i_n \in I, l_1, \ldots, l_n \in L\}$ is again a fractional ideal. It can be shown that the set of nonzero fractional ideals of R equipped with this operation forms a free abelian group and that the set of maximal ideals of R is a free basis of this group. Let us denote this group by $I(R)$. It is easy to see that $P(R) = \{kR \mid 0 \neq k \in K\}$ is a subgroup of $I(R)$. The quotient $C(R) = I(R)/P(R)$ is called the *ideal class group* of R.

As R is a Dedekind domain, every fractional ideal of R is projective. Moreover, it is easy to see that $I_1, I_2 \in I(R)$ are isomorphic as R-modules if and only if $I_1 P(R) = I_2 P(R)$, so the ideal class group of R is a classification of nonzero fractional ideals up to isomorphism. Finally, it is well known that every finitely generated projective module over R is isomorphic to $R^n \oplus I$ where $n \in \mathbb{N}_0$ and I is an ideal of R. Since every fractional ideal of R is isomorphic to an ideal of R, we see that the ideal class group of R is in fact a classification of indecomposable finitely generated projective modules up to isomorphism. Further, it can be shown that $C(R) \simeq K_0(R)/\langle [R] \rangle$, the group on the right hand side is also called reduced K_0 of R (see for example [25, Theorem 1.4.12]). Having a classification of finitely generated projective modules over R one has in fact a classification of all finitely generated R-modules. It is because of the theorem of Steinitz: Every finitely generated module over R is isomorphic to a module of the form $P \oplus \bigoplus_{i=1}^{n} R/(P_i^{p_i})$, where P is a finitely generated projective module, $P_1, \ldots, P_n$ are maximal ideals of R, and $p_1, \ldots, p_n \in \mathbb{N}_0$.

If K is a number field and R is the ring of algebraic integers of K then $C(R)$ is a finite group. The size of $C(R)$ is called the *class number* of K and it is an important invariant of K. The class number of K is 1 if and only if R is a principal ideal domain. Let us recall that Kummer proved the Fermat Last Theorem for exponents which are regular primes. Let us explain this notion. Let p be an odd prime and let $K = \mathbb{Q}[e^{2\pi i/p}]$. The ring of algebraic integers in K is $R = \mathbb{Z}[e^{2\pi i/p}]$. Now p is called regular if p does not divide the class number of K. This can be expressed in terms of projective R-modules in the following way: Take an indecomposable projective R-module I. If I^p (direct sum of p copies of I) is a free module then $I \simeq R$.

Large ideal class groups can be used as platform groups for cryptographic protocols based on discrete logarithm problem. Imaginary quadratic fields seem to be particularly convenient (see [10]).

Finally, let us briefly discuss a connection of finitely generated projective modules and vector bundles. This was discovered by Serre [28] in the setting of algebraic geometry. Here we explain a similar result proved later by Swan. Let X be a topological vector space. A *real vector bundle* ξ on X consists of a topological space $E(\xi)$ and a continuous onto map $p \colon E(\xi) \to X$ such that for every $x \in X$ the fiber $p^{-1}(x)$ has the structure of a real vector space of finite dimension. Moreover, vector bundle ξ has to be locally trivial, that is for every $x \in X$ there exists an open neighborhood U of x, $n \in \mathbb{N}_0$, and homeomorphism $h\colon p^{-1}(U) \to U \times \mathbb{R}^n$ such that

(i) for every $e \in \pi^{-1}(U)$ the equality $\pi_U(h(e)) = p(e)$ holds

(ii) for every $u \in U$ the map h induces by restriction an isomorphism of vector spaces $p^{-1}(u)$ and $\pi_U^{-1}(u)$

As an example, put $E(\xi) = X \times \mathbb{R}^n$, for every $x \in X$ the set $\{x\} \times \mathbb{R}^n$ carries the canonical structure of a vector space and $p\colon X \times \mathbb{R}^n \to X$ is defined to be the canonical projection. Such a vector bundle is called trivial. Of course there are vector bundles that are not trivial, for example the tangent bundle of a real 2-dimensional sphere.

A section of a real vector bundle ξ is a continuous map $s\colon X \to E(\xi)$ such that $ps = \mathrm{id}_X$. Let us denote $\Gamma(\xi)$ the set of all sections of ξ. Observe that this set has a natural structure of an abelian group, one can define $s_1 + s_2\colon x \mapsto s_1(x) + s_2(x)$ for any $s_1, s_2 \in \Gamma(\xi)$. Moreover, if f is a real valued continuous function, then $sf\colon x \mapsto f(x)s(x)$ is a section of ξ again. Therefore if R is the ring of all continuous real valued functions on X, the set $\Gamma(\xi)$ has a structure of an R-module. For example, if ξ is a trivial vector bundle then $\Gamma(\xi)$ is a free module.

One can define a category whose objects are real vector bundles over X and extend Γ to an additive functor to the category Mod-R. Swan [29, Corollary 4] proved that if X is normal then real vector bundles ξ_1, ξ_2 on X are isomorphic if and only if the corresponding R-modules $\Gamma(\xi_1)$ and $\Gamma(\xi_2)$ are isomorphic. If we assume X to be compact Hausdorff then, by [29, Theorem 2], a module M over R is isomorphic to a module $\Gamma(\xi)$ for some real vector bundle ξ on X if and only if M is a finitely generated projective module.

One could also formulate the result that the category of real vector bundles on X and the category of finitely generated projective R-modules are equivalent provided X is compact Hausdorff. Swan [29] apply this theorem to get many interesting examples of projective modules from geometry. For example, let us look at ring $S = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$. It is easy to see that $P = \{(\alpha, \beta, \gamma) \in S^3 \mid \alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} = 0\} \subseteq S^3$ is a projective module (here $\overline{x}, \overline{y}, \overline{z}$ are the images of x, y, z in S). One can see that this module is not free as a consequence of the fact that the tangent bundle on real sphere of dimension 2 is not trivial.

Having said this the reader should be convinced that finitely generated projective modules are modules of particular importance. The case of nonfinitely generated projective modules is quite different.

There are only several important papers devoted to the study of nonfinitely generated projective modules. A fundamental one is Kaplansky [21]. Kaplansky proved that every projective module is a direct sum of countably generated projective modules. Bass [5] showed that nonfinitely generated projective modules are free in many important cases. A complete classification of nonfinitely generated projective modules have been achieved for example by Levy and Robson [19] over hereditary noetherian prime rings. They show that in this case there may exist big projective modules that are not free. On the other hand over any hereditary ring every projective module is a direct sum of finitely generated modules. An example of a projective module without a nonzero finitely generated direct summand was constructed in [15].

This thesis consists of the following articles:

1. P. Příhoda: *Projective modules are determined by their radical factors.* J. Pure Appl. Algebra 210 (2007), no. 3, 827 – 835.

2. P. Příhoda: *Fair-sized projective modules.* Rend. Sem. Mat. Univ. Padova 123 (2010), 141 – 167.

3. P. Příhoda, G. Puninski: *Nonfinitely generated projective modules over generalized Weyl algebras.* J. Algebra 321 (2009), 1326 – 1342.

4. P. Příhoda, G. Puninski: *Classifying generalized lattices. Some examples as an introduction.* J. London Math. Soc. (2009), no. 4, 1326 – 1342.

5. D. Herbera, P. Příhoda: *Big projective modules over noetherian semilocal rings.* J. Reine Angew. Math. 648 (2010), 111 – 148.

6. D. Herbera, P. Příhoda: *Infinitely generated projective modules over pullbacks of rings.* to appear in Trans. Amer. Math. Soc.

All of them are devoted to the study of rings possessing strange nonfinitely generated projective modules. Since such a subject may look suspicious we include a brief, rather informal, summary providing motivation and perspectives (in other words where we got stuck) of this subject.

The main result of the first paper in this thesis says that one can classify countably generated projective modules over a semilocal ring. First let us give a general statement: If P is a projective module over a ring R, its Jacobson radical $\mathrm{rad}(P)$ is defined as the intersection of all maximal submodules of P. We prove that if P and Q are projective modules over any ring R such that $P/\mathrm{rad}(P)$ and $Q/\mathrm{rad}(Q)$ are isomorphic, then P and Q are isomorphic.

This result is very classical in the realm of finitely generated projective modules, as one can argue that canonical projections $\pi_P : P \to P/\mathrm{rad}(P)$ and $\pi_Q : Q \to Q/\mathrm{rad}(Q)$ are projective covers. For nonfinitely generated projective modules it is not possible to use this argument and it is necessary to be a bit more careful.

This result answers a question posed by Dolors Herbera during the problem session of the conference Some Trends in Algebra 2003 (let us quote her "of course, I don't believe it is true in general but ... "). Another motivation comes from the research on direct sum decompositions of serial modules, several results could be explained as a consequence of the theorem.

The result mentioned above is particularly significant over semilocal rings. It says that the notion of dimension known from the linear algebra can be generalized for projective modules over semilocal rings. Let R be a ring, we denote $\mathrm{rad}(R)$ as $J(R)$. It is known that $J(R)$ is a twosided ideal of R. The ring R is called *semilocal* if $R/J(R)$ is semisimple artinian (that is every module over $R/J(R)$ is projective). If R is commutative then semilocal means that it has only finitely many maximal ideals. Noncommutative semilocal rings appear quite naturally in module theory, for example the ring of endomorphisms of any artinian module is semilocal by [11]. Our result says that every projective R-module is up to isomorphism determined by the $R/J(R)$-module $P/\mathrm{rad}(P) = P/PJ(R)$. If $R/J(R)$ is semisimple artinian, the structure of $R/J(R)$-modules is well understood. There exists $k \in \mathbb{N}$ such that $R/J(R)$-modules can be classified by k-tuples of cardinals. We will not go into the details, just note that if $R/J(R)$ is a field then $R/J(R)$-modules are just vector spaces and therefore they are classified by dimension.

Let us give two easy applications to give at least some sketch how it works. Of course, it is not necessary to use our theorem but we hope they illustrate the matter well.

Suppose R is a local ring, that is $R/J(R)$ is a (not necessarily commutative) field and P a projective module over R. Now if $P/PJ(R)$ has dimension κ, then $P/\mathrm{rad}(P) \simeq R^{(\kappa)}/(\mathrm{rad}(R^{(\kappa)}))$. Consequently $P \simeq R^{(\kappa)}$ and every projective R-module is free. Hence we obtained a classical result of Kaplansky (see [21, Theorem 2]) that every projective module over a local ring is free. Some other standard results can be also derived as a direct consequence of the theorem.

Suppose that R is a subring of $\mathbb{Q}$ given by $R = \{\frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus 6\mathbb{Z}\}$. This is a semilocal ring, $J(R) = 6R$, $R/J(R) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. If P is a projective R-module then $P/\mathrm{rad}(P) \simeq P/2P \oplus P/3P$. Observe that $P/2P$ is a vector space over $\mathbb{Z}/2\mathbb{Z}$, $P/3P$ is a vector space over $\mathbb{Z}/3\mathbb{Z}$. So we write $\dim(P) = (x, y)$, where x is a dimension of $P/2P$ over $\mathbb{Z}/2\mathbb{Z}$ and y is the dimension of $P/3P$ over $\mathbb{Z}/3\mathbb{Z}$. Every projective R-module is described up to isomorphism by this pair of cardinals.

Let us give a (rather complicated) way how to prove that every projective R-module is free. Using the theorem of Kaplansky we need to show that every countably generated projective R-module is free that is that every countably generated projective module over R has dimension of the form (λ, λ) for some $\lambda \le \aleph_0$. If it is not the case then using an argument with projective cover we can see that there exists a projective module P such that $\dim(P) = (0, \lambda)$ or $\dim(P) = (\lambda, 0)$ for some $0 < \lambda \le \aleph_0$. Then the trace ideal of P has to be a proper idempotent ideal of R. But R is a commutative noetherian domain and hence cannot have proper idempotent ideals (it follows for example from the Krull's intersection theorem that every idempotent ideal of a commutative noetherian ring is generated by an idempotent element). Applying similar arguments it is possible to prove that every projective module over a commutative noetherian semilocal domain is free, we did not use the fact that R is a Dedekind domain.

The first paper uses this kind of arguments to obtain some less trivial applications. Among other things we give answers to [23, Question 8.1, Question 8.2, Question 8.3].

The second paper was strongly inspired by an unpublished work of Gena Puninski. Puninski was probably motivated by an old question of Peter Linnell: *If G is a finite group is every indecomposable projective module over $\mathbb{Z}G$ finitely generated?* This question appeared in the list of problems from the group theory known as Kourovka notebook [18] (I have to confess I have never seen this reference). We give a positive answer to this question, but to be honest let us stress that the answer has nothing to do with the theory of groups.

Let us explain the historical background of this question. Swan [30] proved that if G is a finite solvable group then every nonfinitely generated projective $\mathbb{Z}G$-module is free. Bass quoted this result in his well-known paper [5] and continued: "and it is undoubtedly true in general". Later Akasaki [1, 2] considered a question when an integral group ring of a finite group contains a nontrivial idempotent ideal. A general solution to his problem was given by Roggenkamp [24]. He proved that if G is a finite group then $\mathbb{Z}G$ has no nontrivial idempotent ideal if and only if G is a solvable group. Later Whitehead [27] considered a problem which idempotent ideals are trace ideals of a projective module. He gave the following striking corollary: If $I \subseteq R$ is an idempotent ideal finitely generated as a left R-module, then I is a trace ideal of a countably generated projective right R-module. Now if P is a projective $\mathbb{Z}G$-module of the trace ideal I such that $0 \ne I \ne \mathbb{Z}G$ then P cannot be free. So if G is a finite group and G is not solvable then $\mathbb{Z}G$ has to contain a nonfinitely generated projective module which is not free. This was noted by Akasaki [3], for a different approach see [20].

Being aware of this, Puninski noticed that if L is a finite dimensional Lie algebra over $\mathbb{C}$ and L is not solvable, then its universal enveloping algebra $U(L)$ has to contain a nontrivial idempotent ideal. So the same argument as in the case of integral group rings shows that there are nonfinitely generated projective $U(L)$-modules that are not free. Puninski conjectured that this property should characterize solvable finite dimensional Lie algebras over $\mathbb{C}$. We confirmed that it is indeed true: If L is a finite dimensional Lie algebra over a field of characteristic 0, then L is solvable if and only if every nonfinitely generated projective module over $U(L)$ is free. We do not know if the same is true for Lie algebras over fields of positive characteristics.

Let us briefly explain the technique introduced in this paper. We consider projective modules that are called fair-sized. As this name suggests the concept generalizes Bass' uniformly big modules from [5]. Suppose that R is a noetherian ring that is every left and right ideal is a finitely generated R-module. The theory works if R satisfies the following condition $(*)$: If $I_1, I_2, \cdots$ is a sequence of twosided ideals of R such that $I_{k+1}I_k = I_{k+1}$ for every $k \in \mathbb{N}$, then there exists $l_0 \in \mathbb{N}$ such that $I_l = I_{l_0}$ for every $l \geq l_0$. This condition is satisfied quite often in the noetherian context. For example, if A is a ring whose underlying abelian group is a free group of finite rank, then A satisfies $(*)$.

Now suppose that we have a noetherian ring R satisfying $(*)$ and suppose we are able to do the following:

a) Find the set of all idempotent ideals of R.

b) For every idempotent ideal $I \subseteq R$ we are able to classify finitely generated projective modules over R/I.

Then we are able to classify all countably generated projective modules over R. Roughly speaking a countably generated projective R-module P is described by the smallest ideal I such that P/PI is finitely generated and by the finitely generated projective R/I-module P/PI. One can ask what kind of information about a finite group G is hidden in the classification of countably generated projective modules over $\mathbb{Z}G$. It appears that for every perfect normal subgroup H of G there is an idempotent ideal of $\mathbb{Z}G$, namely the kernel of the augmentation homomorphism $\mathbb{Z}G \to \mathbb{Z}G/H$. Moreover, the classification has to contain full information about finitely generated projective modules over $\mathbb{Z}G/H$. It seems to be a very hard problem and we think that this is the main drawback of this method. We tried to calculate the case when $G = A_5$. This group has only 2 perfect normal subgroups 1 and A_5, so one would expect only 3 idempotent ideals of $\mathbb{Z}A_5$, namely $0, \mathbb{Z}A_5$ and the augmentation ideal of $\mathbb{Z}A_5$. Surprisingly, the calculations show that there is one more idempotent ideal. Right now we do not have a theoretical explanation for its appearance. We were not able to classify finitely generated modules over $\mathbb{Z}A_5$ but using the theory introduced in the paper all the remaining projective $\mathbb{Z}A_5$-modules were found.

The condition $(*)$ is true in some other classes of noetherian rings that are also studied in this thesis, for example semilocal noetherian rings and generalized Weyl algebras. Unfortunately, it is not true that every noetherian ring satisfies $(*)$. For example take the Lie algebra $L = \mathrm{sl}(2, \mathbb{C})$ and consider its universal enveloping algebra $R = U(L)$. It is well known that R contains strictly descending chains of idempotent ideals and therefore R cannot satisfy $(*)$. What is even worse, one can show that the classification scheme introduced in this paper do not apply in this case, roughly speaking the algebra $R = U(\mathrm{sl}(2, \mathbb{C}))$ possesses too many countably generated projective modules. Even to give a classification of countably but not finitely generated projective modules over $U(\mathrm{sl}(2, \mathbb{C}))$ seems to be a very hard problem.

The following two papers apply the technique of fair-sized modules. The third paper gives a classification of infinitely generated projective modules over a class of rings known as generalized Weyl algebras. Rings in this class are connected to geometry (see Hodges [17]), for example the first Weyl algebra, ring of differential operators on a projective line and some primitive factors of $U(\mathsf{sl}(2,\mathbb{C}))$ belong to this class. Many algebraic properties of these algebras are well known, finitely generated projective modules are usually well understood on the level of K_0, but on the other hand the classification of finitely generated projective modules seems to be unreachable. In this paper we find a classification of nonfinitely generated projective modules over any generalized Weyl algebra. We find finitely generated projective modules $P_1, P_2, \dots, P_m$ such that every nonfinitely generated projective module is of the form $\oplus_{i\in I} Q_i$ where every Q_i is isomorphic to a module of the set $\{P_1, \dots, P_m\}$.

Our strategy relies on the fact that every generalized Weyl algebra A is a noetherian domain with a finite set of two-sided ideals, so the condition $(*)$ holds. Using the results of Bavula [6] we are able to find all idempotent ideals in A and modifying a construction of some indecomposable finitely generated projective A-modules from Hodges [17], we show that every idempotent ideal of A is a trace ideal of a finitely generated projective module. Further we have to show that for every nonzero idempotent ideal $I \subseteq A$ and for every finitely generated projective module P' over A/I there exists a finitely generated projective A-module P such that $P' \simeq P/PI$. In the paper we do this in a bit different way, we will omit the details here. The important thing is that in order to find all nonfinitely generated projective modules over A it is not at all necessary to know everything about finitely generated projective modules. Here it was enough to find only several finitely generated projectives having some particular properties.

The fourth paper deals with generalized lattices. In order to avoid general definitions let us concentrate on one particular example from integral representation theory. The general theory of [9, 26] applies for (locally) lattice finite R-orders in separable algebras where R is a Dedekind domain. Let G be a finite group. A finitely generated $\mathbb{Z}G$-module M is called *$\mathbb{Z}G$-lattice* if the underlying abelian group of M is free. $\mathbb{Z}G$-lattices can be also seen as group homomorphisms from G to $\mathrm{Aut}_{\mathbb{Z}}(\mathbb{Z}^n)$.

Butler at al. [9] studied so called generalized lattices. A *generalized $\mathbb{Z}G$-lattice* is a $\mathbb{Z}G$-module M whose underlying abelian group is free. This research was motivated by problems coming from C^*-algebras and Lie theory where people got interest in generalized $\mathbb{Z}C_2$-lattices, where C_2 stands for the group of order 2. The algebra $\mathbb{Z}G$ is called *lattice finite* if there are only finitely many indecomposable $\mathbb{Z}G$-lattices up to isomorphism. For example, if C_p is the group of prime order p then $\mathbb{Z}C_p$ is lattice finite (see [16]). Suppose $\mathbb{Z}G$ lattice finite, let $L_1, \dots, L_n$ be the list of all indecomposable $\mathbb{Z}G$-lattices up to isomorphism. The lattice $L = L_1 \oplus \cdots \oplus L_n$ is called the *Auslander lattice* and the ring $S = \mathrm{End}_{\mathbb{Z}G}(L_1 \oplus \cdots \oplus L_n)$ is called the *Auslander order* of L. An amazing result [9, Theorem 2.1] says that if $\mathbb{Z}G$ is lattice finite then generalized $\mathbb{Z}G$-lattices are exactly objects of $\mathrm{Add}(L)$, that is every generalized lattice is a direct summand of $L^{(\kappa)}$ for some cardinal κ. Moreover, by a standard result of Dress [12], categories $\mathrm{Add}(L)$ and *Proj-S* are equivalent. So understanding generalized lattices is basically the same as understanding to projective S-modules. Moreover if lattices are understood then we are left to understand nonfinitely generated projective modules.

Butler at. al [9] proved that every generalized lattice over $\mathbb{Z}C_p$ is a direct sum of lattices. Their approach was basically this: Using a theorem of Kaplansky we see that every generalized $\mathbb{Z}C_p$-lattice is a direct sum of countably generated generalized $\mathbb{Z}C_p$-

lattices. Then a careful analysis of countable generalized lattices shows that any of them is a direct sum of lattices.

The proof in our spirit would go differently. This example is not in the paper, but we hope an interested reader can fill in the details easily. Let Γ be the normalization of $\Lambda := \mathbb{Z}C_p$. It is known that $\Gamma = \mathbb{Z} \times \mathbb{Z}[e^{2\pi i/p}]$ and that the ring Λ may be identified with a subring of Γ, namely we can suppose $\Lambda = p\mathbb{Z} \times (1 - e^{2\pi i/p})\mathbb{Z}[e^{2\pi i/p}] \cup (1,1)\mathbb{Z}$. If L is an Auslander lattice of Λ, then by [4, Proposition 7.2] $\mathrm{Add}(L) = \mathrm{Add}(\Lambda \oplus \Gamma)$. So in order to prove that every generalized Λ-lattice is a direct sum of lattices it is enough to show that every projective module over $S = \mathrm{End}_\Lambda(\Lambda \oplus \Gamma)$ is a direct sum of finitely generated modules. Now let C be the largest ideal of Γ contained in Λ. Then the ring S is isomorphic to

$$T = \begin{pmatrix} \Lambda & C \\ \Gamma & \Gamma \end{pmatrix} \subseteq \mathrm{M}_2(\Gamma)\,.$$

One can find all the idempotent ideals in T. It appears that every idempotent ideal of T is a trace ideal of a finitely generated projective T-module. Moreover, for every nonzero idempotent ideal $I \subseteq T$ and for every finitely generated projective T/I-module P' there exists a finitely generated projective T-module P such that $P' \simeq P/PI$. This is all one needs to prove that every projective T-module is a direct sum of finitely generated modules. Further, when these calculations are done carefully, it is possible to obtain a classification of generalized lattices up to isomorphism. In our paper we have a similar example with generalized lattices over quadratic orders.

Further important results on generalized lattices were obtained by Rump [26]. He gave a combinatorial criterion when every generalized lattice over a locally lattice finite order is a direct sum of lattices. His approach does not aim at classification problems, in our paper we investigated several interesting examples from [26]. We show that one of Rump's example contains a superdecomposable generalized lattice.

The last two papers are devoted to the study of $V^*(R)$ of a semilocal ring. The monoid $V^*(R)$ is formed by isoclasses of countably generated projective R-modules. Again, for $[P], [Q] \in V^*(R)$ the sum is defined by $[P] + [Q] := [P \oplus Q]$. Let us define a structure of a commutative monoid on $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$ by extending the standard addition of $\mathbb{N}_0$ with $x + \infty = \infty + x = \infty$. The main result of the first paper says that for every semilocal ring R there exists a positive integer k such that $V^*(R)$ can be embedded to $(\mathbb{N}_0^*)^k$. Moreover, this embedding is quite canonical, for example k can be the number of distinct simple modules possessed by R. In this embedding the image of $V(R)$ is contained in $\mathbb{N}_0^k$ and $[R]$ is mapped to an element having all its coordinates nonzero. Now a natural question is for which submonoids of $(\mathbb{N}_0^*)^k$ there exists a semilocal ring R such that the monoid is the image of $V^*(R)$ in the canonical embedding. This question is still open, the fifth paper gives a satisfactory answer if we restrict the question to semilocal noetherian rings.

Let us explain how the fifth paper is connected to the earlier result of Facchini and Herbera [13]. They considered the same problem restricted to finitely generated projective modules: Which submonoids of $\mathbb{N}_0^k$ arise as images of canonical embeddings of $V(R)$ to $\mathbb{N}_0^k$, where R is a semilocal ring? Their answer is very nice: If we identify $V(R)$ to a submonoid of $\mathbb{N}_0^k$ using the canonical embedding then the following two conditions have to be satisfied: (i) $V(R)$ contains an element having all its coordinates nonzero (for example the element $[R]$) and (ii) if $(x_1, \dots, x_k), (y_1, \dots, y_k) \in V(R)$ are such that $x_i \leq y_i$ for every $i = 1, \dots, k$ then $(y_1 - x_1, \dots, y_k - x_k) \in V(R)$. Moreover, if M is a submonoid of $\mathbb{N}_0^k$ satisfying (i) and (ii) then M arises as a canonical image of $V(R)$ of some semilocal ring R. The construction from [13] is quite complicated (let us quote the review from MathSciNet "I cannot do justice to the proof, which is interesting but rather tricky."). It uses Bergman's results on projective modules over pushouts and then applies Cohn's universal localization. The problem is that this construction produces always semilocal hereditary ring with prescribed $V(R)$. But it is well known that every projective module over a hereditary ring is a direct sum of finitely generated modules. So over hereditary rings the monoid $V(R)^*$ is determined by its submonoid $V(R)$. Further if we try to repeat arguments for rings that are not hereditary, the universal localization seems to be out of control. Fortunately, there is another approach in the literature. Wiegand [31] studied similar monoids formed from finitely generated modules over commutative local noetherian rings of dimension one and their behavior under completion. He proved a similar result to that from [13] using pullbacks. The structure of projective modules over certain pullbacks of rings was described by Milnor [22]. Using his results one can give a quite short proof of the main result from [13]. But what is more important the approach via pullbacks gives some space for projective modules that are not direct sums of finitely generated modules.

It is possible to give another characterization of monoids satisfying (i) and (ii). Fix $k \in \mathbb{N}$. Suppose that E_1, E_2 are matrices $m \times k$ with coefficients in $\mathbb{N}_0$, let D be a matrix of size $l \times k$ with coefficient in $\mathbb{N}_0$ and $(x_1, \dots, x_l) \in \mathbb{N}^l$. These data defines a submonoid M of $\mathbb{N}_0^k$, $M = \{z \in \mathbb{N}_0^k \mid E_1 z^T = E_2 z^T, Dz^T \in (x_1\mathbb{N}_0, \dots, x_l\mathbb{N}_0)^T\}$. We say that M is defined by equations. The characterization from [13] can be reformulated as follows: A submonoid of $\mathbb{N}_0^k$ is identified to a $V(R)$ of a semilocal ring if and only if it is defined by equations and it contains an element having all its coordinates nonzero. What is quite surprising we have almost the same characterization of $V^*(R)$ of semilocal noetherian rings. Suppose that $E_1, E_2, D, (x_1, \dots, x_l)$ are as above. These data define a submonoid of $(\mathbb{N}_0^*)^k$, $M = \{z \in (\mathbb{N}_0^*)^k \mid E_1 z^T = E_2 z^T, Dz^T \in (x_1\mathbb{N}_0^*, \dots, x_l\mathbb{N}_0^*)^T\}$. We take the formalism $0.\infty = 0$. The main result of our paper says that a submonoid of $(\mathbb{N}_0^*)^k$ is identified to a $V^*(R)$ of a semilocal noetherian ring if and only if it is defined by equations and it contains an element having all its coordinates nonzero.

The general case of $V^*(R)$ of a semilocal ring is still an open problem. The last paper of the thesis uses the construction of Gerasimov and Sakhaev [15] together with pullback constructions from the previous paper to realize monoids defined by inequalities. We will not go into details here. One of the examples shows an asymmetry between left and right projective modules. We found a semilocal ring such that every projective left R-module is free but there exists a nonfinitely generated projective right module over R which is not a generator. This answers a question of Fuller and Shutters from [14, page 310].

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2. PROJECTIVE MODULES ARE DETERMINED BY THEIR RADICAL FACTORS

2.1 Introduction

Using a projective cover argument, one can show that two finitely generated projective modules are isomorphic if and only if they have isomorphic factors modulo their Jacobson radicals. This well known result can be used to get information about finitely generated projective modules over semilocal rings. For example, Fuller and Shutters [7] proved that over any semilocal ring there are only finitely many indecomposable finitely generated projective modules up to isomorphism.

The aim of this note is to prove that arbitrary projective modules P, Q are isomorphic whenever they have isomorphic factors modulo their Jacobson radical. Let us briefly recall some related results achieved so far.

For some results saying when an infinitely generated projective module is free see [2] and [12]. In fact, we are interested in when it is not the case. Beck [3] proved that if P is a projective right R-module and $P/\mathrm{rad}(P)$ is free right $R/J(R)$-module, then P is free R-module. Later, Gruson (see appendix of [12]), proved that any free base of $P/\mathrm{rad}(P)$ can be lifted to a free base of P. It follows from Jøndrup [12] that if P, Q are projective modules such that $P/\mathrm{rad}(P)$ is a direct summand of $Q/\mathrm{rad}(Q)$, then P can be embedded to Q. Facchini, Herbera and Sakhajev [6] proved that if P, Q are projective modules and there exists a pure monomorphism from $P/\mathrm{rad}(P)$ to $Q/\mathrm{rad}(Q)$, then there is a pure monomorphism from P to Q.

We prove the result in the title and then we give several immediate corollaries for projective modules over semilocal rings. For example, we show that there are at most countably many indecomposable projective modules over a semilocal ring. As a bit more sophisticated application we show how to use knowledge of objects in Add of a uniserial module to give a classification of right projective modules over an endomorphism ring of a uniserial module. Also we get answers to some problems Puninski posed in [13]. The last part of this note deals with an example of a semilocal ring having a projective module that is not possible to write as a direct sum of indecomposable modules. Let us stress that over commutative semilocal rings the situation is much easier. Indeed, it follows from Hinohara [9, 10] that over commutative weakly noetherian rings (hence also over commutative semilocal rings) every projective module is a direct sum of finitely generated modules.

All basic results about the Jacobson radical can be found, for example, in [1]. Unless otherwise stated, we work inside the category of right modules over a (fixed) associative

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ring R with unit. We denote as $J(R)$ the (Jacobson) radical of R and as $\mathrm{rad}(M)$ the (Jacobson) radical of the module M. If P is a projective module, then $\mathrm{rad}(P) = PJ(R)$. We call $P/\mathrm{rad(P)}$ the *radical factor* of P.

2.2 The result

If P, Q are projective modules and $\pi_P : P \to P/\mathrm{rad}(P)$, $\pi_Q : Q \to Q/\mathrm{rad}(Q)$ are the canonical projections, then for any homomorphism $\overline{f} : P/\mathrm{rad}(P) \to Q/\mathrm{rad}(Q)$ there exists a homomorphism $f : P \to Q$ such that $\pi_Q f = \overline{f}\pi_P$. We say that f is a *lift* of $\overline{f}$. The idea we are going to use in the next lemma is essentially described in [11].

Lemma 2.2.1. *Let P, Q be countably generated projective modules. Suppose that $\overline{f} : P/\mathrm{rad}(P) \to Q/\mathrm{rad}(Q)$ and $\overline{g} : Q/\mathrm{rad}(Q) \to P/\mathrm{rad}(P)$ are mutually inverse isomorphisms. Let $f : P \to Q$ be any lift of $\overline{f}$ and let $X \subseteq P$ be a finite set. Then there exists a lift $g : Q \to P$ of $\overline{g}$ such that $gf(x) = x$ for any $x \in X$.*

Proof. Let P', Q' be projective modules such that $P \oplus P'$ and $Q \oplus Q'$ are countably generated free modules. It is possible to suppose $f' : P' \to Q'$ and $g' : Q' \to P'$ are mutually inverse morphisms. (In fact, we can suppose $P' = Q' = R^{(\omega)}$ because of the Eilenberg's trick.) Let g_0 be any lift of $\overline{g}$ and let us fix some free base $Y = \{e_1, e_2, \dots\}$ of $P \oplus P'$. Consider the homomorphism $h = (g_0 \oplus g') \circ (f \oplus f') : P \oplus P' \to P \oplus P'$. For any $e \in P \oplus P'$ is $h(e) - e \in \mathrm{rad}(P \oplus P')$. Let $n \in \mathbb{N}$ be such that any element of X can be expressed as a combination of $e_1, \dots, e_n$. We claim there is an endomorphism $h' : P \oplus P' \to P \oplus P'$ lifting the identity on $P \oplus P'/\mathrm{rad}(P \oplus P')$ such that $h'h(e_i) = e_i$ for any $i = 1, \dots, n$. In order to see this, express h as a column-finite matrix A (the i-th column is formed by coordinates of $h(e_i)$ determined by the base Y). Let $m \geq n \in \mathbb{N}$ be such that first n columns of A have non-zero values only in the first m rows. Let B be a $m \times m$ matrix given by the top left corner of A. Consider B as an element of the ring $M_m(R)$. Then $B \in 1 + J(M_m(R))$ is an invertible matrix and its inverse C is also an element of $1 + J(M_m(R))$. Replacing the top left $m \times m$ corner in the identical $\mathbb{N} \times \mathbb{N}$ matrix by C we obtain a column-finite matrix A' that represents desired endomorphism h' with respect to the base Y.

Let $\pi_P : P \oplus P' \to P$ be the canonical projection and let $\iota_P : P \to P \oplus P'$ be the canonical inclusion. Then we can put $g = \pi_P h' \iota_P g_0$. □

Lemma 2.2.2. *Let P, Q be countably generated projective modules such that $\overline{f} : P/\mathrm{rad}(P) \to Q/\mathrm{rad}(Q)$ is an isomorphism. Then there exists a lift of $\overline{f}$ which is an isomorphism.*

Proof. Let $\{p_0, p_1, \dots\}$ be generators for P and let $\{q_0, q_1, \dots\}$ be generators for Q. Let $\overline{g} : Q/\mathrm{rad}(Q) \to P/\mathrm{rad}(P)$ be an inverse of $\overline{f}$. We are going to define homomorphisms $f_i : P \to Q$, $g_i : Q \to P$ and finite sets $P_i \subseteq P$, $Q_i \subseteq Q$ for any $i \in \mathbb{N}_0$ as follows:

Put $P_0 = \{p_0\}$ and let f_0 be any lift of $\overline{f}$.

Suppose P_i, f_i were defined, define Q_i, g_i by $Q_i = f_i(P_i) \cup \{q_0, \dots, q_i\}$ and let g_i be a lift of $\overline{g}$ such that $g_i f_i(x) = x$ for any $x \in P_i$.

Suppose Q_i, g_i were defined, define $P_{i+1} = g_i(Q_i) \cup \{p_0, \dots, p_{i+1}\}$ and let f_{i+1} be a lift of $\overline{f}$ such that $f_{i+1} g_i(x) = x$ for any $x \in Q_i$.

Observe that $P_i \subseteq g_i(Q_i) \subseteq P_{i+1}$. If $p \in P_i$, then $p = g_i f_i(p)$ and $f_{i+1}(p) = f_{i+1} g_i f_i(p) = f_i(p)$ since $f_i(p) \in Q_i$. Therefore $f_{i+1}|_{\langle P_i \rangle} = f_i|_{\langle P_i \rangle}$. Thus we can define $f : P \to Q$ by $f(p) = f_i(p)$ if $p \in \langle P_i \rangle$.

Suppose that $f(p) = 0$. Then $p \in \langle P_i \rangle$ for some $i \in \mathbb{N}$. But then $0 = g_i f(p) = g_i f_i(p) = p$. Therefore f is mono. In order to see that f is epi, just observe $f(P_{i+1}) \supseteq Q_i$. Finally it remains to prove that f is a lift of $\overline{f}$. But this is obvious since all f_i's are lifts of $\overline{f}$. □

Theorem 2.2.3. *Let P, Q be projective modules such that $\overline{f}\colon P/\mathrm{rad}(P) \to Q/\mathrm{rad}(Q)$ is an isomorphism. Then there is an isomorphism $f\colon P \to Q$ which is a lift of $\overline{f}$.*

Proof. By the theorem of Kaplansky, there are decompositions $P = \oplus_{i\in I} P_i$ and $Q = \oplus_{j\in J} Q_j$ such that the modules $P_i, Q_j, i \in I, j \in J$ are countably generated. It is well known that $\mathrm{rad}(P) = \oplus_{i\in I}\mathrm{rad}(P_i)$, $\mathrm{rad}(Q) = \oplus_{j\in J}\mathrm{rad}(Q_j)$. As in the proof of [4, Theorem 2.50] we find an ordinal κ and sets $I_\lambda \subseteq I, J_\lambda \subseteq J$, $\lambda \le \kappa$ such that

(i) $I_0 = \emptyset = J_0$,

(ii) $I_{\lambda'} \subseteq I_\lambda$, $J_{\lambda'} \subseteq J_\lambda$ for any $\lambda' < \lambda \le \kappa$

(iii) $I_\lambda = \cup_{\lambda'<\lambda} I_{\lambda'}$ and $J_\lambda = \cup_{\lambda'<\lambda} J_{\lambda'}$, if $\lambda \le \kappa$ is limit

(iv) if $\lambda < \kappa$, then $|I_{\lambda+1} \setminus I_\lambda| \le \omega$ and $|J_{\lambda+1} \setminus J_\lambda| \le \omega$,

(v) $I = I_\kappa$, $J = J_\kappa$,

(vi) $\overline{f}(\oplus_{i\in I_\lambda} P_i/\mathrm{rad}(P_i)) = \oplus_{j\in J_\lambda} Q_j/\mathrm{rad}(Q_j)$.

For any $\lambda \le \kappa$ let $P_\lambda = \oplus_{i\in I_\lambda} P_i$, $Q_\lambda = \oplus_{j\in J_\lambda} Q_j$ and let $\overline{P_\lambda}, \overline{Q_\lambda}$ be the corresponding radical factors. Observe that $\overline{f}|_{\overline{P_\lambda}}$ gives an isomorphism of $\overline{P_\lambda}$ and $\overline{Q_\lambda}$.

By induction on $\lambda \le \kappa$ we construct isomorphisms $f_\lambda\colon P_\lambda \to Q_\lambda$ such that f_λ extends $f_{\lambda'}$ for any $\lambda' \le \lambda \le \kappa$ and f_λ is a lift of $\overline{f}|_{\overline{P_\lambda}}$ for any $\lambda \le \kappa$. We put $f_0 = 0$.

If $\lambda < \kappa$ and f_λ has been defined, we define $f_{\lambda+1}$ as follows: Let $P' = \oplus_{i\in I_{\lambda+1}\setminus I_\lambda} P_i$, $Q' = \oplus_{j\in J_{\lambda+1}\setminus J_\lambda} Q_j$ and let $\overline{P'}$, $\overline{Q'}$ be their radical factors. So $P_{\lambda+1} = P_\lambda \oplus P'$ and $Q_{\lambda+1} = Q_\lambda \oplus Q'$. Consider the restriction $\overline{f}|_{\overline{P'}}\colon \overline{P'} \to \overline{Q_\lambda} \oplus \overline{Q'}$ and put $\overline{\alpha} = \pi_{\overline{Q_\lambda}}\overline{f}|_{\overline{P'}}$ and $\overline{\beta} = \pi_{\overline{Q'}}\overline{f}|_{\overline{P'}}$. Suppose that $\overline{\beta}(p) = 0$ for some $p \in \overline{P'}$. Then, by (vi), $\overline{f}(p) = \overline{f}(p')$ for some $p' \in \overline{P_\lambda}$. Since $\overline{f}$ is a monomorphism, $p = 0$ and $\overline{\beta}$ is a monomorphism. On the other hand, by (vi), $\overline{Q'} = \pi_{\overline{Q'}}(\overline{f}(\overline{P_\lambda}) + \overline{f}(\overline{P'})) = \overline{\beta}(\overline{P'})$ and thus $\overline{\beta}$ is an epimorphism. Since $\overline{\beta}$ is an isomorphism, and P', Q' are countably generated projective according to (iv), there is an isomorphism $\beta\colon P' \to Q'$ lifting $\overline{\beta}$ by Lemma 2.2.2. Since P' is projective, there exists $\alpha\colon P' \to Q_\lambda$ lifting $\overline{\alpha}$. If we put $f_{\lambda+1} = f_\lambda \oplus (\alpha + \beta)$, we can check that $f_{\lambda+1}$ is an isomorphism extending f_λ and lifting $\overline{f}|_{\overline{P_{\lambda+1}}}$.

If λ is limit, and $f_{\lambda'}$ has been defined for every $\lambda' < \lambda$, we put $f_\lambda = \cup_{\lambda'<\lambda} f_{\lambda'}$. By induction, $f_\lambda\colon P_\lambda \to Q_\lambda$ is an isomorphism lifting $\overline{f}|_{\overline{P_\lambda}}$.

Finally, $f = f_\kappa$ is the desired isomorphism. □

Some well known results about projective modules can be seen also as corollaries of Theorem 2.2.3.

Corollary 2.2.4.

(i) Any nonzero projective module has a maximal submodule.

(ii) Let R be a local ring. Then any projective module is free.

(iii) Let R be a semiperfect ring, let $S_1, \dots, S_n$ be representatives of simple modules and let P_i be a projective cover of S_i for any $1 \le i \le n$. Then any projective module can be uniquely decomposed as a direct sum of copies of $P_1, \dots, P_n$.

We hope that Theorem 2.2.3 is a step toward understanding of projective modules over semilocal rings. Recall that a ring is *semilocal* if $R/J(R)$ is semisimple artinian, thus the radical factor of a projective module over a semilocal ring is semisimple R-module (or $R/J(R)$-module.) Facchini and Herbera [5] gave a description of direct sum decompositions of finitely generated projective modules over a semilocal ring. In particular, it is proved that for any semilocal ring R there exists a semilocal hereditary ring R' such that R and R' have the same decomposition theory of finitely generated projective modules. As we shall see this is not true for arbitrary projective modules because any projective module over a hereditary ring is a direct sum of finitely generated modules. However, some well known properties of finitely generated projective modules over a semilocal ring can be generalized.

Corollary 2.2.5. *Let R be a semilocal ring. If P, Q are projective right R-modules, then the following are equivalent*

(i) $P \simeq Q$

(ii) There exist epimorphisms $f\colon P \to Q$ and $g\colon Q \to P$.

(iii) There exist pure monomorphisms $f\colon P \to Q$ and $g\colon Q \to P$.

Moreover, $P^n \simeq Q^n$ implies $P \simeq Q$ for any $n \in \mathbb{N}$.

Proof. Since $P/\mathrm{rad}(P)$ and $Q/\mathrm{rad}(Q)$ are semisimple, each of (ii), (iii) implies $P/\mathrm{rad}(P) \simeq Q/\mathrm{rad}(Q)$. Now Theorem 2.2.3 applies. □

Using Kaplansky's theorem once again we obtain the following

Corollary 2.2.6. *Let R be a semilocal ring. Then there are at most countably many pairwise non-isomorphic indecomposable projective modules.*

We do not know an example of a semilocal ring having infinitely many non-isomorphic indecomposable projective modules (recall that over a semilocal ring there are only finitely many non-isomorphic indecomposable finitely generated projective modules.)

Observe that there would be only finitely many indecomposable projective modules over a semilocal ring if the following was true: If P, Q are projective modules and $P/\mathrm{rad}(P)$ is a direct summand of $Q/\mathrm{rad}(Q)$, then P is a direct summand of Q. Unfortunately, this is not true. Gerasimov and Sakhajev [8] gave an example of a semilocal ring R which possesses an infinitely generated projective module P such that $P/\mathrm{rad}(P)$ is finitely generated. In fact, P can be chosen such that $P/\mathrm{rad}(P)$ is isomorphic to a direct summand of $R/J(R)$. Of course, P cannot be isomorphic to a direct summand in R. Similar phenomena will occur in Sections 4 and 5.

Our last corollary uses a technique of Sakhajev to give an information about the structure of projective modules having radical factor cyclic.

Corollary 2.2.7. *Let P be projective R-module such that $P/\mathrm{rad}(P)$ is cyclic. Then there exists $r \in R$ and $p_1, p_2 \dots \in P$ such that $P = \sum_{i \in \mathbb{N}} p_i R$ and $p_{i+1} r = p_i$ for any $i \in \mathbb{N}$.*

Proof. We shall use the idea of [13, Fact 3.1]. Observe that P is countably generated. Moreover, P can be considered as a pure right ideal of R by [6, Proposition 6.1], so we can suppose that P is a countably generated pure right ideal in R. Take $p \in P$ such that $(pR + \mathrm{rad}(P))/\mathrm{rad(P)} = P/\mathrm{rad}(P)$. Since P is pure in R_R, there exists $q \in P$ such that $qp = p$. By assumption, there is $t \in R$ and $j \in \mathrm{rad}(P) = J(R) \cap P$ such that $q = pt + j$. Now, $p = qp = (pt + j)p$ and $pt = (pt)^2 + jpt$ follows. Since $j \in J(R)$, the element $u = (1 - j)$ is invertible and $upt = (pt)^2$. For any $i \in \mathbb{N}$ put $x_i = u^{-i}ptu^{i-1}$. Then $x_{i+1}x_i = u^{-i-1}(pt)^2u^{i-1} = x_i$ that is $x_1, x_2, ...$ is a right a-sequence. As proved in [13, Fact 3.1], $Q = \sum_{i\in\mathbb{N}} x_iR$ is a pure right ideal and hence projective. We can embedd $P/\mathrm{rad}(P), Q/\mathrm{rad}(Q)$ canonically to $R/J(R)$.

Since, by purity, $\mathrm{rad}(P) = P \cap J(R), \mathrm{rad}(Q) = Q \cap J(R)$, the canonical projection $\pi\colon R \to R/J(R)$ induces embeddings of $P/\mathrm{rad}(P)$ and $Q/\mathrm{rad}(Q)$ into $R/J(R)$. Obviously, $\pi(P) = pR + J(R)/J(R)$. Observe that $u^{-1} = 1 + j'$, where $j' \in J(R)$. Thus $\pi(x_i) = \pi((1 + j')^i ptu^{i-1}) = \pi(ptu^{i-1})$. Moreover, $\pi(p) = \pi((pt + j)p) = \pi(pt)p$ implies $\pi(x_1)p = \pi(p)$. Therefore $\pi(Q) \subseteq pR + J(R)/J(R)$ and since $\pi(x_1)p = \pi(p)$, the equality $\pi(P) = \pi(Q)$ holds. Thus $P/\mathrm{rad}(P) \simeq Q/\mathrm{rad}(Q)$ and $P \simeq Q$ by Theorem 2.2.3. For any $i \in \mathbb{N}$ put $q_i = x_iu^{-i+1}$. Observe that $q_1, q_2, \dots$ generate Q and $q_{i+1}pt = q_i$ for any $i \in \mathbb{N}$, so if $f : Q \simeq P$ is an isomorphism, we can put $p_i = f(q_i)$ and $r = pt$.

□

Remark 2.2.8. Recall that the *trace ideal* of a module M is $\mathrm{Tr}(M) = \sum_{f\in\mathrm{Hom}_R(M,R)} f(M)$. If P is a projective module, then $\mathrm{Tr}(P)$ is the smallest ideal in the set of ideals $\{I \subseteq R \mid PI = P\}$. Suppose that P is a projective module such that there exist $r' \in \mathrm{Tr}(P)$ and $p_1, p_2, \dots \in P$ generating P such that $p_{i+1}r' = p_i$ for any $i \in \mathbb{N}$. Since $Pr'R = P$, we infer that $\mathrm{Tr}(P) = Rr'R$. Analyzing the proof of Corollary 2.2.7, we see that the element $r \in R$ can be chosen in the trace ideal of P. Thus, by Corollary 2.2.7, we get that if P is a projective module with $P/\mathrm{rad}(P)$ cyclic, then $\mathrm{Tr}(P)$ is generated by a single element as a two-sided ideal.

2.3 *Comparing* $\mathrm{Add}(M_R)$ *and* $\mathrm{Proj} - \mathrm{End}_R(M_R)$

We are going to investigate the relation between objects of $\mathrm{Add}(M_R)$ (i.e., direct summands of modules that are direct sums of copies of M) and $\mathrm{Proj} - \mathrm{End}_R(M_R)$ (i.e., projective right modules over the endomorphism ring of M), where M is a nonzero R-module. Probably we shall reinvent a wheel but we were not able to find a convenient reference. One could simply say that the tensor product commutes with direct sums but we need to be more explicit.

Let M be a nonzero right module over a ring R and let I be a non-empty set. Let S denote the endomorphism ring of M. Consider a direct sum decomposition of the free right S-module $A \oplus B = S^{(I)}$, let $\iota_A\colon A \to S^{(I)}$, $\iota_B\colon B \to S^{(I)}$ be canonical inclusions. Applying the tensor product functor $- \otimes_S M\colon \mathrm{Mod} - S \to \mathrm{Mod} - R$, we get $\mathrm{Im}\ (\iota_A \otimes M) \oplus \mathrm{Im}\ (\iota_B \otimes M) = S^{(I)} \otimes M$. The module $S^{(I)} \otimes M$ is isomorphic to $M^{(I)}$ via the isomorphism $\varphi\colon S^{(I)} \otimes M \to M^{(I)}$ given by $\varphi((s_i)_{i\in I} \otimes m) = (s_i(m))_{i\in I}$. Denote $A' = \mathrm{Im}\ \varphi \circ (\iota_A \otimes M)$, $B' = \mathrm{Im}\ \varphi \circ (\iota_B \otimes M)$ and observe that $A' \oplus B' = M^{(I)}$. Let $\iota_{A'}\colon A' \to M^{(I)}, \iota_{B'}\colon B' \to M^{(I)}, \pi_{A'}\colon M^{(I)} \to A', \pi_{B'}\colon M^{(I)} \to B'$ be canonical injections and projections given by this decomposition. Any $i \in I$ also gives the canonical injection $\iota_i\colon M \to M^{(I)}$ and the canonical projection $\pi_i\colon M^{(I)} \to M$. Fix an arbitrary $j \in I$ and consider the element $e_j = (\delta_{i,j})_{i\in I} \in S^{(I)}$, where $\delta_{j,j} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$.

There exist unique $(a_i)_{i\in I} \in A$ and $(b_i)_{i\in I} \in B$ such that $e_j = (a_i)_{i\in I} + (b_i)_{i\in I}$. Then $\iota_j(m) = \varphi(e_j \otimes m) = (a_i(m))_{i\in I} + (b_i(m))_{i\in I}$. Note that $(a_i(m))_{i\in I} \in A'$ and $(b_i(m))_{i\in I} \in B'$. Therefore $\pi_i\iota_{A'}\pi_{A'}\iota_j = a_i$ and $\pi_i\iota_{B'}\pi_{B'}\iota_j = b_i$ for any $i \in I$. Therefore a direct sum decomposition of $S^{(I)}$ in Mod $-$ S induces a decomposition of $M^{(I)}$. Unfortunately, not every decomposition of $M^{(I)}$ can be constructed in such a way because the module $\pi_{A'}(\iota_i(M))$ has a finite support for any $i \in I$. Thus we have defined a map Φ which assigns a decomposition of $M^{(I)}$ to every decomposition of $S^{(I)}$.

Now let us consider the decomposition $A' \oplus B' = M^{(I)}$ such that $\pi_{A'}(\iota_i(M))$ has finite support for any $i \in I$, that is for any $i \in I$ there exist only finitely many $j \in I$ such that $\pi_j\iota_{A'}\pi_{A'}\iota_i \neq 0$. We shall say that A', B' form a *finite support decomposition* of $M^{(I)}$. For any $i, j \in I$ let us denote $a_{j,i} = \pi_j\iota_{A'}\pi_{A'}\iota_i \in S$ and $b_{j,i} = \pi_j\iota_{B'}\pi_{B'}\iota_i \in S$. Since $\pi_{A'}(\iota_i(M))$ (and hence also $\pi_{B'}(\iota_i(M))$) has finite support, we have $a_i = (a_{j,i})_{j\in I} \in S^{(I)}$ and $b_i = (b_{j,i})_{j\in I} \in S^{(I)}$. Put $A = \sum_{i\in I} a_iS \subseteq S^{(I)}$, $B = \sum_{i\in I} b_iS \subseteq S^{(I)}$. Since $a_i + b_i = e_i$, $A + B = S^{(I)}$. Suppose that there exists nonzero $x \in A \cap B$. Then there are $s_i \in S, i \in I$, almost all of them equal zero, such that $x = \sum_{i\in I} a_is_i$. Take some $j \in I$ and $m \in M$ such that $\sum_{i\in I} a_{j,i}s_i(m)$ is nonzero. Observe that $x' = \sum_{i\in I}(a_{k,i}(s_i(m)))_{k\in I} = \sum_{i\in I} \pi_{A'}(\iota_i(s_i(m)))$ is a nonzero element of A'. By our assumption there are $t_i \in S, i \in I$, almost all of them zero, such that $x = \sum_{i\in I} b_it_i$. By the same arguments as above, we infer $x' \in B'$, a contradiction. Therefore $A \oplus B = S^{(I)}$. Now we have defined a map Ψ that assigns a decomposition of $S^{(I)}$ to every finite support decomposition of $M^{(I)}$. It can be easily verified that Φ and Ψ are mutually inverse.

Now we can summarize these observations in

Proposition 2.3.1. *Let M be a nonzero right R-module, let $S = \mathrm{End}_R(M)$, and let I be a nonempty set. Put $\mathcal{C} = \{(A, B) \mid A \subseteq S^{(I)}, B \subseteq S^{(I)}, A \oplus B = S^{(I)}\}$ and $\mathcal{D} = \{(A', B') \mid A', B'$ form finite support decomposition of $M^{(I)}\}$. The maps $\Phi\colon \mathcal{C} \to \mathcal{D}$ and $\Psi\colon \mathcal{D} \to \mathcal{C}$ are mutually inverse bijections.*

If $A \oplus B = A_1 \oplus B_1 = S^{(I)}$ are two decompositions of $S^{(I)}$ and $A \simeq A_1$, then $A' \simeq A'_1$, where $(A', B') = \Phi((A, B)), (A'_1, B'_1) = \Phi((A_1, B_1))$. This is because Φ is "carried" by a functor. But we do not have an analogy to this statement in the opposite direction, so the classification of the projective S-modules can be different from the classification of objects of $\mathrm{Add}(M)$ that arises from finite support decompositions of $M^{(I)}$. But observe that projective modules should be more complex from this point of view because the tensor product can make two non-isomorphic modules isomorphic.

2.4 Projective modules over the endomorphism ring of a biuniform module

Now we apply these general concepts to the particular case of the endomorphism ring of a biuniform module. Recall that a module M is called *biuniform* if it is nonzero, M is not a sum of its two proper submodules and any two nonzero submodules of M have a non-trivial intersection. A module is said to be *uniserial* if its lattice of submodules is a chain. Obviously, any nonzero uniserial module is biuniform. Let $S = \mathrm{End}_R(M)$. By [4, Theorem 9.1] $I = \{f \in S \mid f \text{ is not mono}\}$ and $K = \{f \in S \mid f \text{ is not epi}\}$ are two-sided ideals. If I, K are comparable in inclusion, then S is local and hence all projective modules are free. Therefore we shall consider only the opposite case, I, K incomparable and, by [4, Theorem 9.1], I, K are the only maximal right ideals of S. Then

S is semilocal and $S/J(S) \simeq S/I \times S/K$. Note that simple S-modules $S/I, S/K$ cannot be isomorphic. Following [13], we shall use the following notation: Let P be a countably generated projective S-module. Then $P/\mathrm{rad}(P) \simeq S/I^{(k)} \oplus S/K^{(l)}$ for some $0 \leq k, l \leq \omega$. Since k, l are uniquely determined by P, we can define $\dim(P) = (k, l)$ (the *dimension of* P). In particular $\dim(S) = (1, 1)$; hence if P is a free module, then $\dim(P) = (k, k)$ for some $0 \leq k \leq \omega$.

The following lemma is easy to prove see for example [17, Lemma 2.2].

Lemma 2.4.1. *Let $U_i, i \in I$ be a family of biuniform modules. Suppose $A \oplus B = \oplus_{i \in I} U_i$. If A is nonzero, then there are $i, j \in I$ such that $\pi_j \iota_A \pi_A \iota_i$ is a monomorphism.*

The following lemma answers [13, Question 8.1].

Lemma 2.4.2. *Let P be a countably generated projective S-module. If* $\dim(P) = (0, k)$*, then $P = 0$.*

Proof. We can suppose that $P \oplus Q = S^{(\omega)}$ for some module Q. By [13, Remark 2.3] $P/\mathrm{rad}(P) \simeq P/PI \oplus P/PK$, hence our assumption is equivalent to $P = PI$. Thus if $(s_i)_{i \in \omega} \in P$, then none of the s_i is a monomorphism. Suppose $P \neq 0$. Applying the map Φ of Proposition 2.3.1 to (P, Q) we get a decomposition $P' \oplus Q' = M^{(\omega)}$. Since Φ is mono, P' is nonzero. Moreover, we saw that the endomorphisms $\pi_j \iota_{P'} \pi_{P'} \iota_i$, $i, j \in \omega$ are given as coordinates of elements in P. Hence non of these endomorphisms is a monomorphism, a contradiction to Lemma 2.4.1. □

Proposition 2.4.3. *Let P be a countably generated projective S-module. If* $\dim(P) = (k, l)$*, then $k \geq l$.*

Proof. Suppose there exists a countably generated projective S-module P such that $\dim(P) = (k, l)$ and $k < l$. Then $k < \omega$. Observe that $\dim(S) = (1, 1)$. Since S is a finitely generated projective module, there exists Q such that $P \simeq S^k \oplus Q$. Because the dimension is additive, $\dim(Q) = (0, l')$, $l' \neq 0$. This contradicts Lemma 2.4.2. □

Let us recall a part of the main result of [13].

Fact 2.4.4. [13, Theorem 4.3] *Let M be a biuniform R-module, $S = \mathrm{End}_R(M)$. Then the following are equivalent:*

(i) There is a monomorphism $f \in S$ and an epimorphism $g \in S$ such that $gf = 0$.

(ii) There exists a countably generated projective S-module P such that $\dim(P) = (1, 0)$.

Observe that our results give a classification of projective S-modules in case S satisfies the equivalent conditions of this theorem. Namely, if P is a module of dimension (1,0), then all projective right S-modules are isomorphic to $P^{(X)} \oplus S^{(Y)}$.

Right now we are not able to say more in general. In case M is uniserial one can complete the classification using the following lemma which is just a small modification of [17, Proposition 2.7]. Let us recall that if M is a nonzero uniserial module, S its endomorphism ring and I, K the ideals of S defined above, then we define $M_m = \cap_{f \in S \setminus I} f(M)$ and $M_e = \sum_{f \in S \setminus K} \mathrm{Ker}\, f$. These are fully invariant submodules of M; for some properties of these submodules see [18].

Lemma 2.4.5. [17, Proposition 2.7] *Let M be a nonzero uniserial module such that* $\mathrm{End}_R(M)$ *is not local. Assume there is a decomposition $A \oplus B = M^{(\omega)}$, $A \neq 0$ such that $\pi_j \iota_A \pi_A \iota_i \in \mathrm{End}_R(M)$ is not an epimorphism for any $i, j \in \omega$. Then there are a monomorphism $f \colon M \to M$ and an epimorphism $g \colon M \to M$ such that $gf = 0$.*

Proof. For any $i \in \omega$ let M_i be an isomorphic copy of M and let $N = \oplus_{i\in\omega} M_i$. We consider a decomposition $N = A \oplus B$ such that for any $i, j \in \omega$ $\pi_i \iota_A \pi_A \iota_j(M_j) \neq M_i$, where $\pi_A\colon N \to A$, $\pi_B\colon N \to B$, $\pi_i\colon N \to M_i$ are the canonical projections and $\iota_A\colon A \to N$, $\iota_B\colon B \to N$, $\iota_i\colon M_i \to N$ are the canonical injections. Observe that for any $i \in \omega$ we have $\pi_i \pi_B(M_i) = M_i$ and $\pi_j \pi_B(M_i) \neq M_j$ whenever $i \neq j$. (Since we work inside the module N, we can omit the canonical inclusions and consider the projections as endomorphisms of N.)

The strategy of the proof is following: We find a decomposition of B as a direct sum of modules isomorphic to M, then we prove $A \subseteq \oplus_{i\in\omega}(M_i)_e$ and finally we show $M_m \subsetneq M_e$.

Set $M'_0 = \pi_B(M_0)$. Since $\pi_0\pi_B(M_0) = M_0$, [17, Lemma 2.2], gives that M'_0 is a direct summand of N isomorphic to M. Therefore there exists $B_0 \subseteq B$ such that $B = M'_0 \oplus B_0$. Note that, for any $j > 0, \pi_j(M'_0) \neq M_j$.

Suppose that we have constructed $M'_0, \dots, M'_k$ such that $B = M'_0 \oplus \dots \oplus M'_k \oplus B_k$ for some $B_k \subseteq B$, $\pi_j(M'_0 \oplus \dots \oplus M'_k) \neq M_j$ for any $j > k$ and $\pi_B(M_0 \oplus \dots \oplus M_k) = M'_0 \oplus \dots \oplus M'_k$. Put $M'_{k+1} = \pi_{B_k}(M_{k+1})$ (the projection $\pi_{B_k}: N \to B_k$ is with respect to the decomposition $N = A \oplus M'_0 \oplus \dots \oplus M'_k \oplus B_k$). Now we have $\pi_{k+1}(M'_{k+1}) = M_{k+1}$; therefore, by [17, Lemma 2.2], M'_{k+1} is a direct summand of B_k isomorphic to M and we have $B = M'_0 \oplus \dots \oplus M'_k \oplus M'_{k+1} \oplus B_{k+1}$ for some $B_{k+1} \subseteq B_k$. From the induction argument we have that $M'_0 \oplus \dots \oplus M'_{k+1} = \pi_B(M_0 \oplus \dots \oplus M_{k+1})$ and thus $\pi_j(M'_0 \oplus \dots \oplus M'_{k+1}) \neq M_j$ for any $j > k+1$.

It is easy to check that $B = \oplus_{i\leq\omega} M'_i$, where $\pi_i(M'_i) = M_i$.

