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Mathematical Institute

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Branislav Jurčo

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GERBES IN GEOMETRY AND PHYSICS

BRANISLAV JURČO
MATHEMATICAL INSTITUTE, FACULTY OF MATHEMATICS AND PHYSICS
CHARLES UNIVERSITY IN PRAGUE

Contents

1. Introduction

The thesis consists of a series of seven papers related to higher gauge theories:

[P1] P. Aschieri, B. Jurčo, Gerbes, M5-Brane Anomalies and E8 Gauge Theory, JHEP **0410** 068 (2004)

[P2] P. Aschieri, L. Cantini, B. Jurčo, Nonabelian bundle gerbes, their differential geometry and gauge theory, Commun. Math. Phys. **254**, 367-400 (2005)

Date: February 2015.

[P3] I. Baković, B. Jurčo, The classifying topos of a topological bicategory, Homol. Homotopy Appl. **12(1)**, 279-300 (2010)

[P4] P. Aschieri, I. Baković, B. Jurčo, P. Schupp, Noncommutative gerbes and deformation quantization, J.Geom.Phys. **60**, 1754-1761 (2010)

[P5] B. Jurčo, Nonabelian bundle 2-gerbes, Int. J. Geom. Meth. Mod. Phys. **8(1)**, 49-78 (2011)

[P6] B. Jurčo, Crossed module bundle gerbes; classification, string group and differential geometry, Int. J. Geom. Meth. Mod. Phys. **8(5)**, 1079-1095, (2011)

[P7] B. Jurčo, From simplicial Lie algebras and hypercrossed complexes to differential graded Lie algebras via 1-jets, J. Geom. Phys. **62**, 2389-2400 (2012)

Paper [P2] is on nonabelian bundle gerbes and their geometry in the smooth setting, paper [P1] discusses their twistings by abelian 2-gerbes and relation to anomaly cancellation on M5-branes. Paper [P6] concerns with nonabelian bundle gerbes and their classification based on their simplicial description and the relation to string structures. Paper [P4] introduces noncommutative gerbes as deformation quantization of abelian gerbes and explains in which sense these are true noanbelian gerbes. Paper [P5] is devoted to nonabelian bundle 2-gebres, simultaneously generalizing nonabelian bundle gerbes and their twistings. Quantization of twisted Poisson stuctures is a nice example. Nonabelian gerbes can further be generalized to bicategory bundles, paper [P3] describes the corresponding classifying topos. The simplicial point of view is pursued in paper [P7] leading to the first step in undestanding connective structures on more general nonabelian n-gerbes.

The thesis is organized as follows. In the first part, Section 2, we provide some basic notions and facts relevant to the subject of the thesis. This is done in a rather informal manner; though restraining from formal definitions, propositions and theorem, we still tried to give a concise exposition at a reasonable level of rigor. More details can be found in the above mentioned papers or in cited references.

Results are comprised in the second part, Section 3. In order to keep this text within a reasonable length, we include only some results. In order to make clear what the contributions of the thesis are, we used in this part the formal style of definitions, propositions and theorems. Again, more details can be found in the above listed papers [P1]-[P7], which are appended.

The purpose of Sections 2 and 3 is to give a flavor of the used methods and of the results of included papers. References in these sections are far from complete. A more complete list of references can be found in [P1]-[P7]. Also, since these papers were written and published, many highly relevant papers on the subject of the theses appeared. We apologize to authors of these for not trying to include them.

Papers [P1]-[P7] are included as an appendix.

Let us finish this introduction with a following remark. A gerbe, see e.g. [37, 14, 16, 15, 21, 62], in its full generality is defined as *"a stack in groupoids, which is locally non-empty and locally connected".* In this spirit, for 2-gerbes [15, 16] (more generally, for n-gerbes), we would have to use language of higher stacks and groupoids. Although, all these are intriguing mathematical structures, in this thesis we try to avoid the language of higher categories. Instead we make an attempt, at least for the part of it, to stay as close to the language of classical differential geometry. We believe that such formulation can be the useful one, when considering possible applications in physics.

ACKNOWLEDGMENTS

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2. PRELIMINARIES

Higher gauge theory is a generalization of gauge theory - such as the theory of principal and vector bundles, connections and the parallel transport - from point particles to the higher-dimensional extended objects. In this context, an abelian gerbe can be viewed as the next level after complex line bundles in realizing integral cohomology classes on a manifold. Complex line bundles are classified (in topology) by their Chern classes, which are integral 2-cohomology classes. An abelian bundle gerbe gives geometric meaning to integral 3-cohomology class, [64, 21]. As in the case of line bundles, abelian bundle gerbes can be described in terms of local "transition functions". However, now the "transition functions" are not functions but local complex line bundles satisfying cocycle conditions for tensor products over triple overlaps of open sets. A more global point of view is to think of an abelian gerbe as a principal $PU(\mathcal{H})$ bundle. Here $PU(\mathcal{H})$ is the projective unitary group in a complex Hilbert space $\mathcal{H}$. In contrast to line bundles, gerbes are generically infinite-dimensional objects; only in the case of a torsion 3-cohomology class one can choose $\mathcal{H}$ to be finite-dimensional. Both of the above realizations of abelian gerbes arise in a natural way in quantum field theory. For instance, in [23, 24, 61] they are related to chiral anomalies and in string theory; and, for instance, in [13, 12, 59] they appear in classification of D-branes in a nontrivial background B-field. For a discussion of relevance of abelian gerbes in in WZW model, TQFT and strings see, e.g., [35, 69, 22, 43], respectively. Abelian (bundle) gerbes are not only a realization of the 3rd cohomology class (the Dixmier-Douady class). One can add geometric structures, a gerbe connection, and (local family of) 2-forms (curving). A gerbe with connection and curving (modulo equivalencies) is a Deligne class on the base manifold (for instance, on a D-brane world-volume); its top form part, the 3-form curvature, gives the Dixmier-Douady class. Here we will restrict ourselves only to a very brief description of only few results concerning gerbes and higher gauge theories, which are directly related to the subject of the thesis. Also, we shall explicitly mention only the literature, which has a direct relationship to the results comprised in papers [P1-P7] and presented in the thesis. For introduction on the higher gauge theories, gerbes, abelian bundle gerbes we recommend, e.g., [7, 21, 62, 65, 45], respectively. Finally, we should also mention that nonabelian gerbes arose in the context of nonabelian cohomology, which goes back to Grothendieck [30, 37, 14] (see [16] or [62] for a concise introduction). The (synthetic) differential geometry of nonabelian gerbes – from the algebraic geometry point of view – is discussed thoroughly in the work of Breen and Messing [17].

Below in this section, we collect some prerequisites necessary for introducing various generalizations of abelian gebres in Sec. 3.

2.1. Abelian gerbes.

Line bundles can be described, in a well known manner, using transition functions. Consider a cover $\{O_i\}$ of the manifold M, then a line bundle is given by a set of $U(1)$ valued smooth transition functions $\{\lambda_{ij}\}$ that satisfy $\lambda_{ij} = \lambda_{ji}^{-1}$ and that on triple overlaps $O_{ijk} = O_i \cap O_j \cap O_k$ satisfy the cocycle condition

$$\lambda_{ij}\lambda_{jk} = \lambda_{ik} \ . \tag{2.1