By [18, Observation 2.6], for any $x \in \oplus_{i\in\omega}(M_i)_e \subseteq N$ we have $\pi_B(x) \subseteq \oplus_{i\in\omega}(M'_i)_e \subseteq B$. Further, observe that $\pi_B(M_i) \subseteq \oplus_{j=0}^{i} M'_j$ for any $i \in \omega$. Finally, let $\pi'_i\colon B \to M'_i, i \in \omega$ be the canonical projections given by $B = \oplus_{i\in\omega} M'_i$. Then $\pi'_i \pi_B(M_i) = M'_i$. Suppose that there exists $a \in A \setminus \oplus_{i\in\omega}(M_i)_e$; write $a = m_0 + \dots + m_k, m_i \in M_i$. Let j be the greatest index such that $m_j \notin (M_j)_e$. If $l < j$, then $\pi'_j\pi_B(m_l) = 0$ and if $l > j$, then $\pi'_j\pi_B(m_l) \in (M'_j)_e$. Since, by [18, Lemma 2.3(iv)], for any epimorphism $f\colon M_j \to M'_j$, $f^{-1}((M'_j)_e) = (M_j)_e$, we get $\pi'_j\pi_B(m_j) \notin (M'_j)_e$. Thus $\pi_B(a) \neq 0$, a contradiction to $a \in A$. Therefore we conclude that $A \subseteq \oplus_{i\in\omega}(M_i)_e$.

In order to conclude the proof, let us recall that, by Lemma 2.4.1, $A \neq 0$ implies that there exist $i, j \in \omega$ such that $\pi_j\iota_A\pi_A\iota_i$ is a monomorphism. So $\pi_j(A)$ contains $(M_j)_m$ as a proper submodule according to Lemma [18, Lemma 2.2(ii)]. Thus if $A \neq 0$, then $M_m \subsetneq M_e$. If $x \in M_e \setminus M_m$, then there are a monomorphism $f\colon M \to M$ such that $f(M) \subseteq xR$ and an epimorphism $g : M \to M$ such that $g(x) = 0$. Obviously $gf = 0$. $\square$

Proposition 2.4.6. *Let M be a nonzero uniserial module. Suppose that $gf \neq 0$ for any monomorphism $f\colon M \to M$ and any epimorphism $g\colon M \to M$. Then any projective $S-$module is free.*

Proof. By a classical result of Kaplansky any projective module over a local ring is free (we re-proved this in Corollary 2.2.4). Thus we can suppose that $\mathrm{End}_R(M)$ is not local. Let P be a countably generated projective S-module that is not free. We can suppose that $\dim(P) = (k, 0)$, where $k \neq 0$. Let Q be an S-module such that $P \oplus Q = S^{(\omega)}$. Applying Φ we obtain a decomposition $P' \oplus Q' = M^{(\omega)}$, where P' is nonzero. Since $\dim(P) = (k, 0)$, this decomposition satisfies the assumption of Fact 2.4.5. That implies the existence of a monomorphism $f\colon M \to M$ and an epimorphism $g\colon M \to M$ such that $gf = 0$. $\square$

Thus we reached the classification of projective modules over the endomorphism ring of a uniserial module U that is quite similar to that of modules in $\mathrm{Add}(U)$.

Theorem 2.4.7. *Let U be a nonzero uniserial R-module and let $S = \mathrm{End}_R(U)$. Then every right projective S-module is free if and only if $gf \neq 0$ for a monomorphism $f: U \to U$ and an epimorphism $g: U \to U$. In the opposite case there is a countably (but not finitely) generated projective S module P such that $P/\mathrm{rad}(P)$ is simple and every right projective S-module is isomorphic to $P^{(X)} \oplus S^{(Y)}$.*

Recall that a module U is called *self-small* if for any homomorphism $f: U \to U^{(\omega)}$ there exists a finite set $F \subseteq \omega$ such that the image of f is contained in U^F. A module U is said to be *quasi-small* if for any family $M_i, i \in X$, of modules such that U is a direct summand of $\oplus_{i\in X} M_i$, there exists a finite set $X_0 \subseteq X$ such that U is isomorphic to a direct summand of $\oplus_{i\in X_0} M_i$. For example, any finitely generated module is self-small and quasi-small. It can be shown that any self-small uniserial module is quasi-small.

Remark 2.4.8. Let U be a uniserial module such that there are $f \notin I$ and $g \notin K$ such that $gf = 0$. Suppose that U is quasi-small (for example, it happens if there is $u \in U$ such that $h(u) \neq 0$ for any $h \notin K$). By [17, Theorem 1.1], there exists a unique uniserial module $V \not\simeq U$ such that V is a direct summand of $U^{(\omega)}$. In this case all objects of $\mathrm{Add}(U)$ are isomorphic to direct sums of copies of U and V. It can be proved that $\mathrm{Hom}_R(U, V)$ is a projective right $S = \mathrm{End}_R(U)$-module of dimension $(1, 0)$. Moreover, the Hom - tensor adjunction induces an equivalence of $\mathcal{K}$ and $\mathcal{L}$, where $\mathcal{K}$ is the full subcategory of $\mathrm{Add}(U)$ given by modules of finite Goldie dimension and $\mathcal{L}$ is the full subcategory of $\mathrm{Proj}-S$ given by projective modules with finitely generated radical factor. An example of a uniserial module U of required property can be found in [15] but we do not know whether U can be chosen such that $\mathrm{Hom}_R(U, -)$ does not commute with direct sums.

Question 2.4.9. *We ask whether there is a uniserial module U satisfying the following:*

(i) U is quasi-small.

(ii) U is not self-small.

(iii) There exist a monomorphism $f: U \to U$ and an epimorphism $g: U \to U$ such that $gf = 0$.

Observe that (ii) implies that U is a countably but not finitely generated module. We do not know the answer even if (ii) is replaced by this weaker condition.

2.5 Pure projective modules over an exceptional chain ring

In this section we are going to describe a case where there is a projective module which is not a direct sum of indecomposable modules.

Using Theorem 2.2.3 we can complete the classification of pure projective modules over an exceptional chain ring (see [14, Conjecture 5.9]). As most of the work was already done in [14], we shall be as brief as possible but we will follow an abstract approach introduced in [19]. An interested reader is advised to see [4, 16, 14, 19] for details.

Let R be a ring and let T, U be finitely generated uniserial modules such that

(i) The endomorphism ring of T is local.

(ii) There exists a uniserial module $V \not\simeq U$ such that V is a direct summand of $U^{(\omega)}$. Such V is unique up to isomorphism by [17, Theorem 1.1].

(iii) There exists a module W such that $U \oplus T^{(\omega)} \simeq V \oplus W$.

Let $M = T \oplus U$ and let $S = \mathrm{End}_R(M)$. Since M is finitely generated, categories $\mathrm{Add}(M)$ and $\mathrm{Proj} - S$ are equivalent. Let us denote $P_1 = \mathrm{Hom}_R(M,T)$, $P_2 = \mathrm{Hom}_R(M,U)$, $P_3 = \mathrm{Hom}_R(M,V)$ and $P_4 = \mathrm{Hom}_R(M,W)$ (take some W satisfying (iii), we shall see that it is in fact unique). Now we want to understand radical factors of P_1, P_2, P_3, P_4. These S-modules are countably generated and P_1, P_2 are even finitely generated. Since P_1 is a projective module of local endomorphism ring, $S_1 := P_1/\mathrm{rad}(P_1)$ is simple. Further $\mathrm{End}_S(P_2) \simeq \mathrm{End}_R(U)$, U cannot have the endomorphism ring local by (ii), therefore P_2 has exactly 2 maximal submodules X_1, X_2 such that $S_2 := P_2/X_1$ and $S_3 := P_2/X_2$ are not isomorphic. Namely, set $X_1 = \{f\colon T \oplus U \to U \mid f|_U \text{ is not mono}\}$ and $X_2 = \{f\colon T \oplus U \to U \mid f|_U \text{ is not epi}\}$. Observe that an arbitrary $f\colon U \to U$ is not a monomorphism (resp. not an epimorphism) if and only if $\mathrm{Im}\,\mathrm{Hom}_R(M,f) \subseteq X_1$ (resp. $\mathrm{Im}\,\mathrm{Hom}_R(M,f) \subseteq X_2$). Since T is not a direct summand of U, S_1, S_2, S_3 are pair-wise non-isomorphic simple modules $S/J(S) \simeq S_1 \oplus S_2 \oplus S_3$. Again, we shall write $\dim(P) = (a,b,c)$ if a,b,c are cardinals such that $P/\mathrm{rad}(P) \simeq S_1^{(a)} \oplus S_2^{(b)} \oplus S_3^{(c)}$. There exists a split monomorphism $\nu\colon V \to U^{(\omega)}$ such that $\pi_i\nu\colon V \to U$ is not a monomorphism for any $i > 0$ (π_i stands for the canonical projection $U^{(\omega)} \to U$), see [4, Proof of Proposition 9.30] for details. Then it is easy to check that $\dim(P_3) = (0,1,0)$. Finally, we derive $\dim(P_4) = (\omega,0,1)$ easily from $P_1^{(\omega)} \oplus P_2 \simeq P_3 \oplus P_4$. Note that W is indeed described by (i),(ii),(iii) uniquely up to isomorphism. Let us summarize our calculations:

$$\dim(P_1) = (1,0,0), \dim(P_2) = (0,1,1), \dim(P_3) = (0,1,0), \dim(P_4) = (\omega,0,1)\,.$$

Now we claim that any projective S-module is isomorphic to a direct sum of copies of P_1, P_2, P_3 and P_4. Let Q be a countably generated module of dimension (a,b,c). If $a = \omega$, then $P_3^{(b)} \oplus P_4^{(c)} \simeq Q$. If $a < \omega$, there is an S-module Q' such that $Q = Q' \oplus P_1^a$ because P_1 is finitely generated. If $b \geq c$, then Q' is a direct sum of copies of P_2 and P_3. If $b < c$ then, since P_2 is finitely generated, there exists projective S-module Q'' such that $\dim(Q'') = (0,0,d)$. Since $Q'' \oplus P_3^{(d)} \simeq P_2^{(d)}$ there would be a module in $\mathrm{Add}(U)$ that is not a direct sum of copies of V and U, a contradiction to Theorem 2.4.7. This proves the claim and we are ready to classify pure projectives over some very strange rings.

Recall that a ring R is called a *chain ring* if ${}_RR, R_R$ are (left and right) uniserial R-modules. Following [16], a chain ring is said to be *exceptional* if it has exactly 3 two-sided ideals $0, J(R), R$, $J(R)^2 = J(R)$ and R is prime and contains zero divisors. Henceforth, let R be an exceptional coherent chain ring. By [16, Lemma 3.5] for any $0 \neq r, s \in J(R)$ modules R/rR and R/sR are isomorphic and any pure projective R-module is isomorphic to a direct summand of a direct sum of copies of $R, R/rR$ for some (any) $0 \neq r \in J(R)$. Let $U = R/rR, T = R_R$. Then (i) follows since T is projective and uniserial, (ii) holds by [14, Lemma 4.2] and (iii) holds by [14, Lemma 4.3]. As remarked above, pure projective modules over R are exactly objects of $\mathrm{Add}(U \oplus T)$ and categories $\mathrm{Add}(U \oplus T)$ and $\mathrm{Proj} - \mathrm{End}_R(U \oplus T)$ are equivalent, therefore we have

Theorem 2.5.1. *Let R be an exceptional chain coherent ring. Then any pure projective module is isomorphic to a direct sum of copies of T, U, V, W.*

Remark 2.5.2. It was noted in [19] that W is not a direct sum of uniserial modules, but in fact P_4 cannot be written as a direct sum of indecomposable modules and neither can W. Indeed, by [14, Proposition 4.5] any direct sum decomposition of W is of the form

$W \oplus T^{(\delta)}$, where $0 \leq \delta \leq \omega$. This statement now follows easily from Theorem 2.5.1 and remains valid in the more abstract context as at the beginning of this section.

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3. FAIR-SIZED PROJECTIVE MODULES

3.1 Introduction

This paper is devoted to the study of infinitely generated projective modules over associative unitary rings. We are interested in the case in which the ring has projective modules that are not direct sums of finitely generated modules. Some general results and examples of rings with such modules were given in [12]. Our motivation was to find a technique that could be applied to prove the existence of superdecomposable projective modules over semilocal rings.

Let us briefly explain the main idea of the paper. According to a well known theorem of Kaplansky, any projective right module over a ring R is a direct sum of countably generated right modules, so it suffices to investigate countably generated projectives, that is, direct summands of a countably generated free right module $F = R_R^{(\mathbb{N})}$. Suppose that $P \oplus P' = F$. The canonical projection $\pi\colon F \to P$ is given by a column-finite $\mathbb{N} \times \mathbb{N}$ idempotent matrix A. We say that A represents P (observe that the columns of A generate P). Let I_n be the ideal of R generated by the entries of A that are below the n-th row. Clearly, P is finitely generated if and only if there exists $k \in \mathbb{N}$ such that $I_l = 0$ for every $l \geq k$. The other possible extreme case is when $I_1 = I_2 = \cdots = R$. It is a well-known result of Bass [3, Theorem 3.1] that in this case $P \simeq F$ provided $R/J(R)$ is right noetherian. In this paper, we focus our attention on the case in which the sequence $I_1 \supseteq I_2 \supseteq \ldots$ terminates at an ideal I. It is easy to see that I is idempotent. We show that if R is left and right noetherian and the sequence $I_1 \supseteq I_2 \supseteq \cdots$ terminates at I, then P contains as a direct summand any countably generated projective module having its trace ideal in I. Cf. [3, Theorem 3.1]. The following easy condition assures that any sequence of ideals derived from an idempotent column-finite matrix terminates: If $I_1, I_2, \ldots$ is a sequence of ideals in R such that $I_{k+1}I_k = I_{k+1}$ for any $k \geq 1$, then there exists n such that $I_n = I_{n+1} = \ldots$. Call (*) this condition.

In section 2, we show that over a left and right noetherian ring R satisfying condition (*), the theory of projective modules "reduces" to the theory of idempotent ideals in R and the theory of finitely generated projective modules over the factor rings of R modulo idempotent ideals. This explains and is related to the statement in the introduction of [3], according to which "infinitely generated projective modules invite little interest".

The remaining sections are devoted to presenting some examples. We prove that (*) holds for semilocal noetherian rings, integral group rings of a finite group and universal enveloping algebras of finite solvable Lie algebras over a field of characteristic zero. This allows us to prove that:

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(i) There exists a semilocal noetherian ring with superdecomposable projective modules.

(ii) Indecomposable projective modules over integral group rings of finite groups are finitely generated.

(iii) Any infinitely generated projective module over a finite dimensional solvable Lie algebra over a field of characteristic zero is free.

Notice that (ii) solves [9, Problem 8.34].

Let us briefly recall some notions and fix the notation. The word "ring" always means associative ring with an identity and "module" means unital right module. If M is a module over R, then $\sum_{f\in \mathrm{Hom}_R(M,R)} f(M)$ is an ideal of R called the *trace ideal of* M. We denote it $\mathrm{Tr}(M)$. If P is a projective module over R, then $\mathrm{Tr}(P)$ is the smallest ideal X of R such that $PX = P$, and is an idempotent ideal. Further if X is a subset of a ring R, we denote RXR the (two-sided) ideal generated by X. In case $X = \{x\}$ we denote RxR the smallest ideal of R containing x. Notice that in general the relation $RxR = \{rxs \mid r, s \in R\}$ does not hold. Recall the following important result due to Whitehead:

Fact 3.1.1. [18, Corollary 2.7] *Let I be an idempotent ideal of R finitely generated on the left. Then there exists a countably generated projective right R-module P such that* $\mathrm{Tr}(P) = I$.

To avoid confusion, we will call the rings which have all left ideals and all right ideals finitely generated left and right noetherian rings, although they are often called noetherian rings. Finally, we will call *infinitely generated* projective modules the projective modules that are countably generated but not finitely generated.

3.2 I-big modules

Let P be a countably generated projective module over a ring R and let I be an ideal of R. We say that P is I*-big* if for any countably generated projective module Q with trace ideal contained in I there exists an epimorphism of P onto Q. Hence, in this case, P contains a direct summand isomorphic to Q. Notice that this definition will be applied to countably generated projective modules only.

Remark 3.2.1. (Eilenberg's trick) Let I be an ideal of a ring R and let P be an I-big projective module. If Q is a countably generated projective module with trace ideal contained in I, then $P \oplus Q \simeq P$, because $Q^{(\omega)}$ is a direct summand of P.

Lemma 3.2.2. *Let I be an idempotent ideal that is finitely generated as a left ideal. Then there exists an I-big projective module P such that* $\mathrm{Tr}(P) = I$. *Such a module P is unique up to isomorphism.*

Proof. By [18, Corollary 2.7], there exists a countably generated projective module P with $\mathrm{Tr}(P) = I$. Clearly, $\mathrm{Tr}(P^{(\omega)}) = I$. If Q is a countably generated projective module having the trace ideal contained in I, then $QI = Q$ and Q is a factor of $P^{(\omega)}$. Let P_1, P_2 be I-big modules such that $\mathrm{Tr}(P_1) = \mathrm{Tr}(P_2) = I$. By Remark 3.2.1, $P_1 \oplus P_2 \simeq P_1$. Similarly, $P_1 \oplus P_2 \simeq P_2$. Thus $P_1 \simeq P_2$. ∎

Remark 3.2.3. We have just proved that, for any ideal I that is a trace ideal of a countably generated projective module, there exists a unique countably generated projective module P (up to isomorphism) such that P is I-big and $\mathrm{Tr}(P) = I$. We will make use of I-big modules over left and right noetherian rings. Observe that $R^{(\omega)}$ is an R-big projective module and that any R-big projective module has trace ideal R. Therefore any R-big projective module is isomorphic to $R^{(\omega)}$.

We say that a ring R satisfies Condition (*) if for any sequence $I_1, I_2, \dots$ of ideals in R such that $I_{k+1}I_k = I_{k+1}, k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $I_k = I_n$ for any $n \leq k \in \mathbb{N}$. Notice that such a sequence is necessarily a descending chain.

We will use this condition in the following context: Suppose we have a countably generated projective module P. Thus P is a direct summand of a countably generated free module, $P \oplus P' = R^{(\mathbb{N})}$ say. The canonical projection $\pi\colon R^{(\mathbb{N})} \to P$ can be written with respect to the canonical basis of $R^{(\mathbb{N})}$ as an $\mathbb{N} \times \mathbb{N}$ matrix $A = (a_{i,j})_{i,j \in \mathbb{N}}$ with entries in R. Moreover, A is a column-finite matrix (that is, for any $j \in \mathbb{N}$ there exists $i \in \mathbb{N}$ with $a_{k,j} = 0$ whenever $k \geq i$). Therefore A^2 is defined and it is easy to see that $A^2 = A$ (that is, A is an idempotent matrix). Conversely, given any idempotent column-finite $\mathbb{N} \times \mathbb{N}$ matrix A, the corresponding module $P = AR^{(\mathbb{N})}$ is projective.

Now, let $A = (a_{i,j})_{i,j \in \mathbb{N}}$ be an idempotent column-finite matrix over R and let $I_k = \sum_{k \leq i \in \mathbb{N}, j \in \mathbb{N}} Ra_{i,j}R$, $k \in \mathbb{N}$. For any $k \in \mathbb{N}$ there exists an integer $n_k > k$ such that $a_{i,j} = 0$ whenever $i \geq n_k$ and $j < k$. Since A is idempotent, we have $I_{n_k}I_k = I_{n_k}$. Hence there exist positive integers $m_1 < m_2 < \cdots$ such that $I_{m_{j+1}}I_{m_j} = I_{m_{j+1}}$. If R satisfies (*), then there exists $l \in \mathbb{N}$ such that $I_{m_j} = I_{m_l}$ for any $l \leq j \in \mathbb{N}$, in particular, $I_{m_j} = I_{m_{j+1}} = I_{m_{j+1}}I_{m_j} = I_{m_j}^2$ if $j \geq l$. So if $I = \cap_{j \in \mathbb{N}} I_{n_j}$, then I is an idempotent ideal and $I_j = I$ for almost all $j \in \mathbb{N}$.

We will say that a projective module P over a ring R is *fair-sized* if P is countably generated and the set $I(P) := \{\, I \mid I \text{ is an ideal of } R \text{ such that } P/PI \text{ is finitely generated} \,\}$ has a least element. The following lemma shows that any countably generated projective module over a ring satisfying (*) is fair-sized. Moreover, the proof reveals the relation between the smallest element of $I(P)$ and an idempotent matrix representing P.

Lemma 3.2.4. *Let R be a ring satisfying (*) and let P be a countably generated projective module over R. The set $\{\, I \mid I$ is an ideal of R such that P/PI is finitely generated$\,\}$ has a least element I_0, which is an idempotent ideal.*

Proof. Let $A = (a_{i,j})_{i,j \in \mathbb{N}}$ be an idempotent column-finite matrix representing P, and $I_k, k \in \mathbb{N}$, be the ideals defined above. Set $I_0 = \cap_{k \in \mathbb{N}} I_k$. As remarked above, I_0 is idempotent. Let $\{\, e_i \mid i \in \mathbb{N} \,\}$ be the canonical free basis of $R^{(\mathbb{N})}$ and suppose that $I_0 = I_m = I_{m+1} = \dots$ Then $\sum_{i=1}^{m-1} Ae_iR + PI_0 = P$, so P/PI_0 is finitely generated. Let K be an ideal such that P/PK is a finitely generated module. Assume $P = AR^{(\mathbb{N})} \subseteq R^{(\mathbb{N})}$. Notice that $PK = P \cap K^{(\mathbb{N})}$, that is, the elements of PK are exactly the elements of P having all their components in K. If P/PK is finitely generated, then there exists $k \in \mathbb{N}$ such that $a_{i,j} \in K$ for every $i \geq k$ and $j \in \mathbb{N}$. Therefore $I_0 \subseteq K$. ∎

Thus if R satisfies (*), every countably generated projective module P determines a pair (I, P'), where I is an idempotent ideal and P' is a finitely generated projective module over R/I. More precisely, I is the smallest ideal of R such that P/PI is finitely generated and P' is the module P/PI considered as an R/I-module in the obvious way. If P is a countably generated projective module, then the corresponding idempotent ideal I is given by a matrix representing P as $I = \cap_{k \in \mathbb{N}} I_k$, but the characterization of I in Lemma

3.2.4 implies that I is independent of the choice of the matrix (and of the complement P' in the decomposition $P \oplus P' = R^{(\mathbb{N})}$).

Lemma 3.2.5. *Let I be an idempotent ideal of a ring R such that I is finitely generated as a left module and as a right module. If P and Q are I-big projective modules satisfying $P/PI \simeq Q/QI$, then $P \simeq Q$.*

Proof. Let B be the unique I-big projective module having trace ideal I. Observe that $P \oplus B^{(\omega)} \simeq P$ by Remark 3.2.1. If $f \colon P \to Q$ induces an isomorphism $P/PI \to Q/QI$, then $f(P) + QI = Q$. Since QI is countably generated and $\mathrm{Tr}(B) = I$, we get an epimorphism $h \colon P \oplus B^{(\omega)} \to Q$ such that $h|_P = f$. As f induces a monomorphism $P/PI \to Q/QI$ and $h(B^{(\omega)}) \subseteq QI$, we get $X = \mathrm{Ker}\ h \subseteq PI \oplus B^{(\omega)}$. Thus X is a direct summand of $PI \oplus B^{(\omega)}$. In particular, $XI = X$. Consequently, $\mathrm{Tr}(X) \subseteq I$, so $Q \oplus X \simeq Q$ by Remark 3.2.1. Finally, $Q \simeq Q \oplus X \simeq P \oplus B^{(\omega)} \simeq P$, and $Q \simeq P$ follows. ∎

The following lemma is a straightforward extension of [18, Corollary 2.7].

Lemma 3.2.6. *Let I be a proper idempotent ideal of a ring R. Assume I finitely generated as a left ideal. Let P' be a finitely generated projective module over R/I. Then there exists an I-big projective module P such that $P/PI \simeq P'$.*

Proof. We will find a countably generated projective module P_0 such that $P_0/P_0I \simeq P'$.

Suppose that P' is given by an $n \times n$ matrix X idempotent modulo I. The R-matrix X is a lifting of an idempotent R/I-matrix $\overline{X}$. Let $I = Ii_1 + \cdots + Ii_l, i_1, \dots, i_l \in I$. Construct a sequence of matrices $A_1, A_2, \dots$ as follows: A_1 has $c_1 = n$ columns and $r_1 = ln + n$ rows. The square matrix given by the first n rows of A_1 is X, $(A_1)_{i,j} = 0$ if $n < i \leq n + (j-1)l$ or $i > n + jl$, and the remaining entries in each column are filled with the generators $i_1, \dots, i_l$. That is, the matrix A_1 written in blocks is

$$\begin{pmatrix} & & X & & \\ b & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & b \end{pmatrix},$$

where b is the column $(i_1, \dots, i_l)^T$.

If A_k, r_k, c_k have been defined, then A_{k+1} has $c_{k+1} = r_k$ columns and $r_{k+1} = r_k + lr_k$ rows. The $n \times n$ top left corner of A_{k+1} is given by the matrix X and all the other entries in the first r_k rows of A_{k+1} are zero. The remaining lr_k rows contains $i_1, \dots, i_l$ placed in each column in the same “independent manner” as described for A_1.

We claim that for any $k \in \mathbb{N}$ there is a $c_{k+1} \times r_{k+1}$ matrix B_k such that $B_k A_{k+1} A_k = A_k$. Observe that the $c_k \times c_k$ matrix given by the first c_k rows of A_k is idempotent modulo I. We can find an $r_k \times r_k$ matrix C_k such that $C_k A_k = A_k$: The $n \times n$ top left corner of C_k is given by X, the other entries in the first c_k columns are zero and the matrix C_k can be completed by elements of I because $I = Ii_1 + \cdots + Ii_l$ and $i_1, \dots, i_l$ are placed independently in the bottom part of A_k. This matrix C_k can be written as $D_k A_{k+1}$, where D_k is a suitable $r_k \times r_{k+1}$ matrix. (Again we place X in the top left corner of D_k, and put all the other entries in the first r_k columns of D_k equal to zero. The remaining entries can be completed because the generators of I are placed independently in A_{k+1}.) Now, since $A_k = C_k A_k = D_k A_{k+1} A_k$, put $B_k = D_k$.

View the free module $F_k = R^{c_k}$ as the set of columns of length c_k. Let $f_k\colon F_k \to F_{k+1}$ be the homomorphism given by $f_k(u) = A_k \cdot u$ for every $u \in F_k$. By [18, Theorem 2.1], the colimit of the direct system induced by the f_k's is a projective module P_0. Obviously, P_0 is a countably generated module. Applying the functor $- \otimes_R R/I\colon \text{Mod-}R \to \text{Mod-}R/I$, we see that P_0/P_0I is an R/I-module isomorphic to the colimit of the system $(R/I)^n \xrightarrow{\overline{X}} (R/I)^n \xrightarrow{\overline{X}} \cdots$, which is easily seen to be $\overline{X}(R/I)^n \simeq P'$. Therefore $P_0/P_0I \simeq P'$.

Finally, by Lemma 3.2.2, there exists an I-big projective module B such that $\mathrm{Tr}(B) = I$. Since $BI = I$, $P := P_0 \oplus B$ is an I-big projective module with $P/PI \simeq P_0/P_0I \simeq P'$. ■

Remark 3.2.7. Let us explain the construction in the proof of Lemma 3.2.6 via an example. Suppose that I is a proper idempotent ideal of a ring R such that $I = Ii_1 + Ii_2$ for some $i_1, i_2 \in I$. Let $x \in R$ be such that $x + I$ is an idempotent element of R/I, i.e., $x - x^2 \in I$. Then there are $t_1, t_2 \in I$ such that $x = x^2 + t_1i_1 + t_2i_2$. Further, there are $u_1, u_2, v_1, v_2 \in I$ such that $u_1i_1 + u_2i_2 = i_1$ and $v_1i_1 + v_2i_2 = i_2$. Set

$$A_1 = \begin{pmatrix} x \\ i_1 \\ i_2 \end{pmatrix} \quad C_1 = \begin{pmatrix} x & t_1 & t_2 \\ 0 & u_1 & u_2 \\ 0 & v_1 & v_2 \end{pmatrix} \quad C_1' = \begin{pmatrix} x - x^2 & t_1 & t_2 \\ 0 & u_1 & u_2 \\ 0 & v_1 & v_2 \end{pmatrix}.$$

Obviously, $C_1A_1 = A_1$. Moreover, all entries of C_1' are in I. Therefore there is a 3×6 matrix $T = (t_{i,j})_{1\le i\le 3, 1\le j\le 6}$ satisfying $TA_2' = C_1'$, where

$$A_2' = \begin{pmatrix} i_1 & i_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & i_1 & i_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & i_1 & i_2 \end{pmatrix}^T.$$

All the entries of T can be chosen in I, but this is not important. It is easy to see that $C_1 = B_1A_2$, where

$$B_1 = \begin{pmatrix} x & 0 & 0 & t_{1,1} & t_{1,2} & t_{1,3} & t_{1,4} & t_{1,5} & t_{1,6} \\ 0 & 0 & 0 & t_{2,1} & t_{2,2} & t_{2,3} & t_{2,4} & t_{2,5} & t_{2,6} \\ 0 & 0 & 0 & t_{3,1} & t_{3,2} & t_{3,3} & t_{3,4} & t_{3,5} & t_{3,6} \end{pmatrix}$$

$$A_2 = \begin{pmatrix} x & 0 & 0 & i_1 & i_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & i_1 & i_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i_1 & i_2 \end{pmatrix}^T.$$

The following lemma is, in a sense, a restatement of [3, Theorem 3.1]. We prefer to give a brief but complete proof of the statement for left and right noetherian rings rather than specifying what should be modified in the proof of [3, Theorem 3.1] to get a real generalization.

Lemma 3.2.8. *Let R be a left and right noetherian ring. Let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be an idempotent column-finite matrix. Set $I_k = \sum_{i\ge k, j\in\mathbb{N}} Ra_{i,j}R$. If there exists $n_0 \in \mathbb{N}$ such that $I_m = I_{n_0}$ for every $m \ge n_0$, then the module $P = AR^{(\mathbb{N})} \subseteq R^{(\mathbb{N})}$ is I_{n_0}-big.*

Proof. Set $I = I_{n_0}$ and observe that I is finitely generated as a left ideal. Let a_i be the i-th column of A. We will prove the following claim. For any $n \in \mathbb{N}$ there exist

$m \in \mathbb{N}$ and $r_1, \dots, r_m \in R$ such that if $a_1 r_1 + \cdots + a_m r_m = (c_i)_{i \in \mathbb{N}}$, then $I \subseteq \sum_{i \geq n} Rc_i$. By induction, define positive integers $s_1, \dots, s_k$, $s'_1, \dots, s'_k$ and $x_1, \dots, x_k \in R$ such that $\sum_{i=1}^{l} Rx_i \subsetneq \sum_{i=1}^{l+1} Rx_i$ for every $1 \leq l < k$ and $I \subseteq \sum_{i=1}^{k} Rx_i$.

Put $s_1 = 1$, $s'_1 = n$ and $x_1 = a_{s'_1, s_1}$. If $I \subseteq Rx_1$, we have finished. Otherwise, suppose we have defined positive integers $s_1, \dots, s_l$, $s'_1, \dots, s'_l$ and $x_1, \dots, x_l \in R$ such that $I \not\subseteq \sum_{i=1}^{l} Rx_i$. Since R is right noetherian, there exists $m_l \in \mathbb{N}$ such that $m_l > s_l$ and $\sum_{j \in \mathbb{N}} a_{s'_l, j} R = \sum_{1 \leq j < m_l} a_{s'_l, j} R$. Since A is column-finite, there exists $m'_l \in \mathbb{N}$ with $m'_l > s'_l$ and $a_{i,j} = 0$ whenever $i \geq m'_l$ and $j \leq m_l$. As $I \subseteq I_{m'_l}$, there exist $s'_{l+1} > m'_l$, $s_{l+1} > m_l$ and $t_{l+1} \in R$ such that $a_{s'_{l+1}, s_{l+1}} t_{l+1} \notin \sum_{i=1}^{l} Rx_i$. Put $x_{l+1} = a_{s'_{l+1}, s_{l+1}} t_{l+1}$.

Since R is left noetherian, this process must stop, that is, there exists k such that $I \subseteq \sum_{1 \leq i \leq k} Rx_i$. It follows that there are $r_1, \dots, r_{s_k} \in R$ such that the s'_i-th component of $\sum_{i=1}^{s_k} a_i r_i$ is x_i for any $1 \leq i \leq k$. This is obvious for $k = 1$. If $k > 1$, note that $s_1 < m_1 < s_2 < m_2 < \cdots < m_{k-1} < s_k$ and $s'_1 < m'_1 < s'_2 < m'_2 < \cdots < m'_{k-1} < s'_k$. Further, $\sum_{j \in \mathbb{N}} a_{s'_1, j} R = \sum_{j=1}^{m_1 - 1} a_{s'_1, j} R$ and $\sum_{j \in \mathbb{N}} a_{s'_i, j} R = \sum_{j=m_{i-1}}^{m_i - 1} a_{s'_i, j} R$ if $2 \leq i < k$. Moreover, $a_{i,j} = 0$ for any $1 \leq j \leq m_l$ and $i \geq m'_l$. This concludes the proof of the claim.

Now we can construct a sequence $p_1, p_2, \dots$ of elements in P, $p_i = (c_{j,i})_{j \in \mathbb{N}}$ say, such that there exist integers $1 = i_1 < i_2 < \cdots$ with $I \subseteq Rc_{i_k, k} + \cdots + Rc_{i_{k+1}-1, k}$ for any $k \in \mathbb{N}$ and $c_{l,k} = 0$ for any $l \geq i_{k+1}$. We proceed by induction again. Put $i_1 = 1$. By the claim, there exists p_1 such that $I \subseteq \sum_{j \in \mathbb{N}} Rc_{j,1}$. Of course, there exists $i_2 > i_1$ with $c_{l,1} = 0$ for every $l \geq i_2$.

Suppose we have $p_1, \dots, p_k$ and $i_1, \dots, i_{k+1}$. By the claim, there exists p_{k+1} such that $I \subseteq \sum_{j \geq i_{k+1}} Rc_{j,k+1}$. Let $i_{k+2} > i_{k+1}$ be an integer such that $c_{j,k+1} = 0$ for all $j \geq i_{k+2}$.

Now, let Q be a countably generated projective module with trace ideal contained in I given by a column-finite idempotent matrix B over R (again, we consider Q as a submodule of $R^{(\mathbb{N})}$). Since the trace ideal of Q lies in I, all entries of B are in I. Let C be a matrix such that columns of C are given by $p_1, p_2, \dots$. The shape of C guarantees the existence of a column-finite matrix D having all entries in $\mathrm{Tr}(Q)$ such that $DC = B$ (it is important to realize that the elements of D can be chosen in I). Now, let $f\colon R^{(\mathbb{N})} \to R^{(\mathbb{N})}$ be given by D. Observe, that $Q \subseteq f(P)$ and that if $\pi\colon R^{(\mathbb{N})} \to Q$ is a projection, then $\pi f|_P$ is an epimorphism of P onto Q. Hence P is I-big. ■

Remark 3.2.9. Imitating the proof of [3, Theorem 3.1], we could get the following. Let R be a ring such that $R/J(R)$ is right noetherian. Let P, I_k be as above and suppose that $I = I_n = I_{n+1} = \cdots$ is a finitely generated left ideal such that $I \cap J(R) = J(R)I$. Then P is I-big. (For $I = R$ we get Bass' big projectives theorem). Also we could omit the assumption (*) and prove that P is $\cap_{n \in \mathbb{N}} I_n$-big. We do not give the details because we do not have applications for this version of Lemma 3.2.8.

Comparing the definition of I_0 in the proof of Lemma 3.2.4 and the statement of Lemma 3.2.8, we immediately get

Corollary 3.2.10. *Let R be a left and right noetherian ring satisfying (*). If P is a countably generated projective R-module and I is the least ideal of R such that P/PI is finitely generated, then P is I-big.*

Obviously, Lemma 3.2.8 can be applied to study projective (right) modules over left and right noetherian rings satisfying (*). The following lemma shows that over these rings we can apply Lemma 3.2.8 also for projective left modules. Recall that an $\mathbb{N} \times \mathbb{N}$ matrix

$A = (a_{i,j})_{i,j\in\mathbb{N}}$ is said to be *row-finite* if for any $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ such that $a_{i,k} = 0$ for every $k \geq j$.

Lemma 3.2.11. *Let R be a left and right noetherian ring satisfying (*). Let $A = (a_{i,j})_{i,j\in\mathbb{N}}$ be a row-finite matrix over R such that $A^2 = A$. For any $k \in \mathbb{N}$ let $I_k = \sum_{j>k,i\in\mathbb{N}} Ra_{i,j}R$. Then there exists $n \in \mathbb{N}$ such that $I_m = I_n$ for any $m \geq n$.*

Proof. Throughout the proof, we will work inside the left module $F = {}_RR^{(\mathbb{N})}$. Let $e_1, e_2, \dots$ be the canonical free basis of F. For any $i \in \mathbb{N}$, let a_i be the i-th row of A, that is, $a_i = (a_{i,1}, a_{i,2}, \dots) \in F$. Thus A gives a left projective module $P = FA = \sum_{i\in\mathbb{N}} Ra_i$. For any $l \in \mathbb{N}_0$ let $\pi_l \colon F \to {}_RR^l$ be the projection given by $\pi_l((x_1, x_2, \dots)) = (x_1, \dots, x_l)$ (as usual, ${}_RR^0$ is the trivial left R-module). For any $i \in \mathbb{N}$ let $c_i \colon F \to {}_RR$ be the projection given by $c_i((x_1, x_2, \dots)) = x_i$.

Construct integers $0 = n_1 < n_2 < \cdots$ and ideals $J_1 \supseteq J_2 \supseteq \cdots$ as follows: Put $n_1 = 0$ and $J_1 = \sum_{i,j\in\mathbb{N}} Ra_{i,j}R$. Suppose that n_k and J_k have been defined. Since R is left noetherian, there exists $l \in \mathbb{N}$ such that the module $\pi_{n_k}(P)$ is generated by $\pi_{n_k}(a_1), \dots, \pi_{n_k}(a_l)$. As A is row-finite, there exists $m > n_k$ such that $a_{i,m'} = 0$ for any $1 \leq i \leq l, m' \geq m$. Set $n_{k+1} = m$. Let J_{k+1} be the ideal generated by $\{ r \in R \mid \text{there exist } p \in P \text{ and } i \in \mathbb{N} \text{ such that } \pi_{n_{k+1}}(p) = 0 \text{ and } c_i(p) = r \}$. We claim that $J_{k+1}J_k = J_{k+1}$. In order to prove the claim, it suffices to prove that $S \subseteq J_{k+1}J_k$ for a set S generating J_{k+1}. Let $p \in P$ be such that $\pi_{n_{k+1}}(p) = 0$. Write $p = (0, \dots, 0, r_{n_{k+1}+1}, \dots)$. Then $p = r_{n_{k+1}+1}(e_{n_{k+1}+1}A) + r_{n_{k+1}+2}(e_{n_{k+1}+2}A) + \cdots$ From the construction it follows that for any $i \in \mathbb{N}$ there exists $p_i \in P$ such that $\pi_{n_k}(p_i) = 0$ and $c_{n_{k+1}+j}(p_i) = c_{n_{k+1}+j}(e_{n_{k+1}+i}A)$ for every $j \in \mathbb{N}$. Since $c_{n_{k+1}+j}(p_i) \in J_k$, the equation $r_{n_{k+1}+i} = c_{n_{k+1}+i}(p) = r_{n_{k+1}+1}c_{n_{k+1}+i}((e_{n_{k+1}+1})A) + r_{n_{k+1}+2}c_{n_{k+1}+i}((e_{n_{k+1}+2})A) + \cdots$ implies that $J_{k+1} = J_{k+1}J_k$.

As R satisfies (*), there exists $m \in \mathbb{N}$ such that $J_m = J_{m+1} = \cdots$ Clearly, $J_k \subseteq I_{n_k}$ for any $k \in \mathbb{N}$. On the other hand, $I_{n_{k+1}} \subseteq J_{n_k}$. This concludes the proof of the lemma. ∎

Let R be a ring, let $V_r(R)$ be a set of representatives of the finitely generated projective right R-modules, $V_l(R)$ be a set of representatives of the finitely generated projective left R-modules, $V_r(R)^*$ be a set of representatives of the countably generated projective right modules and $V_l(R)^*$ be a set of representatives of the countably generated projective left R-modules. In the following theorem we consider $V_r(R/R)$ and $V_l(R/R)$ as sets containing one element.

Theorem 3.2.12. *Let R be a left and right noetherian ring satisfying (*). Let $\mathrm{Id}(R)$ be the set of its idempotent ideals and let $\mathcal{S}$ be the disjoint union $\dot\bigcup_{I\in\mathrm{Id}(R)} V_r(R/I)$. Then there is a bijection $\varphi \colon V_r(R)^* \to \mathcal{S}$. Moreover, there exists a bijection between $V_r(R)^*$ and $V_l(R)^*$ extending the classical bijection between $V_r(R)$ and $V_l(R)$ induced by $\mathrm{Hom}_R(-, R_R)$.*

Proof. By Corollary 3.2.10, any countably generated projective right module P is I-big, where I is the least ideal such that P/PI is finitely generated. We know that I is idempotent. This gives a map of $V_r(R)^*$ into $\mathcal{S}$. This map is a bijection by Lemmas 3.2.5 and 3.2.6.

Of course, all the results of this section can be formulated for left modules. We do not know whether condition (*) is equivalent to condition (*'): Let $I_1, I_2, \dots$ be a sequence of ideals such that $I_kI_{k+1} = I_{k+1}$ for any $k \in \mathbb{N}$. Then there exists $n \in \mathbb{N}$ such that $I_n = I_{n+1} = \dots$ Condition (*) is connected to right modules while (*') is connected to left modules. Therefore it would be more precise to talk about condition right (*) instead of (*). In order to be concise, we have omitted the word “right”, but the reader should be aware that this condition has to be changed formulating the versions of the results for left modules. However, we can use Lemma 3.2.11 and the “left version” of Lemma 3.2.8 to see that any countably generated projective left module Q is I-big, where I is the least ideal such that Q/IQ is finitely generated. Again, I is idempotent and finitely generated as a right module, therefore the “left versions” of Lemma 3.2.5 and Lemma 3.2.6 give a bijection of $V_l^*(R)$ and the disjoint union $\dot{\cup}_{I\in \mathrm{Id}(R)} V_l(R/I)$. The bijection between $V_r^*(R)$ and $V_l^*(R)$, then follows from the dualities between finitely generated projective left and right R/I-modules, where I varies in $\mathrm{Id}(R)$. ∎

Remark 3.2.13. Observe that if R is a left and right noetherian ring having (*), then every indecomposable projective module is finitely generated. Although we think that (*) is a very particular property (see [8] for examples of rings having infinite properly descending chains of idempotent ideals), it seems to occur quite often in natural examples of left and right noetherian rings.

3.3 Semilocal noetherian rings

Recall that a ring R is said to be *semilocal*, if the factor of R modulo its Jacobson radical is semisimple artinian. Throughout the paper, $J(R)$ denotes the Jacobson radical of R. If P, Q are projective modules, then $P/PJ(R) \simeq Q/QJ(R)$ if and only if $P \simeq Q$ [13, Theorem 1.3]. In this section, we show that semilocal left and right noetherian rings satisfy (*), so that any countably generated projective module over such a ring is fair-sized. Further, we show a connection between the pair $(I, P/PI)$ defined in the previous section and the semisimple module $P/PJ(R)$. Finally, we give an example of superdecomposable projective module over a semilocal noetherian ring.

Recall that if P is a projective module over R, then the intersection of all maximal submodules of P, called the *radical* of P, is $\mathrm{rad}(P) = PJ(R)$. If R is semilocal and $S_1, \dots, S_k$ are representatives of the simple R-modules (that is, for any simple R-module S there exists exactly one $i \in \{1, \dots, k\}$ with $S \simeq S_i$), then for every projective module P there are cardinals $\lambda_1, \dots, \lambda_k$, uniquely determined, such that $P/PJ(R) = S_1^{(\lambda_1)} \oplus \dots \oplus S_k^{(\lambda_k)}$. We will write $\dim(P) = (\lambda_1, \dots, \lambda_k)$. Clearly, dim depends on the ordering of the representatives of the simple R-modules. Therefore we will always suppose that with any semilocal ring R we have some fixed ordering on the set of representatives of the simple R-modules. By [13, Theorem 1.3], two projective R-modules P and Q are isomorphic if and only if $\dim(P) = \dim(Q)$.

Lemma 3.3.1. *Let R be a right noetherian semilocal ring. If I and K are idempotent ideals of R such that $I + J(R) = K + J(R)$, then $I = K$. In particular, R has only finitely many idempotent ideals.*

Proof. Since $R/J(R)$ has only finitely many (idempotent) ideals, it is enough to show that $I + J(R) = K + J(R)$ implies $I = K$ whenever I and K are idempotent ideals of R.

First suppose that $I \subseteq K$ are idempotent ideals of R. In particular, $KI = I$. Suppose that $I + J(R) = K + J(R)$. Then $K = K(K + J(R)) = K(I + J(R)) = I + KJ(R)$. Since R is right noetherian, Nakayama's Lemma gives $I = K$.

In general, suppose that I and K are idempotent ideals of R with $I+J(R) = K+J(R)$. Then I and $I + K$ are idempotent ideals of R such that $I + J(R) = I + K + J(R)$. By the previous step, $I = I + K$, and therefore $K \subseteq I$. The proof for $I \subseteq K$ is similar. ∎

Corollary 3.3.2. *Let R be a right noetherian semilocal ring. Then R satisfies condition (*).*

PROOF. Let $\pi\colon R \to R/J(R)$ be the natural projection. Consider a descending sequence of ideals in R such that $I_{k+1}I_k = I_{k+1}$. Since $\pi(I_1), \pi(I_2), \dots$ is a descending sequence in an artinian ring $R/J(R)$, there exists $k_0 \in \mathbb{N}$ such that $\pi(I_k) = \pi(I_{k_0})$ for every $k \geq k_0$. Then $I_{k+1} = I_{k+1}(I_{k+1} + J(R)) = I_{k+1}^2 + I_{k+1}J(R)$ for every $k \geq k_0$. By Nakayama's Lemma, we see that I_k is idempotent for any $k > k_0$. Now conclude by Lemma 3.3.1. ∎

The following lemma and its application was suggested by Dolors Herbera.

Lemma 3.3.3. *Let P be a projective R-module with trace ideal I and let S be a simple R-module. The following conditions are equivalent.*

(i) S is a factor of I_R.

(ii) S is a factor of P.

(iii) $SI = S$.

PROOF. (i)⇒(ii) Suppose that $f\colon I \to S$ is nonzero. Then $f(i) \neq 0$ for some $i \in I$. Since I is the trace ideal of P, there are homomorphisms $g_1, \dots, g_k\colon P \to I$ and $p_1, \dots, p_k \in P$ with $g_1(p_1) + \dots + g_k(p_k) = i$. Therefore $fg_j \neq 0$ for some $1 \leq j \leq k$. (Observe that we did not use P projective for this implication.)

(ii)⇒(iii) Follows from $PI = P$.

(iii)⇒(i) Let $f\colon R_R \to S$ be nonzero. Then $f(I) = S$, because $SI = S$. ∎

Proposition 3.3.4. *Let R be a semilocal left and right noetherian ring. Suppose that P is a countably generated projective module. Then there exists a least ideal I in R such that P/PI is finitely generated.*

Moreover, let $\{S_1, \dots, S_k\}$ be a set of representatives of the simple modules, indexed in such a way that $P/PJ(R) \simeq S_1^{n_1} \oplus \dots \oplus S_l^{n_l} \oplus S_{l+1}^{(\omega)} \oplus \dots \oplus S_k^{(\omega)}$, $n_1, \dots, n_l \in \mathbb{N}_0$, $0 \leq l \leq k$. Then:

(i) P is I-big,

(ii) $S_i = S_iI$ if and only if $i > l$,

(iii) $P/PI/\mathrm{rad}(P/PI) \simeq S_1^{n_1} \oplus \dots \oplus S_l^{n_l}$.

PROOF. We have seen in Corollary 3.3.2 that R satisfies (*). By Lemma 3.2.4, there exists I such that P/PI is finitely generated and I is contained in any other ideal K such that P/PK is finitely generated. Moreover, P is I-big according to Corollary 3.2.10. Since I is finitely generated as a left ideal, there exists a unique I-big projective module B with trace ideal I and $P \oplus B^{(\omega)} \simeq P$ according to Remark 3.2.1. By Lemma 3.3.3, if S is a simple module, then $S^{(\omega)}$ is a factor of P (and hence of $P/PJ(R)$) whenever $SI = S$. Choose an enumeration of the simple modules such that $S_1, \dots, S_l$ are annihilated by I

and $S_{l+1}, \dots, S_k$ are factors of I. Let $0 \leq \lambda_1, \dots, \lambda_k \leq \infty$ be such that $P/PJ(R) \simeq S_1^{(\lambda_1)} \oplus \cdots \oplus S_k^{(\lambda_k)}$. As remarked above, $\lambda_{l+1} = \cdots = \lambda_k = \infty$. On the other hand, $S_1^{(\lambda_1)} \oplus \cdots \oplus S_l^{(\lambda_l)}$ is a factor of P annihilated by I, hence a factor of P/PI. Thus $\lambda_1, \dots, \lambda_l$ are finite. Suppose $P/PI/\mathrm{rad}(P/PI) \simeq S_1^{n_1} \oplus \cdots \oplus S_l^{n_l}$. Since $S_1^{\lambda_1} \oplus \cdots \oplus S_l^{\lambda_l}$ is a semisimple factor of P/PI, $\lambda_i \leq n_i$ for any $1 \leq i \leq l$. On the other hand, $S_1^{n_1} \oplus \cdots \oplus S_l^{n_l}$ is a factor of P, so that $n_i \leq \lambda_i$ for every $1 \leq i \leq l$. ■

Recall that a nonzero module is called *superdecomposable* if it has no indecomposable direct summand. The following lemma explains our craving for the existence of superdecomposable projectives over semilocal rings.

Lemma 3.3.5. *Suppose that there exists a superdecomposable projective module over a semilocal ring R. Then R possesses a nonzero decomposable projective module having all its nonzero direct summands isomorphic.*

PROOF. By the theorem of Kaplansky, if there exists a superdecomposable projective module, then there exists a superdecomposable countably generated projective module. It follows easily that then there exists a superdecomposable countably generated projective module Q such that $\dim(Q) = (m_1, \dots, m_k)$, where $m_i = 0$ or $m_i = \omega$ for any $1 \leq i \leq k$ (use the additivity of dim). Let Q' be a superdecomposable module such that $\dim(Q')$ has all components in $\{0, \omega\}$ and the number of nonzero components is as small as possible. Then it is easy to see that $\dim(Q') = \dim(Q'')$ for any nonzero direct summand of Q', so [13, Theorem 1.3] gives that Q' has the required property. ■

The following example discovered by Puninski [12] shows that a superdecomposable projective module may exist even over a semilocal noetherian ring.

(cf. [12, Proposition 7.5]) Let $\Sigma = \mathbb{Z} \setminus 2\mathbb{Z} \cup 3\mathbb{Z} \cup 5\mathbb{Z}$ and let $\mathbb{Z}_\Sigma$ be the localization of integers at Σ. Let A_5 be the group of even permutations on the set of cardinality 5. Then the group ring $\mathbb{Z}_\Sigma[A_5]$ is a semilocal left and right noetherian ring with a superdecomposable projective module.

PROOF. We will repeat general arguments of [12] that show that the ring $R = \mathbb{Z}_\Sigma[A_5]$ is a semilocal left and right noetherian ring. First, R is a finitely generated as a (left and right) module over the commutative noetherian ring $\mathbb{Z}_\Sigma$, therefore R is noetherian on both sides. Further, $R \simeq \mathrm{End}_R(R_R)$, so that there exists an injective homomorphism $\varphi\colon R \to \mathrm{End}_{\mathbb{Z}_\Sigma}(R)$ given by the left multiplication of R on $R_{\mathbb{Z}_\Sigma}$. For any $g \in A_5$, let $\theta_g \in \mathrm{End}_{\mathbb{Z}_\Sigma}(R)$ be given by $\theta_g(r) = rg, r \in R$. Obviously, Im φ consists exactly of the elements of $\mathrm{End}_{\mathbb{Z}_\Sigma}(R)$ that commute with all the endomorphisms of the set $\{\, \theta_g \mid g \in A_5 \,\}$. It follows that φ is a local homomorphism, that is, r is invertible in R if $\varphi(r)$ is invertible in $\mathrm{End}_{\mathbb{Z}_\Sigma}(R)$. Finally, since $\mathrm{End}_{\mathbb{Z}_\Sigma}(R) \simeq \mathrm{M}_{60}(\mathbb{Z}_\Sigma)$ is a semilocal ring, the ring R is also semilocal by [4, Theorem 1].

Let I be the augmentation ideal of R, that is, the kernel of the epimorphism $f\colon R \to \mathbb{Z}_\Sigma$, $f(\sum_{g \in A_5} r_g g) = \sum_{g \in A_5} r_g$. Since $[A_5, A_5] = A_5$, the ideal I is idempotent [1, Theorem 2.1]. By [12], it can be proved that every nonzero finitely generated projective module over R is a generator. In fact, we only need to show that if P is a finitely generated projective R-module, then $\mathrm{Tr}(P)$ cannot be contained in I: Since $\mathbb{Z}_\Sigma$ is a Dedekind ring of zero characteristic and 2,3,5 are not invertible in $\mathbb{Z}_\Sigma$, $P' = P \otimes_{\mathbb{Z}_\Sigma[A_5]} \mathbb{Q}[A_5]$ is a free $\mathbb{Q}[A_5]$-module by [16, Theorem 8.1]. If $\mathrm{Tr}(P) \subseteq I$, then $P'I' = P'$, where I' is the augmentation ideal of $\mathbb{Q}[A_5]$, a contradiction. Let Q be a projective module having trace ideal I. If Q' is a nonzero direct summand of Q, then Q' cannot be finitely generated, and there is a nonzero idempotent ideal K such that Q' is K-big. Therefore Q' cannot be indecomposable. ∎

Remark 3.3.6. In the next section we look closer at the localizations of $\mathbb{Z}[A_5]$ showing that the augmentation ideal of $\mathbb{Z}_\Sigma[A_5]$ contains no nontrivial idempotent ideals.

3.4 Integral group rings, especially $\mathbb{Z}[A_5]$

In this section, we prove that an integral group ring of a finite group satisfies condition (*). The proof presented here is not the quickest one, but it shows how to calculate idempotent ideals in particular examples. We apply this method to $\mathbb{Z}[A_5]$ describing all countably but not finitely generated projective modules up to isomorphism. Our approach will be elementary.

First of all, let us introduce the notation we will use throughout this section. Let G be a finite group and $R = \mathbb{Z}[G], R_p = \mathbb{Z}_{(p)}[G], R_0 = \mathbb{Q}[G]$. For any prime p we have $R \subseteq R_p \subseteq R_0$. If I is an ideal of R, $I_{(p)}$ stands for the ideal in R_p generated by I and $I_{(0)}$ stands for the ideal of R_0 generated by I. That is, $I_{(p)} = \mathbb{Z}_{(p)}I$ and $I_{(0)} = \mathbb{Q}I$. We say that an ideal $I \subseteq R$ (or an ideal $I \subseteq R_p$) *extends to* an ideal $K \subseteq R_0$ if $K = \mathbb{Q}I$. If S is a commutative ring, the *augmentation ideal* of $S[G]$ is the kernel of the canonical homomorphism $f\colon S[G] \to S$ given by $f(\sum_{g\in G} s_g g) = \sum_{g\in G} s_g$. It is denoted by $\mathrm{Aug}(S[G])$.

In the following we summarize the framework of our calculations.

Fact 3.4.1. *Let G be a finite group and let $R = \mathbb{Z}[G]$. Then*

(i) If I is an ideal of R, then $I_{(0)} = \mathbb{Q}I_{(p)}$ for every prime p.

(ii) Let I, K be ideals in R. Then $I = K$ if and only if $I_{(p)} = K_{(p)}$ for every prime p.

(iii) If $I \subseteq R$ is an ideal, then I is idempotent if and only if $I_{(p)}$ is idempotent for every prime p.

(iv) If I, K are idempotent ideals of R and p a prime not dividing $|G|$, then $I_{(p)} = K_{(p)}$ if and only if $I_{(0)} = K_{(0)}$. In this case, all central idempotents of R_0 are contained in R_p and every idempotent ideal of R_p is generated by a central idempotent.

(v) Let e be a central idempotent of R_0 and suppose that, for every prime p that divides $|G|$, there is an idempotent ideal $I_p \subseteq R_p$ with $\mathbb{Q}I_p = eR_0$. Then there exists a unique idempotent ideal $I \subseteq R$ such that $I_{(p)} = I_p$ for any $p \mid |G|$ and $I_{(p)} = eR_p$ for any $p \nmid |G|$.

Proof. Statements (i),(ii),(iii) and (v) are rather standard. Statement (iv) follows from the fact that $\mathbb{Z}_{(p)}[G]$ is a maximal $\mathbb{Z}_{(p)}$-order in $\mathbb{Q}[G]$ if and only if p does not divide $|G|$ (see [5, Proposition 27.1]) and using the machinery of maximal orders.

Here we give another proof of (iv). Let $\mathbb{Q} \subseteq F$ be a finite Galois extension of $\mathbb{Q}$ such that F is a splitting field of G. Recall that if ξ is a complex character of a simple representation of G over F (considered as a function $\xi\colon G \to F$), then $\frac{\xi(1_G)}{|G|}(\sum_{g\in G} \xi(g^{-1})g)$ is a primitive central idempotent of $F[G]$. In order to get the set of primitive central idempotent of $\mathbb{Q}[G]$, consider the usual action of $\mathrm{Gal}(F:\mathbb{Q})$ on the set of primitive central idempotents of $F[G]$ and take sums of the orbits. It follows that if p is a prime and $p \not| |G|$, then any central idempotent of R_0 is in R_p.

Let I be an idempotent ideal of R_p, where p is a prime not dividing $|G|$. Then $\mathbb{Q}I$ is an ideal of R_0 generated by a central idempotent e of R_0. Since $e \in R_p$, $K = eR_p$ is an idempotent ideal of R_p, necessarily $I \subseteq K$ because $eI = I$. Since $\mathbb{Q}I = \mathbb{Q}K$, there exists $k \in \mathbb{N}$ such that $p^k K \subseteq I$. As $\mathbb{Z}_p[G]$ is semisimple, idempotent ideals in $\mathbb{Z}_{p^n}[G]$ are generated by central idempotents for any $n \in \mathbb{N}$ (combine [2, Proposition 27.1] and [10, 22.10]). Moreover, it is easily seen that if K' is an idempotent ideal of $\mathbb{Z}_{p^{2n}}[G]$, then $p^n K'$ is an essential submodule of K'. Now let $\pi\colon R_p \to \mathbb{Z}_{p^{2k}}[G]$ be the canonical projection. Then $p^k\pi(K) \subseteq \pi(I) \subseteq \pi(K)$. By our previous remarks, $\pi(I) = \pi(K)$. Since R_p is a semilocal noetherian ring and π is an epimorphism with $\operatorname{Ker} \pi \subseteq J(R_p)$ (Fact 3.4.3), $I = K$ follows from Lemma 3.3.1. ■

The following result also follows from [15, Theorem 3].

Corollary 3.4.2. *Any integral group ring of a finite group satisfies (*) and has only finitely many idempotent ideals.*

Proof. Since R is a ring of Krull dimension 1, it is enough to see that R has no descending chain of idempotent ideals. Let I be an idempotent ideal, let e be a central idempotent of R_0 such that $eR_0 = \mathbb{Q}I$. Then $I_{(p)} = eR_p$ for every prime p not dividing $|G|$ by Fact 3.4.1(iv). If p is a prime divisor of $|G|$, then we have only finitely many possibilities for $I_{(p)}$ by Lemma 3.3.1. Therefore, by Fact 3.4.1(v), R contains only finitely many idempotent ideals. ■

The proof of Corollary 3.4.2 shows a method of finding idempotent ideals in R. We can proceed as follows: Take an ideal I_0 of R_0. Let P be the set of prime divisors of $|G|$. For any $p \in P$, determine the set M_p consisting of the idempotent ideals of R_p that extend to I_0. Then there is a bijective correspondence between the idempotent ideals of R extending to I_0 and the set $\prod_{p\in P} M_p$.

Thus we can now work in semilocal localizations (see [5] or use the same kind of arguments as in Example 3.3).

Fact 3.4.3. *The natural homomorphism $\pi_p\colon R_p \to \mathbb{Z}_p[G]$ is a local morphism for any prime p. In particular, $pR_p \subseteq J(R_p)$ and R_p is a semilocal ring.*

Let us show the method in the case of $G = A_5$, the alternating group on 5 elements. The usual question "Why A_5?" has a simple answer. By a result of Swan [17], non-finitely generated projective modules over integral group rings of finite solvable groups are free. Therefore there are no proper idempotent ideals in integral group rings of finite solvable groups (a direct proof of this was given by Roggenkamp [14]). On the other hand, it is known [1] that if G contains a perfect normal subgroup H, that is, $[H,H] = H$ and $H \trianglelefteq G$, then the augmentation ideal of H (that is, the kernel of the canonical homomorphism $\mathbb{Z}[G] \to \mathbb{Z}[G/H]$) is idempotent. If there were no other idempotent ideals in $\mathbb{Z}[G]$, then countably generated projective modules over $\mathbb{Z}[G]$ would be induced by finitely generated projective modules over $\mathbb{Z}[G/H]$, where H ranges in the set of perfect normal subgroups of G. So A_5 is the first candidate to check. Unfortunately, we will see that there indeed exists an idempotent ideal that is not the augmentation ideal of a perfect normal subgroup. Hence the structure theory for big projective modules over integral group rings seems to be more complicated.

Throughout the next paragraphs, suppose $G = A_5$. The conjugacy classes of G are the following: c_1 - the conjugacy class of the identity; c_2 - the permutations that are product of two 2-cycles (the conjugacy class of $(1,2)(3,4)$); c_3 - all 3-cycles; c_5 - the conjugacy class of $(1,2,3,4,5)$; and c_5' - the conjugacy class of $(1,3,5,2,4)$.

Let us recall what we know about the semisimple ring R_0. The primitive central idempotents of R_0 are $e_1 = \frac{1}{60}\sum_{g\in G} g$, $e_3 = \frac{1}{20}(6 - 2\sum_{g\in c_2} g + \sum_{g\in c_5\cup c_5'} g)$, $e_2 = \frac{1}{15}(4 + \sum_{g\in c_3} g - \sum_{g\in c_5\cup c_5'} g)$, $e_5 = \frac{1}{12}(5 + \sum_{g\in c_2} g - \sum_{g\in c_3} g)$. Let T_1, T_3, T_2, T_5 be the corresponding simple modules (e_i corresponds to T_i). Their dimensions over $\mathbb{Q}$ are $1, 6, 4, 5$.

We need to calculate the idempotent ideals in R_2, R_3, R_5. Set $S_i = \mathbb{Z}_i[A_5]$ for $i = 2, 3, 5$. By Fact 3.4.3, any simple S_i-module can be considered as a simple R_i-module and there are no other simple R_i-modules except for these. In order to find the number of different simple modules over R_i, one can use the following results proved by Berman and Witt (see [5, Theorem 21.5, Theorem 21.25]).

Fact 3.4.4. *Let G be a finite group of exponent m.*

(i) Let $\sim$ be the relation on G given by $g \sim h$ if g is conjugate to h^t for some $t \in \mathbb{N}, (t,m) = 1$. Then the number of simple $\mathbb{Q}[G]$-modules is equal to $|G/\sim|$.

(ii) Let p be a prime, and $G_{p'}$ the set of p-regular elements of G. On the set $G_{p'}$ consider the equivalence $\sim$ defined by $g \sim h$ if g is conjugate to h^{p^j} for some $j \in \mathbb{N}_0$. Then the number of simple $\mathbb{Z}_p[G]$-modules is equal to $|G_{p'}/\sim|$.

Thus each ring R_2, R_3, R_5 has exactly three non-isomorphic simple modules. Now idempotent ideals in semilocal rings are determined by their simple factors (Lemma 3.3.1). Call a ring T *almost semiperfect* if for any simple T-module M there exists a positive integer n such that M^n has a projective cover. The next lemma describes the distribution of idempotent ideals in R_i, for $i \in \{2,3,5\}$. In all the remaining proofs of this section, I_i stands for $\mathrm{Aug}(R_i)$.

Lemma 3.4.5. *Let $i \in \{2,3,5\}$. The ring R_i has exactly 3 minimal idempotent ideals and any nonzero idempotent ideal of R_i is a sum of minimal idempotent ideals. Moreover, R_i is almost semiperfect and any idempotent ideal of R_i is a trace ideal of a finitely generated projective module. Finally, two minimal idempotent ideals are described as follows: If I_i is the augmentation ideal of R_i, then e_iR_i and $(1-e_i)I_i$ are minimal idempotent ideals of R_i.*

Proof. We give the proof for $i = 5$, the remaining cases are similar. The augmentation ideal $I_5 \subseteq R_5$ is idempotent, because A_5 is perfect. Observe $e_5 \in R_5$. Therefore also e_5R_5 and $(1-e_5)I_5$ are idempotent ideals. Let M_1, M_2, M_3 be the set of representatives of the simple R_5-modules and suppose that M_1 is the module induced by the trivial representation of S_5. Obviously, $M_1I_5 = 0$, so M_1 is not a factor of I_5. Since I_5 must have at least two simple factors (it contains two different nontrivial idempotent ideals), M_2, M_3 are both factors of I_5. Choose the notation in such a way that M_2 is the unique simple factor of $(1-e_5)I_5$ and M_3 is the unique simple factor of e_5R_5.

Obviously, e_5R_5 is the trace ideal of the projective module e_5R_5. Set $g = (1,2)(3,4)$. The idempotent $e' = (1-e_5)(1-\frac{1}{2}(1+g))$ gives a projective R_5-module $P' = e'R_5$ with trace ideal $(1-e_5)I_5$. It follows that $P'/P'J(R_5) = M_2^k$, for some $k \in \mathbb{N}$ (it is necessary to check that $P' \neq 0$, below we calculate $\mathbb{Z}_{(5)}$-rank of P' using the so called Hattori-Stallings map).

On the other hand, the projective module $P = (1 - e_5)R_5$ has the radical factor $P/PJ(R_5) = M_1 \oplus M_2^l$. Therefore P'^l splits in P^k, that is, there exists a projective module Q such that $P^k = P'^l \oplus Q$. Since $Q/QJ(R_5) \simeq M_1^k$, it follows that $\mathrm{Tr}(Q)$ is an idempotent ideal such that M_1 is its only simple factor.

So we have proved that the finitely generated projective modules Q, P', e_5R_5 are the projective covers of convenient finite powers of M_1, M_2, M_3 and R_5 is almost semiperfect. Therefore $\mathrm{Tr}(Q)$, $\mathrm{Tr}(P')$ and $\mathrm{Tr}(e_5R_5)$ are the minimal idempotent ideals of R_5 and any nonzero idempotent ideal of R_5 is a sum of minimal idempotent ideals. ∎

Lemma 3.4.6. *The only idempotent ideals of $R = \mathbb{Z}[A_5]$ contained in* $\mathrm{Aug}(R)$ *are* 0 *and* $\mathrm{Aug}(R)$.

Proof. Set $I = \mathrm{Aug}(R)$ and let $0 \neq K$ be an idempotent ideal of R contained in I. Then $K_{(i)}$ also is a non-zero idempotent ideal of R_i contained in I_i, hence, by Lemma 3.4.5, $\mathbb{Q}K_{(i)}$ is either e_iR_0, $(e_2 + e_3 + e_5 - e_i)R_0$ or $I_{(0)} = (e_2 + e_3 + e_5)R_0$. Now $\mathbb{Q}K_{(2)} = \mathbb{Q}K_{(3)} = \mathbb{Q}K_{(5)} = \mathbb{Q}K$. An easy inspection gives that the only possibility is $K_{(i)} = I_i$ for any $i \in \{2, 3, 5\}$. Therefore $K = I$ by Fact 3.4.1(v). ∎

For any $i \in \{2, 3, 5\}$, let K_i be the (unique) minimal idempotent ideal of R_i that is not contained in the augmentation ideal of R_i. In order to classify the idempotent ideals in R that are not contained in the augmentation ideal of R, we must determine $\mathbb{Q}K_2, \mathbb{Q}K_3$ and $\mathbb{Q}K_5$. Let us prove an auxiliary general result, which is probably well known.

Lemma 3.4.7. *Let $\varphi\colon S \to T$ be a ring homomorphism. If P is a projective S-module with trace ideal I, then $P \otimes_S T$ is a projective T-module with trace ideal $T\varphi(I)T$.*

Proof. Let X be a set and let $\pi\colon S^{(X)} \to S^{(X)}$ be an idempotent endomorphism of $S^{(X)}$ such that $\pi(S^{(X)}) \simeq P$. If π is expressed as a column-finite idempotent matrix A (with respect to the canonical basis), then $\varphi(A)$ is an idempotent matrix corresponding to the endomorphism $\pi'\colon T^{(X)} \to T^{(X)}$ such that $P \otimes_S T \simeq \pi'(T^{(X)})$. Now $\mathrm{Tr}(P)$ (resp. $\mathrm{Tr}(P \otimes_S T)$) is the ideal generated by the entries of A (resp. $\varphi(A)$). ∎

Fact 3.4.8. *Let S be a commutative local ring and let H be a finite group. Suppose that $e = \sum_{h \in H} s_h h$ is an idempotent of $S[H]$. The module $eS[H]$ is free when considered as an S-module. Moreover, $|H|s_1 = n \cdot 1_S$, where $n \in \mathbb{N}_0$ is the rank of the free S-module $eS[H]$.*

Proof. This is a consequence of [7, Example 7]. Let us briefly explain the idea. Let T be a ring and $T/[T,T]$ be the group that is the factor of the additive group of T modulo $[T,T] = \langle \{\, t_1t_2 - t_2t_1 \mid t_1, t_2 \in T \,\} \rangle_{(T,+)}$. There exists a map $r\colon K_0(T) \to T/[T,T]$ defined as follows. Let P be a finitely generated projective module over T and A an idempotent matrix representing P. Then $r([P]) := \mathrm{Tr}(A) + [T,T]$ (here $\mathrm{Tr}(A)$ is the sum of the diagonal entries in A).

Since S is a local ring, $K_0(S) \simeq \mathbb{Z}$. As S is commutative, r is a well defined map of $K_0(S)$ into S. It follows that $\mathrm{Im}\ r \subseteq \mathbb{Z}1_S$. Now view $S[H]$ as a free S-module of rank $|H|$. The left multiplication by e gives an idempotent endomorphism α of this S-module whose image is $eS[H]$. Now compute $r([eS[H]])$. Consider the matrix of α with respect to the basis $\{\, h \mid h \in H \,\}$. All the diagonal entries of this matrix are equal to s_1. Therefore $|H|.s_1 = n \cdot 1_S$, where n is the rank of the free S-module $eS[H]$. ∎

Now we can continue in $\mathbb{Z}[A_5]$. In the following proofs I_i is again the augmentation ideal of R_i and $S_i = \mathbb{Z}_i[A_5]$ for every $i \in \{2, 3, 5\}$.

Lemma 3.4.9. *Let K_5 be the minimal idempotent ideal not contained in* $\mathrm{Aug}(R_5)$. *Then* $\mathbb{Q}K_5 = (e_1 + e_2)R_0$.

Proof. Let M_1, M_2, M_3 be the simple R_5-modules such that M_1 is a unique simple factor of K_5, M_2 is a unique simple factor of $(1-e_5)I_5$ and M_3 is a unique simple factor of e_5R_5. Let $g = (1,2)(3,4)$. Then $e' = (1-e_5)(1-\frac{1}{2}(1+g))$ gives a projective R_5-module $P' = e'R_5$ with trace ideal $(1-e_5)I_5$, so it follows that $P'/P'J(R_5) = M_2^k$ for some $k \in \mathbb{N}$. Moreover, if $P = (1-e_5)R_5$, then $P/PJ(R_5) \simeq M_1 \oplus M_2^l$ for some $l \in \mathbb{N}$. We want to determine k and l. The integer l is given by the multiplicity of M_2 in $S_5/J(S_5)$. Any simple S_5-module is absolutely simple, therefore l is equal to the dimension of the non-trivial simple representation that is annihilated by e_5. By [18, page 201], $l = 3$. Obviously, P' is a direct summand of P, and $k \in \{1,2,3\}$ follows. Using Fact 3.4.8, we have that the $\mathbb{Z}_{(5)}$-rank of P is equal to 35 and the $\mathbb{Z}_{(5)}$-rank of P' is equal to 20. If $k = 1$, then P'^3 would be a direct summand of P, which is not possible. Further, consider the S_5-module $P'/P'5R_5$. This is a vector space over $\mathbb{Z}_5$ of dimension 20. If $k = 3$, then $P'/P'5R_5 \simeq M^3$, where M is an S_5-module which is a projective cover of M_2 if M_2 is considered as a simple S_5-module. Since 3 does not divide 20, this is also impossible. Therefore $k = 2$.

As we have shown in the proof of Lemma 3.4.5, K_5 is the trace ideal of Q, where Q is a projective module defined by the relation $Q \oplus P'^3 \simeq P^2$. By Lemma 3.4.7, $\mathbb{Q}K_5 = \mathrm{Tr}(Q \otimes_{R_5} R_0)$. The module $Q \otimes_{R_5} R_0$ has $\mathbb{Q}$-dimension 10 and contains the trivial representation of R_0 with multiplicity 2. The only possibility (looking at the $\mathbb{Q}$-dimension of the simple R_0-modules) is $Q \otimes_{R_5} R_0 \simeq T_1^2 \oplus T_2^2$. ∎

Lemma 3.4.10. *Let K_3 be the minimal idempotent ideal of R_3 that is not contained in* $\mathrm{Aug}(R_3)$. *Then* $\mathbb{Q}K_3 = e_1R_0 + e_5R_0$.

Proof. Put $e = 1 - e_3$, $g = (1,2)(3,4)$ and $h = (1,2,3,4,5)$. These elements of G give idempotents $e' = e(1-\frac{1}{2}(1+g))$ and $f' = e(1-\frac{1}{5}(1+h+h^2+h^3+h^4))$. Let $P' = e'R_3$, $P'' = f'R_3$ and $P = eR_3$. Let M_1, M_2, M_3 be the simple R_3-modules such that M_1 is a unique simple factor of K_3, M_2 is a unique simple factor of eI_3 and M_3 is a unique simple factor of e_3R_3. Again we want to find $k, l \in \mathbb{N}$ such that $P/PJ(R_3) \simeq M_1 \oplus M_2^l$ and $P'/P'J(R_3) \simeq M_2^k$.

Consider the module M over S_3 given by the obvious action of A_5 on the vector space $\{(z_1, \ldots, z_5) \in \mathbb{Z}_3^5 \mid z_1 + \cdots + z_5 = 0\}$ (that is, if $x \in A_5$, then $(z_1, \ldots, z_5)x = (z_{x(1)}, \ldots, z_{x(5)})$). The module M can be viewed as an absolutely simple representation of A_5 over $\mathbb{Z}_3$ and its dimension is 4. Now consider M as an R_3-module via the canonical epimorphism $\pi\colon R_3 \to S_3$. Then M is a simple R_3-module annihilated by e_3, therefore $M \simeq M_2$. It follows that the multiplicity of M_2 in $R_3/J(R_3)$ is 4, therefore $l = 4$.

Since P' is a direct summand of P, $k \in \{1,2,3,4\}$. Using Fact 3.4.8, we get $\dim_{\mathbb{Z}_3} P/P(3R_3) = 42$, $\dim_{\mathbb{Z}_3} P'/P'(3R_3) = 18$, $\dim_{\mathbb{Z}_3} P''/P''(3R_3) = 36$. Now the only simple factor of P' and P'' is M_2, so that $P'' \simeq P'^2$. Thus P'^2 is a direct summand of P, and therefore $k \in \{1,2\}$. If $k = 1$, then P'^3 would be a direct summand of P and this is not possible, because $42 < 3 \cdot 18$. Therefore $k = 2$ and there exists Q such that $P \simeq P'^2 \oplus Q$. The semisimple module $Q \otimes_{R_3} R_0$ has its $\mathbb{Q}$-dimension equal to 6 and the multiplicity of T_1 in $Q \otimes_{R_3} R_0$ is 1. The only possibility is $Q \otimes_{R_3} R_0 \simeq T_1 \oplus T_5$. Hence $\mathbb{Q}\mathrm{Tr}(Q) = e_1R_0 + e_5R_0$.
∎

Lemma 3.4.11. *Let K_2 be the minimal idempotent ideal of R_2 that is not contained in* $\mathrm{Aug}(R_2)$. *Then* $\mathbb{Q}K_3 = e_1R_0 + e_3R_0 + e_5R_0$.

PROOF. Let M_1, M_2, M_3 be the simple R_2-modules such that M_1 is the simple factor of K_2, M_2 is the simple factor of $(1 - e_2)I_2$ and M_3 is the simple factor of e_2R_2. Let $e = 1 - e_2$, $e' = e(1 - \frac{1}{3}(1 + g + g^2))$, where $g = (1, 2, 3)$. Put $P = eR_2$, $P' = e'R_2$, so that $P/PJ(R_2) \simeq M_1 \oplus M_2^l$ and $P'/P'J(R_2) \simeq M_2^k$.

Let F be a field given by adjoining a primitive fifteenth root of one to $\mathbb{Z}_2$. By [18, page 200], the ring $F \otimes S_2/J(S_2)$ has two 2-dimensional simple modules and they are annihilated by e_2 (because they appear as composition factors of a representation annihilated by e_2). Therefore $F \otimes M_2$ is a direct sum of these two representations. Thus the $\mathbb{Z}_2$-dimension of M_2 is 4, but the multiplicity of M_2 in $S_2/J(S_2)$ is 2. It follows that $l = 2$.

Using Fact 3.4.8, we get that the $\mathbb{Z}_{(2)}$-rank of P is 44 and $\mathbb{Z}_{(2)}$-rank of P' is 32. Therefore P'^2 cannot be a direct summand of P and $k = 2$ follows. Then $P \simeq P' \oplus Q$ for some Q and $K_2 = \mathrm{Tr}(Q)$. By Lemma 3.4.7, $\mathbb{Q}K_2 = \mathrm{Tr}(Q \otimes_{R_2} R_0)$. Observe that $Q \otimes_{R_2} R_0$ has $\mathbb{Q}$-dimension 12 and contains T_1 with multiplicity 1. The only way of writing 11 as a sum of multiples of 6 and 5 is $11 = 6 + 5$. Therefore $Q \otimes_{R_2} R_0 \simeq T_1 \oplus T_5 \oplus T_3$ and $\mathbb{Q}K_2 = (e_1 + e_3 + e_5)R_0$. ■

Now we can finish the classification of the idempotent ideals in $\mathbb{Z}[A_5]$.

Proposition 3.4.12. *The idempotent ideals in $R = \mathbb{Z}[A_5]$ are $0, \mathrm{Aug}(R), X$ and R, where $X \neq R$ and $\mathbb{Q}X = \mathbb{Q}[A_5]$.*

PROOF. The idempotent ideals contained in $\mathrm{Aug}(R)$ were classified in Lemma 3.4.5. Let K be an idempotent ideal of R not contained in $\mathrm{Aug}(R)$. Then for any $i \in \{2, 3, 5\}$, $K_{(i)}$ is an idempotent ideal of R_i not contained in $\mathrm{Aug}(R_i)$. By Lemma 3.4.9, we have $e_2 \in K_{(0)}$, by Lemma 3.4.10, we have $e_5 \in K_{(0)}$ and by Lemma 3.4.11, we have $e_3 \in K_{(0)}$. It follows that $K_{(0)} = \mathbb{Q}[A_5]$.

If L is an idempotent ideal of R_5 such that $\mathbb{Q}L = \mathbb{Q}[A_5]$, then $L = R_5$ by Lemmas 3.4.5 and 3.4.9. Similarly, if L is an idempotent ideal of R_3 such that $\mathbb{Q}L = \mathbb{Q}[A_5]$, then $L = R_3$ by Lemmas 3.4.5 and 3.4.10. But if L is an idempotent ideal of R_2 such that $\mathbb{Q}L = \mathbb{Q}[A_5]$, then either $L = R_2$ or $L = K_2 + e_2R_2$ by Lemmas 3.4.5 and 3.4.11. Therefore there exists an idempotent ideal $X \subseteq R$ such that $X_{(2)} = K_2 + e_2R_2$, $X_{(3)} = R_3$ and $X_{(5)} = R_5$. ■

Finally, we can classify the non-finitely generated projective modules over $\mathbb{Z}[A_5]$.

Theorem 3.4.13. *The countably but not finitely generated projective modules over $R = \mathbb{Z}[A_5]$ are the following: Let $I = \mathrm{Aug}(R)$ and let X be the other non-trivial idempotent ideal of R. Let B_I be the unique I-big projective R-module with trace I, and let B_X be the unique X-big projective module with trace X. Apart from these, there is an X-big projective module P such that P/PX is the unique indecomposable projective module over R/X. Then:*

(i) Any countably generated projective module over R that is neither free nor finitely generated has a unique decomposition as a sum $Q \oplus F$, where $Q \in \{B_X, B_I, P\}$ and F is a finitely generated free module.

(ii) $B_X \oplus B_I \simeq R^{(\omega)}$ and $B_I \oplus P \simeq R^{(\omega)}$.

(iii) $P \oplus B_X \simeq P$ and $P \oplus P \simeq R \oplus B_X$.

Proof. Let M be a countably generated projective module over R. Since R has (*), there exists a least ideal K such that M/MK is finitely generated. If $K = 0$, M is finitely generated. If $K = R$, then M is R-big and hence free. If $K = I$, then $M/MI \simeq \mathbb{Z}^n$ for some $n \in \mathbb{N}_0$, because $R/I \simeq \mathbb{Z}$. Since $N = B_I \oplus R^n$ is a countably generated projective module such that I is the smallest ideal of the set $\{\, L$ ideal of $R \mid N/NL$ is finitely generated$\,\}$ and $N/NI \simeq M/MI$, by Lemma 3.2.5 and Corollary 3.2.10, we have $M \simeq N$. Clearly, for every $m, n \in \mathbb{N}$, one has $B_I \oplus R^n \simeq B_i \oplus R^m$ if and only if $m = n$.

The remaining case is $X = K$. Recall that $X_{(p)} = R_p$ for any prime different from 2. It follows that there exists $k \in \mathbb{N}$ such that $2^k \in X$. Now $R/X \simeq (R/2^kR)/(X/2^kR) \simeq (R_2/2^kR_2)/(X_{(2)}/2^kR_2)$. Let $S = \mathbb{Z}_{2^k}[A_5]$, let $\pi\colon R_2 \to S$ be the canonical epimorphism and let $X' = \pi(X_{(2)})$. From the proof of Lemma 3.4.11, we know that $S/J(S) \simeq M_1 \oplus M_2^2 \oplus M_3^n$ for some $n \in \mathbb{N}$ (in fact $n = 4$, but we do not need this) and the M_1, M_3 are the simple factors of X'. Now $S/J(S)/(X' + J(S))/J(S) \simeq (S/X')/(J(S/X')) \simeq \mathrm{M}_2(\mathrm{End}_S(M_2))$. It follows that R/X is a homogeneous semilocal ring with an indecomposable projective module P' satisfying $P'^2 \simeq R/X$. The module P' gives a unique countably generated projective module P such that P is X-big and $P/PX \simeq P'$. Since $P' \oplus P' \simeq R/X$, we get $P \oplus P \simeq B_X \oplus R$. The relation $B_X \oplus P \simeq P$ holds because P is X-big.

It remains to prove the relations in *(ii)*. Since a direct sum of an X-big module and an I-big module is R-big, these relations follow immediately. ■

3.5 One more application

Finally let us consider universal enveloping algebras. Let g be a Lie algebra over a field k and let X be a basis of g. A *universal enveloping algebra of* g, denoted by $U(\mathsf{g})$, is a factor of the free k-algebra over X modulo the relations $xy - yx = [x, y]$ $(x, y \in X)$. If g is a nilpotent Lie algebra of finite dimension, then $U(\mathsf{g})$ is a left and right noetherian AR-domain (see [11, Section 4.2] for the definition). It follows that all infinitely generated projective modules are free [12, Lemma 8.6]. The AR-property does not hold for solvable Lie algebras in general, but property (*) does. This enables us to prove that infinitely generated projective modules are free over $U(\mathsf{g})$ if g is a solvable Lie algebra of finite dimension and k has characteristic zero. This concludes the proof of [12, Conjecture 8.5], stating that a finite dimensional Lie algebra over a field of characteristic zero is solvable if and only if any (left and right) projective module over $U(\mathsf{g})$ is a direct sum of finitely generated modules.

We say that a ring R satisfies *strong (*)* if every sequence of ideals $I_1, I_2, \dots \subseteq R$ satisfying $I_{k+1}I_k = I_{k+1}$, $k \in \mathbb{N}$ has either $I_k = R$ for every $k \in \mathbb{N}$ or there exists $l \in \mathbb{N}$ such that $I_l = 0$. Let us point out the following straightforward consequence of Bass' theorem [3, Theorem 3.1].

Lemma 3.5.1. *Let R be a left and right noetherian ring satisfying (*). Then the following are equivalent:*

(i) R satisfies strong (*).

(ii) The only idempotent ideals of R are 0 and R.

(iii) Every projective module over R is either finitely generated or free.

Lemma 3.5.2. *Let S be a noetherian domain and let $D\colon S \to S$ be a derivation on S. Let $R = S_D[x]$ be the corresponding skew polynomial ring. If X and Y are ideals of R such that $XY = X$ and X is nonzero, then Y contains a constant polynomial.*

Proof. Let K be the (left and right) quotient field of S and $\overline{D}\colon K \to K$ the derivation extending D. Then R can be considered as a subring of the (left and right) principal ideal domain $\overline{R} = K_{\overline{D}}[x]$. Let $\overline{X}$ be the ideal of $\overline{R}$ generated by X and let $\overline{Y}$ be the ideal of $\overline{R}$ generated by Y. Using the division algorithm one can check that $\overline{X} = \{\, s^{-1}p \mid 0 \neq s \in S, p \in X \,\}$ and $\overline{Y} = \{\, ps^{-1} \mid 0 \neq s \in S, p \in Y \,\}$. Considering the degrees of the polynomials, $\overline{X} \neq 0$ implies $\overline{Y} = \overline{R}$. But then Y must contain a polynomial of degree 0. ■

Proposition 3.5.3. *Let S be a noetherian prime algebra over $\mathbb{Q}$ satisfying strong (*). Suppose that $D\colon S \to S$ is a derivation on S and $R = S_D[x]$ is the corresponding skew polynomial ring. If any prime ideal of S is completely prime, then R satisfies strong (*).*

Proof. Let $I_1, I_2, \dots$ be a sequence of nonzero ideals in R such that $I_{k+1}I_k = I_{k+1}$ for every $k \in \mathbb{N}$. We have to prove that $I_k = R$ for every $k \in \mathbb{N}$. For any ideal $I \subseteq R$, consider the smallest ideal $c(I)$ of S such that $I \subseteq \sum_{i=0}^{\infty} c(I)x^i$. Observe that $c(I_{k+1})c(I_k) = c(I_{k+1})$ and $c(I_k) \neq 0$ for every $k \in \mathbb{N}$. Therefore the strong (*) in S implies $c(I_k) = S$ for every $k \in \mathbb{N}$.

Now let Q be a prime ideal of S invariant under D. On S/Q define $D_Q\colon S/Q \to S/Q$ by $D_Q(s+Q) := D(s) + Q, s \in S$. Consider the ring $R_Q = S/Q_{D_Q}[x]$ and the canonical projection $\pi_Q\colon R \to R_Q$. Observe that π_Q is an epimorphism with kernel $Q' = \sum_{i=0}^{\infty} Qx^i$.

We claim that for any prime ideal $Q \subseteq S$ invariant under D and for any $k \in \mathbb{N}$ we have $\pi_Q(I_k) = R_Q$. Then we conclude applying the claim to $Q = 0$.

Suppose the claim is not true that is the set $M = \{Q \mid Q \text{ is a prime ideal of } R \text{ invariant under } D \text{ such that } \pi_Q(I_l) \neq R_Q \text{ for some } l \in \mathbb{N}\}$ is nonempty. Let P be a maximal ideal of M. Let $\pi\colon S \to S/P$ be the canonical projection. Observe that P cannot be a maximal two-sided ideal of S: Since $c(I_k) = S$, $\pi_P(I_k) \neq 0$ for every $k \in \mathbb{N}$. Applying Lemma 3.5.2 to $\pi_P(I_1), \pi_P(I_2), \dots$ we get $S/P \cap \pi_P(I_k) \neq 0$. Therefore if S/P is a simple ring, then $1 \in \pi_P(I_k)$ for every $k \in \mathbb{N}$.

In general, Lemma 3.5.2 gives $L_k = \pi_P(I_k) \cap S/P \neq 0$. Put $L'_k = \pi^{-1}(L_k)$ and notice that L'_k is an ideal of S invariant under D. If $L'_k = S$ for every $k \in \mathbb{N}$, then $\pi_P(I_k) = R_P$ for every $k \in \mathbb{N}$, a contradiction to the choice of P. Therefore suppose that $L'_l \neq S$ for some $l \in \mathbb{N}$. Let $P_1, \dots, P_m$ be the minimal primes of L'_l. As S is a $\mathbb{Q}$-algebra, applying [6, Lemma 3.3.3], $P_1, \dots, P_m$ are primes of S invariant under D properly containing P. In particular, $\pi_{P_i}(I_l) = R_{P_i}$ or $R = I_l + P'_i$ for every $i = 1, \dots, m$. Then $R = (I_l + P'_1)\cdots(I_l + P'_m) = I_l + P'_1 \cdots P'_m$ also. Further, by [11, Theorem 2.3.7], there exists $n \in \mathbb{N}$ such that $(P_1 \cdots P_m)^n \subseteq L'_l$. Note $R = R^n = I_l + (P'_1 \cdots P'_m)^n$, therefore $R_P = \pi_P(R) \subseteq \pi_P(I_l) + R_P L_l R_P = \pi_P(I_l)$. So $R_P = \pi_P(I_l)$, a contradiction again. ■

Lemma 3.5.4. *Let k be a field of characteristic zero and let g be a solvable Lie algebra of finite dimension over k. Then $U(\mathsf{g})$ satisfies strong (*).*

Proof. First suppose that k is algebraically closed. Then g is completely solvable by [11, Theorem 14.5.3]. That is, there exists a basis $x_1, \dots, x_n$ of g over k such that $\mathsf{g}_m = \mathsf{k}x_1 + \cdots + \mathsf{k}x_m$ is an ideal of g for every $m = 1, \dots, n$. Then $U(\mathsf{g}_{m+1})$ can be seen as a skew polynomial ring over $U(\mathsf{g}_m)$ for $m = 1, \dots, n-1$. Recall that each prime ideal of

$U(\mathsf{g}_m)$ is completely prime by [11, Theorem 14.2.11], therefore we can apply Proposition 3.5.3.

In general, let $\overline{\mathsf{k}}$ be an algebraic closure of k. Let $I_1, I_2, \ldots$ be a sequence of nonzero ideals in $U(\mathsf{g})$ such that $I_{k+1}I_k = I_{k+1}$ for every $k \in \mathbb{N}$. Consider $\overline{R} = U(\mathsf{g}) \otimes \overline{\mathsf{k}} \simeq U(\mathsf{g} \otimes \overline{\mathsf{k}})$ and the ideals $\overline{I_k} = I_k \otimes \overline{\mathsf{k}}$. It is easy to see that $\overline{I_{k+1}} = \overline{I_{k+1}I_k}$ for every $k \in \mathbb{N}$. By the preceding step, $\overline{I_k} = \overline{R}$ for every $k \in \mathbb{N}$. But this is possible only if $I_k = U(\mathsf{g})$. ∎

Corollary 3.5.5. *Let* g *be a finite dimensional solvable Lie algebra over a commutative field of characteristic zero. Then*

(i) Every idempotent ideal of $U(\mathsf{g})$ *is trivial.*

(ii) The universal enveloping algebra of g *satisfies (*).*

(iii) Every projective $U(\mathsf{g})$*-module that is not finitely generated is free.*

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4. NONFINITELY GENERATED PROJECTIVE MODULES OVER GENERALIZED WEYL ALGEBRAS

4.1 Introduction

The theory of finitely generated projective modules is a classical topic in ring theory inspired by rich connections with K-theory, geometry and algebraic topology. However, it is often difficult to classify finitely generated projective modules over a given ring up to isomorphism, and one should be usually content with finding coarser invariants of this class of modules such as its Grothendieck group. For instance, this is certainly the case for projective modules over the first Weyl algebra; and calculating ideal class groups of commutative Dedekind domains is a core problem in algebraic number theory.

On the other hand the theory of infinitely generated projective modules is often essentially easier. For instance, Kaplansky's classical result says that every non-finitely generated projective module over a commutative Dedekind domain is free and later Bass [2] extended this to any indecomposable commutative noetherian ring as a consequence of his theory of big projectives. For instance, it follows from his theory that every non-finitely generated projective module over a simple noetherian ring is free. Thus it is quite often that the theory of infinitely generated projectives is 'trivial', which partly justifies Bass' remark [2, p. 24] that it 'invites little interest'. However, this is not always the case: non-finitely generated projective modules could be truly 'big'. For example, extending early results by Akasaki [1] and Linnell [14], Příhoda [19] found a superdecomposable (that is, without indecomposable direct summands) projective module over a certain localization of the integral group ring of the alternating group A_5.

In fact this result is a consequence of a far reaching development by Příhoda [19] of Bass' theory of big projectives, that leads to a 'rough' classification of infinitely generated projective modules over noetherian rings satisfying one mild additional condition $(*)$; for instance, $(*)$ holds true for any noetherian ring with the d.c.c. on two-sided ideals. Namely, he showed that projective modules over a noetherian ring R with $(*)$ are classified by pairs (I, P), where I is an idempotent ideal of R and P is a finitely generated projective R/I-module. The only drawback of his classification is that it is usually very difficult to understand the structure of the projective module Q corresponding to a given pair (I, P); for instance, to decide whether Q is finitely generated or isomorphic to a direct sum of finitely generated modules.

In this paper we will apply Příhoda's theory to obtain a satisfactory classification of non-finitely generated projective modules over the so-called generalized Weyl algebras (GWAs). This class of algebras was introduced and investigated by Bavula [3], but

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also was studied by Hodges [9] who called the rings in this class deformations of type-A Kleinian singularities; and by Rosenberg [20] under the name of hyperbolic rings. For instance, every GWA is a noetherian domain of Krull dimension 1, and this class of algebras includes the first Weyl algebra and all infinite dimensional primitive quotients of the universal enveloping algebra Usl_2 over a field of characteristic zero. In particular, the global dimension of any GWA is 1, 2 or ∞, and there is a good understanding of the finitely generated projective modules — the Grothendieck group of projectives has been calculated (see [7, 9, 11, 18]) for most GWAs.

Recall that an old result of Kaplansky says that every projective left module over a left hereditary ring is a direct sum of finitely generated modules isomorphic to left ideals. In this paper we will show that something similar is true for projective modules over GWAs. In fact, the result is even more precise: in each GWA we will find finitely many homogeneous left ideals such that every non-finitely generated projective (left) module is a direct sum of copies of those.

In detail, in Section 4.2 we discuss some basic properties of idempotent ideals and will gather, in Section 4.3, certain (mostly folklore) statements on the structure of projective modules and their trace ideals. We will overview, in Section 4.4, the theory of (countably generated) projective modules (called fair-sized projectives in [19]) over noetherian rings with $(*)$, and draw some consequences of this theory. For example, in Theorem 4.4.4 we will give a general criterion for when every projective module over a noetherian ring with $(*)$ is a direct sum of finitely generated modules. For instance, for this to be true, finitely generated projective modules over factors of R by idempotent ideals must lift to finitely generated projectives over R. We also collect in this section some nice examples illustrating the power of the aforementioned theory. For instance, (see Example 4.4) we will classify non-finitely generated projective modules over the ring of differential operators of n-dimensional projective space.

In Section 4.5 we will discuss some (mostly known) facts on the structure of generalized Weyl algebras, the main sources of information being Bavula [3] and Hodges [9]. Note that every GWA A is a noetherian domain with finitely many two-sided ideals (so $(*)$ holds true) and A has a least nonzero ideal $I_{\min}$. We also recall the structure of maximal ideals of GWAs and their simple finite dimensional modules. We will prove that the nonzero idempotent ideals of a GWA A form a finite Boolean algebra $B(A)$ and describe its coatoms.

Finally, in Section 4.7 we will classify infinitely generated projective modules over any GWA A. Using a description of idempotent ideals of A we will show that every such ideal is the trace of a finitely generated projective module; moreover, finitely generated projectives can be lifted modulo idempotent ideals of A. This is the crucial point of the paper, and our choice of finitely generated projective modules (to cover all finitely generated projectives over factor rings) is a bare guess. Certainly we had in mind a family of finitely generated projective modules constructed by Hodges [9], but our situation is essentially more demanding. For instance, the construction of a finitely generated projective A-module whose trace equals $I_{\min}$ (see Lemma 4.7.1) is quite involved. Even more this is true for the construction (in Lemma 4.7.2) of finitely generated projectives whose traces are atoms in $B(A)$.

Having spent a lot of time and space on these technicalities, we are awarded with a relative easy proof of two final results (Theorem 4.7.5 and Proposition 4.7.6). Namely, Theorem 4.7.5 states that every infinitely generated projective module over a GWA A is a direct sum of homogeneous left ideals of A from a prescribed finite family. In Proposition 4.7.6 we will improve this result by finding a canonical form for every infinitely generated projective module over any GWA, thus classifying projectives over GWAs by means of cardinal invariants.

4.2 Idempotent ideals

Most modules in this paper will be left modules over rings with unity. An element e of a ring R is said to be an *idempotent* if $e = e^2$. For instance, $0, 1 \in R$ are trivial idempotents. We say that an ideal I of R is *idempotent* if $I = I^2$, for which $\{0\}$ and R are trivial examples. Furthermore, the (two-sided) ideal ReR generated by an idempotent e (or by any set of idempotents) is idempotent. By [12, Corollary 2.43], every finitely generated idempotent ideal of a commutative ring is generated by an idempotent. However, if I is the augmentation ideal of the integral group ring $\mathbb{Z}A_5$, then (see [1]) I is idempotent, but $\mathbb{Z}A_5$ has no nontrivial idempotents.

If R is a semisimple artinian ring, then every two-sided ideal of R is generated by a central idempotent, therefore idempotent. Furthermore, in this case the set of (idempotent) ideals of R ordered by inclusion forms a finite Boolean algebra whose atoms correspond to minimal (two-sided) ideals of R, therefore to isomorphism classes of simple R-modules.

Note that the sum of any set of idempotent ideals is idempotent. For instance, every ideal I of R contains a largest idempotent ideal $I_{\text{idem}} \subseteq I$. Furthermore, when ordered by inclusion, the set of idempotent ideals of R forms a lattice. The join in this lattice is the usual sum, but the meet of two idempotent ideals I and J equals $(I \cap J)_{\text{idem}}$, which could be a proper subset of $I \cap J$ (see some examples below).

It is often important to describe the lattice of idempotent ideals of a given ring R. For this the following reductions will be useful. Suppose that $I \subseteq J$ are ideals of R such that I is idempotent. Then J is idempotent iff its image J/I is an idempotent ideal of the factor ring R/I. For instance, assume that R has a least nonzero ideal $I_{\min}$ (that is, R is subdirectly irreducible) such that $I_{\min}^2 \neq 0$, therefore $I_{\min}$ is idempotent. It follows from the above remark that the description of idempotent ideals of R boils down to the description of idempotent ideals of $R/I_{\min}$.

To make some further reductions we need the following result.

Fact 4.2.1. *[22, L. 1] If I, K are distinct idempotent ideals of R and $J^m \subseteq I$, $J^n \subseteq K$ for some ideal J of R, then I and K have distinct images in R/J.*

Another way to say this is that $I + J = K + J$ yields $I = K$, that is, every idempotent ideal is uniquely determined by its image in R/J. One obvious instance of this situation is when J is a nilpotent ideal of R, and more can be said in this case. Recall that a ring R is said to be *semiperfect*, if the factor of R by its Jacobson radical J is a semisimple artinian ring and idempotents can be lifted modulo J. A semiperfect ring with a nilpotent Jacobson radical is called *semiprimary.* For instance, every one-sided artinian ring is semiprimary.

Lemma 4.2.2. *Every idempotent ideal of a semiprimary ring R is generated by an idempotent. Furthermore, the lattice of idempotent ideals of R is a finite Boolean algebra with m atoms, where m is the number of simple R-modules.*

Proof. If J denotes the Jacobson radical of R, then J is nilpotent and R/J is a semisimple artinian ring.

Let I be an idempotent ideal of R. Then $\overline{I} = (I + J)/J$ is an idempotent ideal of the semisimple ring R/J, hence $\overline{I}$ is generated by a central idempotent $\bar{e}$. Since J is nilpotent, one can lift $\bar{e}$ modulo J — there exists an idempotent $e \in R$ with $e + J = \bar{e}$. Then $K = ReR$ is an idempotent ideal of R such that $\overline{K} = \bar{e}\,(R/J) = \overline{I}$, therefore $K = I$ by Fact 4.2.1. Thus every idempotent ideal of R is generated by an idempotent.

Now the canonical projection $\pi : R \to R/J$ induces a map (also denoted by π) from the poset of idempotent ideals of R into the poset of idempotent ideals of R/J that preserves sums, hence preserves ordering. Since R/J is semisimple, the latter poset is a Boolean algebra with m atoms. Because J is nilpotent, Fact 4.2.1 yields that π is an injection. Furthermore, by the proof of the first part, π is a surjection, and it is easily seen that π reflects sums, hence reflects the ordering. Thus π is an isomorphism of posets, therefore an isomorphism of lattices. ∎

The following corollary is exactly what we need for further applications.

Corollary 4.2.3. *Suppose that R is a ring with a least nonzero ideal $I_{\min}$, $I_{\min}^2 \neq 0$, such that $R/I_{\min}$ is a semiprimary ring. Then the lattice of nonzero idempotent ideals of R is a finite Boolean algebra with m atoms, where m is the number of simple (non-isomorphic) $R/I_{\min}$-modules.*

4.3 Projective modules

One explanation why idempotent ideals are important is that they are intimately connected with projective modules. Recall that a module P over a ring R is said to be *free* if P is isomorphic to a module $R^{(I)}$ for some set I; and P is called *projective* if it is isomorphic to a direct summand of a free module. For instance, every free module is projective, as is the module Re for an idempotent e; but below we will see less obvious examples of projective modules.

If P is a projective module, then the *trace* of P, $\mathrm{Tr}(P)$, will denote the sum of images of all morphisms from P to ${}_RR$. For instance, if $P = Re$ for an idempotent e, then $\mathrm{Tr}(P) = ReR$ is an idempotent ideal. In fact it is always the case.

Fact 4.3.1. *If P is a projective module, then $\mathrm{Tr}(P)$ is an idempotent ideal such that $P = \mathrm{Tr}(P)\,P$. Furthermore, $\mathrm{Tr}(P)$ is the least among ideals I such that $P = IP$.*

Proof. The first part is a common knowledge (see [12, Proposition 2.40]). The second part is also well known, but somehow avoids any written account. ∎

Clearly (say, from Fact 4.3.1) $\mathrm{Tr}(P) \neq 0$ for any nonzero projective module P and P is said to be a *generator* if $\mathrm{Tr}(P) = R$ (the maximal possible value of the trace). If P is a direct summand of a free module $R^{(I)}$, then P is isomorphic to the module generated by the columns of a column-finite idempotent $I \times I$ matrix E over R, therefore $\mathrm{Tr}(P)$ is a two-sided ideal generated by entries of E.

Given projective modules P and Q, we say that P *generates* Q if, for some α, there is an epimorphism $P^{(\alpha)} \to Q$. Since Q is projective, this is the same as Q being isomorphic to a direct summand of $P^{(\alpha)}$. The following lemma is also folklore, but should be put on the paper, at least once.

Lemma 4.3.2. *Let P and Q be projective modules. Then the following are equivalent.*
1) P generates Q;
2) $Q = \mathrm{Tr}(P)\,Q$;
3) $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$.

Proof. 1) $\Rightarrow$ 2). Let $f : P^{(\alpha)} \to Q$ be an epimorphism, Applying f to $P^{(\alpha)} = \mathrm{Tr}(P)\,P^{(\alpha)}$ (see Fact 4.3.1) we obtain $Q = \mathrm{Tr}(P)\,Q$.

2) $\Rightarrow$ 3). By Fact 4.3.1, $\mathrm{Tr}(Q)$ is the least ideal I such that $Q = IQ$, therefore $Q = \mathrm{Tr}(P)\,Q$ yields $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$.

3) $\Rightarrow$ 2). Since $\mathrm{Tr}(Q)\,Q = Q$ and $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$, we conclude that $\mathrm{Tr}(P)\,Q = Q$.

2) $\Rightarrow$ 1). It suffices to prove that every $q \in Q$ is in the image of a morphism $P^k \to Q$, for some (finite) k. From $Q = \mathrm{Tr}(P)\,Q$ it follows that $q = \sum_{i=1}^{n} r_i q_i$ for some $r_i \in \mathrm{Tr}(P)$, $q_i \in Q$. Clearly we may assume that $n = 1$, that is, $q = rq'$, $r \in \mathrm{Tr}(P)$, $q' \in Q$. Furthermore, $r \in \mathrm{Tr}(P)$ yields that $r = \sum_{j=1}^{k} f_j(p_j)$, where $p_j \in P$ and $f_j : P \to {}_R R$ are morphisms. Let $g = \sum_{j=1}^{k} f_j : P^k \to R$ and let $h : R \to Q$ be given by $h(1) = q'$. Then hg maps P^k into Q and $hg(\sum_{j=1}^{k} p_j) = h(r) = rh(1) = rq' = q$, as desired. ∎

A module M is said to be *countably generated* if it has a finite or infinite countable set of generators. By Kaplansky's theorem (see [8, Corollary 2.48]) every projective module is a direct sum of countably generated modules, thus most (but not all) questions on the structure of projective modules can be reduced to the countably generated case.

The following lemma, which is a version of Eilenberg's trick (see [2, p. 24] or [12, p. 22]), shows that a projective module with a larger trace 'absorbs' another 'smaller' projective module.

Lemma 4.3.3. *Let P and Q be countably generated projective modules with $\mathrm{Tr}(Q) \subseteq \mathrm{Tr}(P)$. If $\alpha \geq \beta, \omega$, then $P^{(\alpha)} \cong P^{(\alpha)} \oplus Q^{(\beta)}$.*

Proof. By Lemma 4.3.2 and because Q is countably generated, Q, hence $Q^{(\beta)}$ is isomorphic to a direct summand of $P^{(\alpha)}$. If $P^{(\alpha)} \cong Q^{(\beta)} \oplus T$ for some module T, then

$$P^{(\alpha)} \cong (P^{(\alpha)})^{(\omega)} \cong (Q^{(\beta)} \oplus T)^{(\omega)} \cong Q^{(\beta)} \oplus (T \oplus Q^{(\beta)})^{(\omega)} \cong Q^{(\beta)} \oplus P^{(\alpha)}. \quad ∎$$

As we have seen in Fact 4.3.1 the trace of a projective module is always an idempotent ideal. Unfortunately, given an idempotent ideal I, it is usually quite difficult to decide whether I is a trace of some projective module. The following is a rare case that provides such an answer.

Fact 4.3.4. *[25, Corollary 2.7] Let I be an idempotent ideal of a ring R such that I is finitely generated as a right ideal. Then there exists a countably generated projective left R-module whose trace equals I.*

However, we do not know much about the structure of this projective module, for instance, whether it can be chosen to be finitely generated or not.

In the next section we will discuss the property of a projective module to decompose into a direct sum of finitely generated modules. Thus the following result of Kaplansky will be useful in this discussion.

Fact 4.3.5. *(see [12, 2.24]) Every projective left module over a left hereditary ring is a direct sum of modules isomorphic to finitely generated left ideals.*

Recall that Kaplansky proved that every projective module over a local ring is free. One more result along this line is worth mentioning.

Fact 4.3.6. *(see [2, Corollary 3.4]) Every infinitely generated projective left module over a left noetherian simple ring is free.*

4.4 The theory of fair-sized projectives

In this section we recall (from [19]) a classification of (countably generated) projective modules over certain classes of noetherian rings. One can consider this theory as a far reaching generalization of Bass' theory of big projectives (see [2]).

We say that a ring R satisfies the condition $(*)$ if the following holds.

$(*)$ Every (descending) chain $I_1, I_2, \dots$ of ideals of R, with $I_{k+1}I_k = I_{k+1}$ for any k, stabilizes.

For instance, if the lattice of (two-sided) ideals of R is finite then R satisfies $(*)$.

Remark 4.4.1. Sakhaev [21] characterized rings R with the following property: Any projective left R-module finitely generated modulo its Jacobson radical is finitely generated. He showed that this condition is connected with the stabilization of the (descending) sequence of left principal ideals of the matrix ring $M_n(R)$ generated by $n \times n$ matrices A_i, where $A_{i+1}A_i = A_{i+1}$ for every i (see condition (t6) in his Theorem 3) for every positive integer n. If I_i denotes the two-sided ideal generated by entries of A_i then we obtain that $I_{i+1}I_i = I_{i+1}$, as in $(*)$. However, it is easy to see that Sakhaev's condition is satisfied in any (left) noetherian ring while there are noetherian rings not satisfying (*). Therefore in this paper we will not pursue this analogy any further.

Proposition 4.4.2. *[19] Suppose that R is a noetherian ring satisfying $(*)$. Then there is a natural one-to-one correspondence between countably generated projective R-modules and pairs (I, P), where I is an idempotent ideal of R and P is a finitely generated projective R/I-module.*

One direction in this correspondence is easy to describe. If Q is a countably generated projective R-module, then $(*)$ implies (see [19] for a proof) that there exists a least ideal $I = I(Q)$ of R such that $P = Q/IQ$ is a finitely generated (projective) R/I-module. Thus we assign to Q the pair (I, P). The opposite direction in the above correspondence is rather an existence theorem. For example, it is usually quite difficult to decide whether the (countably generated) projective module corresponding to a given pair (I, P) is finitely generated or not.

Note that the pairs $(0, P)$ in the above classification correspond to finitely generated projective R-modules, so Proposition 4.4.2 says nothing new about them. Furthermore, if Q is a countably generated projective module, then, using Fact 4.3.1, it is easily seen that $Q^{(\omega)}$ corresponds to the pair $(\mathrm{Tr}(Q), 0)$. In particular, the pair $(R, 0)$ corresponds to the free module $R^{(\omega)}$. For example, it follows that every infinitely generated projective module over a simple noetherian ring is free, a slightly weaker form of Bass' result in Fact 4.3.6.

Now we will show how this theory works in a slightly more elaborate situation.

Proposition 4.4.3. *Suppose that R is a noetherian ring with a unique nonzero proper ideal J and such that $D = R/J$ is a skew field. Further assume that there exists a finitely generated projective module Q such that $\mathrm{Tr}(Q) = J$. Then every infinitely generated projective module is either free or isomorphic to $R^{(\alpha)} \oplus Q^{(\beta)}$, where $\alpha < \beta$, $\beta \geq \omega$, and α, β are uniquely determined by Q.*

PROOF. Since $D = R/J$ is a skew field, every finitely generated projective R/J-module is of the form $(R/J)^k$ for some $k < \omega$. If P is a countably infinitely generated projective module, then $I(P) \neq 0$, hence either $I(P) = R$, and then P is free, or $I(P) = J$. In the latter case P goes to $(J, (R/J)^k)$ in the correspondence of Proposition 4.4.2. But clearly $R^k \oplus Q^{(\omega)}$ also corresponds to this pair, therefore $P \cong R^k \oplus Q^{(\omega)}$.

If P is uncountably generated, then (using Kaplansky's theorem) decompose it into a direct sum of countably infinitely generated modules $P = \oplus_{i \in I} P_i$. By what we have already proved each P_i is either free or isomorphic to $R^{k_i} \oplus Q^{(\omega)}$ for some $k_i < \omega$. Gathering the copies of R and Q together, we obtain $P \cong R^{(\alpha)} \oplus Q^{(\beta)}$. If $\alpha \geq \beta, \omega$ then P is isomorphic to $R^{(\alpha)}$ by Lemma 4.3.3. Otherwise, since P is not finitely generated, $\alpha < \beta$ and $\beta \geq \omega$.

Now $\alpha = \dim_D P/JP$ is uniquely determined by P and the same is true for $\beta = \alpha + \beta$ which equals the uniform dimension of P. ∎

Note that (at least in some cases — see below) a finitely generated projective module Q is not unique. However, if Q' is another finitely generated projective module with $\mathrm{Tr}(Q') = J$, then Proposition 4.4.2 implies that $Q^{(\omega)} \cong Q'^{(\omega)}$, because both modules correspond to the pair $(J, 0)$.

Now we will give some examples showing that the situation described in Proposition 4.4.3 occurs naturally.

Let $R = \mathcal{D}(\mathbb{P}^n)$ denote the ring of differential operators on the projective space $\mathbb{P}^n(k)$, where k is an algebraically closed field of characteristic zero. By [6, p. 213–214] R is a noetherian domain of Krull dimension n (and global dimension $n+1$) with a unique nonzero proper ideal J and $R/J \cong k$ holds true. Thus to apply Proposition 4.4.3 it suffices to find a finitely generated projective module Q such that $\mathrm{Tr}(Q) = J$. Indeed, let $Q = \mathcal{D}(1)$ as in [6, p. 215]. Then, by [6, Cor. 4.8], Q is a finitely generated projective (left) $\mathcal{D}(\mathbb{P}^n)$-module such that $JQ = Q$, hence $\mathrm{Tr}(Q) = J$.

Thus Proposition 4.4.3 gives a classification of infinitely generated projective modules over $\mathcal{D}(\mathbb{P}^n)$.

Let k be a field of characteristic 2 containing a nonzero element λ which is not a root of unity. Let S be obtained by factoring the ring of Laurent polynomials $k[X^{\pm 1}, Y^{\pm 1}]$ by the ideal generated by $XY - \lambda YX$. Let σ be an automorphism of S of order 2 given by $\sigma(X) = X^{-1}$, $\sigma(Y) = Y^{-1}$; and set $R = S^\sigma$, the subring of S fixed by σ.

Then (see [15, Example 1.8] or [10, p. 140–141]) R has a unique (nonzero proper) two-sided ideal J such that $R/J \cong k$ and S is an indecomposable rank 2 projective module whose trace is equal to J.

Thus, by Proposition 4.4.3 again, we obtain a classification of non-finitely generated projective R-modules.

As one more example let us consider the subring $R = k + xA_1(k)$ of the first Weyl algebra over a field k of characteristic zero. By [16, 1.3.10, 5.5.11], R is a hereditary noetherian domain with a unique nonzero proper two-sided ideal $J = xA_1(k)$. Then J is a finitely generated projective module coinciding with its trace. Thus taking $Q = J$ and applying Proposition 4.4.3 we obtain a classification of infinitely generated projective R-modules (though one should be able to extract this from the classification of infinitely generated projective modules over hereditary noetherian prime rings in Levy and Robson [13]).

In this case the finitely generated projective module Q is not unique. Indeed it is well known that $A_1(k)$ has infinitely many non-isomorphic left ideals. Using $\mathrm{End}(J) = xA_1(k)x^{-1} \cong A_1(k)$ one concludes that there are infinitely many nonisomorphic (projective) left ideals of R with trace J.

Next we will investigate an even more advanced example of Stafford [23]. To keep the notation of his paper, in this example we will consider right modules.

Let k be a field of characteristic zero, $C = k[x_1, \ldots, x_n]$ be the ring of polynomials, and δ is a derivation of C given by $\delta(x_1) = 1$ and $\delta(x_i) = x_i x_{i-1} - 1$ for $i > 1$. Let $S = C[y, \delta]$ be the ring of differential polynomials, and take $R = C + x_1 S$. Then (see [23, p. 384–385]) R is a noetherian domain with a least nonzero proper ideal $J = x_1 S$ and $R/J \cong k[x_1, \ldots, x_{n-1}]$. It follows easily that J is the only nonzero proper idempotent ideal of R; and every finitely generated projective R/J-module is isomorphic to $(R/J)^k$ (because every projective $k[x_1, \ldots, x_{n-1}]$-module is free). Furthermore, it is not difficult to check that S is a finitely generated projective R-module whose trace equals J. Thus arguing as in Proposition 4.4.3 we conclude that every infinitely generated projective R-module is either free or isomorphic to $R^{(\alpha)} \oplus S^{(\beta)}$, $\alpha < \beta$, $\beta \geq \omega$.

Note that over rings in Examples 4.4–4.4 every projective module is a direct sum of finitely generated modules, but this is not always the case. Indeed, let $R = \mathbb{Z}A_5$ be the integral group ring of the alternating group A_5 and let I be the augmentation ideal of R. Since (see [1]) I is idempotent, by Fact 4.3.4 there exists a countably generated projective module P whose trace is equal to I. But P cannot contain a finitely generated direct summand because (as follows from [24, Theorem 8.1] — see [1, Corollary 14] for arguments) every finitely generated projective R-module is a generator.

In the next proposition we characterize in the framework of the theory of fair-sized projectives the rings whose projective modules are direct sums of finitely generated modules. As we have already mentioned (see Fact 4.3.5) this holds true for left hereditary rings; for a more thorough treatment of this question see [17].

Theorem 4.4.4. *Let R be a noetherian ring satisfying $(*)$. Then the following are equivalent.*

1) Every projective module is a direct sum of finitely generated modules;

2) a) every idempotent ideal of R is the trace of a finitely generated projective module and

b) if I is an idempotent ideal of R and P is a finitely generated projective R/I-module, then there exists a finitely generated projective module Q such that $Q/IQ \cong P$.

Proof. 1) $\Rightarrow$ 2). a) Suppose that I is an idempotent ideal of R. By Fact 4.3.4, there exists a countably generated projective module Q whose trace is equal to I. By the assumption, $Q = \oplus_{j \in J} Q_j$ is a direct sum of finitely generated modules. Then $\mathrm{Tr}(Q)$ is a directed union of traces of finitely generated projectives $Q_{j_1} \oplus \cdots \oplus Q_{j_k}$, $j_1, \ldots, j_k \in J$. Since R is noetherian, I is the trace of one of such finitely generated modules.

b) Suppose that P is a finitely generated projective R/I-module, where I is an idempotent ideal of R. Let Q be a countably generated projective module that corresponds to the pair (I, P) in Proposition 4.4.2. By the assumption, $Q = \oplus_{j \in J} Q_j$ is a direct sum of finitely generated modules. From the definition of $I = I(Q)$ it follows that $Q_j \neq IQ_j$ for only finitely many $j \in J$. Adding up the Q_j from this finite subset we obtain a finitely generated projective module Q' such that $Q'/IQ' \cong P$.

2) $\Rightarrow$ 1). Let Q be a countably generated projective module and set $I = I(Q)$, $P = Q/IQ$ (see Proposition 4.4.2), therefore I is an idempotent ideal of R and P is a finitely generated projective R/I-module. By the assumption, there are finitely generated projective modules P_1 and P_2 such that $\mathrm{Tr}(P_1) = I$ and $P_2/IP_2 \cong P$. It is easily seen that the module $P_1^{(\omega)} \oplus P_2$ also corresponds to the pair (I, P), therefore $Q \cong P_1^{(\omega)} \oplus P_2$ by Proposition 4.4.2. ∎

Note that 2 b) of the above theorem says that one can 'lift' finitely generated projective modules modulo idempotent ideals.

4.5 Generalized Weyl algebras

Let k be an algebraically closed field of characteristic zero and let σ be an automorphism of the ring of polynomials $k[H]$. In this paper we will consider only the case when $\sigma(H) = H - 1$ (and σ fixes k pointwise); for the case when σ is arbitrary see for example [5]. Let $a(H) \in k[H]$ be a nonconstant polynomial. We say that a k-algebra $A = A(a)$ is a *generalized Weyl algebra*, GWA, if A is generated over $k[H]$ by (noncommuting) variables X, Y subject the following relations.

$$YX = a(H),\ XY = \sigma(a) = a(H-1) \text{ and } HY = Y(H-1),\ HX = X(H+1)\,.$$

Thus for every polynomial $b(H) \in k[H]$ we obtain

$$b(H)\cdot Y = Y\sigma(b) = Y \cdot b(H-1) \quad \text{and} \quad b(H)\cdot X = X\sigma^{-1}(b) = X\cdot b(H+1)\,.$$

For instance, consider the first Weyl algebra $A_1(k)$ as an algebra of differential operators acting on the ring of polynomials $k[x]$ on the left; therefore A_1 is generated by x and ∂ subject to the relation $\partial x - x\partial = 1$. It is easily checked that the map $X \to x$, $Y \to \partial$ and $H \to \partial x$ provides an isomorphism from the generalized Weyl algebra $A(H)$ onto $A_1(k)$.

Furthermore, (see [9] or [5, p. 522]) if G is a cyclic group of order m acting on $A_1(k)$ via $\partial \to \omega\partial$, $x \to \omega^{-1}x$, where ω is a primitive m-th root of unity, then the fixed ring $A_1^G = k\langle \partial^m, \partial x, x^m\rangle$ is a GWA with $a(H) = m^m H(H + 1/m)\cdot \cdots \cdot (H + (m-1)/m)$, where $X \to x^m$, $Y \to \partial^m$ and $H \to \partial x/m$.

Finally, let U be the universal enveloping algebra $Usl_2(k)$ with the usual generators e, f, h (thus $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$). If $C = 4fe + h^2 + 2h$ is the Casimir element, then all infinite dimensional primitive factors of U are of the form $U_\lambda = U/(C-\lambda)U$, $\lambda \in k$. It is straightforward to verify that U_λ is a GWA with $a(H) = \lambda/4 - (H+1)H$, where $X \to e$, $Y \to f$ and $H \to h/2$.

By [4, Theorem 3.28], one can multiply the polynomial $a(H)$ by a nonzero constant and 'shift' it to the left or right without changing the isomorphism type of A. It follows that every GWA with a linear polynomial $a(H)$ is isomorphic to $A_1(k)$, and every GWA with a quadratic polynomial is isomorphic to one of primitive factors U_λ.

Note that some rings we have already considered are GWAs. For instance, from [6, p. 205] it follows that the ring of differential operators $\mathcal{D}(\mathbb{P}^1)$ is isomorphic to U_0, that is, to a GWA with $a(H) = -(H+1)H$. By what we have just said this GWA is isomorphic to the GWA with $a(H) = H(H-1)$. Using [4, Theorem 3.28], it is easily checked that the latter GWA is not isomorphic to the GWA with $a(H) = H(H-2)$. However, using translation functors from [9, Theorem 2.3], one concludes that the last two GWAs are Morita equivalent.

The first crucial fact about GWAs is that they are noetherian.

Fact 4.5.1. *[3, Proposition 1.3, Theorem 2.5] Every GWA is a noetherian domain of Krull dimension* 1.

Furthermore, looking at the roots of $a(H)$ one can decide whether a given GWA is simple and calculate its global dimension. We say that $\lambda, \mu \in k$ are *comparable* if $\lambda - \mu \in \mathbb{Z}$.

Fact 4.5.2. *[3, Theorem 5] Let $A = A(a)$ be a GWA.*

1) A is simple iff $a(H)$ has no comparable (distinct) roots;

2) A is hereditary iff $a(H)$ has neither comparable nor repeated (= multiple) roots;

3) A has global dimension 2 *iff $a(H)$ has comparable roots but no repeated roots;*

4) A is of infinite global dimension iff $a(H)$ has a repeated root.

Thus every GWA has global dimension 1, 2 or ∞. For instance, if $a(H) = H^2$, then A is a simple algebra of infinite global dimension; and if $a(H) = H(H-1)$, then A has global dimension 2 and is not simple.

Recall that every GWA A has a standard $\mathbb{Z}$-grading: setting $\deg(X) = 1$, $\deg(Y) = -1$ and $\deg(H) = 0$, we obtain $A = \oplus_{n\in\mathbb{Z}} A_n$, where $A_n = k[H]Y^n = Y^n k[H]$ if $n < 0$, $A_0 = k[H]$, and $A_n = k[H]X^n = X^n k[H]$ if $n > 0$. Note also that $\operatorname{ad}(H)X^n = [H, X^n] = nX^n$ and $\operatorname{ad}(H)Y^m = -mY^m$. It follows easily that every (two-sided) ideal I of A is homogeneous, $I = \oplus_{n\in\mathbb{Z}} I_n$, where $I_n = I \cap A_n$ is the nth homogeneous component of I; therefore the lattice of two-sided ideals of A is distributive (because the lattice of ideals of $k[H]$ is distributive). In fact more can be said.

Fact 4.5.3. *[3, Proposition 2.2] If I is a nonzero ideal of a GWA A, then the factor A/I is finite dimensional. Furthermore, the lattice of ideals of A is finite and there is a least nonzero ideal $I_{\min}$.*

In the following lemma we will pinpoint this ideal. Note that, for every $n \geq 1$, $X^n Y^n = a(H-1) \cdot \dots \cdot a(H-n)$ is a polynomial $c_n(H)$ such that $Y^n X^n = a(H+n-1) \cdot \dots \cdot a(H) = c_n(H+n)$.

Lemma 4.5.4. *Let n be the maximum of $|\lambda - \mu|$, where λ and μ are comparable roots of $a(H)$. Then $I_{\min}$ is generated by the polynomial $d_n(H) = \gcd(X^n Y^n, Y^n X^n) = \gcd(c_n(H), c_n(H+n))$ and $X^n, Y^n \in I_{\min}$.*

Proof. Let I be a nonzero ideal of A. Since I is homogeneous, it contains a nonzero polynomial $f(H)$, and we may assume that $\deg f \geq 1$. Choose $k \geq n$ such that $f(H)$ and $f(H-k)$ are coprime. Then $f(H)X^k \in I$ and $X^k f(H) = f(H-k)X^k \in I$ implies $X^k \in I$ (and similarly $Y^k \in I$). It follows that $X^k Y^k = c_k(H) \in I$ and $Y^k X^k = c_k(H+k) \in I$, therefore $d_k(H) = \gcd(c_k(H), c_k(H+k)) \in I$.

If λ is a root of $d_k(H)$ then $\lambda - i$ and $\lambda + j$ are roots of $a(H)$ for some $1 \leq i \leq k$ and $0 \leq j \leq k-1$. By the assumption, $i + j = |(\lambda - i) - (\lambda + j)| \leq n$, in particular $i \leq n$ and $j \leq n-1$. It follows easily that $d_n(H) = d_k(H) \in I$. Thus $d_n(H)$ belongs to every nonzero ideal of A, therefore d_n generates $I_{\min}$.

Suppose that $\lambda \leq \mu$ are roots of $d_n(H)$. Then $\lambda - i$ and $\mu + j$ are roots of $a(H)$ for some $1 \leq i$ and $j \geq 0$. By the assumption $|(\mu + j) - (\lambda - i)| = \mu - \lambda + i + j \leq n$, hence

$\mu - \lambda \leq n-1$. Now it is easily checked that $d_n(H)$ and $d_n(H-n)$ are coprime, therefore, by the first part of the proof, $X^n, Y^n \in I$. ∎

For instance, if A is a GWA with $a(H) = H(H-1)$, then $n = 1$, hence $d_1(H) = \gcd(XY, YX) = \gcd((H-1)(H-2), H(H-1)) = H-1$; and $\langle H-1\rangle$ is the unique nonzero proper ideal of A. If A is a GWA with $a(H) = H(H-1)(H-2)$, then $n = 2$ and $d_2(H) = \gcd(X^2Y^2, Y^2X^2) = (H-1)(H-2)$.

Since every maximal ideal of $k[H]$ is generated by $H-\lambda$, $\lambda \in k$, the action of σ on the set of maximal ideals can be identified with the action $\lambda \to \lambda + 1$ on k. The orbits of this action are of the form $\lambda + \mathbb{Z}$, $\lambda \in k$, therefore $\lambda, \mu \in k$ are on the same orbit iff they are comparable. If B is an orbit and $\lambda, \mu \in B$, then we set $\lambda \leq_B \mu$ if $\mu - \lambda \geq 0$, that is $\mu - \lambda$ is a nonnegative integer; clearly $\leq_B$ is a linear ordering.

Let S be the (finite) set of all roots of $a(H)$, and let U denote the set of all orbits containing at least two roots of $a(H)$. If $B \in U$, then $S \cap B$ contains a smallest element x_B and a largest element $y_B \neq x_B$ (with respect to $\leq_B$). Denote by $T_B = (x_B, y_B]$ the semi-interval $\{z \in B \mid x_B < z \leq y_B\}$ and set $T = \cup_{B\in U} T_B$. For instance, if $a(H) = H(H-2)(H-5)$, then $U = \{0 + \mathbb{Z}\}$ and $T = \{1,2,3,4,5\}$, in particular, $1 \in T$ is not a root of $a(H)$. By Fact 4.5.2, T is nonempty iff A is not simple.

For every $\lambda \in T$ let $L_\lambda = \{\mu \in S \mid \lambda - \mu \in \mathbb{N}\}$ and $R_\lambda = \{\mu \in S \mid \mu - \lambda \in \mathbb{N}_0\}$, where $\mathbb{N}$ stands for the set of positive integers and $\mathbb{N}_0$ for the set of nonnegative integers. Thus $\mu \in L_\lambda$ iff μ is strictly to the left of λ within the equivalence class of λ; and $\mu \in R_\lambda$ iff μ is strictly to the right of λ in the equivalence class of λ or $\mu = \lambda$. Let m_μ denote the multiplicity of $H - \lambda$ in $a(H)$ and set $k_\lambda = \min(\sum_{\mu\in L_\lambda} m_\mu, \sum_{\tau\in R_\lambda} m_\tau)$.

We will give an even more algorithmic way (compare with Lemma 4.5.4) to compute $d_n(H)$.

Lemma 4.5.5. $d_n(H) = \prod_{\lambda\in T}(H-\lambda)^{k_\lambda}$.

Proof. By definition, $d_n(H)$ is the greatest common divisor of $c_n(H) = X^nY^n = a(H-1)\cdot\dots\cdot a(H-n)$ and $c_n(H+n) = Y^nX^n = a(H+n-1)\cdot\dots\cdot a(H)$. First notice that $H-\lambda$ divides $d_n(H)$ iff it divides both $c_n(H)$ and $c_n(H+n)$, that is, $\lambda - i$ and $\lambda + j$ are roots of $a(H)$ for some $1 \leq i \leq n$, $0 \leq j \leq n-1$. By the definition of T, it follows that $H - \lambda$ divides $d_n(H)$ iff $\lambda \in T$.

Suppose that $\lambda \in T$ and let us calculate the multiplicity of $H-\lambda$ in $c_n(H)$. Notice that $H-\lambda$ has multiplicity $m_{\lambda-1}$ in $a(H-1)$, ..., and multiplicity $m_{\lambda-n}$ in $a(H-n)$. Thus $H-\lambda$ has multiplicity $\sum_{\mu\in L_\lambda} m_\mu$ in $c_n(H)$. By similar arguments $H-\lambda$ has multiplicity $\sum_{\tau\in R_\lambda} m_\tau$ in $c_n(H+n)$, and the result follows immediately. ∎

For instance, let $a(H) = H^2(H-1)^3(H-2)^4$. Then $T = \{1,2\}$, $L_1 = \{0\}$, $R_1 = \{1,2\}$, hence $\sum_{\mu\in L_1} m_\mu = 2$, $\sum_{\tau\in R_1} m_\tau = 3+4 = 7$ and $k_1 = \min(2,7) = 2$. Similarly $L_2 = \{0,1\}$, $R_2 = \{2\}$, therefore $\sum_{\mu\in L_2} m_\mu = 2+3 = 5$, $\sum_{\tau\in R_2} m_\tau = 4$ and $k_2 = \min(5,4) = 4$. Thus $d_n(H) = (H-1)^2(H-2)^4$ is a generator for $I_{\min}$.

4.6 Idempotent ideals of GWAs

As one may see from Proposition 4.4.2, the description of idempotent ideals is an important ingredient in the classification of projective modules. In this section we describe the idempotent ideals of any GWA. But first we should recall description of maximal ideals of GWAs.

Recall (see Fact 4.5.3) that every GWA A has a least nonzero ideal $I_{\min}$ such that $A/I_{\min}$ is a finite dimensional algebra. It follows that every maximal ideal of A contains $I_{\min}$ and is the annihilator of a simple finite dimensional A-module. A classification of such simples is available from [3]. Suppose that $\lambda < \mu$ are roots of $a(H)$ lying on the same orbit B. We say that λ and μ are *adjacent* if the interval $(\lambda, \mu) = \{\tau \in B \mid \lambda < \tau < \mu\}$ contains no roots of $a(H)$. For instance, if $a(H) = H(H-2)(H-4)$, then $0 < 2$ and $2 < 4$ are the only pairs of adjacent roots. If $\lambda < \mu$ are adjacent roots of $a(H)$, then $S_{\lambda,\mu}$ will denote the cyclic module $A/A(Y^n, X, H-\mu)$. It is easily calculated that this module is n-dimensional with a k-basis given by $Y^{n-1}, \ldots, 1$. Note also that Y^i spans a one-dimensional eigenspace for the action of H, with eigenvalue $\mu - i$.

Lemma 4.6.1. *[3, Theorem 3.2] $S_{\lambda,\mu}$ is a simple (finite dimensional) A-module, and every simple finite dimensional A-module is isomorphic to a module of this form.*

In particular, the number of simple finite dimensional A-modules is the cardinality of $T \cap S$.

Thus if $I_{\lambda,\mu}$ denotes the annihilator of $S_{\lambda,\mu}$, then these ideals form a complete list of maximal ideals of A. Furthermore, (see [3, Lemma 3.3]) if $\mu - \lambda = m$, then the factor $A/I_{\lambda,\mu}$ is isomorphic to the full matrix ring $M_m(k)$, therefore $A/I_{\lambda,\mu}$ is a direct sum of m copies of $S_{\lambda,\mu}$.

In fact one can give a precise formula for a generator of $I_{\lambda,\mu}$. If $\lambda < \mu$ are adjacent roots on an orbit B, then $T_{\lambda,\mu}$ will denote the semi-interval $(\lambda, \mu] = \{\tau \in B \mid \lambda < \tau \leq \mu\}$. For instance, if $a(H) = H(H-2)(H-4)$, then $T_{0,2} = \{1, 2\}$ and $T_{2,4} = \{3, 4\}$. Clearly $T = \cup T_{\lambda,\mu}$, where the union runs over all pairs of adjacent roots of $a(H)$.

Fact 4.6.2. *[3, Lemma 3.3] $I_{\lambda,\mu}$ is generated by $d_{\lambda,\mu}(H) = \prod_{\tau \in T_{\lambda,\mu}} (H - \tau)$.*

For instance, if $a(H) = H(H-2)(H-4)$, then $I_{0,2}$ is generated by $(H-1)(H-2)$, in particular, $X^2, Y^2 \in I_{0,2}$, but $X, Y \notin I_{0,2}$.

Let $J \supseteq I_{\min}$ denote the ideal of A whose image $J/I_{\min}$ is the Jacobson radical of $A/I_{\min}$. It follows that J is the intersection of the ideals $I_{\lambda,\mu}$ when $\lambda < \mu$ run over all pairs of adjacent roots of $a(H)$. Since $T = \cup T_{\lambda,\mu}$, we obtain the following.

Corollary 4.6.3. *The zeroth homogeneous component J_0 of J is generated by $f(H) = \prod_{\tau \in T} (H - \tau)$.*

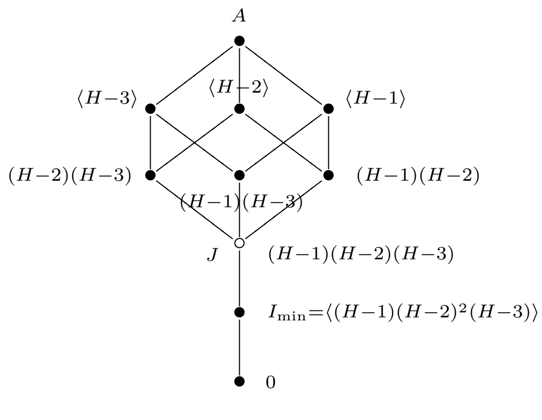
The remaining homogeneous components of J can be calculated using Fact 4.6.2. For instance, if $a(H) = H(H-2)(H-4)$, then $X^2, Y^2 \in J$ (since the maximum of differences between adjacent roots of $a(H)$ is 2), but (see Lemma 4.5.4) X^4 is the first power of X in $I_{\min}$.

Since every GWA has the least nonzero ideal $I_{\min}$ and $A/I_{\min}$ is a finite dimensional algebra, by Corollary 4.2.3 we obtain the following.

Corollary 4.6.4. *Let A be a GWA with m nonisomorphic simple finite dimensional modules. Then the lattice of nonzero idempotent ideals of A is a (finite) Boolean algebra $B(A)$ with m atoms.*

Note that $I_{\min}$ is the least element of $B(A)$, and every element of $B(A)$ but $I_{\min}$ is a sum of atoms (since the join in $B(A)$ is usual sum).

But first let us look at the following example. Let A be a GWA with $a(H) = H(H-1)(H-2)(H-3)$. Then the following is a fragment of the lattice of two-sided ideals of A containing $B(A)$, where idempotent ideals are marked by bullets.



For instance, $I_{0,1} = \langle H-1\rangle$ (that is, generated by $H-1$), $I_{1,2} = \langle H-2\rangle$ and $I_{2,3} = \langle H-3\rangle$ are the only maximal ideals of A, and they are idempotent. However, J, the intersection of all these ideals, is not idempotent and is strictly larger than $I_{\min}$. Indeed, the zeroth component of J is generated by $(H-1)(H-2)(H-3)$ and is it possible to check (it is not so obvious as it seems!) that it is larger than the zeroth component of $I_{\min}$, which is generated by $(H-1)(H-2)^2(H-3)$.

If $\lambda < \mu$ are adjacent roots of $a(H)$, then $m(\lambda, \mu) = \min(m_\lambda, m_\mu)$ will denote the common multiplicity of λ and μ as roots of $a(H)$. The following lemma describes coatoms in $B(A)$, that is, maximal idempotent ideals.

Lemma 4.6.5. *$I_{\lambda,\mu}^{m(\lambda,\mu)}$, where $\lambda < \mu$ run over adjacent roots of $a(H)$, is a complete list of maximal idempotent ideals of A.*

Proof. If I is a maximal idempotent ideal of A, then I is contained in a maximal ideal L; and $L = I_{\lambda,\mu}$ for some adjacent roots $\lambda < \mu$ of $a(H)$, by the description of maximal ideals. Since I is idempotent, it follows that $I \subseteq L^m$ for all m. But, by [3, Proof of Theorem 3.3], $m = m(\lambda, \mu)$ is the smallest number such that the ideal $I_{\lambda,\mu}^m$ is idempotent. ∎

Since $B(A)$ is a Boolean algebra, every idempotent ideal of A is a (unique) intersection (in $B(A)$) of maximal idempotent ideals $I_{\lambda,\mu}^{m(\lambda,\mu)}$. However, since the intersection in $B(A)$ may differ from set-theoretic intersection, this description is not very constructive. In the next section we will list the atoms of $B(A)$, hence obtain another, more handy, description of the idempotent ideals of GWAs.

4.7 projective modules over GWAs

In this section we will classify projective modules over any given GWA. Recall that (by Bass' result) if A is a simple GWA, then every infinitely generated projective module is free. Thus the only interesting case is when A is not simple, hence (by Fact 4.5.2) $a(H)$ has distinct comparable roots (that is, $T \neq \emptyset$). In most statements of this section we will make a default assumption that A is not simple.

Let us make a general (well known) remark. Suppose that I is a left ideal of a GWA A and let $Q = Q(A)$ denote the skew field of quotients of A. Since A is a noetherian domain, every morphism from I to ${}_AA$ is given by right multiplication by some $q \in Q$.

Using the dual basis lemma (see [12, Lemma 2.9]) we conclude that I is projective iff there are $p_1, \dots, p_m \in I$ and $q_1, \dots, q_m \in \mathrm{Hom}(I, A) \subseteq Q$ such that $\sum_{i=1}^m q_i p_i = 1$. In this case right multiplication by the row $(q_1, \dots, q_m)$ defines a morphism from I to ${}_A A^m$ whose one-sided inverse is given by right multiplication by the column $(p_1, \dots, p_m)^t$. Thus I is represented by the idempotent $m \times m$ matrix $(p_i q_j)$, therefore the trace of I is generated by the $p_i q_j$. Moreover, $\mathrm{Tr}(I)$ is also generated by $p'_i q_j$, where $p'_1, \dots, p'_l$ is any set of generators for I, for instance this is the case when $l = m$ and $p_i = r_i p'_i$ for some $r_i \in A$.

First we construct a projective homogeneous left ideal of A whose trace is equal to $I_{\min}$. We will use the notation introduced before Lemma 4.5.5. Recall that if $\lambda \in T$, then R_λ denotes the set of all roots of $a(H)$ that are comparable with λ and lie to the right of λ (including λ). Let $n_\lambda = \sum_{\mu \in R_\lambda} m_\mu$, where m_μ denotes the multiplicity of μ as a root of $a(H)$; and we set $q(H) = \prod_{\lambda \in T} (H - \lambda)^{n_\lambda}$. It is easily seen (see Lemma 4.5.5 for a similar proof) that $H - \lambda$ has multiplicity n_λ in $c_n(H + n) = Y^n X^n$, therefore $q(H) = Y^n X^n \mid_T$, the restriction of $c_n(H + n)$ to T. For instance, if $a(H) = H(H-1)(H-2)(H-3)$, then $q(H) = (H-1)^3 (H-2)^2 (H-3)$.

Recall that n denotes the maximum of $|\lambda - \mu|$, where λ and μ are comparable roots of $a(H)$. For instance, if $\lambda \in T$, then $\lambda - n \notin T$.

Lemma 4.7.1. *$P_{\min} = Aq(H) + AX^n$ is a projective homogeneous left ideal of A whose trace is equal to $I_{\min}$.*

Proof. Recall that Q denotes the classical ring of quotients of A, and let the morphism $f : A \to {}_A Q^{|T|+1}$ be given by right multiplication by the row $(q_0, \dots, q_n) = (1, \dots, Y^n (H - \lambda - n)^{-n_\lambda}, \dots)$, where each $\lambda \in T$ gives an entry. We claim that, when restricted to $P_{\min}$, f provides a morphism from $P_{\min}$ to ${}_A A$. Indeed $f(q(H)) = (q(H), \dots, q(H) Y^n (H - \lambda - n)^{-n_\lambda}, \dots)$. Since $(H - \lambda)^{n_\lambda}$ is a factor of $q(H)$ for each $\lambda \in T$, therefore $q(H) Y^n (H - \lambda - n)^{-n_\lambda} = q(H)(H - \lambda)^{-n_\lambda} Y^n \in A$. It remains to check that each component of $f(X^n)$ belong to A. Indeed, as we have already noticed, $(H - \lambda)^{n_\lambda}$ divides $c_n(H + n) = Y^n X^n$, hence $(H - \lambda - n)^{n_\lambda}$ divides $c_n(H) = X^n Y^n$. Thus $X^n \cdot Y^n (H - \lambda - n)^{-n_\lambda} = c_n(H)(H - \lambda - n)^{-n_\lambda} \in A$.

Now we consider the following polynomials: $q(H)$ and $Y^n (H - \lambda - n)^{-n_\lambda} X^n = Y^n X^n (H - \lambda)^{-n_\lambda}$, $\lambda \in T$. Because $q(H) = \prod_{\lambda \in T} (H - \lambda)^{n_\lambda} = Y^n X^n \mid_T$, therefore $H - \lambda$ does not divide $Y^n X^n (H - \lambda)^{-n_\lambda}$ for any $\lambda \in T$, and the above polynomials are coprime. Thus there are polynomials $p(H)$, $p_\lambda(H)$, $\lambda \in T$ such that $q(H)p(H) + \sum_{\lambda \in T} Y^n (H - \lambda - n)^{-n_\lambda} X^n p_\lambda(H) = 1$. Now $(p_0, \dots, p_n)^t = (q(H)p(H), \dots, X^n p_\lambda(H), \dots)^t$ is the column of $|T| + 1$ elements of $P_{\min}$ such that the right multiplication by this column defines a morphism $g : A^{|T|+1} \to P$ with $gf = 1_{P_{\min}}$, therefore $P_{\min}$ is projective.

It remains to show that $\mathrm{Tr}(P_{\min}) = I_{\min}$. By what we have said at the beginning of the section, the trace of $P_{\min}$ is generated by the images of $q(H)$ and X^n when multiplying them by the q_i on the right. Since $I_{\min}$ is a minimal nonzero ideal, it suffices to check that $q(H), q(H) Y^n (H - \lambda - n)^{-n_\lambda} \in I_{\min}$ and $X^n, X^n Y^n (H - \lambda - n)^{-n_\lambda} \in I_{\min}$. But (see Lemma 4.5.4) $X^n, Y^n \in I_{\min}$, therefore $q(H) Y^n (H - \lambda - n)^{-n_\lambda} = q(H)(H - \lambda)^{-n_\lambda} Y^n \in I_{\min}$, because $(H - \lambda)^{n_\lambda}$ divides $q(H)$. Further, from Lemma 4.5.5 and the definition of $q(H)$ it follows that $d_n(H)$ divides $q(H)$, therefore $q(H) \in I_{\min}$.

Now consider $X^n Y^n (H - \lambda - n)^{-n_\lambda} = c_n(H)(H - \lambda - n)^{-n_\lambda}$. As we have already seen, $(H - \lambda - n)^{n_\lambda}$ divides $c_n(H)$, hence it can be canceled. Recall (see Lemma 4.5.4) that $d_n(H)$ also divides $c_n(H)$ and is a product of polynomials $H - \mu$, $\mu \in T$. If $\lambda \in T$, then $\lambda - n \notin T$, hence $d_n(H)$ still divides $c_n(H)(H - \lambda - n)^{-n_\lambda}$. By Lemma 4.5.4, the latter polynomial belongs to $I_{\min}$, as desired. ∎

For example if $a(H) = H(H-1)(H-2)(H-3)$, then $P_{\min} = A(H-1)^3(H-2)^2(H-3) + AX^3$.

In the next lemma we will extend our supply of projective modules, hence of idempotent ideals. For $\lambda \in T$ we define $q_\lambda(H) = q(H)/(H-\lambda)^{n_\lambda} = \prod_{\mu\in T,\, \mu\neq\lambda}(H-\mu)^{n_\mu}$ and set $P_\lambda = Aq_\lambda(H) + AX^n$.

Lemma 4.7.2. *If $\lambda \in T$, then P_λ is a projective homogeneous left ideal of A whose trace is generated by $q_\lambda(H)$.*

Proof. As in Lemma 4.7.1, let $f : A \to {}_AQ^{|T|}$ be given by right multiplication by the row $(1, \dots, Y^n(H-\mu-n)^{-n_\mu}, \dots)$, where each $\mu \in T$, $\mu \neq \lambda$ gives one entry. We claim that the restriction of f to P_λ gives a morphism from P_λ to ${}_AA$. It suffices to check that $q_\lambda(H) \cdot Y^n(H-\mu-n)^{-n_\mu} \in A$ and $X^n \cdot Y^n(H-\mu-n)^{-n_\mu} \in A$. Indeed $q_\lambda(H)Y^n(H-\mu-n)^{-n_\mu} = q_\lambda(H)(H-\mu)^{-n_\mu}Y^n \in A$, because $\mu \neq \lambda$ yields that $(H-\mu)^{n_\mu}$ divides $q_\lambda(H)$. Since $(H-\mu)^{n_\mu}$ divides $c_n(H+n)$, it follows that $(H-\mu-n)^{n_\mu}$ divides $c_n(H)$, therefore $X^nY^n(H-\mu-n)^{-n_\mu} = c_n(H)(H-\mu-n)^{-n_\mu} \in A$.

Now we consider the following $|T|$ polynomials: $q_\lambda(H)$ and $Y^n(H-\mu-n)^{-n_\mu}X^n$, where each $\mu \in T, \mu \neq \lambda$ gives one polynomial. Since $Y^n(H-\mu-n)^{-n_\mu}X^n = c_n(H+n)(H-\mu)^{-n_\mu}$, from the definition of $q_\lambda(H)$ it follows that these polynomials are coprime. (Indeed, $H-\mu$ is not a root of $c_n(H+n)(H-\mu)^{-n_\mu}$ for every $\lambda \neq \mu \in T$). Thus $q_\lambda(H)p_\lambda(H) + \sum_{\mu\in T,\, \mu\neq\lambda} Y^n(H-\mu-n)^{-n_\mu}X^np_\mu(H) = 1$ for some polynomials $p_\tau(H)$, $\tau \in T$. Now right multiplication by the column $(q_\lambda(H)p_\lambda(H), \dots, X^np_\mu(H), \dots)^t$ of elements of P_λ defines a morphism $g : A^{|T|} \to P$ such that $gf = 1_{P_\lambda}$, therefore P_λ is projective.

It remains to calculate the trace of P_λ. By the remark at the beginning of the section, $\mathrm{Tr}(P_\lambda)$ is generated by the images of $q_\lambda(H)$ and X^n when multiplying them by 1 or $Y^n(H-\mu-n)^{-n_\mu}$, $\lambda \neq \mu \in T$. Thus $q_\lambda(H) = q_\lambda(H) \cdot 1 \in \mathrm{Tr}(P_\lambda)$, and clearly $X^n, Y^n \in \mathrm{Tr}(P_\lambda)$ (because X^n, Y^n belong to every nonzero ideal — see Lemma 4.5.4). Furthermore, $q_\lambda(H)Y^n(H-\mu-n)^{-n_\mu} = q_\lambda(H)(H-\mu)^{-n_\mu}Y^n$ is a multiple of Y^n, hence belongs to $\langle q_\lambda(H)\rangle$. Thus it remains to look at $X^nY^n(H-\mu-n)^{-n_\mu} = c_n(H)(H-\mu-n)^{-n_\mu}$. But in the proof of Lemma 4.7.1 we showed that this polynomial is in $I_{\min} \subseteq \langle q_\lambda(H)\rangle$. ∎

For instance, if $a(H) = H(H-1)(H-2)(H-3)$ and $\lambda = 1 \in T = \{1,2,3\}$, then $q_1(H) = (H-2)^2(H-3)$, hence $P_1 = A(H-2)^2(H-3) + AX^3$ is a projective module whose trace is generated by $(H-2)^2(H-3)$.

Now we are in a position to describe the atoms of $B(A)$.

Lemma 4.7.3. *If $\tau \in T$, then $\langle q_\tau\rangle = Aq_\tau A$ is an atom in $B(A)$, and every atom of $B(A)$ is of this form.*

Proof. By Lemma 4.7.2, $\langle q_\tau\rangle$ is the trace of the projective module P_τ, hence idempotent. Since $\tau \in T$, it follows that $\tau \in T_{\lambda,\mu}$ for the only pair $\lambda < \mu$ of adjacent roots of $a(H)$. From Fact 4.6.2 it follows that $q_\tau \notin I_{\lambda,\mu}$ and $q_\tau \in I_{\rho,\pi}$ for all remaining pairs of adjacent roots $\rho < \pi$ of $a(H)$. Since $\langle q_\tau\rangle$ is idempotent, it equals the intersection in $B(A)$ of maximal idempotent ideals $I_{\rho,\pi}^{m(\rho,\pi)}$ (see Lemma 4.6.5). It follows easily that $\langle q_\tau\rangle$ is an atom in $B(A)$, and every atom of $B(A)$ is of this form. ∎

Thus we have obtained a somehow better (see a remark after Lemma 4.6.5) description of the idempotent ideals of GWAs. Since every nonzero idempotent ideal of A either equals $I_{\min}$ or is a (finite) sum of atoms, it follows from Lemmas 4.7.1 and 4.7.2 that every idempotent ideal of A is the trace of a finitely generated projective module, hence 2 a) of Theorem 4.4.4 holds true. Instead of verifying 2 b) of this theorem, we will proceed directly to the classification of projective modules. But first we need the following lemma.

Lemma 4.7.4. *If $\tau \in T_{\lambda,\mu}$, then P_τ/JP_τ is a simple module isomorphic to $S_{\lambda,\mu}$.*

PROOF. First we will show that P_τ/JP_τ is a cyclic module generated by $\bar{q}_\tau = q_\tau + JP_\tau$. For this it suffices to prove that X^n, the second generator of P_τ, belongs to JP_τ. Indeed, from $X^n \in J$ we obtain $X^n q_\tau(H) = q_\tau(H-n)X^n \in JP_\tau$. Further, if $f(H) = \prod_{\eta\in T}(H-\eta)$ is a generator of the zeroth component of J (see Corollary 4.6.3), then $f(H)X^n \in JP_\tau$. Since all the roots of q_τ are in T (and n is the maximum of differences of comparable roots), it follows that $q_\tau(H-n)$ and $f(H)$ are coprime, hence $X^n \in JP_\tau$.

From the description of maximal ideals of A (see after Fact 4.6.2) we conclude that $q_\tau \notin I_{\lambda,\mu}$ and $q_\tau \in I_{\rho,\pi}$ for all remaining maximal ideals of A. It follows easily that $I_{\lambda,\mu}\bar{q}_\tau = \bar{0}$. Since $I_{\lambda,\mu}$ is the annihilator of $S_{\lambda,\mu}$, this implies that P_τ/JP_τ is a direct sum of copies of $S_{\lambda,\mu}$.

Recall (see before Lemma 4.6.1) that the τ-eigenspace of $S_{\lambda,\mu}$ (when acting by H) is 1-dimensional. Thus to prove that P_τ/JP_τ is simple it suffices to show that its τ-eigenspace is also 1-dimensional. Moreover, since $\bar{q}_\tau$ is a generator for this module, it is enough to check that $(H-\tau)\bar{q}_\tau = \bar{0}$, that is, $(H-\tau)q_\tau \in JP_\tau$. If $f(H) = \prod_{\eta\in T}(H-\eta)$, then (as above) $f(H) \in J$, hence $f(H)q_\tau \in JP_\tau$. Furthermore, $Y^n \in J$ implies $Y^nX^n = c_n(H+n) \in JP_\tau$, therefore $g(H) = \gcd(f(H)q_\tau(H), c_n(H+n)) \in JP_\tau$. Since every root of $f(H)q_\tau(H)$ belongs to T and $Y^nX^n\mid_T = q(H) = (H-\tau)^{n_\tau}q_\tau$, it follows that $g(H) = \gcd(f(H)g_\tau, q) = (H-\tau)q_\tau \in JP_\tau$, as desired. ∎

Note that we have some excess of projective modules 'covering' the same simple module: if $\tau, \eta \in T_{\lambda,\mu}$, then both P_τ/JP_τ and P_η/JP_η are isomorphic to $S_{\lambda,\mu}$. To get uniqueness one can choose one representative τ in each set $T_{\lambda,\mu}$; and the most natural choice would be to take $\tau = \mu$, the utmost right end of $T_{\lambda,\mu}$, which is a root of $a(H)$. Thus simple finite dimensional A-modules, hence the corresponding projective ideals, are parameterized by $T \cap S$.

Let $\lambda_1, \ldots, \lambda_m$ be a complete list of elements of $T\cap S$ (that is, of elements of T which are roots of $a(H)$), where we may assume that $i < j$ implies $\lambda_i <_B \lambda_j$, if λ_i and λ_j are on the same orbit B. Let $S_1, \ldots, S_m$ be the corresponding (complete) list of finite dimensional A-modules. Thus, if $\lambda_i < \lambda_{i+1}$ are adjacent roots of $a(H)$, then $S_{i+1} = S_{\lambda_i,\lambda_{i+1}}$ (in notation before Lemma 4.6.1). For example, if $a(H) = H(H-2)(H-4)$, then $T \cap S = \{2, 4\}$, therefore we set $\lambda_1 = 2 < \lambda_2 = 4$ and $S_1 = S_{0,2}$, $S_2 = S_{2,4}$. By what we have just noticed, then $P_{\lambda_1}, \ldots, P_{\lambda_m}$ are projective homogeneous left ideals of A such that $P_{\lambda_i}/JP_{\lambda_i} \cong S_i$.

Now we are in a position to prove the main result of the paper.

Theorem 4.7.5. *Every infinitely generated projective module Q over a generalized Weyl algebra A is a direct sum of copies of homogeneous left ideals $P_{\min}$ and $P_{\lambda_1}, \ldots, P_{\lambda_m}$.*

PROOF. By Kaplansky's theorem we may assume that Q is countably (infinitely) generated. Let $I = I(Q)$ be a two-sided ideal of A corresponding to Q in Proposition 4.4.2; in particular, I is idempotent and $P = Q/IQ$ is a finitely generated projective A/I-module. Since Q is infinitely generated, therefore $I \neq 0$.

Suppose first that $I = I_{\min}$. Since $I_{\min} \subseteq J$ and J is nilpotent modulo $I_{\min}$, therefore the canonical projection $P/I_{\min}P \to P/JP$ is a projective cover of P/JP as an $A/I_{\min}$-module. Furthermore, because A/J is a semisimple artinian ring, we conclude that P/JP

is a direct sum of simple finite dimensional A-modules, $P/JP \cong S_1^{k_1} \oplus \cdots \oplus S_m^{k_m}$. Then $P' = P_{\lambda_1}^{k_1} \oplus \cdots \oplus P_{\lambda_m}^{k_m}$ is a projective left A-module with $P'/JP' \cong P/JP$. Thus $P/I_{\min}P$ and $P'/I_{\min}P'$ are projective covers of P/JP as an $A/I_{\min}$-module, therefore these modules are isomorphic.

Now it is easy to calculate that the pair corresponding to the projective module $P_{\min}^{(\omega)} \oplus P'$ is $(I_{\min}, P'/I_{\min}P')$, therefore Q is isomorphic to this module by Proposition 4.4.2.

Now assume that $I \supset I_{\min}$ is an idempotent ideal of A. If I_i denotes $\langle q_{\lambda_i} \rangle$, then, by Lemma 4.7.3, $I_1, \ldots, I_m$ is a complete list of atoms of $B(A)$, therefore I admits a (unique) representation as a sum of atoms, $I = \sum_{j \in \Lambda} I_j$, where Λ is a subset of $\{1, \ldots, m\}$ (for instance, if $I = A$, then $\Lambda = \{1, \ldots, m\}$); and let $\Lambda' = \{1, \ldots, m\} \setminus \Lambda$ be the complement of Λ.

Since A/J is semisimple, we conclude that P/JP is a direct sum of copies of simple modules $S_1, \ldots, S_m$. Furthermore, because $I(P) = I$, it follows easily that $Q/JQ \cong \bigoplus_{j \in \Lambda} S_j^{(\omega)} \oplus \bigoplus_{l \in \Lambda'} S_l^{k_l}$, $k_l < \omega$, therefore $Q/IQ \cong \bigoplus_{l \in \Lambda'} S_l^{k_l}$. Let us consider the following projective A-module $Q' = \bigoplus_{j \in \Lambda} P_{\lambda_j}^{(\omega)} \oplus \bigoplus_{l \in \Lambda'} P_{\lambda_l}^{k_l}$. Clearly $I(Q) = \sum_{j \in \Lambda} \mathrm{Tr}(P_{\lambda_j}) = \sum_{j \in \Lambda} I_j = I$ and $Q'/JQ' \cong Q/JQ$. Using projective covers (as in the first part of the proof) we conclude that $Q'/I_{\min}Q' \cong Q/I_{\min}Q$. Since $I_{\min} \subseteq I$, it follows that $Q'/IQ' \cong Q/IQ$, therefore $Q' \cong Q$ by Proposition 4.4.2. ∎

Note that Hodges [9, Lemma 2.4] constructed a family of finitely generated projective modules over a GWA A as follows. Suppose that $a(H) = b(H)c(H)$, where the polynomials $b(H)$ and $c(H)$ are coprime. Then $P_b = Ab(H) + AX$ is a projective homogeneous left ideal of A. It is not difficult to check that $\mathrm{Tr}(P_b)$ is generated by $X, Y, b(H)$ and $c(H-1)$. For instance, if $a(H) = H(H-1)(H-2)$ and $b(H) = H - 1$, then $\mathrm{Tr}(P_b) = \langle H - 1 \rangle$, therefore $\mathrm{Tr}(P_b)$ is a maximal (idempotent) ideal of A. However, $\mathrm{Tr}(P_b)$ is always situated close to the top of $B(A)$, for instance, in most cases one cannot obtain $I_{\min}$ as $\mathrm{Tr}(P_b)$. Thus our approach to idempotent ideals 'from below' seems to have a crucial advantage.

If we take a GWA with $a(H) = H(H-2)$, set $b(H) = H - 2$ and apply Hodges' construction, then (see [9, Theorem 2.3]) $P = A(H-2) + AX$ is a projective generator whose endomorphism ring is isomorphic to the GWA with $a(H) = H(H-1)$, therefore these algebras are Morita equivalent. This is an example of a translation functor we mentioned before Fact 4.5.1.

As one can see from the proof of Theorem 4.7.5, some direct summands of the projective module Q are clearly redundant. For instance, executing this proof for $Q = A^{(\omega)}$, we will end up with representation $Q \cong \oplus_{i=1}^m P_{\lambda_i}^{(\omega)}$. In the next proposition we will get rid of these repetitions, therefore obtain a canonical form for each infinitely generated projective module over a GWA. This also allows us to include uncountably generated projectives.

Proposition 4.7.6. *Let Q be an infinitely generated projective module over a GWA A. Then exactly one of the following holds true.*

1) Q is free;

2) $Q \cong A^{(\alpha)} \oplus \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus P_{\min}^{(\beta)}$, where $\omega \leq \alpha < \alpha_i < \beta$ and Λ is a proper (maybe empty) subset of $\{1, \ldots, m\}$;

3) $Q \cong A^{(\alpha)} \oplus \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)}$, where $\omega \leq \alpha < \alpha_i$ and Λ is a proper nonempty subset of $\{1, \ldots, m\}$;

4) $Q \cong \bigoplus_{i \in \Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus \bigoplus_{j \in M} P_{\lambda_j}^{k_j} \oplus P_{\min}^{(\beta)}$, where $k_j < \omega$, $\omega \leq \alpha_i < \beta$, and Λ, M are disjoint subsets of $\{1, \ldots, m\}$ and Λ is proper and nonempty;

5) $Q \cong \bigoplus_{i\in\Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus \bigoplus_{j\in M} P_{\lambda_j}^{k_j}$, *where* $k_j < \omega$, $\omega \leq \alpha_i$, *and* Λ, M *are disjoint subsets of* $\{1,\dots,m\}$ *and* Λ *is proper and nonempty;*

6) $Q \cong \bigoplus_{j\in M} P_{\lambda_j}^{k_j} \oplus P_{\min}^{(\beta)}$, *where* $k_j < \omega$, $\beta \geq \omega$, *and* M *is a subset of* $\{1,\dots,m\}$.

Furthermore all the exponents $\alpha, \beta, \dots$ *in the above representations are uniquely determined by* Q.

Proof. By Theorem 4.7.5, every infinitely generated projective A-module Q is isomorphic to a direct sum of copies of A, $P_{\lambda_1},\dots,P_{\lambda_m}$ and $P_{\min}$ (clearly there is no harm in adding A!). Separating finite and infinite exponents of the P_{λ_i}, we obtain that

$$Q \cong A^{(\alpha)} \oplus \bigoplus_{i\in\Lambda} P_{\lambda_i}^{(\alpha_i)} \oplus \bigoplus_{j\in M} P_{\lambda_j}^{k_j} \oplus P_{\min}^{(\beta)},$$

where each $\alpha_i \geq \omega$, $k_j < \omega$, and Λ, M are disjoint subsets of $\{1,\dots,m\}$; and choose a representation of Q with a maximal possible α.

Suppose first that $\alpha \geq \omega$. Because $A = \mathrm{Tr}(A) \supset \mathrm{Tr}(P_{\lambda_i}) = I_i \supset \mathrm{Tr}(P_{\min}) = I_{\min}$, therefore, by Lemma 4.3.3, we can absorb projectives $P_{\mu_j}^{k_j}$ into $A^{(\alpha)}$, therefore assume that $M = \emptyset$. Similarly, if $\alpha \geq \alpha_i$ for some $i \in \Lambda$ then $A^{(\alpha)} \oplus P_{\lambda_i}^{(\alpha_i)} \cong A^{(\alpha)}$ (so we can drop $P_{\lambda_i}^{(\alpha_i)}$); and $A^{(\alpha)} \oplus P_{\min}^{(\beta)} \cong A^{(\alpha)}$ if $\alpha \geq \beta$. Furthermore, again by Lemma 4.3.3, $P_{\lambda_i}^{(\alpha_i)} \oplus P_{\min}^{(\beta)} \cong P_{\lambda_i}^{(\alpha_i)}$ if $\alpha_i \geq \beta$.

Thus either Q is free or we may assume either that $\alpha < \alpha_i < \beta$ for each $i \in \Lambda$ (or just $\alpha < \beta$ if $\Lambda = \emptyset$) or $\beta = 0$, $\Lambda \neq \emptyset$ and $\alpha < \alpha_i$ for each $i \in \Lambda$.

Suppose that $\Lambda = \{1,\dots,m\}$ and $\alpha_j = \min_{i\in\Lambda} \alpha_i$. Since $\mathrm{Tr}(P_{\lambda_1} \oplus \dots \oplus P_{\lambda_m}) = A$ it follows that $\oplus_{i\in\Lambda} P_{\lambda_i}^{(\alpha_i)}$ splits off $A^{(\alpha_j)}$ as a direct summand, which can be transferred to $A^{(\alpha)}$. Since $\alpha + \alpha_j = \alpha_j > \alpha$, this contradicts our choice of α. As a result Λ is a proper subset of $\{1,\dots,m\}$, thus we have obtained 2) and 3) of the proposition.

It remains to consider the case when $\alpha = s$ if finite. If $\Lambda \neq \emptyset$ and $j \in \Lambda$ then using Proposition 4.4.2 it is easily seen that $A^s \oplus P_{\lambda_j}^{(\alpha_j)}$ is isomorphic to $P_{\lambda_j}^{(\alpha_j)} \oplus \bigoplus_{i=1}^m P_{\lambda_i}^s$, therefore Q is isomorphic to a module of the form 4) or 5).

Similarly if $\Lambda = \emptyset$ and Q is not finitely generated, we obtain 6).

Arguing as in Proposition 4.4.3 it is easily seen that exponents $\alpha, \beta, \dots$ are uniquely determined by Q. For instance, in 4), α_i is equal to the uniform dimension of Q/KQ, where K is the annihilator of the simple module $S_i = P_{\lambda_i}/JP_{\lambda_i}$. ∎

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5. CLASSYFIYNG GENERALIZED LATTICES. SOME EXAMPLES AS AN INTRODUCTION.

5.1 Introduction

The title of this paper is just a rephrase of Ringel's [17] 'Infinite length modules. Some examples as introduction', which manifest a recent trend in representation theory of finite dimensional algebras to switch an attention from studying just finite dimensional representations to investigating carefully chosen classes of infinite dimensional modules.

This ideology has been recently adopted by people in integral representation theory. If Λ is an order over a Dedekind domain D, then Butler at al. [4] call a D-projective Λ-module a generalized lattice. A remarkable result of [4] is a complete classification of generalized lattices over the group ring $\mathbb{Z}C_p$ of a cyclic group C_p of prime order, therefore a classification of representations of the operator X, $X^p = 1$, by column-finite $\mathbb{Z}$-matrices. An essential step in this classification is to prove that every generalized $\mathbb{Z}C_p$-lattice is a direct sum of (finitely generated) lattices whose classification is well known (this result was later extended by Rump [18] to generalized lattices over $\mathbb{Z}C_{p^2}$).

For orders of (locally) finite lattice type Rump [18] gave a useful combinatorial criterion when every generalized lattice is a direct sum of (finitely generated) lattices. For instance, using this criterion it has been checked that over a Bass' order $\Lambda(6) = \{(m, n) \in \mathbb{Z}^2 \mid 6$ divides $m-n\}$ there exists a generalized lattice that is not a direct sum of finitely generated lattices.

The main objective of this paper is to suggest a strategy of classifying generalized lattices over orders of finite lattice type even in the case when not every generalized lattice is a direct sum of finitely generated ones. The first step in this approach is a standard one: using Auslander's lattice one can reduce the original problem to the classification of (finitely and infinitely generated) projective modules over the Auslander order A of Λ. This order is a module-finite Noetherian algebra — a quick search through P.I.-theory results shows that one technical condition $(*)$ holds true, therefore the theory [14] of fair-sized projectives gives a classification of infinitely generated projective modules over A in terms of idempotent ideals and finitely generated projectives over corresponding factor-rings.

Despite a conceptual clearness the difficulty in completing this classification of projectives and converting it into a classification of generalized lattices over the original order could be enormous. In this paper we will demonstrate on carefully chosen examples (mostly borrowed from [18]) how to carry through this classification program. Thus the emphasis of this text is rather on examples than on general theory.

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Namely, in Section 5.4 we will give a complete classification of generalized lattices over the quadratic orders $\mathbb{Z}[\sqrt{n}]$, n is square-free. The most interesting case is $n \equiv 1 \pmod 8$ in which there exist generalized lattices that are not direct sums of finitely generated lattices, however the classification can be completed.

Even more this is true for one particular example of $\mathbb{Z}$-order Λ considered by Rump [18, Example 2]. In this case we will show in Section 5.6 (answering a question by Rump) that there exists a superdecomposable (i.e. without indecomposable direct summands) projective Λ-module, but also classify all (infinitely generated) projective Λ-modules.

To complete this classification we develop in Section 5.2 (following [14] and [16]) a piece of general theory concerning infinitely generated projective modules over noetherian rings satisfying $(*)$. Namely we will give a criterion when there exists a superdecomposable projective module over such a ring, therefore when there exists a superdecomposable generalized lattice over an order of finite lattice type.

In Section 5.7 we will analyze generalized lattices over a Bass order $\Lambda(6)$ and show how to classify them modulo some straightforward (but tedious) calculations. For instance we will prove that every generalized Λ_6-lattice contains a finitely generated direct summand, but (as also follows from [18]) there exists a generalized lattice that is not a direct sum of finitely generated lattices. Furthermore, this generalized lattice is not isomorphic to a direct sum of indecomposable modules.

5.2 Projective modules over noetherian rings with $(*)$

In this section we recall (from [14], which has been circulated for quite a while, but is yet to be published) the theory of fair-sized projective modules over noetherian rings with $(*)$. Because there is a good explanation of similar things in the follow-up paper [16], we will be quite concise.

Recall that a module P over a ring R is said to be *projective*, if P is isomorphic to a direct summand of a free module $R^{(J)}$. An important invariant of a projective module P is its trace ideal, $\mathrm{Tr}(P)$, which is the sum of images of all morphisms form P to R_R. For instance, if $e \in R$ is an idempotent, then eR is a projective (right) R-module with $\mathrm{Tr}(P) = ReR$ (the two-sided ideal generated by e). It is known that $\mathrm{Tr}(P)$ is an idempotent ideal (that is, $\mathrm{Tr}(P)^2 = \mathrm{Tr}(P)$) such that $P = P\,\mathrm{Tr}(P)$, and $\mathrm{Tr}(P)$ is the least ideal with this property. In what follows we will mostly consider countably generated projective modules over (left and right) noetherian rings. A justification for such a restriction is given by Kaplansky's theorem (see [7, Corollary 2.48]): every projective module (over an arbitrary ring) is a direct sum of countably generated modules.

We say that a noetherian ring R satisfies the condition $(*)$ if the following holds true.

$(*)$ Every (descending) chain $I_1, I_2, \ldots$ of ideals of R, with $I_{k+1}I_k = I_{k+1}$ for any k, stabilizes.

For instance, if R has a (strictly) descending chain of idempotent ideals (which is true for the universal enveloping algebra $Usl_2(k)$ over an algebraically closed field k) then R fails to satisfy $(*)$. Furthermore, as the following lemma shows, these properties are equivalent for rings of Krull dimension 1 (for a definition of Krull dimension of a noetherian ring see [13, Chapter 6]).

Lemma 5.2.1. *Suppose that R is a noetherian ring of (right) Krull dimension 1. Then the following are equivalent.*
1) R satisfies $(*)$;
2) R has a d.c.c. on idempotent ideals.

PROOF. 1) $\Rightarrow$ 2) is obvious.

2) $\Rightarrow$ 1). Otherwise let $I_1 \supset I_2 \supset \dots$ be a (strictly descending) chain of ideals of R such that $I_{k+1}I_k = I_{k+1}$ for every k. We will transform this chain into a descending chain of idempotent ideals of R getting a contradiction to 2).

Since R has Krull dimension 1, almost all (right) modules I_k/I_{k+1} are of finite length. Because the property $I_{k+1}I_k = I_{k+1}$ passes to subchains, we may assume that all the modules I_k/I_{k+1} are of finite length. Observe that $I_{k+1}I_k = I_{k+1}$ implies $I_{k+1}I_k^n = I_{k+1}$ for every n, in particular $I_{k+1} \subseteq I_k^n$, hence the descending chain $I_k \supseteq I_k^2 \supseteq \dots$ stabilizes, say $I_k^n = I_k^{n+1}$. Then $I'_k = I_k^n$ is an idempotent ideal and clearly $I'_1 \supset I'_3 \supset \dots$ is a strictly descending chain of idempotent ideals of R, a contradiction. ■

Note that the proof of Lemma 5.2.1 works for rings whose lattice of two-sided ideals has Krull dimension one.

The next proposition is a classification of projective modules over noetherian rings with $(*)$.

Proposition 5.2.2. *(see [14]) Suppose that R is a noetherian ring satisfying $(*)$. Then there is a natural one-to-one correspondence between (at most) countably generated projective R-modules and pairs (I, P), where I is an idempotent ideal of R (that is, $I^2 = I$) and P is a finitely generated projective R/I-module.*

One direction in this correspondence is easy to describe. If Q is a countably generated projective R-module, then $I = I(Q)$ is the least ideal of R such that Q/QI is finitely generated (I exists by the proof of the proposition) and $P = P(Q) = Q/QI$.

The opposite direction in the above correspondence, though constructive, often works as rather an existence theorem. For instance, it is usually quite difficult to decide whether the (countably generated) projective module corresponding to a given pair (I, P) is a sum of finitely generated modules or not.

Note that finitely generated projective modules correspond in the above classification to pairs $(0, P)$, therefore Proposition 5.2.2 says nothing new about finitely generated projectives. Furthermore the free countable rank module $R^{(\omega)}$ corresponds to the pair $(R, 0)$. If a projective module Q corresponds to the pair (I, P) then clearly $I \subseteq \mathrm{Tr}(Q)$ and this inclusion is proper if $P \neq 0$. Furthermore $Q^{(\omega)}$ corresponds to the pair $(\mathrm{Tr}(Q), 0)$.

If I is an idempotent ideal of R, then a (projective) module Q is said to be I-*big*, if Q contains as a direct summand any countably generated projective module S such that $\mathrm{Tr}(S) \subseteq I$. By [14] over a noetherian ring with $(*)$ every projective module Q is $I(Q)$-big. For instance, if Q is a (countably generated) projective module corresponding to the pair $(I, 0)$ then Q is I-big, $\mathrm{Tr}(Q) = I$ and $Q \oplus P \cong Q$ for every countably generated projective module P with $\mathrm{Tr}(P) \subseteq I$, in particular $Q \cong Q^{(k)}$ for every $1 \leq k \leq \omega$.

For some classes of noetherian rings every projective module is a direct sum of finitely generated modules. For instance, by Kaplansky's result (see [10, Theorem 2.24]) this is the case for hereditary rings; and the same is true for semiperfect rings (see [1, Theorem 27.11]) and for generalized Weyl algebras (see [16, Theorem 7.5]). However (see some examples below) it is not always the case.

The following two results will measure the complexity of direct sum decomposition theory of projective modules over noetherian rings with $(*)$.

Fact 5.2.3. *(see [16, Theorem 4.7]) Let R be a noetherian ring with $(*)$. Then the following are equivalent.*

1) Every projective module is a direct sum of finitely generated modules;

2) a) every idempotent ideal of R is the trace of a finitely generated projective module, and

b) if I is an idempotent ideal of R and P is a finitely generated projective R/I-module, then there exists a finitely generated projective R-module Q such that $Q/QI \cong P$.

Thus 2 b) says that one can lift finitely generated projective modules modulo idempotent ideals of R.

An easy consequence of [14] that will be important for us is the following.

Remark 5.2.4. *An indecomposable projective module Q over a noetherian ring with $(*)$ is finitely generated.*

Below we will give some examples of countably generated projective modules Q such that every (nonzero) direct summand of Q contains a (nonzero) finitely generated direct summand, but Q is not a direct sum of finitely generated modules. As the following lemma shows the failure of this property gives an 'ultimate' level of complexity of direct sum decompositions. Recall that a module is said to be *superdecomposable* if it contains no (nonzero) indecomposable direct summands.

Lemma 5.2.5. *Let R be a noetherian ring with $(*)$. Then the following are equivalent.*

1) Every projective module contains a finitely generated direct summand;

2) every projective module has an indecomposable direct summand, that is, there is no superdecomposable projective module;

3) every nonzero idempotent ideal of R contains the trace of a nonzero finitely generated projective module;

4) every minimal nonzero idempotent ideal of R is the trace of a finitely generated projective module.

Proof. 1) $\Rightarrow$ 2). Because R is noetherian, every finitely generated module contains an indecomposable direct summand.

2) $\Rightarrow$ 3). Let $I \neq 0$ be an idempotent ideal of R and let Q be a countably generated projective module corresponding to the pair $(I, 0)$, in particular $\mathrm{Tr}(Q) = I$. If P is a (nonzero) indecomposable direct summand of Q then, by Remark 5.2.4, P is finitely generated. Thus $0 \neq \mathrm{Tr}(P) \subseteq \mathrm{Tr}(Q)$, as desired.

3) $\Rightarrow$ 4). Let I be a minimal nonzero idempotent ideal of R and let P be a (nonzero) finitely generated projective module such that $\mathrm{Tr}(P) \subseteq I$. Since $0 \neq \mathrm{Tr}(P)$ is an idempotent ideal, we conclude that $\mathrm{Tr}(P) = I$.

4) $\Rightarrow$ 1). Let Q be a nonzero (countably generated) projective module. It follows from [14] that $I(Q)$ is an idempotent ideal and Q is $I(Q)$-big. If $I(Q) = 0$ then Q is finitely generated, as desired. Otherwise $I(Q) \neq 0$. Because R satisfies $(*)$, there exists a minimal nonzero idempotent ideal $I \subseteq I(Q)$. By the assumption there exists a finitely generated projective module P with $\mathrm{Tr}(P) = I \subseteq I(Q)$. Since Q is $I(Q)$-big, we conclude that Q contains P as a direct summand. ■

5.3 From lattices to projective modules

In this section we recall how, using standard equivalences of categories, to convert the classification of lattices over orders of finite lattice type into a classification of (finitely and infinitely generated) projective modules over noetherian rings.

If M is a module over a ring R, then $\mathrm{add}(M)$ will denote the category whose objects are direct summands of direct sums of finitely many copies of M. If we drop the finiteness requirement in this definition we obtain the category $\mathrm{Add}(M)$. Thus $N \in \mathrm{Add}(M)$ iff N is a direct summand of $M^{(J)}$ for some set J. The following is a well known trick converting $\mathrm{Add}(M)$ into a category of projective modules.

Fact 5.3.1. *([7, Theorem 4.7]) If M is a finitely generated module, then $\mathrm{Add}(M)$ is equivalent to the category of projective $S = \mathrm{End}(M)$-modules via the following pair of functors: $N_R \mapsto \mathrm{Hom}_R(M, N)$ and $P_S \mapsto P \otimes_S M$.*

Here is how $\mathrm{Add}(M)$ appears in our setting. Let Λ be an order over a Dedekind domain D in a separable finite dimensional K-algebra, where K is a field of quotients of D. Recall that a finitely generated D-torsion-free (=D-projective) Λ-module is called a *lattice*. For an explanation and basic properties of lattices the reader is referred to [5]. Following [4] we say that a Λ-module M is a *generalized lattice* if M is projective as a D-module (that is, if $D = \mathbb{Z}$ or M_D is infinitely generated, then M is a free D-module).

An order Λ is said to be of *finite lattice type* if Λ has only finitely many (up to an isomorphism) indecomposable lattices. Suppose that $M_1, \dots, M_n$ is a complete list of indecomposable Λ-lattices. Then $M = M_1 \oplus \cdots \oplus M_n$ is usually called the *Auslander lattice* of Λ and $A = \mathrm{End}(M)$ is the *Auslander order* of Λ. By classical result of Auslander (see [4]) A has global dimension at most 2.

The following proposition will be crucial in what follows.

Fact 5.3.2. *([4, Theorem 2.1], see also [18, p. 112, Corollary] for a generalization) Suppose that Λ is an order of finite lattice type. A Λ-module N is a generalized lattice iff $N \in \mathrm{Add}(M)$, where M stands for Auslander's lattice of Λ.*

Thus, by Fact 5.3.1, to classify generalized lattices over an order of finite lattice type is the same as to classify (finitely and infinitely generated) projective modules over its Auslander order A. The following proposition shows that the theory of fair-sized projectives is applicable to A.

Proposition 5.3.3. *Suppose that M is the Auslander lattice of an order Λ of finite lattice type and let $A = \mathrm{End}(M)$ be the Auslander order of Λ. Then A is a noetherian ring of Krull dimension 1 satisfying $(*)$.*

Proof. Since M is a finitely generated Λ-module, its endomorphism ring, A, is a module finite D-algebra. Because D is noetherian of Krull dimension 1, the same is true for A.

By Lemma 5.2.1, to verify $(*)$ it suffices to show that A has a d.c.c. on idempotent ideals. In fact more can be said — since A is a noetherian P.I.-ring, [19, Theorem 3] yields that A has only finitely many idempotent ideals. ∎

Thus we can use the theory of fair-sized projectives (see Section 5.2) to investigate and classify generalized lattices over orders of finite lattice type.

5.4 Quadratic orders

In this section we will classify generalized lattices over quadratic $\mathbb{Z}$-orders $\mathbb{Z}[\sqrt{n}]$, where n is a square-free integer. We will use [9, Section 12] as a reference for properties of these rings.

If $n \equiv 2, 3 \pmod 4$ then $\Lambda = \mathbb{Z}[\sqrt{n}]$ is a Dedekind domain. In this case it is well known (or follows from what it will be said below) that every generalized Λ-lattice is a direct sum of (finitely generated) lattices whose structure is reasonably well understood (modulo some number theory).

Otherwise $n \equiv 1 \pmod 4$ and then the normalization of Λ (that is, its integral closure in the field of quotients $Q = \mathbb{Q}(\sqrt{n})$ is $\widetilde{\Lambda} = \mathbb{Z}[\omega]$, where $\omega = (1 + \sqrt{n})/2$. Furthermore the conductor of $\widetilde{\Lambda}$ in Λ is

$$C = (\widetilde{\Lambda} : \Lambda) = \{q \in Q \mid q\widetilde{\Lambda} \subseteq \Lambda\} = 2\,\widetilde{\Lambda} = 2\,\mathbb{Z}[\omega]\,,$$

$\widetilde{\Lambda}/\Lambda \cong \Lambda/C$ as Λ-modules and $\Lambda/C \cong GF(2) = \mathbb{Z}/2\mathbb{Z}$ is a field. Thus C is a maximal ideal in Λ. To understand a position of C as an ideal of $\widetilde{\Lambda}$ we have to consider two cases.

1) If $n \equiv 5 \pmod 8$, then C is a maximal ideal in $\widetilde{\Lambda}$, therefore $\widetilde{\Lambda}/C$ is the Galois field $GF(4)$.

2) If $n \equiv 1 \pmod 8$, then C is not a maximal ideal in $\widetilde{\Lambda}$ and $\widetilde{\Lambda}/C$ is a product of two fields $GF(2) \oplus GF(2)$.

We will proceed with cases 1) and 2). Since Λ is a Bass ring (see Section 5.7 for explanations), every Λ-lattice is a projective Λ'-module for some ring Λ' between Λ and $\widetilde{\Lambda}$. But $\widetilde{\Lambda}/\Lambda$ is a simple Λ-module, hence either $\Lambda' = \Lambda$ or $\Lambda' = \widetilde{\Lambda}$. Thus, if M is the Auslander lattice of Λ, then $\mathrm{Add}(M) = \mathrm{Add}(\Lambda \oplus \widetilde{\Lambda})$, therefore (see Section 5.3) to classify generalized Λ-lattices is the same as to classify projective modules over $S = \mathrm{End}(\Lambda \oplus \widetilde{\Lambda})$, a 'shrinking' of the Auslander order of Λ.

The following calculations are fairly general. Consider S as a matrix ring $S = \left(\begin{smallmatrix} (\Lambda,\Lambda) & (\widetilde{\Lambda},\Lambda) \\ (\Lambda,\widetilde{\Lambda}) & (\widetilde{\Lambda},\widetilde{\Lambda}) \end{smallmatrix}\right)$ acting on the column $\left(\begin{smallmatrix} \Lambda \\ \widetilde{\Lambda} \end{smallmatrix}\right)$ from the left, where we write (M, N) for $\mathrm{Hom}_\Lambda(M, N)$. Using obvious identifications $(\widetilde{\Lambda}, \Lambda) = C$ and $(\widetilde{\Lambda}, \widetilde{\Lambda}) = \widetilde{\Lambda}$ we conclude that

$$S = \begin{pmatrix} \Lambda & C \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{pmatrix}.$$

Recall (see Proposition 5.2.2) that countably generated projective S-modules are classified by pairs (I, P), where I is an idempotent ideal of S and P is a finitely generated projective S/I-module. Thus our intermediate goal is to classify idempotent ideals of S.

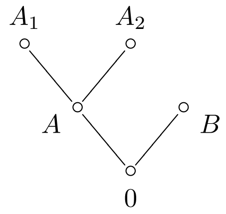
First of all there are two 'obvious' finitely generated projective S-modules. Namely, Λ goes to $e_1 S = \left(\begin{smallmatrix} \Lambda & C \\ 0 & 0 \end{smallmatrix}\right)$ via the correspondence in Fact 5.3.1, and $\widetilde{\Lambda}$ goes to $e_2 S = \left(\begin{smallmatrix} 0 & 0 \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{smallmatrix}\right)$. Their traces are idempotent ideals:

$$A = \mathrm{Tr}(e_1 S) = S e_1 S = \begin{pmatrix} \Lambda & C \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{pmatrix} \cdot \begin{pmatrix} \Lambda & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Lambda & C \\ \widetilde{\Lambda} & C \end{pmatrix}$$

and

$$B = \mathrm{Tr}(e_2 S) = S e_2 S = \begin{pmatrix} \Lambda & C \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{pmatrix} = \begin{pmatrix} C & C \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{pmatrix}.$$

Note that $S/B \cong \Lambda/C$ is a field, hence B is a maximal ideal of S. Since $S/A \cong \widetilde{\Lambda}/C$, therefore A is a maximal ideal of S iff $n \equiv 5 \pmod 8$; and otherwise A is contained in exactly two (maximal) ideals A_1, A_2, where both A_1 and A_2 are idempotent.

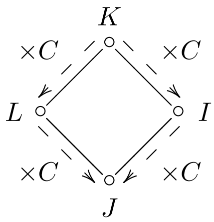


Now we are in a position to describe idempotent ideals of S.

Lemma 5.4.1. *1) If $n \equiv 5 \pmod 8$ then A and B are the only (nonzero proper) idempotent ideals of S;*

2) If $n \equiv 1 \pmod 8$ then A, A_1, A_2, B is a complete list of nonzero proper idempotent ideals of S.

Proof. Let $V = \begin{pmatrix} I & J \\ K & L \end{pmatrix}$ be an idempotent ideal of $S = \begin{pmatrix} \Lambda & C \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{pmatrix}$. Since V is an ideal, $I \subseteq \Lambda$ is an Λ-ideal; $J \subseteq C$ and $K, L \subseteq \widetilde{\Lambda}$ are $\widetilde{\Lambda}$-ideals that satisfy the following scheme of inclusions.



(for instance, $L \subseteq K$ and $KC \subseteq L$).

Since V is idempotent, the following conditions are satisfied.

$$\left(\begin{array}{c|c} I = I^2 + JK & J = IJ + JL \\ \hline K = KI + LK & L = KJ + L^2 \end{array} \right).$$

From $K = KI + KL$ and $I, L \subseteq K$ it follows that $K = K^2$. Since K is an ideal in a commutative noetherian domain $\widetilde{\Lambda}$ we conclude that $K = 0$ or $K = \widetilde{\Lambda}$. The former option leads to $V = 0$, so we may assume the latter. Thus

$$\left(\begin{array}{c|c} I = I^2 + J & J = J(I + L) \\ \hline \widetilde{\Lambda} = I\widetilde{\Lambda} + L & L = L^2 + J \end{array} \right).$$

From $K = \widetilde{\Lambda}$ and $KC \subseteq L, I$ it follows that $C \subseteq L, I$. Furthermore $I\widetilde{\Lambda} + L = \widetilde{\Lambda}$ implies $(I + L)C = (I\widetilde{\Lambda} + L)C = C$. Then $IC, LC \subseteq J$ yields $C \subseteq J$, therefore $J = C$.

Now we consider the conditions $I = I^2 + C$ and $L = L^2 + C$. Since $C \subseteq I \subseteq \Lambda$ are Λ-ideals we conclude that either $I = C$ or $I = \Lambda$. If $I = C$ then $I\widetilde{\Lambda} + L = \widetilde{\Lambda}$ and $C \subseteq L$ implies $L = \widetilde{\Lambda}$, hence $V = B$. Otherwise $I = \Lambda$. If $\widetilde{\Lambda}/C$ is a simple Λ-module, then either $L = C$ yielding $V = A$ or $L = \widetilde{\Lambda}$, which gives $V = S$. Similarly if $\widetilde{\Lambda}/C$ is not simple, there are two more possibilities for L leading to A_1 and A_2. ∎

Before proceeding with the classification let us first calculate some invariants of countably generated projective S-modules.

Remark 5.4.2. *1)* $Q = e_1 S^{(\omega)} \oplus e_2 S^k$, $k < \omega$ *corresponds to the pair* $(A, (S/A)^k)$*;*
2) $e_1 S^l \oplus e_2 S^{(\omega)}$, $l < \omega$ *corresponds to the pair* $(B, (S/B)^l)$.

PROOF. 1) Since $A = \mathrm{Tr}(e_1 S)$ it is clear that A is the least ideal of S such that Q/QA is finitely generated, therefore $I(Q) = A$. It remains to notice that

$$e_2 S \cdot A = \begin{pmatrix} 0 & 0 \\ \widetilde{\Lambda} & \widetilde{\Lambda} \end{pmatrix} \cdot \begin{pmatrix} \Lambda & C \\ \widetilde{\Lambda} & C \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \widetilde{\Lambda} & C \end{pmatrix},$$

therefore $e_2 S / e_2 S \cdot A \cong \widetilde{\Lambda}/C \cong S/A$ yielding $P(Q) = (e_2 S / e_2 S \cdot A)^k \cong (S/A)^k$.
The verification of 2) is similar. ∎

Now we are ready to classify generalized lattices in case 1).

Proposition 5.4.3. *If* $n \equiv 5 \pmod 8$*, then every infinitely countably generated generalized lattice over* $\Lambda = \mathbb{Z}[\sqrt{n}]$ *is isomorphic to one of the lattices* $\Lambda^{(k)} \oplus \widetilde{\Lambda}^{(l)}$, $0 \le k, l \le \omega$, $k + l = \omega$.

PROOF. By Lemma 5.4.1 the only idempotent ideals of $S = \mathrm{End}(\Lambda \oplus \widetilde{\Lambda})$ are $0, A, B$ and S. Since S/A and S/B are fields, the following is a complete list of pairs corresponding to countably generated projective S-modules in Proposition 5.2.2:
$(0, P)$, where P is a finitely generated projective S-module;
$(S, 0)$ corresponding to the free S-module $S^{(\omega)}$;
$(A, (S/A)^k)$, $k < \omega$ corresponding to $e_1 S^{(\omega)} \oplus e_2 S^k$ (see Remark 5.4.2);
$(B, (S/B)^l)$, $l < \omega$ corresponding to $e_1 S^l \oplus e_2 S^{(\omega)}$ (by the same remark).
It remains to turn back to the category $\mathrm{Add}(\Lambda \oplus \widetilde{\Lambda})$ (see Fact 5.3.1) keeping in mind that $e_1 S$ corresponds to Λ and $e_2 S$ corresponds to $\widetilde{\Lambda}$. ∎

Note that we have ignored in Proposition 5.4.3 (and elsewhere) the case when generalized lattices (or projective modules) are uncountably generated. In fact, similar to [16, Proposition 7.6] it is easy to produce canonical forms of such modules, using the fact (see [16, Lemma 3.3]) that a projective module with a larger trace 'absorbs' a module with a smaller trace.

Our next goal is to give a classification of generalized lattices in case when $n \equiv 1 \pmod 8$. But let us make first an easy remark.

Remark 5.4.4. *Let* $\Lambda = \mathbb{Z}[\sqrt{n}]$*,* n *is square-free. Then every generalized* Λ*-lattice contains a finitely generated direct summand.*

PROOF. By Proposition 5.4.3 the only remaining case is $n \equiv 1 \pmod 8$. It suffices to prove that every projective module over $S = \mathrm{End}(\Lambda \oplus \widetilde{\Lambda})$ contains a finitely generated direct summand. Since (by Lemma 5.4.1) minimal nonzero idempotent ideals of S are $A = \mathrm{Tr}(e_1 S)$ and $B = \mathrm{Tr}(e_2 S)$, this is the case by Lemma 5.2.5. ∎

In fact (see below) if (a projective S-module) Q is infinitely generated, it contains $e_1 S$ or $e_2 S$ (even $e_1 S^{(\omega)}$ or $e_2 S^{(\omega)}$) as a direct summand, therefore every generalized Λ-lattice contains $\Lambda^{(\omega)}$ or $\widetilde{\Lambda}^{(\omega)}$ as a direct summand.

Now we consider the remaining case in the classification of generalized lattices over quadratic orders.

Theorem 5.4.5. *Let $\Lambda = \mathbb{Z}[\sqrt{n}]$, $n \equiv 1 \pmod 8$ and let A, A_1, A_2, B be the nonzero proper idempotent ideals of $S = \mathrm{End}(\Lambda \oplus \widetilde{\Lambda})$ (see Lemma 5.4.1). Let P_i, $i = 1, 2$ be a countably generated projective S-module corresponding to the pair $(A, S/A_i)$. Then each P_i is not finitely generated and the following is a complete list of infinitely countably generated projective S-modules with corresponding pairs.*

$P_1^k \oplus P_2^l \mapsto (A, (S/A_1)^k \oplus (S/A_2)^l), \quad 0 \le k, l < \omega;$

$P_1^{(\omega)} \mapsto (A_2, 0);$

$P_2^{(\omega)} \mapsto (A_1, 0);$

$P_1^s \oplus P_2^{(\omega)} \mapsto (A_1, (S/A_1)^s), \quad 1 \le s < \omega;$

$P_1^{(\omega)} \oplus P_2^t \mapsto (A_2, (S/A_2)^t), \quad 1 \le t < \omega;$

$e_1 S^n \oplus e_2 S^{(\omega)} \mapsto (B, (S/B)^n), \quad 1 \le n < \omega;$

$\Lambda^{(\omega)} \mapsto (\Lambda, 0).$

Furthermore we have the following relations:

$P_1 \oplus e_1 S^{(t)} \cong P_1$ *and* $P_2 \oplus e_1 S^{(t)} \cong P_2$ *for every* $1 \le t \le \omega$;

and

$P_1 \oplus P_2 \cong e_1 S^{(\omega)} \oplus e_2 S.$

For instance, every projective S-module has a finitely generated direct summand but neither P_1 nor P_2 admits an indecomposable (direct sum) decomposition.

Proof. By Proposition 5.2.2 there exists a countably generated projective module P_1 corresponding to the pair $(A, S/A_1)$. We will show that $\mathrm{Tr}(P_1) = A_2$, that is, $P_1^{(\omega)}$ corresponds to the pair $(A_2, 0)$. Indeed, $A \subset \mathrm{Tr}(P_1)$ because $S/A_1 \neq 0$ (see a remark after Proposition 5.2.2), therefore $\mathrm{Tr}(P_1) = A_1, A_2$ or Λ. If $\mathrm{Tr}(P_1) = \Lambda$ then P_1 would be a generator, hence the same would be true for the S/A-module $P_1/P_1A \cong S/A_1$, a contradiction (since $S/A \cong S/A_1 \oplus S/A_2$ is a direct sum of nonisomorphic simples). If $\mathrm{Tr}(P_1) = A_1$ then $P_1 = P_1 \cdot A_1$ would imply $P_1/P_1A = (P_1/P_1A) \cdot A_1$, that is, $S/A_1 = (S/A_1) \cdot A_1 = 0$, a contradiction.

Now we prove that P_1 is not a direct sum of finitely generated modules. Indeed otherwise $P_1 \cong \oplus_{i \in I} Q_i$, where each Q_i is finitely generated and indecomposable. Because Λ is Bass, every indecomposable Λ-lattice N is either in $\mathrm{add}(\Lambda)$ or in $\mathrm{add}(\widetilde{\Lambda})$. Since Λ and $\widetilde{\Lambda}$ are commutative noetherian domains it follows that $\Lambda \in \mathrm{add}(N)$ or $\widetilde{\Lambda} \in \mathrm{add}(N)$ accordingly. We conclude that the trace of each Q_i is either A or B, therefore the trace of P_1 equals A, B or $A + B = \Lambda$. But $\mathrm{Tr}(P_1) = A_2$, a contradiction.

Analogously there exists a countably generated projective module P_2 corresponding to the pair $(A, S/A_2)$ and this module is not a direct sum of finitely generated modules. Furthermore, $\mathrm{Tr}(P_2) = A_1$ therefore $P_2^{(\omega)}$ corresponds to the pair $(A_1, 0)$.

Since all possible combinations of pairs (I, P) in Proposition 5.2.2 are taken up, we have completed the classification.

Furthermore the relations are easily checked by calculating corresponding pairs. For instance, $P_1 \oplus P_2 \cong e_1 S^{(\omega)} \oplus e_2 S$ because both modules correspond to the pair $(A, S/A)$. ∎

It may be easier to grasp the above identification by assigning to a projective S-module Q the triple $(\alpha_1(Q), \alpha_2(Q), \beta(Q))$, where $\alpha_i(M)$ is the multiplicity of the simple module S/A_i in Q/QA_i and $\beta(M)$ is the multiplicity of S/B in Q/QB. It clearly follows that

$e_1 S \mapsto (0, 0, 1);$

$e_2 S \mapsto (1, 1, 0);$

$P_1 \mapsto (1, 0, \omega);$

$P_2 \mapsto (0, 1, \omega)$.

Then $P_1 \oplus P_2 \mapsto (1, 1, \omega)$, the same as $e_1 S^{(\omega)} \oplus e_2 S$. Furthermore, it can be checked that infinitely (even uncountably) generated projective S-modules are uniquely determined by their triples. One possible explanation for that is the following. If we localize Λ with respect to the prime ideal $2\mathbb{Z}$ of $\mathbb{Z}$ we obtain a semilocal ring Λ_2 with exactly 3 maximal ideals A'_1, A'_2 and B', hence with exactly 3 simple modules $S_1 = \Lambda_2/A'_1$, $S_2 = \Lambda_2/A'_2$ and $S_3 = \Lambda_2/B'$. If Q is a projective Λ_2-module, then the *dimension* of Q, $\dim(Q)$, is the triple of cardinals $(\alpha_1, \alpha_2, \alpha_3)$, where α_i is the multiplicity of the simple module S_i in the corresponding factor of Q. By [15, Theorem 2.3] projective modules over an arbitrary semilocal ring are uniquely determined by their dimensions. Thus the above result claims that infinitely generated generalized Λ-lattices are classified by their triples. For instance, there is a unique Λ-lattice corresponding to the module $P_1^s \oplus P_2^{(\omega)}$, that is, to the triple (s, ω, ω).

Question 5.4.6. *Is it possible to find a 'nice' representation of the Λ-lattice corresponding to P_i by generators and relations?*

Note that such a generalized lattice can be constructed as in [14] as a direct limit of finitely generated lattices, but the rank of the n'th term in this limit grows as 2^n.

5.5 Package principle

Before analyzing more examples of lattices over orders we recall the so-called package principle (see [11]) that allows to construct a module from a prescribed set of its localizations. For our purposes it will be enough to consider a $\mathbb{Z}$-order Λ, and a module that we are going to construct will be an idempotent ideal I of Λ.

Clearly if I is idempotent then, for every prime p, I_p, the localization of I with respect to a prime ideal $p\mathbb{Z}$ of $\mathbb{Z}$, is an idempotent ideal of the $\mathbb{Z}_p$-order Λ_p. Conversely, suppose that $I(p)$, where p runs over all primes, is a family of idempotent ideals of $\mathbb{Z}_p$-orders Λ_p. Then an instance of the package principle (see [11, Theorem 2.9]) claims that there exists (a necessarily idempotent) ideal I of Λ such that $I(p) = I_p$ for every p iff the following consistency condition holds true: $I(p)_0 = I(q)_0$ for all primes p, q, where $I(p)_0$ denotes the localization of $I(p)$ at the prime ideal 0 of $\mathbb{Z}$, which is an ideal in the $\mathbb{Q}$-algebra Λ_0.

Thus one possible (and extremely effective) strategy to classify idempotent ideals of a given order is to describe them over localizations and then to understand how these localized ideals could be packed into a common ideal of the original order. The main advantage of dealing with localizations is that they are semilocal (say, as module finite algebras over semilocal rings $\mathbb{Z}_p$). Recall that a (noncommutative) ring R is said to be *semilocal* if the factor of R by its Jacobson radical, $\mathrm{Jac}(R)$, is a semisimple artinian ring. Thus the following fact will be very useful.

Fact 5.5.1. *(see [14]) An idempotent ideal I of a semilocal noetherian ring R is uniquely determined by its semisimple factor $I/I\,\mathrm{Jac}(R)$.*

Furthermore, if Λ is a maximal order in a semisimple algebra A, then it is easily seen that every idempotent ideal of Λ is generated by central idempotents, which are the same for Λ and A.

Recall that a semilocal ring whose idempotents can be lifted modulo the Jacobson radical is said to be *semiperfect*. If R is a semiperfect ring, then (see [1, Proposition 27.10

and Theorem 27.11]) there is a direct sum decomposition $R_R = e_1R \oplus \cdots \oplus e_nR$, where e_i are local idempotents (that is, the rings e_iRe_i are local), and every projective R-module is a direct sum of copies of the e_iR. Furthermore, if R is noetherian and semiperfect then (since every idempotent ideal is a trace of a countably generated projective module) its only idempotent ideals are the sums of the ideals Re_iR.

5.6 Rump's example

In this section we will classify projective modules (that is, projective generalized lattices) over the following $\mathbb{Z}$-order

$$\Lambda = \mathbb{Z} \cdot 1 + \begin{pmatrix} 2\mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & 6\mathbb{Z} \end{pmatrix} \times \begin{pmatrix} 6\mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & 3\mathbb{Z} \end{pmatrix}$$

in $M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$ taken from [18, p. 126, Example 2].

Note that Rump [18, p. 127] proved that Λ is of locally finite lattice type. It follows from Jordan-Zassenhaus theorem [5, Theorem 24.1] that Λ is of finite lattice type. Furthermore Rump showed that there exists a (nonzero) projective Λ-lattice L which contains no nonzero finitely generated direct summands. Using the correspondence between generalized lattices and projective modules from Section 5.3 and Remark 5.2.4 we conclude that L is superdecomposable. Thus our classification of projective Λ-modules will include this superdecomposable object.

The following is a more useful description of Λ, which can be verified by direct calculations.

Lemma 5.6.1.

$$\Lambda = \begin{pmatrix} \mathbb{Z} & & 6\mathbb{Z} \\ & \underset{2}{\diagdown\!\!\diagdown} & \\ \mathbb{Z} & & \mathbb{Z} \end{pmatrix} \overset{6}{\diagup\!\!\diagup} \times \begin{pmatrix} \mathbb{Z} & & 6\mathbb{Z} \\ & \underset{3}{\diagdown\!\!\diagdown} & \\ \mathbb{Z} & & \mathbb{Z} \end{pmatrix}$$

Here $\underset{2}{=}$ *means that the difference of* 1×1 *and* 2×2 *entries of the first coordinate of* Λ *is divisible by* 2.

Being a $\mathbb{Z}$-module finite algebra, Λ satisfies $(*)$ (by the proof of the Proposition 5.3.3) therefore we could follow a general strategy of classifying projective modules over noetherian rings with $(*)$. Thus first we have to describe idempotent ideals of Λ. We approach this problem via localizations, that is, using the package principle.

Note that for every $p \neq 2, 3$ the localization Λ_p coincides with the ring $M_2(\mathbb{Z}_p) \times M_2(\mathbb{Z}_p)$ which is a maximal order in $M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$, therefore idempotent ideals of Λ_p are generated by obvious central idempotents (thus there are 4 of them).

If $p = 2$ then Λ_2 is isomorphic to the ring

$$\begin{pmatrix} \mathbb{Z}_2 & & 2\mathbb{Z}_2 \\ & \underset{2}{\diagdown\!\!\diagdown} & \\ \mathbb{Z}_2 & & \mathbb{Z}_2 \end{pmatrix} \overset{2}{\diagup\!\!\diagup} \times \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}$$

for instance, this ring has idempotents $e_2 = 0 \times \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $1 - e_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Easy calculations show that the Jacobson radical of Λ_2 is equal to

$$\begin{pmatrix} 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & 2\mathbb{Z}_2 \end{pmatrix} \times \begin{pmatrix} 2\mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & 2\mathbb{Z}_2 \end{pmatrix}$$

and the factor $\Lambda_2/\operatorname{Jac}(\Lambda_2) \cong GF(2) \oplus GF(2)$. It follows that idempotents can be lifted modulo the Jacobson radical, therefore Λ_2 is a semiperfect ring. Therefore the only (nonzero proper) idempotent ideals of Λ_2 are $\langle e_2 \rangle$ (the two-sided ideal generated by e_2) and

$$\langle 1 - e_2 \rangle = \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix} \overset{2}{\times} \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ \mathbb{Z}_2 & 2\mathbb{Z}_2 \end{pmatrix}$$

Note also that $\langle e_2 \rangle_0 = 0 \times M_2(\mathbb{Q})$ and $\langle 1 - e_2 \rangle_0 = M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$.

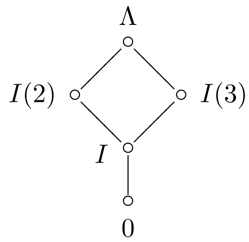
Similarly localizing at 3 we obtain that

$$\Lambda_3 = \begin{pmatrix} \mathbb{Z}_3 & 3\mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} \overset{3}{\times} \begin{pmatrix} \mathbb{Z}_3 & 3\mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix}$$

is a semiperfect ring with $\Lambda_3/\operatorname{Jac}(\Lambda_3) \cong GF(3) \oplus GF(3)$. Furthermore $e_3 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \times 0$ and $1 - e_3 = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ are local idempotents of Λ_3 such that $\langle e_3 \rangle$ and $\langle 1 - e_3 \rangle$ are the only nonzero proper idempotent ideals of Λ_3. Note also that $\langle e_3 \rangle_0 = M_2(\mathbb{Q}) \times 0$ and $\langle 1 - e_3 \rangle_0 = M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$.

Now we are in a position to describe idempotent ideals of Λ.

Proposition 5.6.2. *The following is the lattice of idempotent ideals of* Λ.



Here

$$I(2) = \begin{pmatrix} \mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \overset{6}{\times} \begin{pmatrix} \mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & 2\mathbb{Z} \end{pmatrix}$$

is such that $I(2)_2 = \langle 1 - e_2 \rangle$, $I(2)_3 = \Lambda_3$;

$$I(3) = \begin{pmatrix} 3\mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \overset{6}{\times} \begin{pmatrix} \mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$$

is such that $I(3)_2 = \Lambda_2$, $I(3)_3 = \langle 1 - e_3 \rangle$; *and*

$$I = I_1 \cap I_2 = \begin{pmatrix} 3\mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix} \overset{6}{\times} \begin{pmatrix} \mathbb{Z} & 6\mathbb{Z} \\ \mathbb{Z} & 2\mathbb{Z} \end{pmatrix}$$

is such that $I_2 = \langle 1 - e_2 \rangle$, $I_3 = \langle 1 - e_3 \rangle$.

Proof. Suppose that J is a nonzero idempotent ideal of Λ, in particular $J_2 \neq 0$. If $J_2 = \langle e_2 \rangle$, then $J_0 = 0 \times M_2(\mathbb{Q})$. But, by inspection, there is no idempotent ideal of Λ_3 with the same localization at 0. Thus $J_2 = \langle 1 - e_2 \rangle$ or $J_2 = \Lambda_2$. Similarly $J_3 = \langle 1 - e_3 \rangle$ or $J_3 = \Lambda_3$. According to the package principle all 4 combinations are possible, therefore there are 3 nonzero proper idempotent ideals of Λ. An identification of these ideals in a matrix form is straightforward. ■

From a description of idempotent ideals of Λ it easily follows that $\Lambda/I(2) \cong GF(2)$ and $\Lambda/I(3) \cong GF(3)$. Since $I = I(2) \cap I(3)$ and $I(2) + I(3) = \Lambda$, we conclude that $\Lambda/I \cong GF(2) \oplus GF(3)$. From Proposition 5.2.2 it follows that there exists a countably generated projective module P that corresponds to the pair $(I, GF(2))$; and there exists a countably generated projective module Q corresponding to the pair $(I, GF(3))$. Furthermore, if H is a countably generated projective module corresponding to the pair $(I, 0)$, then $\mathrm{Tr}(H) = I$ and $H \cong H^{(k)}$ for every $1 \leq k \leq \omega$ (see remarks after Proposition 5.2.2 for an explanation).

However to complete the classification of projective Λ-modules, for instance to prove that H is superdecomposable, first we have to develop the theory of finitely generated projectives. In fact we will classify finitely generated projective Λ-modules up to genus. Recall that modules M and N are said to be of the *same genus*, if $M_p \cong N_p$ for every prime p.

Since Λ_2 is a semiperfect ring, the only (finitely generated) indecomposable projective Λ_2-modules are $P(2) = e_2\Lambda_2$ and $Q(2) = (1 - e_2)\Lambda_2$. Let $S_1 \subseteq M_2(\mathbb{Q}) \times 0$ and $S_2 \subseteq 0 \times M_2(\mathbb{Q})$ be simple (right) modules of the ring $M_2(\mathbb{Q}) \times M_2(\mathbb{Q})$. Then clearly $P(2)_0 \cong S_2$ and $Q(2)_0 \cong S_1^2 \oplus S_2$.

Similarly the only indecomposable projective Λ_3-modules are $P(3) = e_3\Lambda_3$ and $Q(3) = (1 - e_3)\Lambda_3$ and clearly $P(3)_0 \cong S_1$, $Q(3)_0 \cong S_1 \oplus S_2^2$.

By the package principle there is a finitely generated projective Λ-module T such that $T_2 \cong Q(2)^2$, $T_3 \cong P(3)^3 \oplus Q(3)$ and $T_0 \cong S_1^4 \oplus S_2^2$. Similarly there exists a finitely generated projective Λ-module U such that $U_2 \cong P(2)^3 \oplus Q(2)$, $U_3 \cong Q(3)^2$ and $U_0 \cong S_1^2 \oplus S_2^4$. In the following proposition we will classify genera of finitely generated projective Λ-modules.

Proposition 5.6.3. *Every finitely generated projective Λ-module is in the genus of $\Lambda^k \oplus T^s$ or $\Lambda^k \oplus U^t$.*

Proof. Let V be a finitely generated projective Λ-module. By what we have already said, $V_2 \cong P(2)^{m_2} \oplus Q(2)^{n_2}$ and $V_3 \cong P(3)^{m_3} \oplus Q(3)^{n_3}$. Localizing at 0 and using the above relations (like $Q(2)_0 \cong S_1^2 \oplus S_2$) we obtain

$$2n_2 = m_3 + n_3 \quad \text{and} \quad m_2 + n_2 = 2n_3\,. \tag{$**$}$$

Consider first the case when $P(2)$ does not occur as a direct summand of V_2, that is, $m_2 = 0$. Then $n_2 = 2n_3$, therefore $3n_3 = m_3$. It follows easily that V is in the genus of T^{n_3} (since they have isomorphic localizations). If $Q(2)$ does not occur as a direct summand of V_2, then $n_2 = 0$. By $(**)$ we obtain $n_3 = m_3 = 0$, therefore $m_2 = 0$ yielding $V = 0$.

Similarly if $P(3)$ does not occur as a direct summand of V_3, that is, $m_3 = 0$, then V is in the genus of U^{n_2}, and the case $n_3 = 0$ leads to $V = 0$.

Thus we may assume that $n_i, m_i > 0$ for $i = 1, 2$. Note that $\Lambda_2 = P(2) \oplus Q(2)$ and $\Lambda_3 = P(3) \oplus Q(3)$. Let $k = \min(m_i, n_i)$. If $m_2 = k$, then from $(**)$ it is easily derived that $n_2 = k + 2s$, $m_3 = k + 3s$ and $n_3 = k + s$ for some $s \geq 0$, therefore V is in the genus of $\Lambda^k \oplus T^s$. Suppose that $n_2 = k$. From $(**)$ it easily follows that $n_i = m_i = k$ for $i = 1, 2$, therefore V is in the genus of Λ^k.

The cases when $m_3 = k$ or $n_3 = k$ are considered similarly. ■

Now we are in a position to complete a classification of countably generated projective Λ-modules.

Theorem 5.6.4. *The following is a complete list of infinitely countably generated projective modules and corresponding pairs over Rump's example* Λ.

$\Lambda^{(\omega)} \mapsto (\Lambda, 0)$;

$P^k \oplus Q^l \mapsto (I, GF(2)^k \oplus GF(3)^l)$, $0 \le k, l < \omega$;

$P^{(\omega)} \mapsto (I(3), 0)$;

$Q^{(\omega)} \mapsto (I(2), 0)$;

$H \mapsto (I, 0)$;

$P^s \oplus Q^{(\omega)} \mapsto (I(2), GF(2)^s)$, $1 \le s < \omega$;

$P^{(\omega)} \oplus Q^t \mapsto (I(3), GF(3)^t)$, $1 \le t < \omega$.

Furthermore H *is a superdecomposable module isomorphic to any of its nonzero direct summands.*

Proof. Recall that P corresponds to the pair $(I, GF(2))$. Note that $e = \left(\begin{smallmatrix} 3 & 0 \\ 0 & 1 \end{smallmatrix}\right) \times \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in I(3) \setminus I(2)$ is an idempotent modulo I such that $e\Lambda + I/I \cong GF(2)$, because $2e \in I$. Thus P is obtained by 'lifting' e modulo I. Because $e \in I(3)$, from the construction of [14] it follows that $\mathrm{Tr}(P) \subseteq I(3)$. Since $I \subset \mathrm{Tr}(P)$, we conclude that $\mathrm{Tr}(P) = I(3)$. Thus $P^{(\omega)}$ corresponds to the pair $(I(3), 0)$. Similarly $\mathrm{Tr}(Q) = I(2)$, therefore $Q^{(\omega)}$ corresponds to the pair $(I(2), 0)$.

Because all the possibilities for the pairs (I, P) (from Proposition 5.2.2) are taken up, we have completed a classification of infinitely countably generated projective Λ-modules.

Recall that H corresponds to the pair $(I, 0)$, in particular $\mathrm{Tr}(H) = I$ and $H \cong H^{(k)}$ for every $1 \le k \le \omega$. Suppose that H' is a nonzero direct summand of H. It follows that $I(H') \subseteq I$, therefore $I(H') = I$ or $I(H') = 0$. If $I(H') = I$, then $P(H) = H/HI = 0$ yields $P(H') = H'/H'I = 0$, therefore $H' \cong H$ by Proposition 5.2.2.

Otherwise $I(H') = 0$, therefore H' is finitely generated. We will prove that this leads to a contradiction. Namely $H' \ne 0$ implies $(H')_2 \cong P(2)^{m_2} \oplus Q(2)^{n_2} \ne 0$. As we have already seen (in the proof of Proposition 5.6.3) that $n_2 = 0$ yields $H' = 0$, a contradiction. Thus we may assume that $n_2 > 0$. If $m_2 > 0$, then $(H')_2$ is a generator. But $\mathrm{Tr}(H) = I$ yields $\mathrm{Tr}(H_2) = I(2)_2 \ne \Lambda_2$, a contradiction. Thus $m_2 = 0$. By similar arguments, $n_3 > 0$ and $m_3 = 0$, which clearly contradicts $(**)$.

Thus we have proved that every direct sum decomposition of H is of the form $H \cong H^{(k)}$, $1 \le k \le \omega$. ■

To include some finitely generated projective Λ-modules into relations let us make first the following remark. Recall that finitely generated projective modules T and U were introduced before Proposition 5.6.3.

Lemma 5.6.5. *1)* $\mathrm{Tr}(T) = I(2)$ *and* $T/TI \cong GF(3)^3$;

2) $\mathrm{Tr}(U) = I(3)$ *and* $U/UI \cong GF(2)^3$.

Proof. 1) By the construction, $T_2 \cong Q(2)^2 = [(1 - e_2)\Lambda_2]^2$, therefore $\mathrm{Tr}(T_2) = \Lambda_2(1 - e_2)\Lambda_2 = I(2)_2$. Furthermore $T_3 \cong P(3)^3 \oplus Q(3)$, therefore $\mathrm{Tr}(T_3) = \Lambda_3$. Because $I(2)_2 = \langle 1 - e_2 \rangle$ and $I(2)_3 = \Lambda_3$, we conclude that $\mathrm{Tr}(T) = I(2)$.

Since $I = I(2) \cap I(3)$, $I(2) + I(3) = \Lambda$ and T is projective, it follows that $T/TI \cong T/TI(2) \oplus T/TI(3) = T/TI(3)$, because $T = TI(2)$. To calculate $T/TI(3)$ it suffices to look at the localization

$$(T/TI(3))_3 = T_3/T_3I(3)_3 = P(3)^3/P(3)^3I(3)_3 \oplus Q(3)/Q(3)I(3)_3 = GF(3)^3 ,$$

since $\mathrm{Tr}(Q(3)) = \langle 1 - e_3 \rangle = I(3)_3$ yields $Q(3)/Q(3)I(3)_3 = 0$.

The verification of 2) is similar. ∎

As a result we can include finitely generated projectives into new relations.

Lemma 5.6.6. *1) $P^3 \cong U \oplus H$ but U is not a direct summand of P^2;*
2) $Q^3 \cong T \oplus H$ but T is not a direct summand of Q^2.

Proof. We will prove only 1).

Since $U/UI \cong GF(2)^3 \cong (P/PI)^3$, both P^3 and $U \oplus H$ correspond to the pair $(I, GF(2)^3)$, therefore these modules are isomorphic (by Proposition 5.2.2). Furthermore, since U/UI has $GF(2)$-dimension 3, therefore U cannot be a direct summand of P^2. ∎

Note that (as it is easily seen from the classification) whether a finitely generated projective Λ-module V is a direct summand of an infinitely generated projective W depends only on the genus of V. Since (by Proposition 5.6.3) we know all genera of finitely generated projectives, it is not difficult to understand all possible direct sum decompositions of countably generated projective modules from Proposition 5.6.4. For instance, P has no finitely generated direct summands, therefore (see a remark before Proposition 5.2.2) P is superdecomposable, and the same is true for Q. On the other hand $P^{(\omega)} \cong P^{(\omega)} \oplus U$ contains U as a direct summand, but $P^{(\omega)}$ cannot be represented as a direct sum of indecomposable modules; for instance because H is a superdecomposable direct summand of P.

5.7 A Bass' order

In this section we will discuss a classification of generalized lattices over a Bass' order $\Lambda = \Lambda(6) = \{(m, n) \in \mathbb{Z}^2 \mid 6 \text{ divides } m - n\}$. Such orders in the commutative case are the topic of a classical Bass' paper [2].

Recall that D denotes a Dedekind domain with a field of quotients K. Suppose that Λ is a D-order in a separable finite-dimensional K-algebra A. We say that Λ is *Gorenstein* if Λ is an injective module in the category of Λ-lattices, that is, every exact sequence $0 \to \Lambda \to M \to N \to 0$ of Λ-lattices splits. If Λ has an additional property that every its overorder is Gorenstein, then Λ is said to be a *Bass order*. In the noncommutative setting this notion was introduced by Drozd, Kirichenko and Roiter [6] (see also [8] for some recent developments). For instance (see [5, Theorem 37.17]) every order whose one-sided ideals are 2-generated is Bass and (by [2]) the converse is true for commutative Bass orders. Furthermore, by [12, Theorem 2.1], a commutative order Λ is Bass iff every faithful Λ-lattice is isomorphic to an invertible (therefore projective) ideal of a ring Λ' between Λ and $\widetilde{\Lambda}$.

Every ideal of our $\mathbb{Z}$-order Λ is clearly 2-generated, therefore Λ is Bass. Furthermore, $\widetilde{\Lambda} = \mathbb{Z} \times \mathbb{Z}$ and the only rings between $\Lambda = \Lambda(6)$ and $\widetilde{\Lambda} = \Lambda(1)$ are $\Lambda(2) = \{(m, n) \in \mathbb{Z}^2 \mid 2 \text{ divides } m - n\}$ and $\Lambda(3) = \{(m, n) \in \mathbb{Z}^2 \mid 3 \text{ divides } m - n\}$. It follows that every indecomposable Λ-lattice M is either isomorphic to $\mathbb{Z}\oplus 0$, $0\oplus\mathbb{Z}$, or is a faithful projective module over $\Lambda(i)$, $i=2,3,6$. Since $\Lambda(i)$ has Krull dimension 1, by Serre's theorem on big projectives M has rank 1, therefore is an element of the Pickard group of $\Lambda(i)$.

Note that $\Lambda=\Lambda(6)$ can be represented as a pullback

$$\begin{tikzcd} \Lambda \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d, "\pi"] \\ \mathbb{Z} \arrow[r, "\pi"'] & \mathbb{Z}/6\mathbb{Z} \end{tikzcd}$$

where π denotes the canonical projection. Using the standard Mayer-Vietoris sequence (see [3, p. 482]) it is easily calculated that the Picard group of $\Lambda=\Lambda(6)$ is trivial, and the same is true for $\Lambda(2)$ and $\Lambda(3)$ (see [12, 4.3.1] for general arguments). It follows that every indecomposable Λ-lattice is isomorphic to $\mathbb{Z}\oplus 0$, $0\oplus\mathbb{Z}$, $\Lambda(2)$, $\Lambda(3)$ or $\Lambda(6)$. Thus we know all (finitely generated) Λ-lattices and our goal is to classify generalized lattices over Λ. According to the general strategy we will form the Auslander lattice $M=\Lambda(1)\oplus\Lambda(2)\oplus\Lambda(3)\oplus\Lambda(6)$ and consider its endomorphism ring $A=\mathrm{End}(M)$, that is, the Auslander order of Λ. If we consider A as acting on the column $\begin{pmatrix}\Lambda(1)\\ \Lambda(2)\\ \Lambda(3)\\ \Lambda(6)\end{pmatrix}$ on the left, then (as an easy calculation shows)

$$A=\begin{pmatrix}\Lambda(1) & \Lambda(1) & \Lambda(1) & \Lambda(1)\\ 2\Lambda(1) & \Lambda(2) & 2\Lambda(1) & \Lambda(2)\\ 3\Lambda(1) & 3\Lambda(1) & \Lambda(3) & \Lambda(3)\\ 6\Lambda(1) & \Lambda(2)\cap 3\Lambda(1) & \Lambda(3)\cap 2\Lambda(1) & \Lambda(6)\end{pmatrix}\subseteq M_4(\mathbb{Z}\times\mathbb{Z})\,.$$

Thus to classify generalized Λ-lattices is the same as to classify (finitely and infinitely) generated projective A-modules; and the first objective towards this goal is to describe idempotent ideals of A. As in Section 5.6 we will approach this problem using localizations. Note that if $p\neq 2,3$ then $A_p=M_4(\mathbb{Z}_p\times\mathbb{Z}_p)$ is a maximal order in $M_4(\mathbb{Q}\times\mathbb{Q})$, therefore every idempotent ideal of A_p is generated by a central idempotent of $M_4(\mathbb{Q}\times\mathbb{Q})$, hence there are 4 of them.

If $p=2$, then the localization A_2 is the following ring

$$A_2=\left(\begin{array}{cc|cc}\mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2\\ 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2 & 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2\\ \hline \mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2\\ 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2 & 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2\end{array}\right),$$

where $\Lambda(2)_2=\{(a/b,c/b)\in\mathbb{Q}^2\mid b\in\mathbb{Z}\setminus 2\mathbb{Z}\text{ and }2\text{ divides }a-c\}$. Thus A_2 is isomorphic to the full matrix ring $M_2(A_2')$, where

$$A_2'=\begin{pmatrix}\mathbb{Z}_2\times\mathbb{Z}_2 & \mathbb{Z}_2\times\mathbb{Z}_2\\ 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2\end{pmatrix}.$$

We would like to describe idempotent ideals of A_2'. It is easily seen that the Jacobson radical of A_2', $\mathrm{Jac}(A_2')$, equals

$$\begin{pmatrix}2(\mathbb{Z}_2\times\mathbb{Z}_2) & \mathbb{Z}_2\times\mathbb{Z}_2\\ 2(\mathbb{Z}_2\times\mathbb{Z}_2) & 2(\mathbb{Z}_2\times\mathbb{Z}_2)\end{pmatrix},$$

therefore $A_2'/\operatorname{Jac}(A_2') = \left(\begin{smallmatrix} GF(2)\oplus GF(2) & 0 \\ 0 & GF(2) \end{smallmatrix}\right)$.

Since A_2' is semilocal and idempotents can be lifted modulo the Jacobson radical, therefore it is a semiperfect ring. It follows that every (nonzero) idempotent ideal of A_2' is a sum of two-sided ideals generated by the following idempotents: $e_1 = \left(\begin{smallmatrix} (1,0) & (0,0) \\ (0,0) & (0,0) \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} (0,1) & (0,0) \\ (0,0) & (0,0) \end{smallmatrix}\right)$ and $e_3 = \left(\begin{smallmatrix} (0,0) & (0,0) \\ (0,0) & (1,1) \end{smallmatrix}\right)$. For instance $\langle e_1\rangle = \left(\begin{smallmatrix} \mathbb{Z}_2\times 0 & \mathbb{Z}_2\times 0 \\ 2\mathbb{Z}_2\times 0 & 2\mathbb{Z}_2\times 0 \end{smallmatrix}\right)$ $\langle e_2\rangle = \left(\begin{smallmatrix} 0\times\mathbb{Z}_2 & 0\times\mathbb{Z}_2 \\ 0\times 2\mathbb{Z}_2 & 0\times 2\mathbb{Z}_2 \end{smallmatrix}\right)$ and $\langle e_3\rangle = \left(\begin{smallmatrix} 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \mathbb{Z}_2\times\mathbb{Z}_2 \\ 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2 \end{smallmatrix}\right)$ are idempotent ideals of A_2'.

Turning back to A_2, we obtain that every nonzero idempotent ideal of A_2 is a sum of ideals generated by idempotents $f_1 = \left(\begin{smallmatrix} e_1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, $f_2 = \left(\begin{smallmatrix} e_2 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $f_3 = \left(\begin{smallmatrix} e_3 & 0 \\ 0 & 0 \end{smallmatrix}\right)$; for instance

$$\langle f_3\rangle = \left(\begin{array}{cc|cc} 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \mathbb{Z}_2\times\mathbb{Z}_2 & 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \mathbb{Z}_2\times\mathbb{Z}_2 \\ 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2 & 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2 \\ \hline 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \mathbb{Z}_2\times\mathbb{Z}_2 & 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \mathbb{Z}_2\times\mathbb{Z}_2 \\ 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2 & 2(\mathbb{Z}_2\times\mathbb{Z}_2) & \Lambda(2)_2 \end{array}\right).$$

Furthermore, $\langle e_1\rangle_0 = M_4(\mathbb{Q})\times 0$, $\langle e_2\rangle_0 = 0\times M_4(\mathbb{Q})$ and $\langle e_3\rangle_0 = M_4(\mathbb{Q})\times M_4(\mathbb{Q})$.

An obvious explanation for the appearance of the matrix structure in A_2 is the following. When localizing with respect to $2\mathbb{Z}$, the lattices $\Lambda(1)$ and $\Lambda(3)$, and also $\Lambda(2)$ and $\Lambda(6)$ are gotten isomorphic. A similar thing is happening when localizing at 3, but now the lattices $\Lambda(1)$ and $\Lambda(2)$, and also $\Lambda(3)$ and $\Lambda(6)$ are identified. Thus (by straightforward calculations)

$$A_3 = \left(\begin{array}{cc|cc} \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 \\ \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 \\ \hline 3(\mathbb{Z}_3\times\mathbb{Z}_3) & 3(\mathbb{Z}_3\times\mathbb{Z}_3) & \Lambda(3)_3 & \Lambda(3)_3 \\ 3(\mathbb{Z}_3\times\mathbb{Z}_3) & 3(\mathbb{Z}_3\times\mathbb{Z}_3) & \Lambda(3)_3 & \Lambda(3)_3 \end{array}\right),$$

where $\Lambda(3)_3 = \{(a/b, c/b)\in\mathbb{Q}^2 \mid b\in\mathbb{Z}\setminus 3\mathbb{Z} \text{ and } 3 \text{ divides } a-c\}$. Thus A_3 is isomorphic to the full matrix ring $M_2(A_3')$, where

$$A_3' = \begin{pmatrix} \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 \\ 3(\mathbb{Z}_3\times\mathbb{Z}_3) & \Lambda(3)_3 \end{pmatrix}.$$

As above nonzero idempotent ideals of A_3 are the sums of the ideals generated by the following idempotents: $g_1 = \left(\begin{smallmatrix} (1,0) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$, $g_2 = \left(\begin{smallmatrix} (0,1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$ and $g_3 = \left(\begin{smallmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & (1,1) & 0 \\ 0 & 0 & 0 & 0 \end{smallmatrix}\right)$, for instance,

$$\langle g_3\rangle = \left(\begin{array}{cc|cc} 3(\mathbb{Z}_3\times\mathbb{Z}_3) & 3(\mathbb{Z}_3\times\mathbb{Z}_3) & \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 \\ 3(\mathbb{Z}_3\times\mathbb{Z}_3) & 3(\mathbb{Z}_3\times\mathbb{Z}_3) & \mathbb{Z}_3\times\mathbb{Z}_3 & \mathbb{Z}_3\times\mathbb{Z}_3 \\ \hline 3(\mathbb{Z}_3\times\mathbb{Z}_3) & 3(\mathbb{Z}_3\times\mathbb{Z}_3) & \Lambda(3)_3 & \Lambda(3)_3 \\ 3(\mathbb{Z}_3\times\mathbb{Z}_3) & 3(\mathbb{Z}_3\times\mathbb{Z}_3) & \Lambda(3)_3 & \Lambda(3)_3 \end{array}\right).$$

Furthermore, $\langle g_1\rangle_0 = M_4(\mathbb{Q}\times 0)$, $\langle g_2\rangle_0 = M_4(0\times\mathbb{Q})$ and $\langle g_3\rangle_0 = M_4(\mathbb{Q}\times\mathbb{Q})$.

Now we are in a position to classify idempotent ideals of A. Let I be a nonzero idempotent ideal of A. Then the localization I_2 is a nonzero idempotent ideal of A_2. Suppose first that $I_2 = \langle f_1\rangle$, therefore $I_0 = M_4(\mathbb{Q}\oplus 0)$. Since $I_3 \neq 0$, a search through a list of idempotent ideals of A_3 shows that the only possibility is $I_3 = \langle g_1\rangle$. It follows easily that I is a two-sided ideal generated by $\left(\begin{smallmatrix} (1,0) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right) \in A$.

Similarly, if $I_2 = \langle f_2\rangle$, then $I_3 = \langle g_2\rangle$, therefore I is a two-sided ideal of A generated by $\begin{pmatrix} (0,1) & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \\ 0 & 0\,0\,0 \end{pmatrix} \in A$.

Otherwise it is easily seen that $I_0 = M_4(\mathbb{Q}\times\mathbb{Q})$, therefore I_2 is one of the following ideals: $\langle f_3\rangle$, $\langle f_1, f_2\rangle$, $\langle f_1, f_3\rangle$, $\langle f_2, f_3\rangle$, A_2 and $I_3 = \langle g_3\rangle$, $\langle g_1, g_2\rangle$, $\langle g_1, g_3\rangle$, $\langle g_2, g_3\rangle$, or A_3. Furthermore by the package principle, any of possible 25 combinations of I_2 and I_3 occurs, say, there exists a unique idempotent ideal I of A such that $I_2 = \langle f_1, f_3\rangle$ and $I_3 = \langle g_2, g_3\rangle$. Thus altogether A has 28 idempotent ideals and any factor of A by a nonzero idempotent ideal is an artinian ring.

To complete a classification of projective A-modules (that is, of generalized lattices over this Bass' order) one should calculate projective modules over factors A/I, where I is an idempotent ideal of A. We will leave this straightforward (but tedious) calculations to the interested reader, but extract only some useful facts.

Proposition 5.7.1. *(see [18, Proposition 7]) Every generalized lattice over the Bass order* $\Lambda = \Lambda(6)$ *contains a finitely generated direct summand.*

Proof. By Remark 5.2.4 and Lemma 5.2.5, it suffices to prove that every nonzero idempotent ideal I of A contains the trace J of a nonzero finitely generated projective A-module. By localizing it suffices to check that $J_p \subseteq I_p$ for every prime p. For this let us recall a list of indecomposable finitely generated projective A-modules (or rather corresponding Λ-lattices) and localizations of their traces at 2 and 3.

$\mathbb{Z}\oplus 0 \mapsto \langle f_1\rangle, \langle g_1\rangle$;
$0\oplus\mathbb{Z} \mapsto \langle f_2\rangle, \langle g_2\rangle$;
$\Lambda(2) \mapsto \langle f_3\rangle, \langle g_1, g_2\rangle$;
$\Lambda(3) \mapsto \langle f_1, f_2\rangle, \langle g_3\rangle$;
$\Lambda \mapsto \langle f_3\rangle, \langle g_3\rangle$.

Now the result follows by an easy inspection. For instance, if $I_2 = \langle f_1, f_3\rangle$ and $I_3 = \langle g_2, g_3\rangle$, then I contains a trace of the finitely generated projective A-module corresponding to the lattice Λ. ∎

The following question is about a general version of Proposition 5.7.1.

Question 5.7.2. *Does there exist a Bass order with a superdecomposable generalized lattice?*

Despite Proposition 5.7.1, not every generalized lattice over Λ is a direct sum of (finitely generated) lattices. Namely let a projective A-module Q correspond to the pair $(I, 0)$ (see Proposition 5.6.3), where I is an idempotent ideal of A with $I_2 = \langle f_1, f_3\rangle$ and $I_3 = \langle g_2, g_3\rangle$. Then $\mathrm{Tr}(Q) = I$ and $Q \cong Q^{(k)}$ for every $1 \le k \le \omega$. If P is a finitely generated direct summand of Q, then $\mathrm{Tr}(P) \subseteq \mathrm{Tr}(Q) = I$, therefore, by inspection, P is a direct sum of copies of a projective A-module corresponding to the lattice $\Lambda = \Lambda(6)$, in particular Q is not a direct sum of finitely generated modules. Furthermore it can be shown that every direct sum decomposition of Q is of the form $Q \cong Q^{(k)} \oplus P^{(l)}$, $1 \le k, l \le \omega$.

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6. INFINITELY GENERATED PROJECTIVE MODULES OVER NOETHERIAN SEMILOCAL RINGS

A theorem of Kaplansky states that, for any ring R, a projective right R-module is a direct sum of countably generated projective right R-modules. This reduces the study of direct summands of $R^{(I)}$, where I denotes an arbitrary set, to the study of direct sum decomposition of $R^{(\aleph_0)}$ or, equivalently, to the study of countably generated projective right R-modules.

The commutative monoid $V(R)$ of isomorphism classes of finitely generated projective right R-modules, with the addition induced by the direct sum of modules, encodes the direct sum behavior of finite direct sums of finitely generated projective right R-modules. Similarly, the monoid $V^*(R)$ of isomorphism classes of countably generated projective right R-modules, with the addition induced by the direct sum of modules, encodes the direct-sum behavior of countably generated projective modules. In this paper we characterize the monoids that can be realized as $V^*(R)$ for R a noetherian semilocal ring.

It is well known that finitely generated projective modules are isomorphic if and only if they are isomorphic modulo the Jacobson radical. Recently, Příhoda in [20] proved that the same holds true for arbitrary projective modules. Hence, if R is a ring with Jacobson radical $J(R)$, we can see not only $V(R)$ as a submonoid of $V(R/J(R))$ but also $V^*(R)$ is a submonoid of $V^*(R/J(R))$. This is an essential tool in this paper.

A ring R is said to be semilocal if it is semisimple artinian modulo its Jacobson radical $J(R)$. To fix notation, we assume that $R/J(R) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ where $D_1, \ldots, D_k$ are division rings uniquely determined up to isomorphism. The monoids $V(R)$ and $V^*(R)$ can be viewed as submonoids of $V(R/J(R)) \cong \mathbb{N}_0^k$ and $V^*(R/J(R)) \cong (\mathbb{N}_0 \cup \{\infty\})^k = (\mathbb{N}_0^*)^k$, respectively; the class of R corresponds to the element $(n_1, \ldots, n_k) \in \mathbb{N}_0^k$. The submonoids of $\mathbb{N}_0^k$ containing $(n_1, \ldots, n_k)$ that can be realized as $V(R)$ for a semilocal ring R were characterized in [8] as the set of solutions in $\mathbb{N}_0^k$ of systems of diophantine equations of the form

$$D \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad \text{and} \qquad E_1 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \qquad (1)$$

where the coefficients of the matrices D, E_1 and E_2 as well as $m_1, \ldots, m_n$ are elements of $\mathbb{N}_0$. Such submonoids of $\mathbb{N}_0^k$ are called full affine submonoids (cf. Definition 6.1.5 and Proposition 6.6.2). This terminology was introduced by Hochster in [17], however full affine monoids appear in different contexts with different names. In the setting of commutative noetherian rings they are also called positive normal monoids, see [1]. Such

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monoids also appear in generalizations of the multiplicative ideal theory where they are called finitely generated Krull monoids see, for example, [5].

In this paper we show that the submonoids of $(\mathbb{N}_0^*)^k$ that can be realized as $V^*(R)$ for a noetherian semilocal ring R are precisely the sets of solutions in $(\mathbb{N}_0^*)^k$ of systems of type (1). We refer the reader to Theorem 6.2.6 for the precise statement. Most of the paper is devoted to the proof of Theorem 6.2.6 which has, essentially, two quite different parts. A more ring theoretical one, in which we provide the necessary tools to construct noetherian semilocal rings with prescribed monoid $V^*(R)$. Our key idea is to use a well known theorem due to Milnor to determine a context in which the category of right projective modules over a pullback of rings is equivalent to the pullback of the categories of projective modules. Surprisingly enough, just considering suitable pullbacks of semilocal principal ideal domains (or just noetherian semilocal rings such that all projective modules are free) and semisimple artinian rings a rich supply of noetherian semilocal rings R with non-trivial $V^*(R)$ is obtained.

The second part of the paper (and of the proof of Theorem 6.2.6) deals with submonoids of $(\mathbb{N}_0^*)^k$. Our starting point are the results in [21] where it was proven that, for a noetherian semilocal ring, $V^*(R)$ is built up from a finite collection of full affine submonoids of $\mathbb{N}_0^{r_1}, \dots, \mathbb{N}_0^{r_m}$, respectively, where $r_i \leq k$, chosen in a *compatible* way. These monoids are placed in the finite supports of the elements of $V^*(R)$ or, better saying, in the complementary of the infinite supports of the elements of $V^*(R)$, see Definition 6.1.2 for the unexplained terminology. In the paper, we make an abstraction of this type of monoid by introducing the concept of *(full affine) system of supports* in Definition 6.7.1, then $V^*(R)$, viewed as a submonoid of $(\mathbb{N}_0^*)^k$, is given by a *full affine system of supports*. Our main result in this part of the paper shows that the monoids given by a full affine system of supports are precisely the solutions in $(\mathbb{N}_0^*)^k$ of systems of the form (1). We stress the fact that though the description of these submonoids of $(\mathbb{N}_0^*)^k$ as sets of solutions of a system of equations is very elegant, and it extends nicely the characterization for the case of finitely generated projective modules, the one given by the systems of supports seems to give a better idea of the complexity of the monoids we are working with.

Going back to the module theoretic point of view, the contrast with the commutative situation is quite striking as all projective modules over a commutative semilocal indecomposable ring are free [16]. On the other hand, the noetherian situation is simpler than the general one. In the noetherian case it follows from [21] that the monoids that can appear are always finitely generated and that $V^*({}_RR) \cong V^*(R_R)$. We do not know whether, for a general semilocal ring, $V^*(R_R)$ is still finitely generated, and in [14] we construct a semilocal ring R such that all projective left R-modules are free while R has a nonzero (infinitely generated) right projective module that is not a generator, and it is not a direct sum of finitely generated projective modules. This shows that the monoid $V^*(R_R)$, in general, is not isomorphic to $V^*({}_RR)$.

Our interest on semilocal rings stems from the fact that many classes of *small modules* have a semilocal endomorphism ring. For example, artinian modules or, more generally, modules with finite Goldie and dual Goldie dimension [15], finitely presented modules over a local ring or, more generally, finitely presented modules over a semilocal ring are classes of modules with a semilocal endomorphism ring [12]. We refer to the monograph [7] as a source to read about, the good and the not so good, properties of modules with a semilocal endomorphism ring.

A description of the projective modules over the endomorphism ring is a first step towards understanding (part of) the category $\mathrm{Add}\,(M)$ of direct summands of any direct sum of copies of a module M. Wiegand in [24] proved that all monoids of solutions in $\mathbb{N}_0^k$ of systems of the form (1) can be realized as $V(R)$ when R is the endomorphism ring of a finitely presented module over a noetherian semilocal ring, or the endomorphism ring of an artinian module. Yakovlev [26, 27] proved the same kind of result for semilocal endomorphism rings of certain classes of torsion free abelian groups of finite rank. For further information, see the survey paper [25]. Our results give a new twist to the situation, as they indicate that when considering countable direct sums of such modules a rich supply of new direct summands might appear.

Let us mention a connection between projective modules over noetherian semilocal rings and integral representation theory. In [3] the so called generalized lattices were investigated. For a Dedekind domain D with a quotient field K, we consider an order R in a separable K-algebra. An R-module M is said to be a generalized R-lattice provided it is projective as a D-module. If M is also finitely generated, M is a lattice over R. R is said to be of finite lattice type if there exist only finitely many indecomposable lattices up to isomorphism. Suppose that R is of finite lattice type and let A be the direct sum of a representative set of isomorphism classes of indecomposable lattices. By [3], the category of generalized lattices over R and the category of projective modules over $\mathrm{End}_R(A)$ are equivalent. For any maximal ideal $\mathcal{M}$ of D let $R_{(\mathcal{M})}$ be the localization of R in $D \setminus \mathcal{M}$ and let $R_{(0)}$ be the localization of R in $D \setminus \{0\}$. Then monoid homomorphisms

$$V^*(\mathrm{End}_R(A)) \to V^*(\mathrm{End}_{R_{(\mathcal{M})}}(A \otimes_R R_{(\mathcal{M})})) \to V^*(\mathrm{End}_{R_{(0)}}(A \otimes_R R_{(0)}))$$

give approximations of generalized lattices over R by projective modules over the noetherian semilocal rings $\mathrm{End}_{R_{(\mathcal{M})}}(A \otimes_R R_{(\mathcal{M})})$ and an artinian ring $\mathrm{End}_{R_{(0)}}(A \otimes_R R_{(0)})$. For further results on generalized lattices see [23].

The paper is structured as follows, in §6.1 we introduce the basic language used throughout the paper. We describe the monoids of projective modules, specializing to a semisimple artinian ring, we recall the results needed to understand the relation between these monoids when considered over R and over $R/J(R)$ emphasizing on the particular case of semilocal rings. In §6.2 we specialize to the noetherian case; we recall the results from [21] essential for our investigation and we state our main Theorem 6.2.6. Sections 6.3, 6.4 and 6.5 deal with the realization part of the proof of Theorem 6.2.6; §6.3 shows how to construct principal ideal domains with prescribed semisimple factor modulo the Jacobson radical, in §6.4 we provide all the results we need on ring pullbacks in order to be able to realize the monoids we want as $V^*(R)$ of semilocal noetherian algebras in §6.5.

In section 6.6 we turn towards monoids. We recall some basics on full affine monoids, and we prove the auxiliary results that will allow us to conclude the proof of Theorem 6.2.6 in §6.7.

The potential of pullback constructions in order to give semilocal rings with prescribed semigroup of countably generated projective modules goes beyond the noetherian situation. In [14] we further investigate this direction.

All rings have 1, ring morphisms and modules are unital. We shall usually consider right modules.

Our convention is $\mathbb{N} = \{1, 2, \dots\}$, and we denote the nonnegative integers by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Another basic object in this paper is the monoid $(\mathbb{N}_0^*, +, 0)$ whose underlying set is $\mathbb{N}_0 \cup \{\infty\}$, the operation $+$ is the extension of addition of non-negative integers by the rule $\infty + x = x + \infty = \infty$. Sometimes we will be also interested in extending the product of $\mathbb{N}_0$ to $\mathbb{N}_0^*$ by setting $\infty \cdot 0 = 0$ and $\infty \cdot n = \infty$ for any $n \in \mathbb{N}_0^* \setminus \{0\}$.

For any right R-module M the *trace* of M in R is the two sided ideal of R

$$\mathrm{Tr}_R(M) = \mathrm{Tr}\,(M) = \sum_{f \in \mathrm{Hom}_R(M,R)} f(M).$$

If $X \subseteq M$ then we denote the *right annihilator* of X by

$$r_R(X) = \{r \in R \mid mr = 0 \text{ for any } m \in X\}.$$

If N is a left R-module and $Y \subseteq N$ then we denote the *left annihilator* of Y by

$$l_R(Y) = \{r \in R \mid rn = 0 \text{ for any } n \in Y\}.$$

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6.1 Monoids of projective modules

Definition 6.1.1. *Let $(M, +, 0)$ be a commutative additive monoid. An element $x \in M$ is said to be an* order unit *or* an archimedean element *of M if for any $y \in M$ there exists $n \in \mathbb{N}$ and $z \in M$ such that $nx = y + z$.*

The monoid M is said to be reduced *if for any $x \in M$, $x + y = 0$ implies $x = 0 = y$.*

Let $x, y \in M$. The relation $x \leq y$ if and only if there exists $z \in M$ such that $x + z = y$ is a preorder order on M that is called the algebraic order *or, more properly, the* algebraic preorder.

For example, any $x \in \mathbb{N}_0^*$ satisfies that $x \leq \infty$. If $k \geq 1$ the algebraic preorder of $\mathbb{N}_0^k$ and in $(\mathbb{N}_0^*)^k$ is the component-wise order and it is a partial order.

Note that if M is a monoid preordered with the algebraic preorder then all the elements must be positive, that is, bigger or equal than zero.

Let R be a ring. We denote by $V(R)$ the monoid of isomorphism classes of finitely generated projective right R-modules with the operation induced by the direct sum. That is, if P_1 and P_2 are finitely generated projective right R-modules then $\langle P_1 \rangle + \langle P_2 \rangle = \langle P_1 \oplus P_2 \rangle$. The monoid $V(R)$ is commutative, reduced and it has an order unit $\langle R \rangle$. We usually think on $V(R)$ as a monoid preordered by the algebraic preorder.

Similarly, we define $V^*(R)$ to be the monoid of isomorphism classes of countably generated projective right R-modules with the sum induced by the direct sum. Clearly $V(R)$ is a preordered submonoid of the preordered monoid $V^*(R)$.

The functor $\mathrm{Hom}_R(-, R)$ induces a monoid isomorphism between $V(R) = V(R_R)$ and the monoid of isomorphism classes of finitely generated left projective modules $V({}_RR)$. This is no longer true for countably generated projective modules so, in general, $V^*(R) = V^*(R_R)$ is not isomorphic to $V^*({}_RR)$, cf. [14].

If $\varphi\colon R_1 \to R_2$ is a ring morphism then the functor $- \otimes_{R_1} R_2$ induces a morphism of monoids with order unit $V(\varphi)\colon V(R_1) \to V(R_2)$ and a morphism of monoids

$V^*(\varphi)\colon V^*(R_1)\to V^*(R_2)$. Both morphisms are given by the formula $\langle P\rangle\mapsto\langle P\otimes_{R_1}R_2\rangle$. Another useful way to describe these monoid morphisms is describing projective modules via idempotent matrices.

Let P_{R_1} be a finitely generated projective right R_1-module. There exist $n\in\mathbb{N}$ and an idempotent matrix $E\in M_n(R_1)$ such that $P\cong ER_1^n$, then $V(\varphi)(\langle ER_1^n\rangle)=\langle M_n(\varphi)(E)R_2^n\rangle$ where $M_n(\varphi)\colon M_n(R_1)\to M_n(R_2)$ is the map defined by $M_n(\varphi)(a_{ij})=(\varphi(a_{ij}))$. One proceeds similarly with the countably generated projective right R_1-modules taking instead of finite matrices elements in $\mathrm{CFM}(R_1)$ and $\mathrm{CFM}(R_2)$ the rings of (countable) column finite matrices with entries in R_1 and R_2, respectively.

6.1.1 The semisimple artinian case

Let R be a semisimple artinian ring. Then by the Artin-Wedderburn theorem, there exist $k\in\mathbb{N}$, $n_1,\dots,n_k\in\mathbb{N}$, $D_1,\dots,D_k$ division rings, and a ring isomorphism $\varphi\colon R\to M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$.

Let $(V_1,\dots,V_k)$ be an ordered set of representatives of the isomorphism classes of simple right R-modules such that $\mathrm{End}_R(V_i)\cong D_i$ and, hence, $\dim({}_{D_i}V_i)=n_i$ for $i=1,\dots,k$. If P_R is a finitely generated projective module then $P_R\cong V_1^{x_1}\oplus\cdots\oplus V_k^{x_k}$. The assignment $\langle P\rangle\mapsto(x_1,\dots,x_k)\in\mathbb{N}_0^k$ induces an isomorphism of monoids $\dim_\varphi\colon V(R)\to\mathbb{N}_0^k$. Since $\dim_\varphi(\langle R\rangle)=(n_1,\dots,n_k)$, taking $(n_1,\dots,n_k)$ as the order unit of $\mathbb{N}_0^k$, $\dim_\varphi$ becomes an isomorphism of monoids with order unit. We call $\dim_\varphi(\langle P\rangle)$ or, by abuse of notation $\dim_\varphi(P)$, *the dimension vector of the (finitely generated) projective module P.*

The morphism $\dim_\varphi$ extends to a monoid morphism $\dim_\varphi\colon V^*(R)\to(\mathbb{N}_0^*)^k$ by setting $\dim_\varphi(\langle V_i^{(\aleph_0)}\rangle)=(0,\dots,\infty^{i)},\dots,0)$ for $i=1,\dots,k$. Again, we call $\dim_\varphi(\langle P\rangle)$ *the dimension vector of the (countably generated) projective module P.*

Throughout the paper it is important to keep in mind how to compute dimension vectors in terms of idempotent matrices. If P is a finitely generated (countably generated) right projective module such that $\dim_\varphi(P)=(x_1,\dots,x_k)$ then $P\cong(E_1,\dots,E_k)\cdot F$ where F is a finitely generated (countably generated) free right R-module and E_i are idempotent matrices over $M_{n_i}(D_i)$ (over $\mathrm{CFM}(D_i)$) such that $\mathrm{rank}_{D_i}(E_i)=x_i$ for $i=1,\dots,k$.

Notice that $\dim_\varphi$ depends on the ordering of the isomorphism classes of the simple right modules. Therefore when we refer to a $\dim_\varphi$ function or to dimension vectors we implicitly assume that we have chosen an ordering of the simple modules. If we explicitly state that the semisimple artinian ring R is isomorphic to $M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$, for $D_1,\dots,D_k$ division rings, then we assume we are choosing an ordered family of representatives of the isomorphism classes of simple right (or left) R-modules $(V_1,\dots,V_k)$ such that $\mathrm{End}_R(V_i)\cong D_i$ for $i=1,\dots,k$.

To ease the work with the elements in $\mathbb{N}_0^*$ we shall use the following definitions.

Definition 6.1.2. *Let $\mathbf{x}=(x_1,\dots,x_k)\in(\mathbb{N}_0^*)^k$. We define*

$$\mathrm{supp}\,(\mathbf{x})=\{i\in\{1,\dots,k\}\mid x_i\neq 0\}$$

and we refer to this set as the support of $\mathbf{x}$. *We also define*

$$\text{inf-supp}\,(\mathbf{x})=\{i\in\{1,\dots,k\}\mid x_i=\infty\},$$

we refer to this set as the infinite support of $\mathbf{x}$.

6.1.2 Passing modulo the Jacobson radical

First we recall the following well known Lemma

Lemma 6.1.3. *Let R be a ring with Jacobson radical $J(R)$. Let P and Q be projective right R-modules.*

(i) Assume P and Q are finitely generated. If there exists a projective right $R/J(R)$-module X such that $P/PJ(R) \cong Q/QJ(R) \oplus X$ then there exists a projective right R-module Q' such that $P \cong Q \oplus Q'$ and $Q'/Q'J(R) \cong X$.

(ii) Assume only that Q is finitely generated. If $f\colon P/PJ(R) \to Q/QJ(R)$ is an onto module homomorphism then Q is isomorphic to a direct summand of P.

If, in the situation of the above Lemma, neither P nor Q are finitely generated then even the weaker divisibility property (ii) is lost. It was shown in [20] that it is still true that projective modules isomorphic modulo the Jacobson radical are isomorphic. We recall this fundamental result in the next statement together with a weaker property on lifting pure monomorphisms.

We recall that a right module monomorphism $f\colon M_1 \to M_2$ is said to be a pure monomorphism if, for any left module N, $f \otimes_R N\colon M_1 \otimes_R N \to M_2 \otimes_R N$ remains a monomorphism. For example, if f is a (locally) split monomorphism then it is pure. If f is a monomorphism between two projective modules then f is pure if and only if $\operatorname{coker} f$ is a flat module if and only if f is locally split.

Theorem 6.1.4. *Let R be any ring, and let P and Q be projective right R-modules.*

(i) [11, Proposition 6.1] *A module homomorphism $f\colon P \to Q$ is a pure monomorphism if and only if so is the induced map $\overline{f}\colon P/PJ(R) \to Q/QJ(R)$.*

(ii) [20, Theorem 2.3] *If $f\colon P/PJ(R) \to Q/QJ(R)$ is an isomorphism of right $R/J(R)$-modules then there exists an isomorphism of right R-modules $g\colon P \to Q$ such that the induced morphism $\overline{g}\colon P/PJ(R) \to Q/QJ(R)$ coincides with f.*

Theorem 6.1.4(ii) allows us to see the monoids $V(R)$ and $V^*(R)$ as submonoids of $V(R/J(R))$ and of $V^*(R/J(R))$, respectively. To give the assertion in a more precise way we shall use the following notion (cf. [9]).

Definition 6.1.5. *A submonoid A of a monoid C is said to be a* full submonoid *of C if for any $x \in A$ and any $t \in C$, $x+t \in A$ implies $t \in A$. If $f\colon A \to C$ is an injective monoid homomorphism and* $\operatorname{im}(f)$ *is a full submonoid of C we say that f is a* full embedding.

A full affine monoid *is a full submonoid of a finitely generated free commutative monoid, and a* full affine embedding *of a monoid A is a full embedding of A into a finitely generated free commutative monoid.*

See Proposition 6.6.2 for a characterization of full affine submonoids of $\mathbb{N}_0^k$.

We note that in the terminology of [1] a full affine embedding is a pure embedding of monoids.

Now we are ready to state the announced result for the monoids of projective modules.

Corollary 6.1.6. *Let R be a ring with Jacobson radical $J(R)$, and let $\pi\colon R \to R/J(R)$ denote the canonical projection. Then:*

(i) *$V(\pi)\colon V(R) \to V(R/J(R))$ is a full embedding of monoids with order unit. In particular, the algebraic preorder on $V(\pi)(V(R))$ coincides with the one induced by the algebraic preorder on $V(R/J(R))$.*

(ii) *$V^*(\pi)\colon V^*(R) \to V^*(R/J(R))$ is an injective monoid morphism.*

There is an interesting intermediate submonoid between $V(R)$ and $V^*(R)$.

Definition 6.1.7. *Let R be a ring. Set $W(R) = W(R_R)$ to be the additive monoid of isomorphism classes of countably generated projective right R-modules that are pure submodules of a finitely generated free right R-module. The addition on $W(R)$ is induced by the direct sum of modules.*

Analogously, $W({}_RR)$ is the additive monoid of isomorphism classes of projective left R-modules that are pure submodules of a finitely generated free right R-module.

For example, if $R = \mathcal{C}([0,1])$ is the ring of real valued continuous functions defined on the interval $[0,1]$ then the ideal

$$I = \{f \in R \mid \text{ there exists } \varepsilon > 0 \text{ such that } f([0,\varepsilon]) = 0\}$$

is countably generated, projective and pure inside R. Therefore $\langle I\rangle \in W(R) \setminus V(R)$.

If $\varphi\colon R \to S$ is a ring homomorphism then there is a homomorphism of monoids $W(\varphi)\colon W(R) \to W(S)$ defined, as usual, by $\langle P\rangle \mapsto \langle P \otimes_R S\rangle$.

The notation $W(R)$ is borrowed from the C^*-algebra world, as we think on $W(R)$ as the discrete analogue of the Cuntz monoid (cf. [6])

The following result, which is a consequence of [11, Theorem 7.1] and Theorem 6.1.4, describes one way to obtain elements in $W(R) \setminus V(R)$ and which is the only one when the ring R is semilocal.

Proposition 6.1.8. *Fix $n \in \mathbb{N}$. Let R be a ring. Let P_1, P_2 be finitely generated projective right $R/J(R)$-modules such that $(R/J(R))^n \cong P_1 \oplus P_2$. Then the following statements are equivalent*

(i) *There exists a projective right R-module P such that $P/PJ(R) \cong P_1$.*

(ii) *There exists a pure right submodule M of R^n such that $M/MJ(R) \cong P_1$.*

(iii) *There exists a projective left R-module Q such that $Q/J(R)Q \cong \mathrm{Hom}_{R/J(R)}(P_2, R/J(R))$.*

When the above statements hold P and Q are countably generated pure submodules of R^n, and they are finitely generated if and only if there exists a projective right R-module P' such that $P'/P'J(R) \cong P_2$.

Observe that, by Theorem 6.1.4, the isomorphism class of the module P appearing in Proposition 6.1.8 is an element of $W(R_R)$ and the isomorphism class of Q gives an element of $W({}_RR)$. Therefore if P is not finitely generated, $\langle P/PJ(R)\rangle \le \langle (R/J(R))^n\rangle$ in $W(R/J(R))$ but P is not a direct summand of R^n. So that, in general, the algebraic preorder on $W(R)$ does not coincide with the order induced by the algebraic preorder on $W(R/J(R))$.

We study in more detail the monoid $W(R)$ in [14]. If R is noetherian then, clearly, $V(R) = W(R)$. Results of Lazard [18] show that this also holds just assuming ascending chain condition on annihilators.

Proposition 6.1.9. *Let R be a ring such that, for any $n \in \mathbb{N}$, $M_n(R)$ has the ascending chain condition on right annihilators of elements. Then a pure submodule of a finitely generated free right R-module is finitely generated and, in particular $V(R) = W(R)$.*

Proof. Combine [18, Lemme 2(i)] with the argument in [10, Corollary 3.6]. ■

6.1.3 Semilocal rings

Let R be a semilocal ring such that $R/J(R) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for suitable division rings $D_1, \dots, D_k$. Fix an onto ring homomorphism $\varphi\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ such that $\operatorname{Ker}\varphi = J(R)$. Then there is an induced ring isomorphism $\overline{\varphi}\colon R/J(R) \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, so that we have a dimension function $\dim_{\overline{\varphi}}$, cf. § 6.1.1. For any countably generated projective right R-module P, set

$$\dim_{\varphi}(\langle P\rangle) := \dim_{\overline{\varphi}}(\langle P \otimes_R R/J(R)\rangle) = \dim_{\overline{\varphi}}(\langle P/PJ(R)\rangle).$$

By Corollary 6.1.6, $\dim_{\varphi}(V(R))$ is a full affine submonoid of $\mathbb{N}_0^k$ with order unit $(n_1, \dots, n_k)$ and $\dim_{\varphi}(V^*(R))$ is a submonoid of $(\mathbb{N}_0^*)^k$.

It was shown in [8] that the full affine property characterizes the monoids A with order unit that can be realized as $V(R)$ of some semilocal ring R. More precisely, if A is a full affine submonoid of $\mathbb{N}_0^k$ with order unit $(n_1, \dots, n_k)$ then there exist a semilocal hereditary ring R, $D_1, \dots, D_k$ division rings and an onto ring homomorphism $\varphi\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ with kernel $J(R)$ such that $\dim_{\varphi} V(R) = A$.

Since over a hereditary ring any projective module is a direct sum of finitely generated projective modules, it follows that for a hereditary ring R as above

$$\dim_{\varphi} V^*(R) = A + \infty \cdot A \subseteq (\mathbb{N}_0^*)^k \qquad (*)$$

(see also Corollary 6.7.9 and Proposition 6.6.7) where $\infty \cdot A = \{\infty \cdot a \mid a \in A\}$. Indeed, it is always true that $\dim_{\varphi} V^*(R) \supseteqq A + \infty \cdot A$; the other equality holds because full affine submonoids of $\mathbb{N}_0^k$ are finitely generated.

6.2 Semilocal rings: The noetherian case

We start this section recalling some results on projective modules over noetherian semilocal rings from [21] and adapting them to our purposes. We also state in 6.2.6 our main characterization theorem.

It is well known that the trace ideal of a projective module is an idempotent ideal. Whitehead in [22] characterized idempotent ideals that are trace ideals of countably generated projective modules. His results yield that in a noetherian ring any idempotent ideal is a trace ideal of a countably generated projective module. In [21], Příhoda noted that Whitehead's ideas can be extended to prove that if I is an idempotent ideal of a noetherian ring R then any finitely generated projective R/I-module can be extended to a projective R-module. For further quotation we state these results.

Proposition 6.2.1. *Let R be a noetherian ring. Then the following statements are equivalent for a two sided ideal I*

(i) $I^2 = I$.

(ii) *There exists a countably generated projective right R-module P such that* $\mathrm{Tr}\,(P) = I$.

(iii) *For any finitely generated projective right R/I-module* P' *there exists a countably generated projective right R-module P such that* $P/PI \cong P'$ *and* $I \subseteq \mathrm{Tr}\,(P)$.

(iv) *There exists a countably generated projective left R-module Q such that* $\mathrm{Tr}\,(Q) = I$.

(v) *For any finitely generated projective left R/I-module* Q' *there exists a countably generated projective left R-module Q such that* $Q/IQ \cong Q'$ *and* $I \subseteq \mathrm{Tr}\,(Q)$.

PROOF. Combine [22, Corollary 2.7] with [21, Lemma 2.6]. ■

Trace ideals of projective modules keep memory of the semisimple factors of the projective module.

Lemma 6.2.2. *Let R be a semilocal ring, and let P be a projective right module with trace ideal I. Let V_R be a simple right R-module with endomorphism ring D, and let $W = \mathrm{Hom}_D(V, D) \cong \mathrm{Hom}_R(V, R/J(R))$ be its dual simple left R-module. Then the following statement are equivalent:*

(i) *V is a quotient of P.*

(ii) *V is a quotient of I.*

(iii) $I + r_R(V) = I + l_R(W) = R$.

(iv) *W is a quotient of I.*

In particular, if I is also the trace ideal of a left projective module Q then the above statements are also equivalent to the fact that W is a quotient of Q.

PROOF. The equivalence of (i) and (ii) is a particular case of [21, Lemma 3.3]. It is clear that (iii) is equivalent to (ii) because, for a semilocal ring, $r_R(V) = l_R(W)$ is a maximal two-sided ideal of R. Statements (iii) and (iv) are equivalent by the symmetry of (iii). ■

Theorem 6.2.3. [21] *Let R be a noetherian semilocal ring. Let $V_1, \dots, V_k$ be an ordered set of representatives of the isomorphism classes of simple right R-modules.*

For $i = 1, \dots, k$, let $D_i = \mathrm{End}_R(V_i)$ and $W_i = \mathrm{Hom}_{D_i}(V_i, D_i) \cong \mathrm{Hom}_R(V_i, R/J(R))$. So that $W_1, \dots, W_k$ is an ordered set of representatives of the isomorphism classes of simple left R-modules. Let S be a subset of $\{1, \dots, k\}$. Assume that there exists a countably generated projective right R-module P such that

$$P/PJ(R) \cong \left(\oplus_{i\in\{1,\dots,k\}\setminus S} V_i^{n_i}\right) \oplus \left(\oplus_{i\in S} V_i^{(\aleph_0)}\right),$$

where $n_i \in \mathbb{N}_0$. Then the following statements hold:

(1) *There exists a countably generated projective right R-module P' such that $P'/P'J(R) \cong \oplus_{i\in S} V_i^{(\aleph_0)}$. Hence $P \cong P \oplus P'$.*

(2) *Let I be the trace ideal of P'. Then P/PI is a finitely generated right R/I-module such that*

$$P/PI \otimes_{R/I} (R/I)\,/J(R/I) \cong P/P(I + J(R)) \cong \oplus_{i\in\{1,\dots,k\}\setminus S} V_i^{n_i}.$$

(3) There exists a countably generated projective left R-module Q such that

$$Q/J(R)Q \cong \left(\oplus_{i\in\{1,\dots,k\}\setminus S} W_i^{n_i}\right) \oplus \left(\oplus_{i\in S} W_i^{(\aleph_0)}\right)$$

(4) There exists a countably generated projective left R-module Q' such that $Q'/J(R)Q' \cong \oplus_{i\in S} W_i^{(\aleph_0)}$. Hence $Q \cong Q \oplus Q'$.

Therefore, $V^*(R_R) \cong V^*({}_RR)$ and, fixing $\varphi\colon R \to M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$ an onto ring homomorphism with kernel $J(R)$, we obtain that $\dim_\varphi V^*(R_R) = \dim_\varphi V^*({}_RR)$ and that $\dim_\varphi V(R) = (\dim_\varphi V^*(R)) \cap \mathbb{N}_0^k$.

Proof. (1). The existence of P' follows from [21, Proposition 3.4]. The isomorphism $P \cong P \oplus P'$ follows from Theorem 6.1.4(ii).

Statement (2) is also part of [21, Proposition 3.4].

By Proposition 6.2.1, I is also the trace ideal of a projective left R-module M. As $M^{(\aleph_0)}/J(R)M^{(\aleph_0)}$ is semisimple and contains all semisimple factors of M it follows from Lemma 6.2.2 that $M^{(\aleph_0)}/J(R)M^{(\aleph_0)} \cong \oplus_{i\in S}(W_i)^{(\aleph_0)}$. Therefore taking $Q' = M^{(\aleph_0)}$ we deduce that the first statement of (4) holds.

By (2), P/PI is a finitely generated R/I-module. Therefore $\overline{Q} = \mathrm{Hom}_{R/I}(P/PI, R/I)$ is a finitely generated projective left R/I-module such that $\overline{Q}/J(\overline{Q}) \cong \oplus_{i\in\{1,\dots,k\}\setminus S} W_i^{n_i}$. By Proposition 6.2.1, there exists a projective left R-module Q_1 such that $Q_1/IQ_1 \cong \overline{Q}$. Then $Q = Q_1 \oplus Q'$ fulfills the requirements of statement (3) and the second half of statement (4).

Finally, note that the assignment $\langle P\rangle \mapsto \langle Q\rangle$ induces an isomorphism between $V^*(R_R)$ and $V^*({}_RR)$ such that $\dim_\varphi(V^*(R_R)) = \dim_\varphi(V^*({}_RR))$. The claim on $\dim_\varphi V(R)$ follows either from (2) or from combining Proposition 6.1.8 with Proposition 6.1.9. ∎

As a corollary of Theorem 6.2.3 we note that, in the context of noetherian semilocal rings, the divisibility property of Lemma 6.1.3(ii) still holds for general projective modules.

Corollary 6.2.4. *Let R be a noetherian semilocal ring. Let P and Q be projective right R-modules such that $P/PJ(R)$ is isomorphic to a direct summand of $Q/QJ(R)$ then P is isomorphic to a direct summand of Q.*

Proof. Since any projective module is a sum of countably generated projective modules we may assume that P and Q are countably generated [7, Proposition 2.50].

Let $V_1, \dots, V_k$ be an ordered set of representatives of the isomorphism classes of simple right R-modules. Since $P/PJ(R)$ is a homomorphic image of $Q/QJ(R)$ there exist $S' \subseteq S \subseteq \{1,\dots,k\}$ such that

$$Q/QJ(R) \cong \left(\oplus_{i\in\{1,\dots,k\}\setminus S} V_i^{n_i}\right) \oplus \left(\oplus_{i\in S} V_i^{(\aleph_0)}\right)$$

and

$$P/PJ(R) \cong \left(\oplus_{i\in\{1,\dots,k\}\setminus S'} V_i^{m_i}\right) \oplus \left(\oplus_{i\in S'} V_i^{(\aleph_0)}\right)$$

where n_i and m_j are in $\mathbb{N}_0$, and $n_i - m_i \in \mathbb{N}_0$ for any $i \in \{1,\dots,k\}\setminus S$.

By Theorem 6.2.3, there exists a countably generated projective module Q' such that $Q'/Q'J(R) \cong \oplus_{i\in S} V_i^{(\aleph_0)}$. Let I be the trace ideal of Q'. Again by Theorem 6.2.3, $\overline{P} = P/PI$ and $\overline{Q} = Q/QI$ are finitely generated projective right R/I-modules.

Now
$$\overline{Q}/\overline{Q}J(R/I) \cong Q/Q(I+J(R)) \cong \oplus_{i\in\{1,\dots,k\}\setminus S} V_i^{n_i}$$
and
$$\overline{P}/\overline{P}J(R/I) \cong P/P(I+J(R)) \cong \oplus_{i\in\{1,\dots,k\}\setminus S} V_i^{m_i}$$
By Corollary 6.1.6, there exists a finitely generated projective right R/I-module $\overline{X}$ such that $\overline{X}/\overline{X}J(R/I) \cong \oplus_{i\in\{1,\dots,k\}\setminus S} V_i^{n_i-m_i}$. By Proposition 6.2.1, there exists a countably generated projective right R-module X such that $X/XI \cong \overline{X}$. By Theorem 6.1.4(ii), $Q \cong P \oplus Q' \oplus X$. ■

After some amount of work, it will turn out that Proposition 6.2.1 and Theorem 6.2.3 contain all the information needed to describe $V^*(R)$ for R a noetherian semilocal ring.

Definition 6.2.5. *Let $k \geq 1$. A submonoid M of $(\mathbb{N}_0^*)^k$ is said to be a monoid* defined by a system of equations *provided that there exist $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times k}(\mathbb{N}_0)$ and $m_1,\dots,m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1,\dots,n\}$, such that M is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system of equations*

$$D\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \quad (*) \qquad \text{and} \qquad E_1\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \quad (**)$$

where ℓ, $n \geq 0$. By convention, ℓ or n equal to zero means that either $(*)$ or $(**)$ are empty systems.

As we shall recall in Proposition 6.6.2, any full affine monoid of $\mathbb{N}_0^k$ is of the form $M \cap \mathbb{N}_0^k$ where M is a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations.

Now we can state our main theorem,

Theorem 6.2.6. *Let $k \in \mathbb{N}$. Let M be a submonoid of $(\mathbb{N}_0^*)^k$ containing $(n_1,\dots,n_k) \in \mathbb{N}^k$. Then the following statements are equivalent:*

(1) M is defined by a system of equations.

(2) For any field F there exist a noetherian semilocal F-algebra R, a semisimple F-algebra $S = M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$, where $D_1,\dots,D_k$ are division rings, and an onto morphism of F-algebras $\varphi\colon R \to S$ with $\operatorname{Ker}\varphi = J(R)$ such that $\dim_\varphi V^*(R) = M$. In particular, $\dim_\varphi V(R) = M \cap \mathbb{N}_0^k$.

(3) There exist a noetherian semilocal ring R, a semisimple ring $S = M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$, where $D_1,\dots,D_k$ are division rings, and an onto ring morphism $\varphi\colon R \to S$ with $\operatorname{Ker}\varphi = J(R)$ such that $\dim_\varphi V^*(R) = M$. Therefore, $\dim_\varphi V(R) = M\cap\mathbb{N}_0^k$.

Remark 6.2.7. *We follow the notation of Definition 6.2.5. As it is done for full affine monoids in [2, Exercise 6.4.16] or [1, Proof of Theorem 2.29], if $M \subseteq (\mathbb{N}_0^*)^k$ is defined by a system of equations as in 6.2.5 then it is isomorphic to the submonoid M' of $(\mathbb{N}_0^*)^{k+n}$ defined by system of linear diophantine equalities*

$$D\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} m_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_n \end{pmatrix}\begin{pmatrix} t_{k+1} \\ \vdots \\ t_{k+n} \end{pmatrix} \qquad \text{and} \qquad E_1\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}$$

The isomorphism is given by the assignment

$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_k, \frac{1}{m_1} \sum_{j=1}^{k} d_{1j} x_j, \dots, \frac{1}{m_n} \sum_{j=1}^{k} d_{nj} x_j),$$

where we make the convention $\frac{\infty}{m_i} = \infty$.

Therefore it is important to take into account that we are considering our monoids always inside some fixed $(\mathbb{N}_0^*)^k$ *or, in the ring context, that we are viewing* $V^*(R)$ *as a submonoid of* $V^*(R/J(R))$.

The monoid $\mathbb{N}_0^*$ is not cancellative, therefore the solutions of two systems of equations may coincide over $\mathbb{N}_0$ but be different when considered over $\mathbb{N}_0^*$. We illustrate this phenomena with an easy example.

Example 6.2.8. *The set of solutions of the equation* $x = y$ *in* $\mathbb{N}_0^2$ *is* $M = \{(n,n) \mid n \in \mathbb{N}_0\}$, *and the set of solutions in* $(\mathbb{N}_0^*)^2$ *is* $M + \infty \cdot M = M \cup \{(\infty, \infty)\}$.

The set of solutions of $2x = x + y$ *in* $\mathbb{N}_0^2$ *is, of course, also* M *but in* $(\mathbb{N}_0^*)^2$ *is* $M_1 = M \cup \{(\infty, n) \mid n \in \mathbb{N}_0^*\}$.

Finally, the set of solutions of $2x + y = x + 2y$ *in* $(\mathbb{N}_0^*)^2$ *is* $M_1 \cup \{(n, \infty) \mid n \in \mathbb{N}_0^*\}$.

Theorem 6.2.6 shows that, for noetherian semilocal rings, the description of $V^*(R)$ viewed inside $V^*(R/J(R))$ nicely extends the one of $V(R)$ inside $V(R/J(R))$ (cf. Proposition 6.6.2). In [14] we give examples showing that the picture for general semilocal rings must be more complicated.

6.3 Semilocal principal ideal domains

We recall that a ring R is a *principal ideal domain* if R is a right and left principal ideal domain, that is, if every right ideal of R has the form aR for some $a \in R$ and every left ideal of R has the form Ra for some $a \in R$.

Semilocal principal ideal domains are a source of semilocal noetherian rings such that all projective modules are free. Our aim in this section is to construct semilocal PID's with certain types of semisimple factors.

Let R be a commutative ring. Let $k \geq 1$, and let $\mathcal{M}_1, \dots, \mathcal{M}_k$ be different maximal ideals of R. The localization of R at the set $\Sigma = R \setminus (\mathcal{M}_1 \cup \dots \cup \mathcal{M}_k)$ is a semilocal ring such that modulo its Jacobson radical is isomorphic to $R/\mathcal{M}_1 \times \dots \times R/\mathcal{M}_k$.

Fuller and Shutters observed in [13] that the same procedure to construct semilocal rings can be extended to, non necessarily commutative, principal ideal domains by using Ore localization.

Proposition 6.3.1. [13, Proposition 4] *Let* $\varphi \colon R \to S$ *be a surjective ring homomorphism of a principal ideal domain* R *onto a semisimple artinian ring* S. *Let* $\Sigma = \{a \in R \mid \varphi(a)$ *is invertible in* $S\}$. *Then:*

(i) Σ *is a right and left Ore set.*

(ii) The Ore localization R_Σ *of* R *with respect to* Σ *is a semilocal principal ideal domain, and the extension* $\overline{\varphi} \colon R_\Sigma \to S$ *of* φ *induces an isomorphism* $R_\Sigma / J(R_\Sigma) \cong S$.

Next result gives a source of examples where to apply Proposition 6.3.1.

Let E be any ring, and let $\alpha\colon E \to E$ a (unital) ring morphism. The skew polynomial ring or the twisted polynomial ring is the ring

$$E[x;\alpha] = \{p(x) = p_0x^m + \cdots + p_m \mid m \in \mathbb{N}_0 \text{ and } p_i \in E \text{ for } i = 0,\dots,m\}$$

with componentwise addition and multiplication induced by the rule $xr = \alpha(r)x$ for any $r \in E$.

It is well known that if E is a division ring and α is an automorphism then $E[x;\alpha]$ has a right and a left division algorithm, hence, it is a principal ideal domain.

Proposition 6.3.2. *Let E be a field. Let $\alpha\colon E \to E$ be a field automorphism of order n with fixed field $E^\alpha = \{a \in E \mid \alpha(a) = a\}$. Then the skew polynomial ring $R = E[x;\alpha]$ has a simple factor isomorphic to $M_n(E^\alpha)$.*

Moreover, if E is infinite then, for any $k \in \mathbb{N}$, R has a factor isomorphic to $M_n(E^\alpha)^k$.

Proof. We may assume that $n > 1$.

Note the following fact that will be useful throughout the proof:

(*) *Let $p(x) = x^m + p_1x^{m-1} + \cdots + p_m \in R$ be such that $p_m \neq 0$. If $a \in E$ satisfies that $ap(x) \in p(x)R$ then $\alpha^m(a) = a$.*

As α has order n, the center of R contains (in fact coincides with) $E^\alpha[x^n]$. Therefore, for any $0 \neq t \in E^\alpha$, the right ideal $(x^n - t)R$ is two-sided. As R is a right principal ideal domain, (*) yields that $(x^n - t)R$ is a maximal two-sided ideal, so that $R/(x^n - t)R$ is a simple artinian ring. We claim that if $t = r^n$ for some $r \in E^\alpha$ then $R/(x^n - t)R \cong M_n(E^\alpha)$. To prove this we need to find a simple right $R/(x^n - t)R$-module such that its endomorphism ring is E^α and its dimension over E^α is n.

In $E^\alpha[x] \subseteq R$ we have a decomposition $x^n - r^n = (x - r)q(x)$. As $x^n - r^n$ is central in R, $V = R/(x - r)R$ is a right $R/(x^n - r^n)R$-module. It is readily checked that V is a right E-vector space of dimension 1, therefore it is a simple right $R/(x^n - r^n)R$-module.

$\mathrm{End}_R(V) = \mathcal{I}/(x-r)R$, where $\mathcal{I} = \{p(x) \in R \mid p(x)(x-r) \in (x-r)R\}$ is the idealizer of $(x - r)R$ in R. As any $p(x) \in R$ can be written in a unique way as $a + (x - r)q(x)$ for $a \in E$,

$$\mathrm{End}_R(V) \cong \mathcal{I} \cap E = \{a \in E \mid a(x - r) \in (x - r)R\}$$

By (*), $\mathcal{I} \cap E = E^\alpha$ and, hence, $\mathrm{End}_R(V) \cong E^\alpha$. Since ${}_{E^\alpha}V \cong {}_{E^\alpha}E$ and, by Artin's Theorem, $[E : E^\alpha] = n$ we deduce that the dimension of V over its endomorphism ring is n as desired.

Now assume that E, and hence E^α, is infinite. Fix $k \in \mathbb{N}$. Let $r_1,\dots,r_k \in E^\alpha$ be such that $r_1^n,\dots,r_k^n$ are k different elements. Consider the ring homomorphism

$$\Phi\colon R \to R/(x^n - r_1^n)R \times \cdots \times R/(x^n - r_k^n)R$$

defined by $\Phi(p(x)) = (p(x) + (x^n - r_1^n)R,\dots,p(x) + (x^n - r_k^n)R)$. By the Chinese remainder Theorem, Φ is also onto. Therefore, by the first part of the proof, Φ induces an isomorphism $R/\mathrm{Ker}\,\Phi \cong M_n(E^\alpha)^k$. ∎

In Theorem 6.5.3 we will use the following examples.

Examples 6.3.3. *Let $n, k \in \mathbb{N}$.*

(i) There exists a semilocal $\mathbb{Q}$-algebra R that is a principal ideal domain such that $R/J(R) \cong M_n(\mathbb{Q})^k$.

(ii) Let F be any field, and consider the transcendental extension of F, $E = F(t_1, \dots, t_n)$. Let $\alpha\colon E \to E$ be the automorphism of E that fixes F and satisfies that $\alpha(t_i) = t_{i+1}$ for $i = 1, \dots, n-1$ and $\alpha(t_n) = t_1$. Then there exists a semilocal F-algebra R, that is a principal ideal domain, such that $R/J(R) \cong M_n(E^\alpha)^k$.

In both cases all projective right or left modules over R are free.

PROOF. (i) Let $\mathbb{Q} \subseteq E$ be a Galois field extension with Galois group $G \cong \mathbb{Z}/n\mathbb{Z}$. Let $\alpha\colon E \to E$ be a generator of G. By Proposition 6.3.2, there exists an onto ring homomorphism $\varphi\colon E[x;\alpha] \to M_n(\mathbb{Q})^k$. By Proposition 6.3.1, $\Sigma = \{a \in R \mid \varphi(a) \text{ is invertible in } M_n(\mathbb{Q})^k\}$ is a right and left Ore set and $(E[x;\alpha])_\Sigma$ has the desired properties.

(ii) Proceed as in (i) combining Proposition 6.3.2 with Proposition 6.3.1. ∎

6.4 Pullbacks of rings

We shall use ring pullbacks to construct noetherian semilocal rings with prescribed $V^*(R)$. In this section we study when ring pullbacks are semilocal and noetherian. We start fixing some notation.

Notation 6.4.1. *Let R_1, R_2 and S be rings with ring homomorphisms $j_i\colon R_i \to S$, for $i = 1, 2$. Let R be the pullback of these rings. That is, R fits into the pullback diagram*

$$\begin{tikzcd} R_1 \arrow[r, "j_1"] & S \\ R \arrow[u, "i_1"] \arrow[r, "i_2"'] & R_2 \arrow[u, "j_2"'] \end{tikzcd}$$

So that it can be described as

$$R = \{(r_1, r_2) \in R_1 \times R_2 \mid j_1(r_1) = j_2(r_2)\}$$

and the maps i_1 and i_2 are just the canonical projections.

A ring homomorphism $\varphi\colon R \to S$ is said to be *local* if, for any $r \in R$, $\varphi(r)$ is a unit of S if and only if r is a unit of R. The following deep result by Rosa Camps and Warren Dicks (see [4, Theorem 1] or [7, Theorem 4.2]) characterizes semilocal rings in terms of local morphisms.

Theorem 6.4.2. *A ring R is semilocal if and only if it has a local ring homomorphism into a semilocal ring.*

We note the following elegant corollary of Theorem 6.4.2.

Corollary 6.4.3. *In the situation of Notation 6.4.1, R is a local subring of $R_1 \times R_2$. In particular, the pullback of two semilocal rings is a semilocal ring.*

PROOF. Note that if $(r_1, r_2) \in R \subseteq R_1 \times R_2$ is a unit of $R_1 \times R_2$ then its inverse (r_1^{-1}, r_2^{-1}) also satisfies that $j_1(r_1^{-1}) = j_2(r_2^{-1})$. Hence $(r_1^{-1}, r_2^{-1}) \in R$, and we deduce that the inclusion $R \to R_1 \times R_2$ is a local ring homomorphism. ∎

Lemma 6.4.4. *Let T be a subring of a ring R, and assume that there exists a two-sided ideal I of R such that $I \subseteq T$ and that R/I is finitely generated as a left T/I-module. Then:*

(i) ${}_T R$ *is finitely generated.*

(ii) If R and T/I are left noetherian rings then so is T.

Proof. (i) Let $x_1, \dots, x_n \in R$ be such that $x_1 + I, \dots, x_n + I$ generate R/I as a left T/I-module. Then $R = Tx_1 + \cdots + Tx_n + T \cdot 1$ is finitely generated as a left T-module.

(ii) Let J be a left ideal of T. As IJ is a left ideal of R, it is finitely generated as a left R-module. By (i), it is finitely generated as a left T-module.

The left R-module RJ/IJ is also a finitely generated left R/I-module, hence it is a noetherian left T/I-module. Therefore $J/IJ \subseteq RJ/IJ$ is a finitely generated left T/I-module. Since

$$0 \to IJ \to J \to J/IJ \to 0$$

we can conclude that ${}_T J$ is finitely generated. This proves that T is left noetherian. ∎

Proposition 6.4.5. *In the situation of Notation 6.4.1, assume that j_1 is surjective and that ${}_{R_2}S$ is finitely generated. If R_i is a left noetherian ring, for $i = 1, 2$, then R is left noetherian.*

Proof. As j_1 is onto, i_2 is also an onto ring homomorphism with kernel $I = \operatorname{Ker} j_1 \times \{0\}$. Let $T = i_1(R)$, and note that $\operatorname{Ker} j_1$ is a two-sided ideal of R_1 that is contained in T. As a first step we shall prove that T is left noetherian and that ${}_T R_1$ is finitely generated. Observe that $T/\operatorname{Ker} j_1 \simeq j_2(R_2)$ is left noetherian. In view of Lemma 6.4.4 we only need to prove that ${}_T S \cong R_1/\operatorname{Ker} j_1$ is finitely generated.

By assumption, there exist $s_1, \dots, s_n \in S$ such that $S = \sum_{i=1}^n R_2 s_i$. Fix an element $s \in S$, there exist $r_2^1, \dots, r_2^n \in R_2$ such that $s = \sum_{i=1}^n r_2^i \cdot s_i = \sum_{i=1}^n j_2(r_2^i) s_i$. Since j_1 is onto, for $i = 1, \dots, n$, there exists $r_1^i \in R_1$ such that $j_1(r_1^i) = j_2(r_2^i)$. Hence $s = \sum_{i=1}^n j_1(r_1^i) s_i = \sum_{i=1}^n r_1^i \cdot s_i$. Since $(r_1^i, r_2^i) \in R$, $r_1^i \in T$ for $i = 1, \dots, n$. This shows that $S = \sum_{i=1}^n T s_i$, so that ${}_T S$ is finitely generated.

We want to prove that any left ideal of R is finitely generated. Let I be a left ideal of R contained in $\operatorname{Ker} j_1 \times \{0\}$. Hence $I = I_1 \times \{0\}$ with I_1 a left ideal of T, as T is left noetherian I is finitely generated.

Now, let I be any left ideal of R. Since i_2 is onto and R_2 is left noetherian, $i_2(I)$ is a left ideal of R_2, finitely generated by elements $r_2^1, \dots, r_2^n \in R_2$ say. Fix $r_1^1, \dots, r_1^n \in R_1$ such that $(r_1^i, r_2^i) \in I$. If $x \in I$ then there exist $y_1, \dots, y_n \in R$ such that $x - \sum_{i=1}^n y_i (r_1^i, r_2^i) \in I \bigcap (\operatorname{Ker} j_1 \times \{0\})$. Therefore $I = I \bigcap (\operatorname{Ker} j_1 \times \{0\}) + \sum_{i=1}^n R(r_1^i, r_2^i)$. By the previous case, $I \bigcap (\operatorname{Ker} j_1 \times \{0\})$ is finitely generated, therefore I is finitely generated. ∎

In the next result we compute the Jacobson radical for some pullbacks of rings.

Lemma 6.4.6. *In the situation of Notation 6.4.1, assume that j_1 is an onto ring homomorphism such that $\operatorname{Ker} j_1 \subseteq J(R_1)$ and $j_1(J(R_1)) \supseteq j_2(J(R_2))$. Then $J(R)$ fits into the induced pullback diagram*

$$\begin{tikzcd}
J(R_1) \arrow[r, "j_1"] & j_1(J(R_1)) \\
J(R) \arrow[u, "i_1"] \arrow[r, "i_2"'] & J(R_2) \arrow[u, "j_2"']
\end{tikzcd}$$

and $R/J(R) \cong R_2/J(R_2)$*. In particular, if* $J(R_1) = \operatorname{Ker} j_1$ *and* $J(R_2) = \operatorname{Ker} j_2$ *then* $J(R) = J(R_1) \times J(R_2)$*.*

Proof. Let J be the pullback of the induced maps $j_1\colon J(R_1) \to j_1(J(R_1))$ and $j_2\colon J(R_2) \to j_1(J(R_1))$. Since, by Corollary 6.4.3, R is a local subring of $R_1 \times R_2$ it follows that $J \subseteq J(R)$.

In order to prove the reverse inclusion consider $(r_1, r_2) \in J(R)$. Being j_1 onto, i_2 is also onto, hence $r_2 \in J(R_2)$. Since $j_1(r_1) = j_2(r_2) \subseteq j_1(J(R_1))$, we deduce that $r_1 \in \operatorname{Ker} j_1 + J(R_1) = J(R_1)$. Therefore $(r_1, r_2) \in J$.

Since $\operatorname{Ker} i_2 = \operatorname{Ker} j_1 \times \{0\} \subseteq J(R)$ and i_2 is onto, it immediately follows that the map

$$R \xrightarrow{i_2} R_2 \xrightarrow{\pi} R_2/J(R_2),$$

where π denotes the canonical projection, induces an isomorphism between $R/J(R)$ and $R_2/J(R_2)$.

The claim when $J(R_i) = \operatorname{Ker} j_i$ follows from the fact that the pullback of zero homomorphisms is the product. ∎

In the situation of Notation 6.4.1 and assuming that j_1 is onto, Milnor in [19] characterized the category of projective right (or left) modules over R in the following way,

Remark 6.4.7. *We follow Notation 6.4.1 and we assume that* j_1 *is onto. Let* $\mathcal{P}$ *be the category with objects the triples* $M(P_1, P_2, h)$ *where, for* $i = 1, 2$, P_i *is a projective right* R_i*-module, and* $h\colon P_1 \otimes_{R_1} S \to P_2 \otimes_{R_2} S$ *is an isomorphism. A morphism* $f\colon M(P_1, P_2, h) \to M(Q_1, Q_2, g)$ *in* $\mathcal{P}$ *is defined as a pair* $f = (f_1, f_2)$ *where, for* $i = 1, 2$, $f_i\colon P_i \to Q_i$ *is a morphism of right* R_i*-modules and* $g(f_1 \otimes S) = (f_2 \otimes S)h$*. Finally we say that an object* $M(P_1, P_2, h)$ *of* $\mathcal{P}$ *is finitely generated (countably generated) if* P_1 *and* P_2 *are finitely generated (countably generated).*

If P, Q *are projective right modules over* R *and* $f\colon P \to Q$ *is a homomorphism then the assignment*

$$P \mapsto M(P \otimes_R R_1, P \otimes_R R_2, \mathrm{Id}_P \otimes S)$$
$$Q \mapsto M(Q \otimes_R R_1, Q \otimes_R R_2, \mathrm{Id}_Q \otimes S)$$
$$f \mapsto (f \otimes_R R_1, f \otimes_R R_2)$$

defines a functor from the category of projective right R*-modules and the category* $\mathcal{P}$*. Milnor proved [19, Theorems 2.1, 2.2 and 2.3] that this functor is an equivalence of categories. He also observed that the equivalence can be restricted to the full subcategories of finitely generated objects, the same proof shows that the equivalence can be also restricted to the full subcategories of countably generated objects.*

In general, the category $\mathcal{P}$ is not just the *pullback* of the categories of projective right R_i-modules. Next Theorem describes a situation where not only this is true, but it also follows that the isomorphism class of a projective right R-module P only depends on the isomorphism class of the projective right R_2-module $P \otimes_R R_2$. The result is a combination of Milnor's characterization with Theorem 6.1.4(ii).

Theorem 6.4.8. *In the situation of Notation 6.4.1, assume that* j_1 *is an onto ring homomorphism and that* $\operatorname{Ker} j_1 \subseteq J(R_1)$*. Then the assignment* $P_R \mapsto P \otimes_R R_2$*, where* P *denotes a projective right* R*-module, induces monoid isomorphisms*

$$V(i_2)\colon V(R) \to \{\langle P_2 \rangle \in V(R_2) \mid \langle P_2 \otimes_{R_2} S \rangle \in \operatorname{Im} V(j_1)\}$$

and

$$V^*(i_2)\colon V^*(R) \to \{\langle P_2\rangle \in V^*(R_2) \mid \langle P_2 \otimes_{R_2} S\rangle \in \operatorname{Im} V^*(j_1)\}.$$

Proof. We shall use Milnor's characterization described in Remark 6.4.7.

Our hypothesis implies that $\operatorname{Ker} i_2 \subseteq J(R_1) \times \{0\}$ and, hence, $\operatorname{Ker} i_2 \subseteq J(R)$. This allows us to use Theorem 6.1.4(ii) to deduce that $P \otimes_R R_2 \cong Q \otimes_R R_2$ implies $P \cong Q$ for any pair of projective right R-modules P, Q. Therefore, $V(i_2)$ and $V^*(i_2)$ are injective. Then the results [19, Theorems 2.1, 2.2 and 2.3] recalled in Remark 6.4.7, show that the image of these maps is as claimed in the statement. ∎

Now we state the precise result we will be using in §6.5.

Corollary 6.4.9. *In the situation of Notation 6.4.1, let R_1 and R_2 be semilocal rings, and let $S = M_{m_1}(D_1') \times \cdots \times M_{m_\ell}(D_\ell')$ for suitable division rings $D_1', \dots, D_\ell'$. Assume that j_1 is an onto ring homomorphism with kernel $J(R_1)$, and that $J(R_2) \subseteq \operatorname{Ker} j_2$. If $R_2/J(R_2) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for $D_1, \dots, D_k$ division rings, and $\pi\colon R_2 \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ is an onto morphism with kernel $J(R_2)$ then*

(i) i_2 induces an onto ring homomorphism $\overline{i_2}\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ with kernel $J(R)$. In particular, R is a semilocal ring and $R/J(R) \cong R_2/J(R_2)$.

(ii) Let $\alpha\colon \dim_\pi V^*(R_2) \to (\mathbb{N}_0^*)^\ell$ be the monoid homomorphism induced by j_2. Then

$$\dim_{\overline{i_2}} V^*(R) = \{x \in \dim_\pi V^*(R_2) \mid \alpha(x) \in \dim_{j_1} V^*(R_1)\}.$$

Moreover, if R_1 and R_2 are noetherian, and S is finitely generated, both as a left and as a right $j_2(R_2)$-module, then R is noetherian and $\dim_{\overline{i_2}}(V(R)) = \big(\dim_{\overline{i_2}} V^*(R)\big) \cap \mathbb{N}_0^k$.

Proof. Statement (i) follows from Lemma 6.4.6 and Corollary 6.4.3. Statement (ii) is a consequence of (i) and Theorem 6.4.8.

The final part of the Corollary follows from Proposition 6.4.5 and the fact that over a noetherian semilocal ring a projective module is finitely generated if and only if it is finitely generated modulo the Jacobson radical (cf. Proposition 6.1.9 or Theorem 6.2.3). ∎

We single out the following particular case of Corollary 6.4.9.

Corollary 6.4.10. *In the situation of Notation 6.4.1, for $i = 1, 2$, assume that j_i is onto and $\operatorname{Ker} j_i = J(R_i)$. Assume $S = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ where $D_1, \dots, D_k$ are division rings. Then*

(i) i_2 (and i_1) induces an onto ring homomorphism $\bar{i}\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ with kernel $J(R)$.

(ii) $\dim_{\bar{i}} V^*(R) = \dim_{j_1} V^*(R_1) \cap \dim_{j_2} V^*(R_2)$.

6.5 Noetherian semilocal rings with prescribed $V^*(R)$

Now we have all the elements to construct noetherian semilocal rings with prescribed $V^*(R)$ and to prove the realization part of Theorem 6.2.6. We explain the basic constructions in the following two examples.

Example 6.5.1. *Let $k, m \in \mathbb{N}$, and let $a_1, \dots, a_k \in \mathbb{N}_0$. Assume $(n_1, \dots, n_k) \in \mathbb{N}^k$ is such that $a_1n_1 + \cdots + a_kn_k = m\ell \in \mathbb{N}$. Let F be a field, and let $F \subseteq F_2$ be a field extension such that there exists a semilocal principal ideal domain R_1, that is also an F-algebra, with $R_1/J(R_1) \cong M_m(F_2)$. Then for any intermediate field $F \subseteq F_1 \subseteq F_2$ such that $[F_2 : F_1] < \infty$ there exist a noetherian semilocal F-algebra R and an onto morphism of F-algebras $\varphi\colon R \to M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1)$ with $\operatorname{Ker}\varphi = J(R)$ such that $\dim_\varphi V^*(R)$ is exactly the set of solution in $(\mathbb{N}_0^*)^k$ of the congruence $a_1t_1 + \cdots + a_kt_k \in m\mathbb{N}_0^*$.*

Note that $\dim_\varphi(\langle R\rangle) = (n_1, \dots, n_k)$.

Proof. Fix $j_1\colon M_\ell(R_1) \to M_{m\ell}(F_2)$ an onto morphism of F-algebras with kernel $J(M_\ell(R_1)) = M_\ell(J(R_1))$.

Set $R_2 = M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1)$, and consider the morphism of F-algebras

$$\begin{array}{cccc} j_2\colon & R_2 & \longrightarrow & M_{m\ell}(F_2) \\ & (r_1, \dots, r_k) & \mapsto & \begin{pmatrix} \begin{smallmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots_{a_1)} & \vdots \\ 0 & \cdots & r_1 \end{smallmatrix} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \begin{smallmatrix} r_k & \cdots & 0 \\ \vdots & \ddots_{a_k)} & \vdots \\ 0 & \cdots & r_k \end{smallmatrix} \end{pmatrix} \end{array}$$

where, for $i = 1, \dots, k$, a_i is the size of the i-th block of the matrix $j_2(r_1, \dots, r_k)$. Note that $(V(R_2), \langle R_2\rangle) \cong (\mathbb{N}_0^k, (n_1, \dots, n_k))$ and $V^*(R_2) \cong (\mathbb{N}_0^*)^k$; $V(M_{m\ell}(F_2))$ is isomorphic to the monoid $\mathbb{N}_0$ with order unit $m \cdot \ell$ and $V^*(M_{m\ell}(F_2)) \cong (\mathbb{N}_0)^*$. Then j_2 induces the morphism of monoids $f\colon (\mathbb{N}_0^*)^k \to \mathbb{N}_0^*$ defined by $f(x_1, \dots, x_k) = a_1x_1 + \cdots + a_kx_k$, cf. §6.1.1.

Let R be the ring defined by the pullback diagram

$$\begin{array}{ccc} M_\ell(R_1) & \xrightarrow{j_1} & M_{m\ell}(F_2) \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle j_2} \\ R & \xrightarrow[\varphi]{} & M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1) \end{array}$$

Since $[F_2 : F_1] < \infty$ we can apply (i) and the final part of Corollary 6.4.9 to deduce R is a noetherian semilocal F-algebra and that φ is an onto morphism of F-algebras with kernel $J(R)$.

Now we compute $\dim_\varphi V^*(R)$ using Corollary 6.4.9(ii). We have chosen R_1 such that $\dim_{j_1} V^*(M_\ell(R_1)) = m\mathbb{N}_0^*$. Therefore $(x_1, \dots, x_k) \in \dim_\varphi V^*(R)$ if and only if $f(x_1, \dots, x_k) = a_1x_1 + \cdots + a_kx_k \in m\mathbb{N}_0^*$. That is, $\dim_\varphi V^*(R)$ is exactly the set of solutions in $(\mathbb{N}_0^*)^k$ of the congruence $a_1t_1 + \cdots + a_kt_k \in m\mathbb{N}_0^*$ as desired. ∎

Example 6.5.2. *Let $k \in \mathbb{N}$, and let $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{N}_0$. Let $(n_1, \dots, n_k) \in \mathbb{N}^k$ be such that $a_1n_1 + \cdots + a_kn_k = b_1n_1 + \cdots + b_kn_k \in \mathbb{N}$. For any field extension $F \subseteq F_1$, there exist a noetherian semilocal F-algebra R and an onto morphism of F-algebras $\varphi\colon R \to M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1)$ with kernel $J(R)$ such that $\dim_\varphi V^*(R)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the equation $a_1t_1 + \cdots + a_kt_k = b_1t_1 + \cdots + b_kt_k$.*

Note that $\dim_\varphi(\langle R\rangle) = (n_1, \dots, n_k)$.

Proof. Set $m = a_1n_1 + \cdots + a_kn_k = b_1n_1 + \cdots + b_kn_k$.

Let R_1 be a noetherian semilocal F-algebra such that $R_1/J(R_1) \cong F_1 \times F_1$, and all projective modules over R_1 are free. For example, we could take the commutative principal ideal domain $R_1 \cong F_1[x]_\Sigma$ with $\Sigma = (F_1[x]) \setminus (xF_1[x] \cup (x-1)F_1[x])$.

Let $j_1 \colon M_m(R_1) \to M_m(F_1) \times M_m(F_1)$ be an onto morphism of F-algebras with kernel $J(M_m(R_1))$. Set $R_2 = M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1)$. Consider the morphism of F-algebras $j_2 \colon R_2 \longrightarrow M_m(F_1) \times M_m(F_1)$ defined by

$$j_2(r_1,\dots,r_k) = \left(\begin{pmatrix} \begin{smallmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots^{a_1)} & \vdots \\ 0 & \cdots & r_1 \end{smallmatrix} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \begin{smallmatrix} r_k & \cdots & 0 \\ \vdots & \ddots^{a_k)} & \vdots \\ 0 & \cdots & r_k \end{smallmatrix} \end{pmatrix}, \begin{pmatrix} \begin{smallmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots^{b_1)} & \vdots \\ 0 & \cdots & r_1 \end{smallmatrix} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \begin{smallmatrix} r_k & \cdots & 0 \\ \vdots & \ddots^{b_k)} & \vdots \\ 0 & \cdots & r_k \end{smallmatrix} \end{pmatrix} \right)$$

Note that j_2 induces the morphism of monoids $f \colon (\mathbb{N}_0^*)^k \to \mathbb{N}_0^* \times \mathbb{N}_0^*$ defined by $f(x_1,\dots,x_k) = (a_1x_1 + \cdots + a_kx_k, b_1x_1 + \cdots + b_kx_k)$, cf. §6.1.1. Hence, $f(n_1,\dots,n_k) = (m,m)$.

Let R be the ring defined by the pullback diagram

$$\begin{array}{ccc} M_m(R_1) & \xrightarrow{j_1} & M_m(F_1) \times M_m(F_1) \\ {\scriptstyle i_1}\big\uparrow & & \big\uparrow{\scriptstyle j_2} \\ R & \xrightarrow[\varphi]{} & M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1) \end{array}$$

Applying (i) and the final part of Corollary 6.4.9, we can deduce that R is a noetherian semilocal F-algebra and that φ is an onto morphism of F-algebras with kernel $J(R)$. We have chosen R_1 such that $\dim_{j_1} V^*(M_\ell(R_1)) = \{(x,x) \mid x \in \mathbb{N}_0^*\}$. Also by Corollary 6.4.9(ii), $(x_1,\dots,x_k) \in \dim_\varphi V^*(R)$ if and only if $f(x_1,\dots,x_k) \in \dim_{j_1} V^*(M_\ell(R_1))$ if and only if $a_1x_1 + \cdots + a_kx_k = b_1x_1 + \cdots + b_kx_k$ as desired. ∎

Theorem 6.5.3. *Let $k \geq 1$, and let F be a field. Let M be a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations and containing an element $(n_1,\dots,n_k) \in \mathbb{N}^k$. Then there exist a noetherian semilocal F-algebra R, a semisimple F-algebra $S = M_{n_1}(E) \times \cdots \times M_{n_k}(E)$, where E is a field extension of F, and an onto morphism of F-algebras $\varphi \colon R \to S$ with* $\operatorname{Ker}\varphi = J(R)$ *satisfying that* $\dim_\varphi V^*(R) = M$.

Note that, in this situation, $\dim_\varphi(\langle R \rangle) = (n_1,\dots,n_k)$.

Proof. Let M be defined by the system of equations,

$$D\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \quad (*) \qquad \text{and} \qquad E_1\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \quad (**)$$

where $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times k}(\mathbb{N}_0)$, $n, \ell \geq 0$ and $m_1,\dots,m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1,\dots,n\}$.

Step 1. *There exist a field E containing F, a noetherian semilocal F-algebra R_1 and an onto morphism of F-algebras $\varphi_1 \colon R_1 \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ such that* $\dim_{\varphi_1} V^*(R_1)$ *is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system of congruences $(*)$.*

If $n = 0$, that is, if $(*)$ is empty we set $E = F$, $R_1 = M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ and $\varphi_1 = \mathrm{Id}$. Assume $n > 0$, therefore we may also assume that all the rows of D are nonzero.

Consider the field extension $F \subseteq F' = F(t_j^i \mid i = 1, \ldots, n,\, j = 1, \ldots, m_i)$. For each $i = 1, \ldots, n$ consider the automorphism α_i of F' that fixes $F_i = F(t_j^s \mid s \neq i) \subseteq F'$, maps t_j^i to t_{j+1}^i for $1 \leq j < m_i$, and maps $t_{m_i}^i$ to t_1^i. Note that α_i has order m_i. Let G be the group of permutations of $m_1 + \cdots + m_k$ variables. Then G acts on F'. Set $E = (F')^G$, and note that $E \subseteq F'$ is a finite field extension.

By Example 6.3.3(ii), for each $i = 1, \ldots, n$, we can construct a principal ideal domain such that modulo the Jacobson radical is isomorphic to $M_{m_i}((F')^{\alpha_i})$. By Example 6.5.1, for $i = 1, \ldots, n$, there exist a noetherian semilocal F-algebra L_i and an onto morphism of F-algebras $\pi_i \colon L_i \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ with kernel $J(L_i)$ and such that $\dim_{\pi_i} V^*(L_i)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the i-th congruence in $(*)$.

Let R_1 be the pullback of the π_i, $i = 1, \ldots, n$. By Corollary 6.4.9, R_1 is a noetherian semilocal F-algebra. By Corollary 6.4.10, there exists an onto morphism of F-algebras $\varphi_1 \colon R_1 \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$, with kernel $J(R_1)$, such that $\dim_{\varphi_1} V^*(R_1)$ is the monoid of solutions of (*). This concludes the proof of the first step.

Step 2. *There exist a noetherian semilocal F-algebra R_2 and an onto morphism of F-algebras $\varphi_2 \colon R_2 \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ such that $\dim_{\varphi_2} V^*(R_2)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system of equations $(**)$.*

If $\ell = 0$, that is, if $(**)$ is empty we set $R_2 = M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ and $\varphi_2 = \mathrm{Id}$. Assume $\ell > 0$. Therefore, we can assume that none of the rows in E_1 and, hence, in E_2 are zero.

By Example 6.5.2, for $i = 1, \ldots, \ell$, there exist a noetherian semilocal F-algebra T_i and an onto morphism of F-algebras $\pi_i \colon T_i \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ with kernel $J(T_i)$ and such that $\dim_{\pi_i} V^*(T_i)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the i-th equation defined by the matrices E_1 and E_2.

Let R_2 be the pullback of π_i, $i = 1, \ldots, \ell$. By Corollary 6.4.9, R_2 is a noetherian semilocal F-algebra with an onto morphism of F-algebras $\varphi_2 \colon R_2 \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ with kernel $J(R_2)$. By Corollary 6.4.10, $\dim_{\varphi_2} V^*(R_2)$ is the set of solutions of $(**)$. This concludes the proof of Step 2.

Finally, set R to be the pullback of $\varphi_i \colon R_i \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$, $i = 1, 2$. By Corollary 6.4.9, R is a noetherian semilocal F-algebra with an onto morphism of F-algebras $\varphi \colon R \to M_{n_1}(E) \times \cdots \times M_{n_k}(E)$ with kernel $J(R)$. By Corollary 6.4.10, the elements in $\dim_{\varphi} V^*(R)$ are the solutions of $(*)$ and $(**)$. ∎

6.6 Solving equations in $\mathbb{N}_0$ and in $\mathbb{N}_0^*$: Supports of solutions

In this section we study the supports of elements of a full affine submonoid of $\mathbb{N}_0^k$ and the supports of elements of a submonoid of $(\mathbb{N}_0^*)^k$ defined by systems of equations. Our main aim is to show in Proposition 6.6.7 that if A is a full affine submonoid of $\mathbb{N}_0^k$ then $A + \infty \cdot A$ is a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations.

We recall that a full affine monoid is finitely generated.

Next result is quite easy but it is very important to keep in mind, for example, in Definition 6.7.1. It shows that full affine submonoids are closed by projections over the complementary of supports of elements.

Proposition 6.6.1. *Let $k \geq 1$. Let $A \subseteq \mathbb{N}_0^k$ be a full affine submonoid. Let $I \subsetneq \{1, \dots, k\}$ be the support of an element of A, and denote by $p_I \colon \mathbb{N}_0^k \to \mathbb{N}_0^{\{1,\dots,k\}\setminus I}$ the canonical projection. Then $p_I(A)$ is a full affine submonoid of $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$.*

Proof. Clearly $p_I(A)$ is a submonoid of $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$. We need to check the full affine property.

Let a, $b \in A$ be such that there exists $z \in \mathbb{N}_0^{\{1,\dots,k\}\setminus I}$ satisfying that $p_I(a) + z = p_I(b)$. Let $d \in A$ be such that $\operatorname{supp}(d) = I$. There exists $n \in \mathbb{N}_0$ such that $a + c = b + nd$ for some $c \in \mathbb{N}_0^k$. As A is full affine, $c \in A$ and, hence, $z = p_I(c) \in p_I(A)$. ■

Let $C \subseteq \mathbb{Q}^k$, and let $C^\perp = \{v \in \mathbb{Q}^k \mid \langle v, c\rangle = 0 \text{ for any } c \in C\}$ where $\langle -, -\rangle$ denotes the standard scalar product. If $X \subseteq \mathbb{Q}^k$, the support of X is defined by

$$\operatorname{supp}(X) = \bigcup_{x \in X} \operatorname{supp}(x)$$

Let A be a submonoid of $\mathbb{N}_0^k$, and let B be the subgroup of $\mathbb{Z}^k$ generated by A. Then A is full affine if and only if $B \cap \mathbb{N}_0^k = A$ (see, for example, [8, Lemma 3.1]). So assume A is full affine and consider $B' = (A^\perp)^\perp \bigcap \mathbb{Z}^k$ which is a subgroup of $\mathbb{Z}^k$ defined by a system of diophantine linear equations

$$E \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

with $E \in M_{(k-\ell)\times k}(\mathbb{Z})$ and ℓ is the rank of the group B'.

By construction $B \subseteq B'$. Since the rank of B is also ℓ, there exist $d_1, \dots, d_\ell \geq 1$ such that $B'/B \cong \mathbb{Z}/d_1\mathbb{Z} \times \cdots \times \mathbb{Z}/d_\ell\mathbb{Z}$. Equivalently, there exists a basis $\{v_1, \dots, v_\ell\}$ of B' such that $\{d_1 v_1, \dots, d_\ell v_\ell\}$ is a basis of B. Therefore, an element $x \in B'$

$$x = (x_1, \dots, x_k) = \alpha_1 v_1 + \cdots + \alpha_\ell v_\ell$$

is in B if and only if, for any $i = 1, \dots, \ell$, $\alpha_i \in d_i\mathbb{Z}$. Since each α_i can be written as a $\mathbb{Q}$-linear combination of $x_1, \dots, x_k$, by clearing denominators and eliminating trivial congruences, we deduce that, for any $x \in B'$, $x \in B$ if and only if it is a solution of

$$D \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{Z} \\ \vdots \\ m_n\mathbb{Z} \end{pmatrix}$$

where $0 \leq n \leq \ell$, $D \in M_{n\times k}(\mathbb{Z})$ and $m_i > 1$ for $i = 1, \dots, n$.

Adding to D a suitable integral matrix in $\begin{pmatrix} m_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & m_n \end{pmatrix} \cdot M_{n,k}(\mathbb{Z})$, we can also assume that $D \in M_{n\times k}(\mathbb{N}_0)$.

In the next Proposition we collect the consequences of this discussion.

Proposition 6.6.2. ([2, Exercise 6.4.16] or [1, Proof of Theorem 2.29]) *Let $k \geq 1$. Let A be a full affine submonoid of $\mathbb{N}_0^k$, and let ℓ be the rank of the group generated by A. Then there exist $0 \leq n \leq \ell$, $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{(k-\ell)\times k}(\mathbb{N}_0)$ and $m_1, \dots, m_n$ integers strictly bigger than one such that* *(i)* $x = (t_1, \dots, t_k) \in \mathbb{N}_0^k$ *is an element of* A *if and only if it is a solution of*

$$D \cdot T \in \begin{pmatrix} m_1\mathbb{N}_0 \\ \vdots \\ m_n\mathbb{N}_0 \end{pmatrix} \qquad \textit{and} \qquad E_1 \cdot T = E_2 \cdot T$$

where $T = (t_1, \dots, t_k)^t$.

For $j = 1, 2$, *let* r_i^j *denote the* i*-th row of* E_j. *Then* E_1 *and* E_2 *can be chosen such that, for* $i = 1, \dots, k - \ell$, $\operatorname{supp}(r_i^1) \cap \operatorname{supp}(r_i^2) = \emptyset$.

(ii) *The set of solutions of* $E_1 \cdot T = E_2 \cdot T$ *is* $A' = (A^\perp)^\perp \cap \mathbb{N}_0^k$.

(iii) *There exists* $d \in \mathbb{N}$ *such that* $dA' \subseteq A$. *In particular,*

$$\{I \subseteq \{1, \dots, k\} \mid \textit{ there exists } a \in A \textit{ such that } \operatorname{supp}(a) = I\} =$$

$$= \{I \subseteq \{1, \dots, k\} \mid \textit{ there exists } a \in A' \textit{ such that } \operatorname{supp}(a) = I\}.$$

Proof. Following with the notation in the remarks before Proposition 6.6.2, we can write the matrix $E = E_1 - E_2$ where E_1 and E_2 are in $M_{(k-\ell)\times k}(\mathbb{N}_0)$. Clearly, E_1 and E_2 can be chosen in a way such that the i-th row of E_1 has disjoint support with the i-th row of E_2. Then (i) follows from the fact that $A = B \cap \mathbb{N}_0^k$.

The rest of the statement is clear. ∎

Now we prove an auxiliary (and probably known) result that will be useful to determine the supports of positive solutions of linear diophantine equations.

Lemma 6.6.3. *Let* $k \geq 1$, *and let* V *be a subspace of* $\mathbb{Q}^k$. *Then the following statements are equivalent,*

(i) $V^\perp \cap \mathbb{N}^k \neq \emptyset$;

(ii) $\operatorname{supp}(V^\perp \cap \mathbb{N}_0^k) = \{1, \dots, k\}$;

(iii) $V \cap \mathbb{N}_0^k = \{0\}$.

Proof. It is clear that (i) and (ii) are equivalent statements and also that (i) implies (iii). We will show that (iii) implies (ii).

The assumption in (iii) is equivalent to say that any element $0 \neq v \in V$ has a component strictly bigger than zero and another one strictly smaller than zero. As a first step we show that $V^\perp$ cannot satisfy this condition.

Assume, by the way of contradiction, that k is the minimal dimension in which the conclusion of (ii) fails. So that there exists $V \leq \mathbb{Q}^k$ such that $V \cap \mathbb{N}_0^k = V^\perp \cap \mathbb{N}_0^k = \{0\}$. Note that k and the dimension of V must be strictly bigger than 1.

Let $v_1, \dots, v_n$ be a basis of V such that there exists $i \in \operatorname{supp}(v_1) \setminus \operatorname{supp}(\{v_2, \dots, v_n\})$. Let $\pi\colon \mathbb{Q}^k \to \mathbb{Q}^{\{1,\dots,k\}\setminus\{i\}}$ denote the canonical projection.

If $\pi(V) \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus\{i\}} = \{0\}$ then, by the minimality of k, there exists $v \in \mathbb{Q}^k$ such that $0 \neq \pi(v) \in \pi(V)^\perp \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus\{i\}}$. Since v can be chosen satisfying that $i \notin \operatorname{supp}(v)$, we would get $0 \neq v \in V^\perp \cap \mathbb{N}_0^k$, a contradiction. Let $0 \neq \lambda_1\pi(v_1) + \dots + \lambda_n\pi(v_n) \in \pi(V) \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus\{i\}}$. Then $w = \lambda_1 v_1 + \dots + \lambda_n v_n \in V$. Since $V \cap \mathbb{N}_0^k = \{0\}$, $\lambda_1 \neq 0$.

Therefore, replacing v_1 by w if necessary, we may assume that $0 \neq \pi(v_1) \in \mathbb{N}_0^{\{1,\dots,k\}\setminus\{i\}}$ and that the i-th component of v_1 is < 0. Let $-a$ be such component.

Let W be the subspace of $\mathbb{Q}^{k\setminus\{i\}}$ generated by $\pi(v_2), \dots, \pi(v_n)$. Our hypothesis imply that $W \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus\{i\}} = \{0\}$. By the minimality of k, there is $v \in \mathbb{Q}^k$ such that $0 \neq \pi(v) \in W^\perp \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus\{i\}}$. Therefore, as $b = \langle \pi(v), \pi(v_1) \rangle \geq 0$, picking v such that its i-th component is b/a we find that $0 \neq av \in V^\perp \cap \mathbb{N}_0^k$, a contradiction. Therefore $V^\perp \cap \mathbb{N}_0^k \neq \{0\}$ for any V such that $V \cap \mathbb{N}_0^k = \{0\}$, as claimed.

Now assume that V is a $\mathbb{Q}$-vector space satisfying (iii). Observe first that $\operatorname{supp}(V^\perp) = \{1, \dots, k\}$, since otherwise there would exist $i \in \{1, \dots, k\}$ such that the i-th component of any element in $V^\perp$ is zero. Therefore, $e_i = (0, \dots, 0, 1^{i)}, 0, \dots, 0) \in (V^\perp)^\perp = V$, which is a contradiction with the assumption.

Let $I = \operatorname{supp}(V^\perp \cap \mathbb{N}_0^k)$ and let $J = \{1, \dots, k\} \setminus I$. We already know that $I \neq \emptyset$ and we want to show $J = \emptyset$. Assume, by the way of contradiction, that $J \neq \emptyset$. Set $\pi_I \colon \mathbb{Q}^k \to \mathbb{Q}^I$ and $\pi_J \colon \mathbb{Q}^k \to \mathbb{Q}^J$ to be the canonical projections.

Pick $x \in V^\perp \cap \mathbb{N}_0^k$ such that $\operatorname{supp}(x) = I$. Then for any $v \in V^\perp$ there exists $n \in \mathbb{N}_0$ such that $\pi_I(nx + v) \geq 0$. By the definition of I, this implies that $\pi_J(v) = \pi_J(nx + v)$ is either zero or it has a component > 0 and another one < 0. Therefore, $\pi_J(V^\perp) \cap \mathbb{N}_0^J = \{0\}$. By the first part of the proof, there exists $w \in \mathbb{Q}^k$ such that $0 \neq \pi_J(w) \in \pi_J(V^\perp)^\perp \cap \mathbb{N}_0^J$. Choosing w such that $\pi_I(w) = 0$, we obtain that $0 \neq w \in (V^\perp)^\perp \cap \mathbb{N}_0^k = V \cap \mathbb{N}_0^k$ which contradicts (iii). Therefore $J = \emptyset$. ∎

Lemma 6.6.3 yields a first characterization of the supports of the elements in a full affine monoid.

Corollary 6.6.4. *Let $k \geq 1$. Let A be a full affine submonoid of $\mathbb{N}_0^k$. Let $\emptyset \neq I \subseteq \{1, \dots, k\}$, and denote by $\pi_I \colon \mathbb{Q}^k \to \mathbb{Q}^I$ the canonical projection. Then there exists $a \in A$ such that $I = \operatorname{supp}(a)$ if and only if $\pi_I(A^\perp) \cap \mathbb{N}_0^I = \{0\}$.*

Proof. If $a \in A$ is such that $\operatorname{supp}(a) = I$ then, as $\langle x, \pi_I(a) \rangle = 0$ for any $x \in \pi_I(A^\perp)$, it follows that $\pi_I(A^\perp) \cap \mathbb{N}_0^I = \{0\}$.

Conversely, if $\pi_I(A^\perp) \cap \mathbb{N}_0^I = \{0\}$ then, by Lemma 6.6.3, there exists $u \in \pi_I(A^\perp)^\perp \cap \mathbb{N}^I$. Let $x = (x_1, \dots, x_k) \in \mathbb{N}_0^k$ be such that $\pi_I(x) = u$ and $x_i = 0$ for any $i \in \{1, \dots, k\} \setminus I$. Then $x \in (A^\perp)^\perp \cap \mathbb{N}_0^k$. By Proposition 6.6.2 (iii), there exists $d \in \mathbb{N}$ such that $a = dx \in A$. By construction, $\operatorname{supp}(a) = I$. ∎

A further characterization is the following.

Corollary 6.6.5. *Let $k \geq 1$. Let A be a full affine submonoid of $\mathbb{N}_0^k$. For any $\emptyset \neq I \subseteq \{1, \dots, k\}$ denote by $\pi_I \colon \mathbb{Q}^k \to \mathbb{Q}^I$ the canonical projection. Let $v_1, \dots, v_r \in \mathbb{Q}^k$ be a finite subset of $A^\perp$. Then, there exist $v_{r+1}, \dots, v_s \in A^\perp$ such that $v_1, \dots v_s$ generate the $\mathbb{Q}$-vector space $A^\perp$ and the set*

$$\mathcal{S}(v_1, \dots, v_s) = \{\emptyset \neq I \subseteq \{1, \dots, k\} \mid \textit{ for any } i = 1, \dots, s,\ \pi_I(v_i) \textit{ is either zero}$$
$$\textit{or it has a component} < 0 \textit{ and a component} > 0\}$$

coincides with

$$\operatorname{Supp}(A \setminus \{0\}) = \{I \subseteq \{1, \dots, k\} \mid I = \operatorname{supp}(a) \textit{ for some } 0 \neq a \in A\}.$$

Proof. Pick $v_{r+1},\dots,v_s \in A^\perp$ such that $v_1,\dots,v_s$ generate $A^\perp$ and the set $\mathcal{S}(v_1,\dots,v_s)$ has minimal cardinality.

We claim that $v_1,\dots,v_s$ have the desired property. If $\emptyset \neq I \subseteq \{1,\dots,k\}$ is such that there exists $a \in A$ such that $\operatorname{supp}(a) = I$ then $I \in \mathcal{S}(v_1,\dots,v_s)$ because $\langle a, v_i\rangle = 0$ for any $i = 1,\dots,s$.

Let $I \in \mathcal{S}(v_1,\dots,v_s)$ and assume, by the way of contradiction, that $I \notin \operatorname{Supp}(A)$. Then $\pi_I(A^\perp) \cap \mathbb{N}_0^I \neq \{0\}$, by Corollary 6.6.4.

Let $v \in A^\perp$ be such that $0 \neq \pi_I(v) \in \pi_I(A^\perp) \cap \mathbb{N}_0^I$. Notice that, $\mathcal{S}(v_1,\dots,v_s,v) \subseteq \mathcal{S}(v_1,\dots,v_s)$ and $I \in \mathcal{S}(v_1,\dots,v_s) \setminus \mathcal{S}(v_1,\dots,v_s,v)$. This contradicts the minimality of the cardinality of $\mathcal{S}(v_1,\dots,v_s)$. This finishes the proof of the claim and of the Corollary.
∎

Now we consider the solutions over $(\mathbb{N}_0^*)^k$. In the next Lemma we see how to determine the set of nonempty supports of such monoids (which coincides with the set of infinite supports) for the special kind of systems that appears in Proposition 6.6.2(i).

Lemma 6.6.6. *Let $k \geq 1$. Let M be a submonoid of $(\mathbb{N}_0^*)^k$ defined by the system of equations*

$$D \cdot T \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \quad (*) \qquad \text{and} \qquad E_1 \cdot T = E_2 \cdot T \quad (**)$$

where $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times k}(\mathbb{N}_0)$ and $m_1,\dots,m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1,\dots,n\}$. For $j = 1,2$ and $i = 1,\dots,\ell$, let r_i^j denote the i-th row of E_j. For $\emptyset \neq I \subseteq \{1,\dots,k\}$, let $\pi_I\colon (\mathbb{N}_0^*)^k \to (\mathbb{N}_0^*)^I$ denote the canonical projection. Then:

- *(i) Let N be the submonoid of $(\mathbb{N}_0^*)^k$ whose elements are the solutions of the system of congruences $(*)$. Then, for any $i \in \{1,\dots,k\}$, the element $(0,\dots,0,\infty,0,\dots,0) \in N$.*
- *(ii) If $x \in M$ then also $\infty \cdot x \in M$.*
- *(iii) If $x \in M$ then the element $x^* \in (\mathbb{N}_0^*)^k$ uniquely determined by the property* $\operatorname{supp}(x^*) = \operatorname{inf-supp}(x^*) = \operatorname{inf-supp}(x)$ *also belongs to M.*
- *(iv) Assume that, for $i = 1,\dots,\ell$,* $\operatorname{supp}(r_i^1) \cap \operatorname{supp}(r_i^2) = \emptyset$, *and let $\emptyset \neq I \subseteq \{1,\dots,k\}$ then there exists $x \in M$ such that* $\operatorname{supp}(x) = I$ *if and only if, for any $i = 1,\dots,\ell$, $\pi_I(r_i^1 - r_2^2)$ is either 0 or it has a component > 0 and another one < 0.*

Proof. Statement (i) is trivial, and it allows us to prove the rest of the statement just for the monoid M defined by the system of linear diophantine equations $(**)$.

Let $x \in M$. Fix $i \in \{1,\dots,\ell\}$, there are three possible situations. The first one is $0 = \langle x, r_i^1\rangle = \langle x, r_i^2\rangle$ which happens if and only if, for $j = 1,2$, $\operatorname{supp}(x) \cap \operatorname{supp}(r_i^j) = \emptyset$. The second one is $0 \neq \langle x, r_i^1\rangle = \langle x, r_i^2\rangle \in \mathbb{N}_0$ which happens if and only if, for $j = 1,2$, $\operatorname{inf-supp}(x) \cap \operatorname{supp}(r_i^j) = \emptyset$ but $\operatorname{supp}(x) \cap \operatorname{supp}(r_i^j) \neq \emptyset$. Finally, $\langle x, r_i^1\rangle = \langle x, r_i^2\rangle = \infty$ if and only if $\operatorname{inf-supp}(x) \cap \operatorname{supp}(r_i^j) \neq \emptyset$ for $j = 1,2$. Then, in the three situations, it also follows that $\langle \infty\cdot x, r_i^1\rangle = \langle \infty \cdot x, r_i^2\rangle$ and $\langle x^*, r_i^1\rangle = \langle x^*, r_i^2\rangle$. This shows that (ii) and (iii) hold.

To prove statement (iv) assume that $\operatorname{supp}(r_i^1) \cap \operatorname{supp}(r_i^2) = \emptyset$ for $i = 1,\dots,\ell$. Let $\emptyset \neq I \subseteq \{1,\dots,k\}$ have the property required in the statement. Let $x \in (\mathbb{N}_0^*)^k$ be such that $\operatorname{supp}(x) = \operatorname{inf-supp}(x) = I$. If $i \in \{1,\dots,\ell\}$ is such that $\pi_I(r_i^1 - r_i^2)$ is zero then

$0 = \langle x, r_i^1 \rangle = \langle x, r_i^2 \rangle$, if $\pi_I(r_i^1 - r_i^2)$ has a positive component and a negative component then $\infty = \langle x, r_i^1 \rangle = \langle x, r_i^2 \rangle$. This shows that x satisfies $(**)$, therefore it is an element of M.

To prove the converse, let $x \in M$. By *(ii)* we may assume that $x = \infty \cdot x$. Let $I = \operatorname{supp}(x)$, then one can proceed as in the proof of *(ii)* and *(iii)* to show that I has the required property. ∎

Proposition 6.6.7. *Let $k \geq 1$, and let A be a full affine submonoid of $\mathbb{N}_0^k$. Then*

$$M = A + \{\infty \cdot a \mid a \in A\}$$

is a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations.

Proof. We divide the proof into a couple of steps.

Step 1. *Let $k \geq 1$, and let A be a full affine submonoid of $\mathbb{N}_0^k$. Then there exists a submonoid M' of $(\mathbb{N}_0^*)^k$ defined by a system of equations such that $M' \cap \mathbb{N}_0^k = A$ and if $x \in M'$ then there exists $a \in A$ such that $\infty \cdot x = \infty \cdot a \in M'$.*

By Proposition 6.6.2, there exist $D \in M_{n \times k}(\mathbb{N}_0)$, $E_1, E_2 \in M_{\ell \times k}(\mathbb{N}_0)$ and $m_1, \dots, m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1, \dots, n\}$ such that A is the set of elements in $\mathbb{N}_0^k$ that satisfy the system

$$D \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \quad (*) \qquad \text{and} \qquad E_1 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \quad (**).$$

For $j = 1, 2$, let r_i^j denote the i-th row of E_j. By Proposition 6.6.2, we can assume that $\operatorname{supp}(r_i^1) \cap \operatorname{supp}(r_i^1) = \emptyset$ for $i = 1, \dots, \ell$. Set $v_1 = r_1^1 - r_1^2, \dots, v_\ell = r_\ell^1 - r_\ell^2$. Notice that $v_1, \dots, v_\ell \in A^\perp \cap \mathbb{Z}^k$ and, in fact, generate the $\mathbb{Q}$-vector space $A^\perp$.

By Corollary 6.6.5, there exist $v_{\ell+1}, \dots, v_s \in A^\perp$ such that the set of supports of nonzero elements in A coincides with

$$\mathcal{S}(v_1, \dots, v_s) = \{\emptyset \neq I \subseteq \{1, \dots, k\} \mid \text{ for any } i = 1, \dots, s,\ \pi_I(v_i) \text{ is either zero}$$
$$\text{or it has a component} < 0 \text{ and a component} > 0\}$$

We can assume that $v_{\ell+1}, \dots, v_s \in A^\perp \cap \mathbb{Z}^k$. For $i = \ell+1, \dots, s$, write $v_i = r_i^1 - r_i^2$ where $r_i^j \in \mathbb{N}_0^k$ and $\operatorname{supp}(r_i^1) \cap \operatorname{supp}(r_i^2) = \emptyset$. For, $j = 1, 2$, let F_j be the matrix whose i-th row is $r_{\ell+i}^j$. Now, add to the initial system defining A the equations defined by

$$F_1 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = F_2 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}.$$

Let M' be the set of solutions in $(\mathbb{N}_0^*)^k$ of the resulting system. The monoid A is still the set of solutions in $\mathbb{N}_0^k$ of this new system, so that $M' \cap \mathbb{N}_0^k = A$. By Lemma 6.6.6(iv), the set of supports of elements of M' is exactly $\mathcal{S}(v_1, \dots, v_s)$ which, by construction, coincides with the set of supports of elements in A. This implies that if $x \in M'$ then there exists $a \in A$ such that $\infty \cdot x = \infty \cdot a$. By Lemma 6.6.6(ii), if $x \in M'$ then $\infty \cdot x \in M'$. This finishes the proof of the first step.

Step 2. *The monoid M in the statement is defined by a system of equations*
Let
$$\mathcal{S} = \{I \subsetneq \{1, \dots, k\} \mid \text{there exists } a \in A \text{ such that supp } (a) = I\}.$$
Notice that since $(0, \dots, 0) \in A$ then $\emptyset \in \mathcal{S}$.

For any $I \in \mathcal{S}$, denote by $p_I \colon \mathbb{N}_0^k \to \mathbb{N}_0^{\{1,\dots,k\}\setminus I}$ the canonical projection. If $I \in \mathcal{S}$ then, by Proposition 6.6.1, $A_I = p_I(A)$ is a full affine submonoid of $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$. In particular $p_\emptyset = \mathrm{Id}$ so that $A = A_\emptyset$.

By Step 1, there is a monoid $M_I' \subseteq (\mathbb{N}_0^*)^{\{1,\dots,k\}\setminus I}$ defined by a system of equations and such that $\mathbb{N}_0^{\{1,\dots,k\}\setminus I} \cap M_I' = A_I$ and if $x \in M_I'$ then there exists $a \in A_I$ such that $\infty \cdot x = \infty \cdot a$.

Set $M_I = \pi_I^{-1}(M_I')$. Notice that M_I is defined by the same system of equations defining M_I' but considered over $\mathbb{N}_0^k$. Notice that $x \in \mathbb{N}_0^k \cap M_I$ if and only if $\pi_I(x) \in A_I$. In particular, $A = \mathbb{N}_0^k \cap M_\emptyset$ and $A \subseteq M_I$ for any $I \in \mathcal{S}$.

Since for any $I \in \mathcal{S}$, M_I is defined by a system of equations so is $\cap_{I\in\mathcal{S}} M_I$. We claim that $M = \cap_{I\in\mathcal{S}} M_I$. We already know that $A \subseteq \cap_{I\in\mathcal{S}} M_I$, so that, by Lemma 6.6.6(ii), $M \subseteq \cap_{I\in\mathcal{S}} M_I$. To prove the other inclusion, let $x \in \cap_{I\in\mathcal{S}} M_I$. Let $I_1 = \text{inf-supp } (x)$ and consider the element $x^* \in (\mathbb{N}_0^*)^k$ such that $I_1 = \text{inf-supp } (x^*) = \text{supp}\,(x^*)$. By Lemma 6.6.6(iii), $x^* \in \cap_{I\in\mathcal{S}} M_I$. Since $x^* \in M_\emptyset = M_\emptyset'$, there exists $a \in A$ such that $x^* = \infty \cdot a$, therefore $x^* \in M$. Since $\text{supp}\,(a) = I_1$, we deduce that $I_1 \in \mathcal{S}$. Therefore, there exists $a_1 \in A$ such that $p_{I_1}(a_1) = p_{I_1}(x) \in M_{I_1}' \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus I_1}$. This implies that $x = x^* + a_1$, so that $x \in M$. This finishes the proof of the claim and the proof of the Proposition. ■

6.7 Systems of supports

In order to conclude the proof of Theorem 6.2.6, we need to show that the monoids that appear as $V^*(R)$ for noetherian semilocal rings are defined by a system of equations. To this aim we abstract the following class of submonoids of $(\mathbb{N}_0^*)^k$.

Definition 6.7.1. *Fix $k \in \mathbb{N}$ and an order unit $(n_1, \dots, n_k) \in \mathbb{N}^k$. A* system of supports *$\mathcal{S}(n_1, \dots, n_k)$ consists of a collection $\mathcal{S}$ of subsets of $\{1, \dots, k\}$ together with a family of commutative monoids $\{A_I, I \in \mathcal{S}\}$ such that the following conditions hold*

(i) $\emptyset \in \mathcal{S}$, and $(n_1, \dots, n_k) \in A_\emptyset$.

(ii) For any $I \in \mathcal{S}$ the monoid A_I is a submonoid of $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$. The monoid $A_{\{1,\dots,k\}}$ is the trivial monoid.

(iii) $\mathcal{S}$ is closed under unions, and if $x \in A_I$ for some $I \in \mathcal{S}$ then $I \cup \text{supp}\,(x) \in \mathcal{S}$. In particular $\{1, \dots, k\} \in \mathcal{S}$.

(iv) Suppose that $I, K \in \mathcal{S}$ are such that $I \subseteq K$ and let $p \colon \mathbb{N}_0^{\{1,\dots,k\}\setminus I} \to \mathbb{N}_0^{\{1,\dots,k\}\setminus K}$ be the canonical projection. Then $p(A_I) \subseteq A_K$.

If in addition, for any $I \in \mathcal{S}$, the submonoids A_I are full affine submonoids of $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$ then $\mathcal{S}(n_1, \dots, n_k)$ is said to be a full affine system of supports*.*

In the next Lemma we show that systems of supports are, in some sense, closed under projections.

Lemma 6.7.2. *Let $k > 1$, and let $(n_1, \dots, n_k) \in \mathbb{N}^k$. Let $\mathcal{S}(n_1, \dots, n_k) = (\mathcal{S}; A_I, I \in \mathcal{S})$ be a system of supports. Fix $I \in \mathcal{S} \setminus \{\{1, \dots, k\}\}$, and let $p_I \colon (\mathbb{N}_0^*)^k \to (\mathbb{N}_0^*)^{\{1,\dots,k\}\setminus I}$ denote the canonical projection.*

If we define $\mathcal{S}_I = \{K \setminus I \mid K \in \mathcal{S}$ and $I \subseteq K\}$ and for each $K \setminus I \in \mathcal{S}_I$ we take $A_{I,K\setminus I} = A_K$, then:

(1) $\mathcal{S}_I(p_I(n_1, \dots, n_k)) = (\mathcal{S}_I; A_{I,K\setminus I}, K\setminus I \in \mathcal{S}_I)$ is a system of supports of $(\mathbb{N}_0^*)^{\{1,\dots,k\}\setminus I}$.

(2) If $I \neq \emptyset$ then $\mid \mathcal{S}_I \mid < \mid \mathcal{S} \mid$.

(3) If $\mathcal{S}(n_1, \dots, n_k)$ is a full affine system of supports then so is $\mathcal{S}_I(p_I(n_1, \dots, n_k))$.

Proof. It is routine to check that $\mathcal{S}_I(p_I(n_1, \dots, n_k))$ satisfies the conditions of a system of supports. Statements (2) and (3) are immediate from the definitions. ■

The following Proposition shows that systems of supports is just a way to describe a particular class submonoids of $(\mathbb{N}_0^*)^k$,

Proposition 6.7.3. *Fix $k \in \mathbb{N}$ and $(n_1, \dots, n_k) \in \mathbb{N}^k$. For any $I \subseteq \{1, \dots, k\}$, let $p_I \colon (\mathbb{N}_0^*)^k \to (\mathbb{N}_0^*)^{\{1,\dots,k\}\setminus I}$ denote the canonical projection. Let $\mathcal{S}(n_1, \dots, n_k)$ be a system of supports. Consider the subset $M(\mathcal{S})$ of $(\mathbb{N}_0^*)^k$ defined by $x \in M(\mathcal{S})$ if and only if $I = \operatorname{inf-supp}(x) \in \mathcal{S}$ and $p_I(x) \in A_I$. Then $M(\mathcal{S})$ is a submonoid of $(\mathbb{N}_0^*)^k$ such that $(n_1, \dots, n_k) \in \mathbb{N}^k \cap M(\mathcal{S})$ and satisfying the properties:*

(M1) if $I \subseteq \{1, \dots, k\}$ is an infinite support of some $x \in M(\mathcal{S})$ then the element x^* determined by $\operatorname{supp}(x^*) = \operatorname{inf-supp}(x^*) = \operatorname{inf-supp}(x)$ belongs to $M(\mathcal{S})$.

(M2) If $x \in M(\mathcal{S})$ then $\infty \cdot x \in M(\mathcal{S})$.

Moreover, $M(\mathcal{S})$ is finitely generated whenever all monoids $A_I, I \in \mathcal{S}$ are finitely generated.

Any submonoid M of $(\mathbb{N}_0^*)^k$ with $M \cap \mathbb{N}^k \neq \emptyset$ and satisfying (M1) and (M2) is of the form $M = M(\mathcal{S})$ for some system of supports $\mathcal{S}$. In this situation, for any $I \in \mathcal{S}$, $A_I = p_I(M) \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus I}$.

Proof. Since $A_\emptyset \subseteq M(\mathcal{S})$, $0 \in M(\mathcal{S})$ and $(n_1, \dots, n_k) \in M(\mathcal{S})$. To see that $M(\mathcal{S})$ is a monoid it remains to see that it is closed under addition.

Let $x, y \in M(\mathcal{S})$. Set $I = \operatorname{inf-supp}(x)$, $J = \operatorname{inf-supp}(y)$ and $K = I \cup J$. By Definition 6.7.1(iii), $K \in \mathcal{S}$. Let $x^* \in (\mathbb{N}_0^*)^k$ be such that $\operatorname{supp}(x^*) = \operatorname{inf-supp}(x^*) = K$. Notice that $x^* \in M(\mathcal{S})$ because $0 \in A_K$, and that $x + x^*$ and $y + x^* \in M(\mathcal{S})$ by Definition 6.7.1(iv). Then $x + y = (x + x^*) + (y + x^*)$ and, since $\operatorname{inf-supp}(x + x^*) = \operatorname{inf-supp}(y + x^*) = K$, we deduce that $x + y \in M(\mathcal{S})$ by applying again Definition 6.7.1(iv)..

Since $0 \in A_I$ for any I, $M(\mathcal{S})$ satisfies (M1). Property (M2) follows combining conditions (iii) and (i) in Definition 6.7.1.

Now suppose that A_I is a finitely generated monoid for every $I \in \mathcal{S}$. Fix $I \in \mathcal{S}$. Let x_I^* be the element of $M(\mathcal{S})$ determined by $\operatorname{supp}(x_I^*) = \operatorname{inf-supp}(x_I^*) = I$. Let $x_I^1, \dots, x_I^{n_I}$ be elements in $M(\mathcal{S})$ such that, for $i = 1, \dots, n_I$, $\operatorname{inf-supp}(x_I^i) = I$ and satisfying that $p_I(x_I^1), \dots, p_I(x_I^{n_I})$ form a set of generators of A_I.

By the construction of $M(\mathcal{S})$, $\bigcup_{I \in \mathcal{S}} \{x_I^*, x_I^1, \dots, x_I^{n_I}\}$ is a (finite) set of generators of $M(\mathcal{S})$, so that $M(\mathcal{S})$ is a finitely generated monoid.

To prove the final part of the statement, let M be a submonoid of $(\mathbb{N}_0^*)^k$ satisfying (M1) and (M2) and such that $(n_1,\dots,n_k)\in M\cap\mathbb{N}^k$. Set

$$\mathcal{S}=\{I\subseteq\{1,\dots,k\}\mid \text{ there exists } x\in M \text{ such that inf-supp}\,(x)=I\}$$

as the collection of subsets of $\{1,\dots,k\}$. Moreover, set $A_I=p_I(M)\cap\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$ for any $I\in\mathcal{S}$. It is easy to check that the properties of M ensure that $\mathcal{S}(n_1,\dots,n_k)$ is a system of supports such that $M(\mathcal{S})=M$. ∎

Remark 6.7.4. *With the notation as in Proposition 6.7.3 and Lemma 6.7.2, assume that $|\mathcal{S}|>2$ and let $\mathcal{T}=\{I_1,\dots,I_\ell\}$ be the set of minimal elements in $\mathcal{S}\setminus\{\emptyset\}$. For each $i\in\{1,\dots,\ell\}$, let S_{I_i} be the system of supports given by Lemma 6.7.2 and let $M(S_{I_i})$ be the associated monoid. Then*

$$M(\mathcal{S})=A_\emptyset\bigcup\left(\cup_{i=1}^{\ell}M'(S_{I_i})\right)=M_0\bigcup\left(\cup_{i=1}^{\ell}M'(S_{I_i})\right)$$

where $M_0=A_\emptyset+\{\infty\cdot x\mid x\in A_\emptyset\}$ and

$$M'(S_{I_i})=\{x\in(\mathbb{N}_0^*)^k\mid p_{I_i}(x)\in M(S_{I_i}) \text{ and } \text{inf-supp}\,(x)\supseteq I_i\}.$$

Now we give a couple of crucial examples of monoids given by a full affine system of monoids.

Example 6.7.5. *Let $k\geq 1$. Let R be a noetherian semilocal ring with an onto ring homomorphism $\varphi\colon R\to M_{n_1}(D_1)\times\cdots\times M_{n_k}(D_k)$ with kernel $J(R)$ and where $D_1,\dots,D_k$ are division rings. Let $M=\dim_\varphi V^*(R)\subseteq(\mathbb{N}_0^*)^k$. Then M is a finitely generated submonoid of $(\mathbb{N}_0^*)^k$ given by a full affine system of supports.*

Proof. Notice that $(n_1,\dots,n_k)\in M$. By Theorem 6.2.3(1), M satisfies condition (M1) in Proposition 6.7.3.

If P is a countably generated projective right R-module then

$$\dim_\varphi(\langle P^{(\aleph_0)}\rangle)=\infty\cdot\dim_\varphi(\langle P\rangle).$$

Hence M also satisfies condition (M2) in Proposition 6.7.3 and we can conclude that M is a submonoid of $(\mathbb{N}_0^*)^k$ given by a system of supports.

Let $x\in M$, and let P be a countably generated projective right module such that $\dim_\varphi(\langle P\rangle)=x$. Let $I=\text{inf-supp}\,(x)$. By Theorem 6.2.3(1), there exists a countably generated projective right R module P' such that

$$\text{supp}\,(\dim_\varphi(\langle P'\rangle))=\text{inf-supp}\,(\dim_\varphi(\langle P'\rangle))=I.$$

Let J be the trace ideal of P'. Then R/J is a semilocal ring with Jacobson radical $J+J(R)/J$ and, by Lemma 6.2.2, φ induces an onto ring homomorphism $\overline{\varphi}\colon R/J\to\prod_{i\in\{1,\dots,k\}\setminus I}M_{n_i}(D_i)$ with kernel $J(R/J)$. Moreover, by Theorem 6.2.3(2),

$$\dim_{\overline{\varphi}}(\langle P/PJ\rangle)=p_I(x)$$

and P/PJ is a finitely generated projective right R/J-module. This shows that $A_I=p_I(M)\cap\mathbb{N}_0^{\{1,\dots,k\}\setminus I}\subseteq\dim_{\overline{\varphi}}V(R/J)$. We claim that $\dim_{\overline{\varphi}}V(R/J)=A_I$. Equivalently, for any finitely generated projective right R/J-module $\overline{P}$ there exists a countably generated projective right R/J-module P_1 such that $P_1/P_1J \cong \overline{P}$. Therefore the claim follows from Proposition 6.2.1(iii).

By Corollary 6.1.6, $A_I = \dim_{\overline{\varphi}} V(R/J)$ is a full affine submonoid of $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$. Therefore the monoid M is given by a full affine system of supports. As full affine monoids are finitely generated, M is also finitely generated by Proposition 6.7.3. ∎

Next example is a consequence of Example 6.7.5 and Theorem 6.5.3. We prefer to give a proof just in the monoid context.

Example 6.7.6. *Let $D \in M_{n\times k}(\mathbb{N}_0)$ and $E_1,\ E_2 \in M_{\ell\times k}(\mathbb{N}_0)$. Let $m_1,\dots,m_n \in \mathbb{N}$ be such that $m_i \geq 2$ for any $i \in \{1,\dots,n\}$.*

Let $M \subseteq (\mathbb{N}_0^*)^k$ be the set of solutions in $(\mathbb{N}_0^*)^k$ of the system

$$D\begin{pmatrix} t_1\\ \vdots\\ t_k\end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^*\\ \vdots\\ m_n\mathbb{N}_0^*\end{pmatrix} \qquad \text{and} \qquad E_1\begin{pmatrix} t_1\\ \vdots\\ t_k\end{pmatrix} = E_2\begin{pmatrix} t_1\\ \vdots\\ t_k\end{pmatrix}.$$

Assume also that there exists $(n_1,\dots,n_k) \in M\cap\mathbb{N}^k$. Then there exists a full affine system of supports $\mathcal{S}(n_1,\dots,n_k)$ such that $M = M(\mathcal{S})$. In particular, M is finitely generated.

Proof. By Lemma 6.6.6, M satisfies conditions (M1) and (M2) in Proposition 6.7.3. Hence, by Proposition 6.7.3 and following the notation there, M is given by a system of supports in which

$$\mathcal{S} = \{I \mid \text{ there exists } x \in M \text{ such that inf-supp}\,(x) = I\}$$

as a collection of subsets of $\{1,\dots,k\}$ and, for any $I \in \mathcal{S}$, $A_I = p_I(M) \cap \mathbb{N}_0^{\{1,\dots,k\}\setminus I}$. We want to show that, for each $I \in \mathcal{S}$, the monoid A_I is full affine in $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$. To this aim we prove that A_I is the set of solutions in $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$ of a certain subsystem of the initial one.

Fix $I \in \mathcal{S}$. Let D_I be the matrix with entries in $\mathbb{N}_0$ obtained from $D = (d_{ij})$ by first deleting the rows i such that there exists $j \in I$ with $d_{ij} \neq 0$, and then deleting in the remaining matrix the j-th column for any $j \in I$. Let us denote K the subset of $\{1,\dots,n\}$ indicating which rows of D were deleted and let $p_K\colon {\mathbb{N}_0^*}^n \to {\mathbb{N}_0^*}^{\{1,\dots,n\}\setminus K}$ be the canonical projection.

As $I \in \mathcal{S}$, for any $i \in \{1,\dots,\ell\}$, either $e^1_{i,j} = 0 = e^2_{i,j}$ for all $j \in I$ or there are $j_1, j_2 \in I$ such that $e^1_{i,j_1} \neq 0$ and $e^2_{i,j_2} \neq 0$. Let E_1^I and E_2^I be the matrices with entries in $\mathbb{N}_0$ obtained from $E_1 = (e^1_{ij})$ and $E_2 = (e^2_{ij})$ by first deleting the rows i such that there exists $j \in I$ satisfying that e^1_{ij} is different from zero; after we also delete to each of the remaining matrices the j-th column for any $j \in I$.

Then the monoid A_I is the set of solutions in $\mathbb{N}_0^{\{1,\dots,k\}\setminus I}$ of the system

$$D_I\cdot p_I\begin{pmatrix} t_1\\ \vdots\\ t_k\end{pmatrix} \in p_K\begin{pmatrix} m_1\mathbb{N}_0^*\\ \vdots\\ m_n\mathbb{N}_0^*\end{pmatrix} \qquad \text{and} \qquad E_1^I\cdot p_I\begin{pmatrix} t_1\\ \vdots\\ t_k\end{pmatrix} = E_2^I\cdot p_I\begin{pmatrix} t_1\\ \vdots\\ t_k\end{pmatrix}.$$

∎

Theorem 6.7.7. *Let $k \in \mathbb{N}$, and let $(n_1,\dots,n_k) \in \mathbb{N}^k$. Let M be a submonoid of $(\mathbb{N}_0^*)^k$ such that $(n_1,\dots,n_k) \in M$. Then the following statements are equivalent* *(i) There exists a full affine system of supports $\mathcal{S}(n_1, \dots, n_k)$ such that $M = M(\mathcal{S})$.*

(ii) The submonoid M is defined by a system of equations.

To easy the proof of the theorem we first prove an auxiliary result.

Lemma 6.7.8. *Let $(A, (n_1, \dots, n_k)) \subseteq (\mathbb{N}_0^*)^k$ be a monoid defined by a system of equations $\mathcal{E}_A$. Let I be a proper nonempty subset of $\{1, \dots, k\}$ and set $I_c = \{1, \dots, k\} \setminus I$. Let $p\colon (\mathbb{N}_0^*)^k \to (\mathbb{N}_0^*)^{I_c}$ denote the canonical projection. Assume also $B \subseteq (\mathbb{N}_0^*)^{I_c}$ is a monoid defined by a system of equations $\mathcal{E}_B$ and such that $p(n_1, \dots, n_k) \in B$.*

Then the set $A' \subseteq (\mathbb{N}_0^*)^k$ defined by $x = (x_1, \dots, x_k) \in A'$ if and only if either

(1) $x \in A$ and $p(x) \in B$

or

(2) $x_i = \infty$ for any $i \in I$ and $p(x) \in B$

is a monoid defined by a system of equations. Moreover, $(n_1, \dots, n_k) \in A'$.

Proof. Let

$$D \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad \text{and} \qquad E_1 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}$$

be a system of equations defining A. Fix $i \in I$. Let $\mathcal{E}'_A$ be the system of equations

$$D \begin{pmatrix} t_1 + n_1 t_i \\ \vdots \\ t_k + n_k t_i \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad \text{and} \qquad E_1 \begin{pmatrix} t_1 + n_1 t_i \\ \vdots \\ t_k + n_k t_i \end{pmatrix} = E_2 \begin{pmatrix} t_1 + n_1 t_i \\ \vdots \\ t_k + n_k t_i \end{pmatrix}.$$

Notice that if $x \in (\mathbb{N}_0^*)^k$ and $i \notin \text{inf-supp}\,(x)$ then it is a solution of $\mathcal{E}'_A$ if and only if $x \in A$; while any $x \in (\mathbb{N}_0^*)^k$ such that $i \in \text{inf-supp}\,(x)$ is a solution of $\mathcal{E}'_A$.

Let $A_i \subseteq (\mathbb{N}_0^*)^k$ be the solutions of the system $\mathcal{E}_i = \mathcal{E}'_A \cup \mathcal{E}_B$, where $\mathcal{E}_B$ is the trivial extension of the system defining B. Then $x \in \mathbb{N}_0^k$ belongs to A_i if and only if $x \in A$ and $p(x) \in B$. In particular, $(n_1, \dots, n_k) \in A_i$. While if $x \in (\mathbb{N}_0^*)^k$ is such that $i \in \text{inf-supp}\,(x)$ then $x \in A_i$ if and only if $p(x) \in B$.

Now the submonoid $A' = \cap_{i \in I} A_i \subseteq (\mathbb{N}_0^*)^k$ defined as the solutions of the system $\cup_{i \in I} \mathcal{E}_i$, satisfies the properties required in the conclusion of the statement. ■

Proof of the Theorem 6.7.7. In view of Example 6.7.6 we only need to prove that (i) implies (ii).

Let $k \in \mathbb{N}$, and let $\mathcal{S}(n_1, \dots, n_k)$ be a full affine system of supports. We proceed by induction on $|\mathcal{S}|$. If $|\mathcal{S}| = 2$ then the only sets in $\mathcal{S}$ are $\emptyset$ and $\{1, \dots, k\}$. So that the only nontrivial full affine semigroup is $A_\emptyset$ and, for any $x \in A_\emptyset \setminus \{0\}$, $\text{supp}\,(x) = \{1, \dots, k\}$. By [9, Example 2.5], this implies that there exists $y = (y_1, \dots, y_k) \in A_\emptyset$ such that $A_\emptyset = y\mathbb{N}_0$. Therefore $M(\mathcal{S}) = A \cup \{\infty \cdot y\}$. By Proposition 6.6.7, $M(\mathcal{S})$ is defined by a system of equations.

Now assume that $|\mathcal{S}| > 2$ and that the statement is true for full affine systems of supports such that the set of supports has smaller cardinality. Let $\mathcal{T} \subseteq \mathcal{S}$ be the set of all minimal elements of $\mathcal{S} \setminus \{\emptyset\}$. Note that, since $|\mathcal{S}| > 2$, no element in $\mathcal{T}$ is equal to $\{1, \dots, k\}$. For any $I \in \mathcal{T}$ we construct the full affine system of supports $\mathcal{S}_I$ given by Lemma 6.7.2. As, for each $I \in \mathcal{T}$, $|\mathcal{S}_I| < |\mathcal{S}|$, we know that the monoid $M_I(\mathcal{S}_I)$ is given by a system of equations. Moreover, by Proposition 6.6.7, $M_0 = A_\emptyset + \{\infty \cdot x \mid x \in A_\emptyset\}$ is a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations.

Let $\mathcal{T} = \{I_1, \dots, I_\ell\}$. We complete M_0 to a chain $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_\ell$ of submonoids of $(\mathbb{N}_0^*)^k$ given by a system of equations, inductively, in the following way: If $i < \ell$ is such that M_i is constructed then M_{i+1} is the monoid given by applying Lemma 6.7.8 to $A = M_i$, $I = I_{i+1}$ and $B = M_{I_{i+1}}(\mathcal{S}_{I_{i+1}})$. Notice that, following the notation of Lemma 6.7.8 and by the definition of a system of supports, $p_{I_{i+1}}(M_i) \subseteq M_{I_{i+1}}(\mathcal{S}_{I_{i+1}})$, therefore $M_{i+1} = M_i \bigcup M'_{I_{i+1}}$ where

$$M'_{I_{i+1}} = \{x \in (\mathbb{N}_0^*)^k \mid p_{I_{i+1}}(x) \in M_{I_{i+1}}(\mathcal{S}_{I_{i+1}}) \text{ and inf-supp}\,(x) \supseteq I_{i+1}\}.$$

Therefore

$$M_\ell = M_0 \cup M'_{I_1} \cup \cdots \cup M'_{I_\ell} = M(\mathcal{S})$$

by Remark 6.7.4. This allows us to conclude that $M(\mathcal{S})$ is a monoid given by a system of equations. ∎

Now Theorem 6.2.6 follows by just patching together our previous results.

Proof Theorem 6.2.6. (1) ⇒ (2) follows from Theorem 6.5.3. It is clear that (2) ⇒ (3).

Finally, assume (3). By Example 6.7.5, M is given by a full affine system of supports. By Theorem 6.7.7, M is defined by a system of equations, and (1) follows. ∎

We close the paper characterizing the monoids corresponding to semilocal rings such that any projective right R-module is a direct sum of finitely generated modules. They are precisely the ones arising in Proposition 6.6.7.

Corollary 6.7.9. *Let $k \in \mathbb{N}$. Let M be a submonoid of $(\mathbb{N}_0^*)^k$ containing $(n_1, \dots, n_k) \in \mathbb{N}^k$. Let $A = M \cap \mathbb{N}_0^k$. Then the following statements are equivalent:*

(i) A is a full affine submonoid of $\mathbb{N}_0^k$ and $M = A + \{\infty \cdot a \mid a \in A\}$.

(ii) There exists a (noetherian) semilocal ring R such that all projective right R modules are direct sum of finitely generated modules, and an onto morphism $\varphi\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ with $\operatorname{Ker}\varphi = J(R)$, where $D_1, \dots, D_k$ are suitable division rings, satisfying that $\dim_\varphi V^*(R) = M$.

Proof. Assume (i). By Proposition 6.6.7, M is given by a system of equations. By Theorem 6.2.6, (ii) holds.

Assume (ii). So that R is a, not necessarily noetherian, semilocal ring such that all projective right modules are direct sum of finitely generated ones. By Corollary 6.1.6, A is a full submonoid of $\mathbb{N}_0^k$. It is clear that $A + \{\infty \cdot a \mid a \in A\} \subseteq M$. Let $P_1, \dots, P_s$ be a set of representatives of the indecomposable (hence finitely generated) projective right modules. For $i = 1, \dots, s$, let $a_i = \dim_\varphi(\langle P_i \rangle)$. As any projective module is a direct sum of finitely generated projective modules, any $x \in M$ satisfies that $x = \alpha_1 a_1 + \cdots + \alpha_s a_s$ for some $\alpha_i \in \mathbb{N}_0^*$, hence $x \in A + \{\infty \cdot a \mid a \in A\}$. This shows that (i) holds. ∎

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7. INFINITELY GENERATED PROJECTIVE MODULES OVER PULLBACKS OF RINGS

After the paper of Bass [2] there seemed to be the general belief that the theory of infinitely generated projective modules *invited little interest.* However some of the developments in the representation theory of finite dimensional algebras [21] and subsequent ones in integral representation theory have drawn the attention to the infinite dimensional representations [22], [3]. Also the study of the direct sum decomposition of infinite direct sums of modules over general rings requires a good knowledge of the behavior of all projective modules [20]. As a result of this pressure, interesting general theory on projective modules has recently appeared [17], [18] and it has been shown that examples of rings such that not all projective modules are direct sum of finitely generated are relatively frequent [19] and the behavior can be quite complex even for noetherian rings [7]. In this paper we continue this line of work by providing further examples of such rings. All of them are semilocal rings, that is, rings that are semisimple artinian modulo the Jacobson radical.

Our study makes essential use of the result proved by P. Příhoda in [17] that, over an arbitrary ring, projective modules are isomorphic if and only if they are isomorphic modulo the Jacobson radical. For a semilocal ring R this implies that the monoid of isomorphism classes of countably generated projective right (or left) R-modules can be seen as a submonoid of $(\mathbb{N}_0 \cup \{\infty\})^k$ for a suitable $k \geq 1$, cf. §7.1 for the precise definitions.

In [12], we characterized the class of monoids that can be realized as monoid of isomorphism classes of countably generated projective right (or left) modules over a noetherian semilocal ring as *essentially* the set of solutions in $\mathbb{N}_0 \cup \{\infty\}$ of finite homogeneous systems of diophantine linear equations. In Theorem 7.1.6 we show that any monoid M which is the set of solutions in $\mathbb{N}_0 \cup \{\infty\}$ of a finite homogeneous system of diophantine linear inequalities can also be realized as monoid of isomorphism classes of countably generated projective right modules over a suitable semilocal ring R. In the examples we construct, the monoid of isomorphism classes of countably generated projective left R-modules is the set of solutions in $\mathbb{N}_0 \cup \{\infty\}$ of the system obtained by reversing the inequalities of the system defining M. While in the noetherian case the monoid of countably generated projective right modules is isomorphic to the one of countably generated projective left modules, as we show in this paper, this is no longer true for general semilocal rings.

In this paper we emphasize in the study of projective modules that are not finitely generated but that they are finitely generated modulo the Jacobson radical. The first example of this kind was provided by Gerasimov and Sakhaev in [11], and the construction was further developed by Sakhaev in [23]. Other examples appear when studying the direct sum decomposition of infinite direct sums of uniserial modules [20], [8] and [17]. From these examples it seemed that the existence of such projective modules is rare and very difficult to handle. With our methods we can produce a wide variety of examples where such projectives exist and where their behavior is *under control.* In our examples, the countably generated projective modules that are finitely generated modulo the Jacobson radical, correspond to the solutions in $\mathbb{N}_0$ of the system of inequalities. Between them we distinguish the finitely generated ones as the ones that fulfill the

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equality.

The techniques we use in this paper are an extension of the ones in [12]. As the title indicates, our rings are constructed as pullbacks of suitable rings, and we take advantage of [16, Theorems 2.1, 2.2 and 2.3] in which Milnor describes *all* projective modules over a class of ring pullbacks. A key ingredient is the Gerasimov-Sakhaev example mentioned above and the computation of its monoid of isomorphism classes of countably generated projective right (and left) modules done in [6].

In §7.1 we give an overview of the paper: we introduce the monoids of projective modules, we define in a precise way the class of monoids that we will realize in section 7.5 as monoids of countably generated projective right modules and of countably generated projective left modules over suitable semilocal rings, and we state our main realization Theorem 7.1.6.

In section 7.2 we develop some theory on projective modules that are finitely generated modulo the Jacobson radical which essentially follows [23]. Theorem 7.2.9 is a slight generalization of the main result in [9].

In section 7.3 we compute some particular examples to illustrate the consequences of Theorem 7.1.6. For instance, in 7.3.6, we construct a semilocal ring such that all projective left R-modules are free while R has a nonzero (infinitely generated) right projective module that is not a generator. Such an example also shows that the notion of p-connected ring is not left-right symmetric; this answers in the negative a question in [10, page 310]. Recall that, following Bass [2], a ring is (left) p-connected if every nonzero left projective module is a generator.

We also provide examples showing that if R is a semilocal ring such that $R/J(R) \cong D_1 \times D_2$ and R has a countably generated, but not finitely generated, projective module that is finitely generated modulo the Jacobson radical then there is still room for countably generated (right and left, or just right) projective modules that are not direct sums of projective modules that are finitely generated modulo the Jacobson radical. This answers in the negative a question formulated in [6, page 3261].

In section 7.4 we develop some properties of the monoids defined by inequalities. Finally, in section 7.5 we prove Theorem 7.1.6.

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7.1 Preliminaries and overview

All our rings are associative with 1, and ring morphism means unital ring morphism.

7.1.1 Monoids of projective modules

Let R be a ring. Let $V^*(R_R) = V^*(R)$ ($V^*({}_RR)$) be the set of isomorphism classes of countably generated projective right (left) R-modules. If P and Q are countably generated projective right R-modules then the direct sum induces an addition on $V^*(R)$ by setting $\langle P\rangle + \langle Q\rangle = \langle P \oplus Q\rangle$, so that $V^*(R)$ is an additive monoid. Similarly, $V^*({}_RR)$ is also an additive monoid.

Let $V(R)$ be the set of isomorphism classes of finitely generated right (or left) R-modules. Again $V(R)$ is an additive monoid, which can be identified with a submonoid of $V^*(R)$. Since the functor $\mathrm{Hom}_R(-, R)$ induces a duality between the category of finitely generated projective right R-modules and the category of finitely generated projective left R-modules we identify $V({}_RR)$ with $V(R)$. So that, we also see $V(R)$ as a submonoid of $V^*({}_RR)$.

Another interesting submonoid of $V^*(R)$ is $W(R_R) = W(R)$ which we define as the set of isomorphism classes of countably generated projective right R-modules that are pure submodules of finitely generated projective modules. The submonoid of $V^*({}_RR)$, $W({}_RR)$ is defined in a similar way. Clearly, $V(R) \subseteq W(R) \subseteq V^*(R)$, and $V(R) \subseteq W({}_RR) \subseteq V^*({}_RR)$. Notice that $W(R) \setminus V(R)$ is also a semigroup.

Along the paper we will find many examples of (semilocal) rings R with non trivial $W(R)$. Now we give a different kind of example.

Example 7.1.1. [2] *Let R denote the ring of continuous real valued functions over the interval* $[0,1]$. *Let*

$$I = \{f \in R \mid \text{ there exists } \varepsilon > 0 \text{ such that } f([0,\varepsilon]) = 0\}$$

then I is a projective pure ideal of R, cf. [8, Example 3.3] or [6, p. 3263].

The notation $W(R)$ is borrowed from the C^*-algebra world, as we think on this monoid as an algebraic analogue of the Cuntz monoid defined in C^*-algebras.

7.1.2 The semilocal case

A ring R is said to be semilocal if modulo its Jacobson radical $J(R)$ is semisimple artinian, that is, $R/J(R) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for suitable division rings $D_1, \dots, D_k$. For the rest of our discussion we fix an onto ring homomorphism $\varphi\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ such that $\operatorname{Ker}\varphi = J(R)$.

Let $V_1, \dots, V_k$ denote a fixed ordered set of representatives of the isomorphism classes of simple right R-modules such that $\operatorname{End}_R(V_i) \cong D_i$. Let us also fix $W_1, \dots, W_k$, where $W_i = \operatorname{Hom}_R(V_i, R/J(R))$ for $i = 1, \dots, k$, as an ordered set of representatives of simple left R-modules.

If P_R is a countably generated projective right R-module then $P/PJ(R) \cong V_1^{(I_1)} \oplus \cdots \oplus V_k^{(I_k)}$ and the cardinality of the sets $I_1, \dots, I_k$ determines the isomorphism class of $P/PJ(R)$. By [17] (cf. Theorem 7.2.2) projective modules are determined, up to isomorphism, by its quotient modulo the Jacobson radical. So that, for a semilocal ring R, to describe $V^*(R)$ we only need to record the cardinality of the sets I_i for $i = 1, \dots, k$. A similar situation holds for projective left R-modules.

Note that, by Theorem 7.2.2(i), in the case of semilocal rings

$$W(R) = \{\langle P \rangle \in V^*(R) \mid P/PJ(R) \text{ is finitely generated}\}.$$

Similarly, for $W({}_RR)$.

7.1.3 The dimension monoids for semilocal rings

Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We also consider the monoid $\mathbb{N}_0^* = \mathbb{N}_0 \cup \{\infty\}$ with the addition determined by the addition on $\mathbb{N}_0$ extended by the rule $n + \infty = \infty + n = \infty$ for any $n \in \mathbb{N}_0^*$.

Following the notation of §7.1.2, if P is a countably generated projective right R-module such that $P/PJ(R) \cong V_1^{(I_1)} \oplus \cdots \oplus V_k^{(I_k)}$ we set $\dim_\varphi(\langle P \rangle)) = (m_1, \dots, m_k) \in (\mathbb{N}_0^*)^k$ where, for $i = 1, \dots, k$, $m_i = |I_i|$ if I_i is finite and $m_i = \infty$ if I_i is infinite. Therefore $\dim_\varphi\colon V^*(R) \to (\mathbb{N}_0^*)^k$ is a monoid morphism. Similarly, we define a monoid morphism $\dim_\varphi\colon V^*({}_RR) \to (\mathbb{N}_0^*)^k$.

By Theorem 7.2.2(ii), $\dim_\varphi\colon V^*(R) \to (\mathbb{N}_0^*)^k$ and $\dim_\varphi\colon V^*({}_RR) \to (\mathbb{N}_0^*)^k$ are monoid monomorphisms. Note that $\dim_\varphi(\langle R \rangle) = (n_1, \dots, n_k) \in \mathbb{N}^k$ and that $\dim_\varphi(W(R)) = \mathbb{N}_0^k \cap \dim_\varphi(V^*(R))$ while $\dim_\varphi(W({}_RR)) = \mathbb{N}_0^k \cap \dim_\varphi(V^*({}_RR))$.

Definition 7.1.2. *A submonoid A of $\mathbb{N}_0^k$ is said to be full affine if whenever a, $b \in A$ are such that $a = b + c$ for some $c \in \mathbb{N}_0^k$ then $c \in A$.*

The class of full affine submonoids of $\mathbb{N}_0^k$ containing an element $(n_1, \dots, n_k) \in \mathbb{N}^k$ is the precise class of monoids that can be realized as $\dim_\varphi(V(R))$ for a semilocal ring R such that $\dim_\varphi(\langle R \rangle) = (n_1, \dots, n_k)$ [7].

The general problem we are interested in is determining which submonoids of $(\mathbb{N}_0^*)^k$ can be realized as dimension monoids, that is, as $\dim_\varphi(V^*(R))$ for a suitable semilocal ring R. For the case $k = 1$ the solution is easy and was first given in [4]; right now we have tools to justify it very briefly: let P a finitely generated projective module such that $\dim_\varphi(\langle P\rangle)$ is a minimal element of $\dim_\varphi(V^*(R))$ it follows from Theorem 7.2.2(ii) that any projective right R module is isomorphic to a direct sum of copies of P.

For $k \geq 2$ the situation is much more involved and we do not know a complete solution of the problem. In the next definition we single out some classes of monoids that can be realized as dimension monoids of semilocal ring.

Definition 7.1.3. *Let $k \geq 1$.*

(i) A submonoid M of $(\mathbb{N}_0^*)^k$ is said to be a monoid defined by a system of equations *if it is the set of solutions in $(\mathbb{N}_0^*)^k$ of a system of the form*

$$D\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad (*) \qquad \textit{and} \qquad E_1\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} = E_2\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \qquad (**)$$

where $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times k}(\mathbb{N}_0)$, $m_1, \dots, m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1, \dots, n\}$ and ℓ, $n \geq 0$.

(ii) A submonoid M of $(\mathbb{N}_0^*)^k$ is said to be a monoid defined by a system of inequalities *provided that there exist $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times k}(\mathbb{N}_0)$, ℓ, $n \geq 0$, and $m_1, \dots, m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1, \dots, n\}$, such that M is the set of solutions in $(\mathbb{N}_0^*)^k$ of*

$$D\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad \textit{and} \qquad E_1\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \geq E_2\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}.$$

(iii) If $M \leq (\mathbb{N}_0^*)^k$ is defined by a system of inequalities as in (ii) we define its dual monoid $D(M)$ as the set of solutions in $(\mathbb{N}_0^*)^k$ of

$$D\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \qquad \textit{and} \qquad E_1\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \leq E_2\begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix}$$

In the situation of (i) or in the situation of (ii) we shall refer to $M \cap \mathbb{N}_0^k$ as a submonoid of $\mathbb{N}_0^k$ defined by a system of equations or by a system of inequalities, respectively.

Remarks 7.1.4. *1) It is important to notice that $\mathbb{N}_0^*$ is no longer a cancellative monoid. So that, for example, the set of solutions in $(\mathbb{N}_0^*)^2$ of the equation $x = y$ is not the same as the set of solutions of $2x = y + x$.*

2) If M is a monoid defined by a system of inequalities then the monoid $D(M)$ depends on the particular system fixed to define M. For an easy example see Examples 7.3.6(ii) and (iii).

3) Let A be a submonoid of $\mathbb{N}_0^k$ containing $(n_1, \dots, n_k) \in \mathbb{N}^k$. It was observed by Hochster that A is full affine if and only if A is a submonoid of $\mathbb{N}_0^k$ defined by a system of equations (cf. [12, §6]).

In this case, the monoid $M = A + \infty \cdot A$ is a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations [12, Corollary 7.9].

7.1.4 Realization results. Main result

For further quoting we recall the main result in [12] which characterized the monoids M that can be realized as $V^*(R)$ for a semilocal noetherian ring R. For this class of rings a projective module that is finitely generated modulo $J(R)$ must be finitely generated so that $W(R) = V(R)$ (see, for example, Proposition 7.2.7), and also, by [18], $V^*({}_RR) \cong V^*(R)$.

Theorem 7.1.5. *Let $k \in \mathbb{N}$. Let M be a submonoid of $(\mathbb{N}_0^*)^k$ containing $(n_1, \dots, n_k) \in \mathbb{N}^k$. Then the following statements are equivalent:*

(1) M is a monoid defined by a system of equations.

(2) There exist a noetherian semilocal ring R, a semisimple ring $S = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$, where $D_1, \dots, D_k$ are division rings, and an onto ring morphism $\varphi\colon R \to S$ with $\operatorname{Ker}\varphi = J(R)$ such that $\dim_\varphi V^*(R) = M$. Therefore, $\dim_\varphi V(R) = M \cap \mathbb{N}_0^k$.

In the above statement, if F denotes a field, R can be constructed to be an F-algebra such that $D_1 = \cdots = D_k = E$ is a field extension of F.

In this paper we shall prove the following realization result

Theorem 7.1.6. *Let $k \geq 1$, and let F be a field. Let M be a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of inequalities. Let $D(M)$ denote its dual monoid. Assume that $M \cap D(M)$ contains an element $(n_1, \dots, n_k) \in \mathbb{N}^k$. Then there exist a semilocal F-algebra R, a semisimple F-algebra $S = M_{n_1}(E) \times \cdots \times M_{n_k}(E)$, where E is a suitable field extension of F, and an onto morphism of F-algebras $\varphi\colon R \to S$ with $\operatorname{Ker}\varphi = J(R)$ satisfying that $\dim_\varphi V^*(R_R) = M$ and $\dim_\varphi V^*({}_RR) = D(M)$.*

Moreover, $\dim_\varphi W(R_R) = M \cap \mathbb{N}_0^k$, $\dim_\varphi W({}_RR) = D(M) \cap \mathbb{N}_0^k$, and $\dim_\varphi V(R) = M \cap D(M) \cap \mathbb{N}_0^k$.

For any semilocal ring $V(R)$ is a finitely generated monoid, so is $V^*(R)$ for R noetherian and semilocal. As we will show in §7.4, monoids defined by a system of inequalities are still finitely generated. But, in general, we do not know whether a monoid that can be realized as $V^*(R)$ for some semilocal ring R must be finitely generated.

7.2 Projective modules, monoids of projectives and Jacobson radical

In this section we want to explain the relation between $W(R_R)$ and $W({}_RR)$ completing the results in [9]. We also take the opportunity to state in a (too) precise way results on lifting maps between projective modules modulo an ideal contained in the Jacobson radical.

Let I be a two-sided ideal of a ring R, let M and N be right R-modules, and let $f\colon M \to N$ denote a module homomorphism. By the induced homomorphism $\overline{f}\colon M/MI \to N/NI$ we mean the map defined by $\overline{f}(m + MI) = f(m) + NI$ for any $m \in M$.

Recall the following well known result.

Lemma 7.2.1. *Let R be any ring, and let $I \subseteq J(R)$ be a two-sided ideal of R. Let $f\colon P \to Q$ be a morphism between finitely generated projective right R-modules. Then f is an isomorphism if and only if the induced homomorphism $\overline{f}\colon P/PI \to Q/QI$ is an isomorphism.*

In contrast, for general projective modules we have.

Theorem 7.2.2. *Let R be any ring, let P and Q be projective right R-modules, and let $I \subseteq J(R)$ be a two-sided ideal of R.*

(i) [9, Proposition 6.1] *A module homomorphism $f\colon P \to Q$ is a pure monomorphism if and only if so is the induced map $\overline{f}\colon P/PI \to Q/QI$.*

(ii) [17, Theorem 2.3 and its proof] *Let $\alpha\colon P/PI \to Q/QI$ be an isomorphism of right R/I-modules. Let $f\colon P \to Q$ be a module homomorphism such that $\overline{f} = \alpha$, and let X be a finite subset of P. Then there exists an isomorphism $g\colon P \to Q$ such that $\overline{g} = \alpha$ and $g(x) = f(x)$ for any $x \in X$.*

In particular, P and Q are isomorphic if and only if they are isomorphic modulo the Jacobson radical.

For further applications we note the following corollary of Theorem 7.2.2.

Corollary 7.2.3. *Let R be a ring, and let $I \subseteq J(R)$ be a two-sided ideal. Let P be a countably generated projective right R-module. Let $f\colon P \to P$ be a homomorphism such that the induced map $\overline{f}\colon P/PI \to P/PI$ is the identity, and let X be a finite subset of P. Then there exists a bijective homomorphism $h\colon P \to P$ such that the induced homomorphism $\overline{h} = \mathrm{Id}_{P/PI}$ and such that $hf(x) = x$ for any $x \in X$.*

Proof. By Theorem 7.2.2(ii), there exists an isomorphism $g\colon P \to P$ such that $\overline{g} = \mathrm{Id}_{P/PI}$ and $g(x) = f(x)$ for any $x \in X$. Set $h = g^{-1}$ to conclude. ∎

Lemma 7.2.4. *Let R be a ring, let P and Q be projective right R-modules. Let I be a two-sided ideal of R contained in $J(R)$, and let $\alpha\colon Q/QI \to P/PI$ and $\beta\colon P/PI \to Q/QI$ be homomorphisms such that $\beta \circ \alpha = \mathrm{Id}_{Q/QI}$. Let $f\colon Q \to P$ and $g\colon P \to Q$ be module homomorphisms such that $\overline{f} = \alpha$ and $\overline{g} = \beta$.*

If $f \circ g$ is idempotent then $P \cong Q \oplus Q'$ and $Q'/Q'I \cong (\mathrm{Id}_{P/PI} - \alpha\beta)(P/PI)$.

Proof. Since $fg(P)$ is a direct summand of P,

$$fg(P)/fg(P)I = fg(P)/(fg(P) \cap PI) \cong (fg(P) + PI)\,/PI.$$

Since, for any $x \in P$, $\beta(fg(x) + PI) = \beta(x + PI)$ we deduce that $\beta\colon fg(P)/fg(P)I \to Q/QI$ is bijective. By Theorem 7.2.2, we conclude that $Q \cong fg(P)$.

Since $((\mathrm{Id}_P - fg)(P) + PI)\,/PI = (\mathrm{Id}_{P/PI} - \alpha\beta)\,(P/PI)$, it follows that $Q' = (\mathrm{Id}_P - fg)(P)$ has the claimed properties. ∎

Corollary 7.2.5. *Let R be a ring with Jacobson radical $J(R)$. Let $I \subseteq J(R)$ be a two-sided ideal. Let P and Q be projective right R-modules such that Q is finitely generated. If there exists a projective right R/I-module X such that $P/PI \cong Q/QI \oplus X$ then there exists a projective right R-module Q' such that $P \cong Q \oplus Q'$ and $Q'/Q'I \cong X$.*

Proof. Since Q is finitely generated, the split exact sequence of R/I-modules

$$0 \to X \to P/PI \xrightarrow{\beta} Q/QI \to 0$$

lifts to a split exact sequence

$$0 \to \mathrm{Ker}\, g \to P \xrightarrow{g} Q \to 0$$

where $\overline{g} = \beta$. Therefore $P \cong Q \oplus \mathrm{Ker}\, g$. We want to show that $\mathrm{Ker}\, g/(\mathrm{Ker}\, g)I \cong X$.

Let $\alpha\colon Q/QI \to P/PI$ be such that $\beta\alpha = \mathrm{Id}_{Q/QI}$, and let $f\colon Q \to P$ be such that $\overline{f} = \alpha$. Since Q is finitely generated and $\overline{gf} = \beta\alpha = \mathrm{Id}_{Q/QI}$, $gf\colon Q \to Q$ is invertible (cf. Lemma 7.2.1). So that, there exists an invertible endomorphism h of Q satisfying that $\overline{h} = \mathrm{Id}_{Q/QI}$, and such that $g(fh) = \mathrm{Id}$. Therefore, $(fh)g$ is an idempotent endomorphism of P and since $(\mathrm{Id} - (fh)g)P = \mathrm{Ker}\, g$ we conclude, by the second part of Lemma 7.2.4, that $Q' = \mathrm{Ker}\, g$ has the claimed properties. ∎

In the following lemma we recall the properties of sequences $\{f_n\}_{n\geq 1}$ satisfying that $f_{n+1}f_n = f_n$. Lazard in [14] realized the importance of them to describe pure ideals of a ring. They play a fundamental rôle in constructing finitely generated flat modules over semilocal rings that are not projective or, equivalently, in constructing non-finitely generated projective modules that are finitely generated modulo the Jacobson radical.

They were very well analyzed by Sakhaev in several papers, see for example [23]. Recently, they have been extensively re-studied [8], [9] and [6].

Lemma 7.2.6. *Let R be any ring. Let P be a right R-module and let $f_1, \dots, f_n, \dots$ be a sequence of endomorphisms of P satisfying that, for each $n \geq 1$, $f_{n+1}f_n = f_n$ then,*

(i) $\bigcup_{n\geq 1} f_n \cdot \mathrm{End}_R(P)$ is a projective pure right ideal of $\mathrm{End}_R(P)$.

(ii) $Q = \bigcup_{n\geq 1} f_n(P)$ *is a pure submodule of* P *isomorphic to a direct summand of* $P^{(\mathbb{N})}$*. In particular, if* P *is projective then so is* Q*.*

Proof. (i). This is due to Lazard [14].

(ii). The purity of I inside S gives $I \otimes_S P \hookrightarrow S \otimes_S P$. Using the identification $S \otimes_S P \simeq P$, we get $\bigcup_{n\geq 1} f_n(P) \simeq I \otimes_S P$. Hence the purity of Q inside P follows from the associativity of the tensor product and (i).

Consider the countable direct system

$$P_1 \xrightarrow{f_1} P_2 \cdots P_n \xrightarrow{f_n} P_{n+1} \cdots$$

where $P = P_n$ for any $n \geq 1$. Since $f_{n+1}f_n = f_n$, the sequence $\{f_n\}_{n\geq 1}$ induces an injective map $f\colon \varinjlim P_n \to P$ such that $\operatorname{Im} f = Q$. Therefore, Q fits into the (pure) exact sequence

$$0 \to \oplus_{n\geq 1} P_n \xrightarrow{\Phi} \oplus_{n\geq 1} P_n \to Q \to 0$$

where, for each $n \geq 1$ and letting $\varepsilon_n\colon P_n \to \oplus_{n\geq 1} P_n$ denote the canonical embedding, the map Φ is determined by $\Phi\varepsilon_n(x) = \varepsilon_n(x) - \varepsilon_{n+1} f_n(x)$ for each $x \in P_n$.

The properties of the sequence of maps $\{f_n\}_{n\geq 1}$ imply that Φ splits (see, for example, [1, Proposition 2.1]). ∎

Proposition 7.2.7. *Let* R *be a ring. Let* P_R *and* Q_R *be projective right* R*-modules such that* P_R *is finitely generated. Let* $\alpha\colon Q/QJ(R) \to P/PJ(R)$ *and* $\beta\colon P/PJ(R) \to Q/QJ(R)$ *be such that* $\beta\alpha = \mathrm{Id}_{Q/QJ(R)}$*. Let* $\varepsilon\colon Q \to P$ *be any module homomorphism such that* $\overline{\varepsilon} = \alpha$*. Then there exists a sequence* $f_1, \dots, f_n, \dots$ *of endomorphisms of* P *such that, for each* $n \geq 1$*,* $f_{n+1}f_n = f_n$*,* $\overline{f_n} = \alpha \circ \beta$ *and* $Q \cong \varepsilon(Q) = \bigcup_{n\geq 1} f_n(P)$*.*

Moreover Q *is finitely generated if and only if there exists* n_0 *such that* $f_{n_0}^2 = f_{n_0}$*. In this case,* $f_{n_0+k}^2 = f_{n_0+k}$ *for any* $k \geq 0$*.*

Proof. Let $\varphi\colon P \to Q$ be a lifting of β.

Note that Q_R must be a countably generated projective module, so that we can fix an ascending chain $\emptyset = X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_n \subseteq \dots$ of finite subsets of Q such that $X = \bigcup_{n\geq 1} X_n$ generates Q.

Since P is finitely generated and using Corollary 7.2.3, we can construct, inductively, a sequence $\mathrm{Id}_Q = h_1, \dots, h_n, \dots$ of (auto)morphisms of Q such that if, for each $n \geq 1$, we set $f_n = \varepsilon h_n h_{n-1} \cdots h_1 \varphi$ then $h_{n+1} h_n \cdots h_1 \varphi f_n = h_n \cdots h_1 \varphi$ and $h_{n+1} h_n \cdots h_1 \varphi \varepsilon(x) = x$ for any $x \in X_{n+1}$. It can be easily checked that the homomorphisms $\{f_n\}_{n\geq 1}$ satisfy the desired properties.

If Q is finitely generated there exists n_0 such that $\varepsilon(Q) = f_{n_0-1}(P)$. Observe that $f_{n_0} f_{n_0-1} = f_{n_0-1}$ says $f_{n_0}(x) = x$ for any $x \in \varepsilon(Q)$. In particular, $f_{n_0+k}^2 = f_{n_0+k}$ for any $k \in \mathbb{N}$.

Conversely, in view of Lemma 7.2.4, if there exists n_0 such that $f_{n_0}^2 = f_{n_0}$ then Q is isomorphic to $f_{n_0}(P)$ which is a direct summand of P. In particular, Q is finitely generated and $f_{n_0}(P) = f_{n_0+k}(P)$ for any $k \geq 0$. Since f_{n_0} is idempotent, for any $k \geq 0$, $f_{n_0+k} = f_{n_0} f_{n_0+k}$ so that $f_{n_0+k}^2 = f_{n_0+k} f_{n_0} f_{n_0+k} = f_{n_0+k}$. ∎

Remark 7.2.8. *In the situation of Proposition 7.2.7, fix* $n \geq 1$*. Notice that* $(f_{n+1} - f_n)f_n = f_n - f_n^2$*. Since* $\overline{f_{n+1} - f_n} = \overline{0} \in \mathrm{End}_R(P/PI)$ *and* P *is a finitely generated projective module,* $u = \mathrm{Id}_P - (f_{n+1} - f_n)$ *is a unit such that* $uf_n = f_n^2$*.*

For any $m \in \mathbb{Z}$*, set* $g_m = u^{-(m+1)} f_n u^m \in \mathrm{End}_R(P)$*. It easily follows that, for any* $m \in \mathbb{Z}$*,* $g_{m+1} g_m = g_m$ *and also that* $(\mathrm{Id}_P - g_{m+1})(\mathrm{Id}_P - g_m) = \mathrm{Id}_P - g_{m+1}$ *so that, by Lemma 7.2.6,* $P_n' = \bigcup_{m\geq 0} g_m P$ *is a projective pure submodule of* P *and* $Q_n' = \bigcup_{m\leq 0} \mathrm{Hom}_R(P, R)(\mathrm{Id}_P - g_m)$ *is a projective pure submodule of the projective left* R*-module* $\mathrm{Hom}_R(P, R)$*.*

Notice that, for any m*,* $\overline{g_m} = \alpha \circ \beta$ *and* $\overline{\mathrm{Id}_P - g_m} = \mathrm{Id}_{P/PI} - \alpha \circ \beta$*. Therefore,* $P_n'/P_n' I \cong Q/QI$*, hence* $P_n' \cong Q$*, and*

$$Q_n'/IQ_n' \cong \mathrm{Hom}_{R/RI}((\mathrm{Id}_{P/PI} - \alpha \circ \beta)P/PI, R/I).$$

In particular, the isomorphism classes of P'_n and Q'_n, respectively, do not depend on n.

Combining Proposition 7.2.7 with Remark 7.2.8 we obtain the following theorem which is a slight refinement of [9, Theorem 7.1].

Theorem 7.2.9. *Let R be a ring, let P be a finitely generated projective right R-module, and let $I \subseteq J(R)$ be a two-sided ideal of R. Assume that there is a split exact sequence of right R/I modules*

$$0 \to X \to P/PI \to X' \to 0.$$

Then the following statements are equivalent,

(i) There exists a (countably generated) projective right R-module Q such that $Q/QI \cong X$.

(ii) There exists a (countably generated) projective left R-module Q' such that $Q'/IQ' \cong \operatorname{Hom}_{R/I}(X', R/I)$.

When the above equivalent statement hold Q is isomorphic to a pure submodule of P, and Q' is isomorphic to a pure submodule of $\operatorname{Hom}_R(P, R)$. Moreover, Q is finitely generated if and only if Q' is finitely generated if and only if there exists a projective right R-module P' such that $P'/P'I \cong X'$.

Now we are going to state some of the results above in terms of monoids of projectives. More precisely, in terms of pre-ordered monoids of projectives.

We recall that over a commutative monoid M there is a pre-order relation called *the algebraic preorder* on M defined by $x \geq y$, for x, $y \in M$, if and only if $x = y + z$ for some $z \in M$.

For example, over $(\mathbb{N}_0^*)^k$ the algebraic order is the component-wise order, which is even a partial order. When the monoid is $V^*(R)$ for some ring R, $\langle Q\rangle \leq \langle P\rangle$ if and only if Q is isomorphic to a direct summand of P.

In terms of monoids of projective modules Corollary 7.2.5 essentially says that for elements in $V(R)$ the algebraic preorder is respected modulo $J(R)$. We state this in a precise way in the next result.

Corollary 7.2.10. *Let R be a ring, and let I be a two-sided ideal of R contained in $J(R)$. Let $\pi\colon R \to R/I$ denote the projection, and let $\widetilde{\pi}\colon V^*(R) \to V^*(R/I)$ denote the induced homomorphism of monoids. If $x \in V^*(R)$, $y \in V(R)$ are such that there exist $c \in V^*(R/I)$ satisfying that $\widetilde{\pi}(x) = \widetilde{\pi}(y) + c$ then there exists $z \in V^*(R)$ such that $\widetilde{\pi}(z) = c$ and $x = y + z$.*

In general, for a semilocal ring R, the monoid $V^*(R)$ is isomorphic to a submonoid of $(\mathbb{N}_0^*)^k$. In view of Theorem 7.2.2, the algebraic order of $(\mathbb{N}_0^*)^k$ induces an order on $V^*(R)$ that is translated in terms of projective modules over R by $\langle Q\rangle \leq \langle P\rangle$ if and only if there exists a pure monomorphism $f\colon Q \to P$ if and only if $Q/QJ(R)$ is a direct summand of $P/PJ(R)$. By [17], the relation $\leq$ is antisymmetric. This partial order relation defined on $V^*(R)$ restricts to the usual algebraic order over $V(R)$, but not on $W(R)$ when $V(R) \subsetneq W(R)$.

Corollary 7.2.11. *Let R be a semilocal ring, fix $\varphi\colon R \to S$ an onto ring homomorphism to a semisimple artinian ring S such that $\operatorname{Ker}\varphi = J(R)$. Then*

(i) $x \in W(R) \setminus V(R)$ if and only if x is incomparable (with respect to the algebraic order) with $n\langle R\rangle$ for any $n \geq 1$ if and only if there exist $n \geq 1$ such that $n \cdot \dim_\varphi\langle R\rangle - \dim_\varphi(x) \in \dim_\varphi W({}_RR) \setminus \dim_\varphi V(R)$.

(ii) $V(R) = W(R) \cap W({}_RR)$.

Proof. Since over a semisimple artinian ring any exact sequence splits, the statement follows by applying Theorem 7.2.9. ■

Remark 7.2.12. *Corollary 7.2.11 implies that, if* $\dim_\varphi V^*(R) \subseteq (\mathbb{N}_0^*)^k$ *is a monoid defined by inequalities and*

$$\dim_\varphi(\langle R\rangle) \in \dim_\varphi V^*(R) \cap D(\dim_\varphi V^*(R)),$$

the elements of the semigroup $\dim_\varphi W(R) \setminus \dim_\varphi V(R)$ *must be the elements of* $\mathbb{N}_0^k$ *such that some of the inequalities they satisfy are strict. So that*

$$\dim_\varphi V(R) = \dim_\varphi V^*(R) \cap D(\dim_\varphi V^*(R)) \cap \mathbb{N}_0^k = \dim_\varphi W(R) \cap \dim_\varphi W({}_RR).$$

In terms of order relations on the monoids we have the following Corollary.

Corollary 7.2.13. *Let R be a semilocal ring. Consider the following relation over $V^*(R)$, $\langle P\rangle \leq \langle Q\rangle$ if and only if $P/PJ(R)$ is isomorphic to a direct summand of $Q/QJ(R)$. Then*

(i) $\langle P\rangle \leq \langle Q\rangle$ *if and only if there exists a pure embedding* $f\colon P \to Q$.

(ii) $\leq$ *is a partial order relation that refines the algebraic order on* $V^*(R)$.

(iii) If, in addition, R is noetherian then the partial order induced by $\leq$ over $V^*(R)$ is the algebraic order.

Proof. (*i*). If $\langle P\rangle \leq \langle Q\rangle$ then there exists a splitting monomorphism $\overline{f}\colon P/PJ(R) \to Q/QJ(R)$ which by Theorem 7.2.2(i) lifts to a pure monomorphism $f\colon P \to Q$. Conversely, if $f\colon P \to Q$ is a pure monomorphism of right R-modules then the induced map $f\otimes_R R/J(R)\colon P\otimes_R R/J(R) \to Q\otimes_R R/J(R)$ is a pure monomorphism of $R/J(R)$-modules. Since $R/J(R)$ is semisimple, $f\otimes_R R/J(R)$ is a split monomorphism.

(*ii*). It is clear that $\leq$ is reflexive and transitive. As it is already observed in [17], Theorem 7.2.2 implies that $\leq$ is also antisymmetric.

If P is isomorphic to a direct summand of Q, then $P/PJ(R)$ is also isomorphic to a direct summand of $Q/QJ(R)$. Hence $\langle P\rangle \leq \langle Q\rangle$, that is, $\leq$ refines the algebraic order on $V^*(R)$.

(*iii*). It is a consequence of the realization Theorem 7.1.5. Indeed, it is not difficult to check that a submonoid M of $(\mathbb{N}_0^*)^k$ defined by a system of equations satisfies that whenever there is an equality $x+y=z$ in $(\mathbb{N}_0^*)^k$ with x and $z \in M$, then there exists $y' \in M$ such that $x+y'=z$. ∎

We shall see in Examples 7.3.6 that the monoid $V^*(R)$ does not determine $V^*({}_RR)$. Theorem 7.2.9, or [9, Theorem 7.1], combined with Theorem 7.2.2(ii) implies that for a semilocal ring $W(R)$ does determine $W({}_RR)$.

Corollary 7.2.14. *For $i=1,2$, let R_i be a semilocal ring and let $\varphi_i\colon R_i \to M_{n_1}(D_1^i)\times\cdots\times M_{n_k}(D_k^i)$ be an onto ring homomorphism such that $\operatorname{Ker}\varphi_i = J(R_i)$ and $D_1^i,\dots,D_k^i$ are division rings.*

Then $\dim_{\varphi_1} W(R_1) = \dim_{\varphi_2} W(R_2)$ *if and only if* $\dim_{\varphi_1} W({}_{R_1}R_1) = \dim_{\varphi_2} W({}_{R_2}R_2)$.

Proof. By symmetry, it is enough to prove that if $\dim_{\varphi_1} W(R_1) = \dim_{\varphi_2} W(R_2)$ then $\dim_{\varphi_1} W({}_{R_1}R_1) \subseteq \dim_{\varphi_2} W({}_{R_2}R_2)$.

Let $x \in \dim_{\varphi_1} W({}_{R_1}R_1)$. There exists $m \in \mathbb{N}$ such that $x \leq m(n_1,\dots,n_k)$. By Theorem 7.2.9, $y = m(n_1,\dots,n_k) - x \in \dim_{\varphi_1} W(R_1) = \dim_{\varphi_2} W(R_2)$. Applying again Theorem 7.2.9, we deduce that $x = m(n_1,\dots,n_k) - y \in \dim_{\varphi_2} W({}_{R_2}R_2)$. ∎

7.3 Some examples

Gerasimov and Sakhaev gave the first example of a semilocal ring such that $V(R) \subsetneqq W(R)$. The final step for the computation of $V^*(R)$ was made in [6]. We want to start this section stating the main properties of this example as it is one of the basic tools to prove our realization Theorem 7.1.6.

Theorem 7.3.1. ([11], [6]) *Let F be any field. There exists a semilocal F-algebra R with an onto ring morphism $\varphi\colon R \to F \times F$ with $\operatorname{Ker}\varphi = J(R)$ and such that all finitely generated projective modules are free but*

$$\begin{aligned}\dim_{\varphi} W(R_R) &= \{(x,y) \in \mathbb{N}_0 \mid x \geq y\} = (1,1)\mathbb{N}_0 + (1,0)\mathbb{N}_0 \\ \dim_{\varphi} V^*(R_R) &= (\dim_{\varphi} W(R_R))\,\mathbb{N}_0^* = \{(x,y) \in \mathbb{N}_0^* \mid x \geq y\}\end{aligned}$$

and

$$\begin{aligned}\dim_{\varphi} W({}_RR) &= \{(x,y) \in \mathbb{N}_0 \mid y \geq x\} = (1,1)\mathbb{N}_0 + (0,1)\mathbb{N}_0 \\ \dim_{\varphi} V^*({}_RR) &= (\dim_{\varphi} W({}_RR))\,\mathbb{N}_0^* = \{(x,y) \in \mathbb{N}_0^* \mid y \geq x\}\end{aligned}$$

In particular, any projective module over R is a direct sum of indecomposable projective modules that are finitely generated modulo $J(R)$.

It is quite an interesting question to determine the structure of $V^*(R)$ for a general semilocal ring. But right now it seems to be too challenging even for semilocal rings R such that $R/J(R) \cong D_1 \times D_2$ where D_1, D_2 are division rings. Now we provide some examples of such rings to illustrate Theorem 7.1.6 and the difficulties that appear in the general case. We first observe that, since $k = 2$ and $\dim_{\varphi}(\langle R\rangle) = (1,1)$, to have some room for interesting behavior of countably generated projective modules all finitely generated projective modules must be free.

Lemma 7.3.2. *Let R be a semilocal ring such that $R/J(R) \cong D_1 \times D_2$ for suitable division rings D_1 and D_2. Fix $\varphi\colon R \to D_1 \times D_2$ an onto ring homomorphism such that $\operatorname{Ker}\varphi = J(R)$. If R has non-free finitely generated projective right (or left) modules then there exists $n \in \mathbb{N}$ such that $\dim_{\varphi} V(R)$ is the submonoid of $\mathbb{N}_0^2$ generated by $(1,1)$, $(n,0)$ and $(0,n)$. In this case,*

$$\dim_{\varphi} V^*(R) = (1,1)\mathbb{N}_0^* + (n,0)\mathbb{N}_0^* + (0,n)\mathbb{N}_0^* = \{(x,y) \in \mathbb{N}_0^* \mid x + (n-1)y \in n\mathbb{N}_0^*\}.$$

Therefore, all projective modules are direct sum of finitely generated projective modules.

Proof. Note that $\dim_{\varphi}(\langle R\rangle) = (1,1)$. So that $(1,1) \in A = \dim_{\varphi} V(R)$.

Let P be a non-free finitely generated projective right R-module, and let $\dim_{\varphi}(\langle P\rangle) = (x,y)$. As P is not free, either $x > y$ or $x < y$. Assume $x > y$, then

$$(x,y) = (x-y,0) + y(1,1) \in A \qquad (*).$$

Since, by Corollary 7.2.5 or its monoid version Corollary 7.2.10, A is a full affine submonoid of $\mathbb{N}_0^2$ we deduce that $(x-y,0) \in A$ and also that $(0,x-y) = (x-y)(1,1) - (x-y,0) \in A$. If $x < y$ we deduce, in a symmetric way that $(y-x,0)$ and $(0,y-x)$ are elements of A.

Choose $n \in \mathbb{N}$ minimal with respect to the property $(n,0) \in A$, and note that then also $(0,n) \in A$. We claim that

$$A = (1,1)\mathbb{N}_0 + (n,0)\mathbb{N}_0 + (0,n)\mathbb{N}_0.$$

We only need to prove that if $(x,y) \in A$ then it can be written as a linear combination, with coefficients in $\mathbb{N}_0$ of $(1,1)$, $(n,0)$ and $(0,n)$. In view of the previous argument, it suffices to show that if $(x,0) \in A$ then $(x,0) \in (n,0)\mathbb{N}_0$. By the division algorithm $(x,0) = (n,0)q + (r,0)$ with $q \in \mathbb{N}_0$ and $0 \leq r < n$. As A is a full affine submonoid of $\mathbb{N}_0^2$ we deduce that $(r,0) \in A$. By the choosing of n, $r = 0$ as desired.

Let P_1 be a finitely generated right R-module such that $\dim_{\varphi}(\langle P_1\rangle) = (n,0)$, and let P_2 be a finitely generated right R-module such that $\dim_{\varphi}(\langle P_2\rangle) = (0,n)$.

Let Q be a countably generated projective right R-module that is not finitely generated. Let $\dim_{\varphi}(\langle Q\rangle) = (x,y) \in \mathbb{N}_0^*$. We want to show that

$$(x,y) \in (1,1)\mathbb{N}_0^* + (n,0)\mathbb{N}_0^* + (0,n)\mathbb{N}_0^*$$

If $x = y$ then $(x,y) = x(1,1)$ and, by Theorem 7.2.2(ii), Q is free. If $x > y$ then $y \in \mathbb{N}_0$ and $(x,y) = (x-y,0) + y(1,1)$, combining Theorem 7.2.2(ii) with Lemma 7.2.5 we deduce that

$Q = yR \oplus Q'$ with Q' such that $\dim_{\varphi}(\langle Q' \rangle) = (z, 0)$ where $z = x - y$. If $z < \infty$ then, by Theorem 7.2.2(ii), $nQ' \cong zP_1$ hence Q', and Q, are finitely generated. If $z = \infty$, by Theorem 7.2.2(ii), $Q' \cong P_1^{(\omega)}$. Hence $(x, y) = \infty \cdot (n, 0) + y(1, 1)$. The case $x < y$ is done in a symmetric way.

It is not difficult to check that the elements of $\dim_{\varphi} V^*(R)$ are the solutions in $\mathbb{N}_0^*$ of $x + (n-1)y \in n\mathbb{N}_0^*$. ∎

Now we will list all the possibilities for the monoid $V^*(R)$ viewed as a submonoid of $V^*(R/J(R))$ when R is a noetherian ring such that $R/J(R) \cong D_1 \times D_2$, for D_1 and D_2 division rings, and all finitely generated projective modules are free. In view of Theorem 7.1.5 this is equivalent to classify the submonoids of $(\mathbb{N}_0^*)^2$ containing $(1, 1)$ and that are defined by a system of equations. Though the presentation of the monoid as solutions of equations is quite attractive there is an alternative one that, even being technical, is more useful to work with.

Definition 7.3.3. *Fix $k \in \mathbb{N}$ and $(n_1, \dots, n_k) \in \mathbb{N}^k$.* A system of supports $\mathcal{S}(n_1, \dots, n_k)$ *consists of a collection $\mathcal{S}$ of subsets of $\{1, \dots, k\}$ together with a family of commutative monoids $\{A_I, I \in \mathcal{S}\}$ such that the following conditions hold*

(i) $\emptyset$ and $\{1, \dots, k\}$ are elements of $\mathcal{S}$.

(ii) For any $I \in \mathcal{S}$, A_I is a submonoid of $\mathbb{N}_0^{\{1,\dots,k\} \setminus I}$. The monoid $A_{\{1,\dots,k\}}$ is the trivial monoid and $(n_1, \dots, n_k) \in A_{\emptyset}$.

(iii) $\mathcal{S}$ is closed under unions, and if $x \in A_I$ for some $I \in \mathcal{S}$ then $I \cup \mathrm{supp}\,(x) \in \mathcal{S}$. In particular $\{1, \dots, k\} \in \mathcal{S}$.

(iv) Suppose that $I, K \in \mathcal{S}$ are such that $I \subseteq K$ and let $p\colon \mathbb{N}_0^{\{1,\dots,k\} \setminus I} \to \mathbb{N}_0^{\{1,\dots,k\} \setminus K}$ be the canonical projection. Then $p(A_I) \subseteq A_K$.

If in addition, for any $I \in \mathcal{S}$, the submonoids A_I are full affine submonoids of $\mathbb{N}_0^{\{1,\dots,k\} \setminus I}$ then $\mathcal{S}(n_1, \dots, n_k)$ is said to be a full affine system of supports.

Remark 7.3.4. *Given a system of supports $\mathcal{S}(n_1, \dots, n_k) = \{A_I, I \in \mathcal{S}\}$ we can associate to it a monoid. Consider the subset $M(\mathcal{S})$ of $(\mathbb{N}_0^*)^k$ defined by $x \in M(\mathcal{S})$ if and only if $I = \text{inf-supp}\,(x) \in \mathcal{S}$ and $p_I(x) \in A_I$, where if $x = (x_1, \dots, x_k)$ then*

$$\text{inf-supp}\,(x) = \{i \in \{1, \dots, k\} \mid x_i = \infty\},$$

and $p_I\colon (\mathbb{N}_0^*)^k \to (\mathbb{N}_0^*)^{\{1,\dots,k\} \setminus I}$ denotes the canonical projection.

By [12, Theorem 7.7], $\mathcal{S}(n_1, \dots, n_k)$ is a full affine system of supports if and only if $M(\mathcal{S})$ is a monoid defined by equations and containing $(n_1, \dots, n_k)$.

We recall that a module is superdecomposable if it has no indecomposable direct summand. By Theorem 7.1.5 and Lemma 7.2.5, in our context superdecomposable modules are relatively frequent as they correspond to the elements $x \in M \subseteq (\mathbb{N}_0^*)^k$ such that, for any $y \in M \cap \mathbb{N}_0^k$, $\mathrm{supp}\,(y) \not\subseteq \mathrm{supp}\,(x)$.

Example 7.3.5. *Let R be a semilocal noetherian ring such that there exists $\varphi\colon R \to D_1 \times D_2$, an onto ring morphism with $\mathrm{Ker}\,\varphi = J(R)$, where D_1 and D_2 are division rings. Assume that all finitely generated projective right R-modules are free. Hence $\dim_{\varphi} V(R) = (1, 1)\mathbb{N}_0$, and $\dim_{\varphi}(\langle R \rangle) = (1, 1)$. Then there are the following possibilities for $\dim_{\varphi} V^*(R)$:*

(0) All projective modules are free, so that $M_0 = \dim_{\varphi} V^*(R) = (1, 1)\mathbb{N}_0^*$. Note that M_0 is the set of solutions $(x, y) \in (\mathbb{N}_0^*)^2$ of the equation $x = y$.

(1) $M_1 = \dim_{\varphi} V^*(R) = (1, 1)\mathbb{N}_0^* + (0, \infty)\mathbb{N}_0^*$. So that, M_1 is the set of solutions $(x, y) \in (\mathbb{N}_0^*)^2$ of the equation $x + y = 2y$.

Note that for such an R there exists a countably generated superdecomposable projective right R-module P such that $\dim_{\varphi}(\langle P \rangle) = (0, \infty)$. Then any countably generated projective right R module Q is isomorphic to $R^{(n)} \oplus P^{(m)}$ for suitable $n \in \mathbb{N}_0^*$ and $m \in \{0, 1\}$.

(1') $M_1' = \dim_\varphi V^*(R) = (1,1)\mathbb{N}_0^* + (\infty,0)\mathbb{N}_0^*$. *So that, M_1' is the set of solutions $(x,y) \in (\mathbb{N}_0^*)^2$ of the equation $x+y=2x$.*

(2) $M_2 = \dim_\varphi V^*(R) = (1,1)\mathbb{N}_0^* + (\infty,0)\mathbb{N}_0^* + (0,\infty)\mathbb{N}_0^*$. *So that, M_2 is the set of solutions $(x,y) \in (\mathbb{N}_0^*)^2$ of the equation $2x+y=x+2y$.*

Note that for such an R there exist two countably generated superdecomposable projective right R-modules P_1 and P_2 such that $\dim_\varphi(\langle P_1\rangle) = (0,\infty)$ and $\dim_\varphi(\langle P_2\rangle) = (\infty,0)$. Any countably generated projective right R module Q satisfies that there exist $n \in \mathbb{N}_0$ and $m_1, m_2 \in \{0,1\}$ such that $Q = R^{(n)} \oplus P_1^{(m_1)} \oplus P_2^{(m_2)}$.

Proof. In view of Theorem 7.1.5 and Remark 7.3.4 we must describe all the possibilities for full affine systems of supports of $\{1,2\}$ such that $A_\emptyset = (1,1)\mathbb{N}_0$. Since the set of supports of a system of supports at least contains $\emptyset$ and $\{1,2\}$ there are just four possibilities.

Since the image of the projections of $A_\emptyset$ on the first and on the second component is $\mathbb{N}_0$, all the monoids A_I in the definition of system of supports are determined by $A_\emptyset$.

Case (0) is the one in which $M_0 = A_\emptyset + \infty \cdot A_\emptyset$. According to Remark 7.1.4 (3), in this case all projective modules are direct sum of finitely generated (indecomposable) modules.

In cases (1) and (1$'$) there are 3 different supports for the elements in the monoid, and in case (2) there are 4. ∎

Now we give some examples whose existence is a direct consequence of Theorem 7.1.6.

Example 7.3.6. *Let F be any field. In all the statements R denotes a semilocal F-algebra, and $\varphi\colon R \to E \times E$ denotes an onto ring homomorphism such that $\operatorname{Ker}\varphi = J(R)$ and E is a suitable field extension of F. Fix $n \in \mathbb{N}$. Then there exist R and φ such that*

(i)

$$N = \dim_\varphi V^*(R_R) = (1,1)\mathbb{N}_0^* + (n,0)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid x \ge y \text{ and } x + (n-1)y \in n\mathbb{N}_0^*\}$$

$$D(N) = \dim_\varphi V^*({}_RR) = (1,1)\mathbb{N}_0^* + (0,n)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid x \le y \text{ and } x + (n-1)y \in n\mathbb{N}_0^*\}$$

For $n=1$, we recover the situation in [11]. Note that over R all projective modules are direct sum of indecomposable projective modules.

(ii)

$$\dim_\varphi V^*(R_R) = N + (0,\infty)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid 2x+y \ge 2y+x \text{ and } x+(n-1)y \in n\mathbb{N}_0^*\}$$

$$\dim_\varphi V^*({}_RR) = D(N) + (\infty,0)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid 2x+y \le 2y+x \text{ and } x+(n-1)y \in n\mathbb{N}_0^*\}$$

In this case R has a superdecomposable projective right R-module and a superdecomposable projective left R-module.

(iii)

$$\dim_\varphi V^*(R_R) = N + (0,\infty)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid x+y \ge 2y \text{ and } x+(n-1)y \in n\mathbb{N}_0^*\}$$

$$\dim_\varphi V^*({}_RR) = D(N) = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid x+y \le 2y \text{ and } x+(n-1)y \in n\mathbb{N}_0^*\}$$

In this situation R has a superdecomposable projective right R-modules but every projective left R-module is a direct sum of indecomposable modules.

(iv)

$$\dim_\varphi V^*(R_R) = (1,1)\mathbb{N}_0^* + (\infty,0)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid 2x = x+y \text{ and } x \ge y\}$$

and

$$\dim_\varphi V^*({}_RR) = (1,1)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid 2x = x+y \text{ and } x \le y\}.$$

Therefore, all projective left R-modules are free hence they are a direct sum of finitely generated modules but this is not true for projective right R-modules. In particular, $V^*(R_R)$ and $V^*({}_RR)$ are not isomorphic.

In the first three examples $V(R) \subsetneq W(R) = (1,1)\mathbb{N}_0 + (n,0)\mathbb{N}_0 \cong W({}_RR)$. *In the fourth example, as Theorem 7.2.9 implies,* $V(R) = W(R) = W({}_RR)$.

PROOF. After Theorem 7.1.6 what is left to do is to check the generating sets of the monoids. But all the computations are straightforward.

In (iv) to prove that $V^*(R)$ is not isomorphic to $V^*({}_RR)$ just count the number of idempotent elements in both monoids. ∎

Remark 7.3.7. *Examples 7.3.6(ii) and (iii) answer a problem mentioned in [6, page 3261], and Example 7.3.6(iv) answers a problem in [10, page 310].*

Following the notation of Examples 7.3.6 and under the same hypothesis, the first place where it was shown that there could be a non finitely generated projective module P *such that* $\dim_\varphi(\langle P\rangle) = (n,0)$ *for a given* $n > 1$ *was in [23].*

The monoid $M = N + (0,\infty)\mathbb{N}_0^*$ *is described in Examples 7.3.6(ii) and (iii) in two different ways as a monoid given by a system of inequalities. Both descriptions result in different monoids* $D(M)$.

Now we give an example such that $W(R) \ncong W({}_RR)$ and $V^*(R) \ncong V^*({}_RR)$. It also shows that Corollary 7.2.5 fails also for the semigroup $W(R)\setminus V(R)$, so that in Theorem 7.2.9 we cannot just assume that P is finitely generated modulo the Jacobson radical.

Example 7.3.8. *Fix* $1 \le n \in \mathbb{N}$. *Let* F *be any field. There exist a semilocal* F*-algebra* R, *a suitable field extension* E *of* F *and an onto ring homomorphism* $\varphi\colon R \to E \times M_n(E)$ *such that* $\operatorname{Ker}\varphi = J(R)$ *and*

$$\dim_\varphi V^*(R) = (1,n)\mathbb{N}_0^* + \cdots + (1,0)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid nx \ge y\}$$

$$\dim_\varphi V^*({}_RR) = (1,n)\mathbb{N}_0^* + (0,1)\mathbb{N}_0^* = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid nx \le y\}.$$

Therefore $W(R) = (1,n)\mathbb{N}_0 + \cdots + (1,0)\mathbb{N}_0$ *and* $W({}_RR) = (1,n)\mathbb{N}_0 + (0,1)\mathbb{N}_0$ *which are non isomorphic monoids provided* $n \ge 2$.

Notice that the $(1,0),\ldots,(1,n-1)$ *are minimal elements of* $W(R)$ *and of* $W(R)\setminus V(R)$ *so that they are incomparable.*

PROOF. The existence of the semilocal ring follows from Theorem 7.1.6. We show that the two monoids have the required set of generators.

Let $M = \{(x,y) \in (\mathbb{N}_0^*)^2 \mid nx \ge y\}$. It is clear that $(1,n)\mathbb{N}_0^* + \cdots + (1,0)\mathbb{N}_0^* \subseteq M$. If $(x,y) \in M \cap \mathbb{N}_0^k$ then $y = n\cdot k + y'$ for some $k, y' \in \mathbb{N}_0$ and $0 \le y' < n$. Therefore, if $x = k$, $(x,y) = k(1,n)$. If $x > k$ then $(x,y) = k(1,n) + (x-k-1)(1,0) + (1,y')$ provided $y' > 0$, otherwise $(x,y) = k(1,n) + (x-k)(1,0)$. In the three cases we conclude that $(x,y) \in (1,n)\mathbb{N}_0 + \cdots + (1,0)\mathbb{N}_0$. For elements with nonempty infinite support the inclusion is clear.

If $(x,y) \in D(M) \cap \mathbb{N}_0^k$ then $(x,y) = x(1,n) + (y-nx)(0,1)$ which proves that $D(M) = (1,n)\mathbb{N}_0^* + (0,1)\mathbb{N}_0^*$.

The monoids $W(R)$ and $W({}_RR)$ have the same number of minimal elements if and only if $n = 1$. Therefore they cannot be isomorphic for $n \ge 2$. ∎

7.4 Monoids defined by inequalities

We think on $(\mathbb{N}_0^*)^k$ and of $\mathbb{N}_0^k$ as ordered monoids with the order relation given by the algebraic order. That is, $(x_1,\ldots,x_k) \le (y_1,\ldots,y_k)$ if and only if $x_i \le y_i$ for any $i = 1,\ldots,k$.

We recall that a monoid M is said to be unperforated if, for every $n \in \mathbb{N}$ and any $x, y \in M$, $nx \le ny$ implies $x \le y$; where $\le$ denotes the algebraic preordering on M.

Proposition 7.4.1. ([15, Proposition],[13, Proposition 2]) *Let* A *be a commutative cancellative monoid without non-trivial units. Then the following statements are equivalent;*

(i) *A is finitely generated and unperforated.*

(ii) *There exist $k \geq 1$, a monoid embedding $f\colon A \to \mathbb{N}_0^k$ and $E \in M_{\ell\times k}(\mathbb{Z})$ such that $f(A)$ is the set of solutions in $\mathbb{N}_0^k$ of the system $E \cdot T = 0$ where $T = (t_1, \dots, t_k)^t$.*

(iii) *There exist $m \geq 1$ and a monoid embedding $g\colon A \to \mathbb{N}_0^m$ such that $g(A)$ is a submonoid of $\mathbb{N}_0^m$ defined by a system of equations.*

(iv) *There exist $s \geq 1$ and a monoid embedding $h\colon A \to \mathbb{N}_0^s$ such that $h(A)$ is the set of solutions in $\mathbb{N}_0^s$ of a system of inequalities.*

Proof. For further quoting we give the proof of the equivalence of *(iii)* and *(iv)*. It is clear that the monoids in *(iii)* can be described as the set of solutions of a system of congruences and inequalities as the ones appearing in *(iv)*.

Conversely, let A be a submonoid of $\mathbb{N}_0^s$ that is the set of solutions of the system of inequalities

$$D\begin{pmatrix} t_1\\ \vdots\\ t_s\end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0\\ \vdots\\ m_n\mathbb{N}_0\end{pmatrix} \qquad \text{and} \qquad E_1\begin{pmatrix} t_1\\ \vdots\\ t_s\end{pmatrix} \geq E_2\begin{pmatrix} t_1\\ \vdots\\ t_s\end{pmatrix}$$

where $D \in M_{n\times s}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times s}(\mathbb{N}_0)$ and $m_1, \dots, m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1, \dots, n\}$. Consider the monoid morphism $g\colon A \to \mathbb{N}_0^{s+\ell}$ defined by

$$g(a_1, \dots, a_s) = \left(a_1, \dots a_s, \sum_{i=1}^{s} e^1_{1i}a_i - \sum_{i=1}^{s} e^2_{1i}a_i, \dots, \sum_{i=1}^{s} e^1_{\ell i}a_i - \sum_{i=1}^{s} e^2_{\ell i}a_i\right)$$

where $(a_1, \dots, a_s) \in A$ and, for $k = 1, 2$, e^k_{ij} denotes the i-j-entry of the matrix E_k.

Then $g(A)$ is the set of solutions in $\mathbb{N}_0^{s+l}$ of the system

$$D\begin{pmatrix} t_1\\ \vdots\\ t_s\end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0\\ \vdots\\ m_n\mathbb{N}_0\end{pmatrix} \qquad \text{and} \qquad E_1\begin{pmatrix} t_1\\ \vdots\\ t_s\end{pmatrix} = E_2\begin{pmatrix} t_1\\ \vdots\\ t_s\end{pmatrix} + \begin{pmatrix} t_{s+1}\\ \vdots\\ t_{s+\ell}\end{pmatrix}.$$

So that A is also a monoid of the type appearing in *(iii)*. ∎

The embeddings of *(iii)* are the full affine embeddings. We recall that if $(n_1, \dots, n_m) \in \mathbb{N}^m \cap g(A)$ then $g(A)$ can be realized as $\dim_\varphi(V(R))$ for some semilocal ring R such that $R/J(R) \cong M_{n_1}(D_1) \times \cdots \times M_{n_m}(D_m)$ for suitable division rings $D_1, \dots, D_m$ [7].

We stress that not all finitely generated submonoids of $\mathbb{N}_0^k$ are unperforated. Consider, for example, $N = 2\mathbb{N}_0 + 3\mathbb{N}_0 \subseteq \mathbb{N}_0$. In N, $2 \cdot 2 \leq 2 \cdot 3$ but 2 and 3 are incomparable in N. Note however that N cannot be realized as $\dim_\varphi(W(R))$ for a semilocal ring R, because for $k = 1$ such monoids are principal (cf. remarks before Definition 7.1.3).

We do not know whether, for any semilocal ring R, $\dim_\varphi(W(R))$ is an unferforated submonoid of $\mathbb{N}_0^k$. For example, $(1, 1)\mathbb{N}_0 + (2, 0)\mathbb{N}_0 + (3, 0)\mathbb{N}_0$ is a submonoid of $\mathbb{N}_0^2$ that is not unperforated and we do not know whether it can be realized as $\dim_\varphi(W(R))$ for a suitable semilocal ring R.

In the next lemma we study monoids defined by a system of equations and monoids defined by a system of inequalities.

Lemma 7.4.2. *Let M be a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of inequalities*

$$D \cdot T \in \begin{pmatrix} m_1\mathbb{N}_0^*\\ \vdots\\ m_n\mathbb{N}_0^*\end{pmatrix} \quad (*) \qquad \textit{and} \qquad E_1 \cdot T \geq E_2 \cdot T \quad (**)$$

where $T=(t_1,\dots,t_k)^t$, $D\in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2\in M_{\ell\times k}(\mathbb{N}_0)$ and $m_1,\dots,m_n\in\mathbb{N}$, $m_i\geq 2$ for any $i\in\{1,\dots,n\}$. Let A be the submonoid of M whose elements are the solutions in $\mathbb{N}_0^k$ of

$$D\cdot T\in\begin{pmatrix} m_1\mathbb{N}_0\\ \vdots\\ m_n\mathbb{N}_0\end{pmatrix}\qquad \textit{and}\qquad E_1\cdot T=E_2\cdot T$$

Then,

(i) M and $D(M)$ are finitely generated monoids.

(ii) $A=M\cap D(M)\cap\mathbb{N}_0^k$.

(iii) For any $m\in M$ and $a\in A$, if there exists $m'\in(\mathbb{N}_0^*)^k$ such that $m=a+m'$ then $m'\in M$.

Proof. (i) Consider the monoid N defined the system of equations

$$D'\cdot T'\in\begin{pmatrix} m_1\mathbb{N}_0^*\\ \vdots\\ m_n\mathbb{N}_0^*\end{pmatrix}\quad(*)\qquad\text{and}\qquad E_1\cdot T=E_2\cdot T+\begin{pmatrix} t_{k+1}\\ \vdots\\ t_{k+\ell}\end{pmatrix}\quad(**)$$

where $T'=(t_1,\dots,t_k,t_{k+1},\cdots,t_{k+\ell})^t$ and $D'=(D|0)\in M_{n\times(k+\ell)}(\mathbb{N}_0)$. By [12, Example 7.6], N is a finitely generated monoid.

Let $p\colon(\mathbb{N}_0^*)^{k+\ell}\to(\mathbb{N}_0^*)^k$ denote the projection onto the first k components. It is easy to see that $p(N)=M$, so that M is finitely generated.

Statements (ii) and (iii) are clear. ∎

In contrast to the results proved in [15], that we have recalled in Proposition 7.4.1, the monoid N appearing in the proof of Lemma 7.4.2 need not be isomorphic to M.

In general, as the following basic example shows, a monoid defined by inequalities may not be isomorphic to a monoid defined by a system of equations. Therefore the equivalence of statements (ii), (iii) and (iv) in Proposition 7.4.1 does not extend to submonoids on $(\mathbb{N}_0^*)^k$.

Example 7.4.3. *Let M be the submonoid of $(\mathbb{N}_0^*)^2$ that is the set of solutions of $x\geq y$. Then M is not isomorphic to a monoid defined by a system of equations.*

Proof. In order to be able to manipulate this monoid we need to think on the language of system of supports, see Definition 7.3.3 and Remark 7.3.4.

First note that $M=(1,1)\mathbb{N}_0+(1,0)\mathbb{N}_0+(\infty,0)\mathbb{N}_0+(\infty,\infty)\mathbb{N}_0$. The elements $c=(\infty,0)$ and $d=(\infty,\infty)$ are nonzero elements satisfying that $2c=c$, $2d=d$ and $d+c=d$. Therefore, if $h\colon M\to N$ is a monoid morphism and N is a submonoid of $(\mathbb{N}_0^*)^k$ defined by a system of equations, $h(c)$ and $h(d)$ must be elements such that its support coincides with its infinite support and, moreover, $\operatorname{supp}h(c)\subseteq\operatorname{supp}h(d)$. If h is bijective, then $h(c)$ and $h(d)$ are the only non-zero elements of N such that its support coincides with its infinite support. So that if we think on the presentation of N as a system $\mathcal{S}$ of supports, we deduce that there are only three different sets in $\mathcal{S}$, that is $\emptyset$, $\operatorname{supp}h(c)$ and $\operatorname{supp}h(d)$. Moreover, $\operatorname{supp}h(c)\subsetneq\operatorname{supp}h(d)$

On the other hand, since $(1,0)+c=c$, we deduce $\infty\cdot h(1,0)=h(c)$. Similarly, $\infty\cdot h(1,1)=h(d)$. Moreover, $h(1,0)+h(1,0)\neq h(1,0)$ and $h(1,1)+h(1,1)\neq h(1,1)$ and then it follows that $\infty\cdot h(1,0)\neq h(1,0)$ and $\infty\cdot h(1,1)\neq h(1,1)$. Since there are only three elements in $\mathcal{S}$, $h(1,1)$ and $h(1,0)$ have empty infinite support. As $\infty\cdot h(1,1)+\infty\cdot h(1,0)=\infty\cdot h(1,1)$, there exists $n\in\mathbb{N}$ and $0\neq z\in\mathbb{N}_0^k$ such that $nh(1,1)=h(1,0)+z$; taking n to be minimal with this property we may assume that the support of z is strictly contained in the support of $h(1,1)$. Since N is defined by a system of equations, and $nh(1,1)$ and $h(1,0)\in N$ we deduce that $z\in N$. Then $\infty\cdot z\in N$, but the infinite support of $\infty\cdot z$ is not a set of $\mathcal{S}$, a contradiction. Therefore, M cannot be isomorphic to a monoid defined by a system of equations. ∎

Finally, we draw some consequences for monoids of projective modules of the results obtained in this section.

Corollary 7.4.4. *Let R be a semilocal ring, let $\varphi\colon R \to S$ be an onto ring homomorphism such that* $\operatorname{Ker}\varphi = J(R)$ *and $S \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for suitable division rings $D_1, \dots, D_k$. Assume that $\dim_\varphi V^*(R)$ can be defined by a system of inequalities such that $\dim_\varphi V^*(_RR) = D(\dim_\varphi V^*(R))$.*

Then the monoids $W(R)$, $W(_RR)$, $V^*(R)$ and $V^*(_RR)$ are finitely generated. In addition, $W(R)$ and $W(_RR)$ are cancellative and unperforated.

If P is a projective right module such that $\langle P\rangle \in W(R)$ then $V(\operatorname{End}_R(P))$ is a cancellative, finitely generated and unperforated monoid.

Proof. By Corollary 7.2.11 and Remark 7.2.12, the elements of $W(R)$ are the solutions in $\mathbb{N}_0^k$ of the system of inequalities defining M. By Proposition 7.4.1, $W(R)$ is finitely generated and unperforated. Being isomorphic to a submonoid of $\mathbb{N}_0^k$, $W(R)$ is also cancellative. The statement on $W(_RR)$ follows by symmetry.

By Lemma 7.4.2, it follows that $V^*(R)$ and $V^*(_RR)$ are finitely generated.

Let P be a projective right R-module such that $P/PJ(R)$ is finitely generated. Let $S = \operatorname{End}_R(P)$. The functors $\operatorname{Hom}_R(P, -)$ and $- \otimes_S P$ induce an equivalence between the category of modules that are direct summands of P^n, for some n and the category of finitely generated projective right modules over $\operatorname{End}_R(P)$ (cf. [5, pp. 984–985]). Therefore

$$V(\operatorname{End}_R(P)) \cong \{x \in W(R) \mid \text{ there exists } n \text{ such that } x \le n\langle P\rangle\} = M$$

Since $W(R)$ is finitely generated, cancellative and unperforated then so is M. ∎

Remark 7.4.5. *Observe that if $R/J(R)$ is right noetherian then $\langle P\rangle \in W(R)$ if and only if $P/PJ(R)$ is finitely generated. In this case $W(R)$ is finitely generated whenever $V^*(R)$ is finitely generated.*

For a general semilocal ring we do not know whether the endomorphism ring of a projective right R-module P such that it is finitely generated modulo the Jacobson radical must be again a semilocal ring. We do not even know whether this happens for the rings appearing in Theorem 7.1.6. On the positive side, Corollary 7.4.4 shows that, at least, the monoid $V(\operatorname{End}_R(P))$ is of the *correct type*, cf. Proposition 7.4.1.

7.5 Realizing monoids defined by inequalities

We use the following result to construct semilocal rings with prescribed $V^*(R)$.

Theorem 7.5.1. [12] *Let R_1 and R_2 be semilocal rings, and let $S = M_{m_1}(D'_1) \times \cdots \times M_{m_\ell}(D'_\ell)$ for suitable division rings $D'_1, \dots, D'_\ell$. For $i = 1, 2$, let $j_i\colon R_i \to S$ be ring homomorphisms. Let R be the ring that fits into the pullback diagram*

$$\begin{array}{ccc} R_1 & \xrightarrow{j_1} & S \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle j_2} \\ R & \xrightarrow[i_2]{} & R_2 \end{array}$$

Assume that j_1 is an onto ring homomorphism with kernel $J(R_1)$, and that $J(R_2) \subseteq \operatorname{Ker} j_2$. If $R_2/J(R_2) \cong M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ for $D_1, \dots, D_k$ division rings, and $\pi\colon R_2 \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ is an onto morphism with kernel $J(R_2)$ then

(i) i_2 induces an onto ring homomorphism $\overline{i_2}\colon R \to M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$ with kernel $J(R)$. In particular, R is a semilocal ring and $R/J(R) \cong R_2/J(R_2)$.

(ii) Let $\alpha\colon \dim_{\pi} V^*(R_2) \to (\mathbb{N}_0^*)^{\ell}$ *be the monoid homomorphism induced by* j_2*. Then*

$$\dim_{\overline{i_2}} V^*(R) = \{x \in \dim_{\pi} V^*(R_2) \mid \alpha(x) \in \dim_{j_1} V^*(R_1)\},$$

and

$$\dim_{\overline{i_2}} V^*({}_RR) = \{x \in \dim_{\pi} V^*({}_{R_2}R_2) \mid \alpha(x) \in \dim_{j_1} V^*({}_{R_1}R_1)\}.$$

Example 7.5.2. *Let* $k \in \mathbb{N}$*, and let* $a_1, \dots, a_k, b_1, \dots, b_k \in \mathbb{N}_0$*. Let* $(n_1, \dots, n_k) \in \mathbb{N}^k$ *be such that* $a_1n_1 + \cdots + a_kn_k = b_1n_1 + \cdots + b_kn_k \in \mathbb{N}$*. For any field extension* $F \subseteq F_1$*, there exist a semilocal* F*-algebra* R *and an onto morphism of* F*-algebras* $\varphi\colon R \to M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1)$ *with kernel* $J(R)$ *such that* $\dim_{\varphi} V^*(R_R)$ *is the set of solutions in* $(\mathbb{N}_0^*)^k$ *of the inequality* $a_1t_1 + \cdots + a_kt_k \geq b_1t_1 + \cdots + b_kt_k$ *and* $\dim_{\varphi} V^*({}_RR)$ *is the set of solutions in* $(\mathbb{N}_0^*)^k$ *of the inequality* $a_1t_1 + \cdots + a_kt_k \leq b_1t_1 + \cdots + b_kt_k$*.*

Note that $\dim_{\varphi}(\langle R \rangle) = (n_1, \dots, n_k)$*.*

Proof. Set $m = a_1n_1 + \cdots + a_kn_k = b_1n_1 + \cdots + b_kn_k$.

Let T be a semilocal F-algebra with an onto algebra morphism $j_1\colon T \to F_1 \times F_1$ with $\mathrm{Ker}(j_1) = J(T)$, and such that $\dim_{j_1} V^*(T_T) = \{(x, y) \in (\mathbb{N}_0^*)^2 \mid x \geq y\}$ and $\dim_{j_1} V^*({}_TT) = \{(x, y) \in (\mathbb{N}_0^*)^2 \mid y \geq x\}$. Such T exists by Theorem 7.3.1. Let $M_m(j_1)\colon M_m(T) \to M_m(F_1) \times M_m(F_1)$ be the induced morphism.

Set $R_2 = M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1)$. Consider the morphism of F-algebras $j_2\colon R_2 \longrightarrow M_m(F_1) \times M_m(F_1)$ defined by

$$j_2(r_1, \dots, r_k) = \left(\begin{pmatrix} \begin{matrix} r_1 & \cdots & 0 \\ \vdots & \ddots_{a_1)} & \vdots \\ 0 & \cdots & r_1 \end{matrix} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \begin{matrix} r_k & \cdots & 0 \\ \vdots & \ddots_{a_k)} & \vdots \\ 0 & \cdots & r_k \end{matrix} \end{pmatrix}, \begin{pmatrix} \begin{matrix} r_1 & \cdots & 0 \\ \vdots & \ddots_{b_1)} & \vdots \\ 0 & \cdots & r_1 \end{matrix} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \begin{matrix} r_k & \cdots & 0 \\ \vdots & \ddots_{b_k)} & \vdots \\ 0 & \cdots & r_k \end{matrix} \end{pmatrix} \right)$$

The morphism j_2 induces the morphism of monoids $f\colon (\mathbb{N}_0^*)^k \to \mathbb{N}_0^* \times \mathbb{N}_0^*$ defined by $f(x_1, \dots, x_k) = (a_1x_1 + \cdots + a_kx_k, b_1x_1 + \cdots + b_kx_k)$. Hence, $f(n_1, \dots, n_k) = (m, m)$.

Let R be the ring defined by the pullback diagram

$$\begin{array}{ccc} M_m(T) & \xrightarrow{M_m(j_1)} & M_m(F_1) \times M_m(F_1) \\ {\scriptstyle i_1}\uparrow & & \uparrow{\scriptstyle j_2} \\ R & \xrightarrow[\varphi]{} & M_{n_1}(F_1) \times \cdots \times M_{n_k}(F_1) \end{array}$$

Applying Theorem 7.5.1 (i), we conclude that R is a semilocal F-algebra and that φ is an onto morphism of F-algebras with kernel $J(R)$. By Theorem 7.5.1(ii), $(x_1, \dots, x_k) \in \dim_{\varphi} V^*(R_R)$ if and only if $f(x_1, \dots, x_k) \in \dim_{M_m(j_1)} V^*(M_m(T))$ if and only if $a_1x_1 + \cdots + a_kx_k \geq b_1x_1 + \cdots + b_kx_k$. Similarly, $(x_1, \dots, x_k) \in \dim_{\varphi} V^*({}_RR)$ if and only if $a_1x_1 + \cdots + a_kx_k \leq b_1x_1 + \cdots + b_kx_k$. ∎

Now we are ready to prove Theorem 7.1.6.

Proof of Theorem 7.1.6. Let M be the monoid defined by the system of inequalities,

$$D \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \in \begin{pmatrix} m_1\mathbb{N}_0^* \\ \vdots \\ m_n\mathbb{N}_0^* \end{pmatrix} \quad (*) \qquad \text{and} \qquad E_1 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \leq E_2 \begin{pmatrix} t_1 \\ \vdots \\ t_k \end{pmatrix} \quad (**)$$

where $D \in M_{n\times k}(\mathbb{N}_0)$, E_1, $E_2 \in M_{\ell\times k}(\mathbb{N}_0)$, $n, \ell \geq 0$ and $m_1, \dots, m_n \in \mathbb{N}$, $m_i \geq 2$ for any $i \in \{1, \dots, n\}$.

By [12, Theorem 5.3] we know the following.

Step 1. *There exist a field extension E of F, a (noetherian) semilocal F-algebra R_1 and an onto morphism of F-algebras $\varphi_1\colon R_1 \to M_{n_1}(E)\times\cdots\times M_{n_k}(E)$ such that $\dim_{\varphi_1} V^*(R_1)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system of congruences $(*)$.*

Now we need to prove,

Step 2. *There exist a semilocal F-algebra R_2 and an onto morphism of F-algebras $\varphi_2\colon R_2 \to M_{n_1}(E)\times\cdots\times M_{n_k}(E)$ such that $\dim_{\varphi_2} V^*(R_2)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system of inequalities $(**)$ and $\dim_{\varphi_2} V^*({}_{R_2}R_2)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the system of inequalities $D(**)$.*

If $\ell = 0$, that is, if $(**)$ is empty we set $R_2 = M_{n_1}(E)\times\cdots\times M_{n_k}(E)$ and $\varphi_2 = \mathrm{Id}$. Assume $\ell > 0$. Therefore, we can assume that none of the rows in E_1 and, hence, in E_2 are zero.

By Example 7.5.2, for $i = 1,\dots,\ell$, there exist a semilocal F-algebra T_i and an onto morphism of F-algebras $\pi_i\colon T_i \to M_{n_1}(E)\times\cdots\times M_{n_k}(E)$ with kernel $J(T_i)$ and such that $\dim_{\pi_i} V^*(T_i)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the i-th inequality defined by the matrices E_1 and E_2, and $\dim_{\pi_i} V^*({}_{T_i}T_i)$ is the set of solutions in $(\mathbb{N}_0^*)^k$ of the reversed inequality.

For $i = 1,\dots,\ell$, we construct inductively semilocal F-algebras R_i' and maps $\varphi_i'\colon R_i' \to M_{n_1}(E)\times\cdots\times M_{n_k}(E)$ such that $\operatorname{Ker}\varphi_i' = J(R_i')$. Set $R_1' = T_1$ and $\varphi_1' = \pi_1$. Assume $1 < i \le \ell$ and that R_{i-1}' and φ_{i-1}' are already constructed, let R_i' to be the pullback of φ_{i-1}' and π_i. By Theorem 7.5.1, R_i' is a semilocal F-algebra with an onto morphism of F-algebras $\varphi_i'\colon R_i' \to M_{n_1}(E)\times\cdots\times M_{n_k}(E)$ with kernel $J(R_i')$. This finishes the construction of R_i' and φ_i'.

Set $R_2 = R_\ell'$ and $\varphi_2 = \varphi_\ell'$. By repeatedly applying Theorem 7.5.1 to R_i', we deduce that $\dim_{\varphi_2} V^*(R_2)$ is the set of solutions of the inequalities $(**)$ and that $\dim_{\varphi_2} V^*({}_{R_2}R_2)$ is the set of solutions of the inequalities $D(**)$. This concludes the proof of Step 2.

Finally, set R to be the pullback of $\varphi_i\colon R_i \to M_{n_1}(E)\times\cdots\times M_{n_k}(E)$, $i = 1,2$. By Theorem 7.5.1, R is a semilocal F-algebra with an onto morphism of F-algebras $\varphi\colon R \to M_{n_1}(E)\times\cdots\times M_{n_k}(E)$ with kernel $J(R)$. The elements in $\dim_\varphi V^*(R_R)$ are the solutions of $(*)$ and $(**)$, and the ones in $\dim_\varphi V^*({}_RR)$ are the elements of $D(M)$.

The description of the images via $\dim_\varphi$ of $V(R)$, $W(R)$ and $W({}_RR)$ follows from Remark 7.2.12. ∎

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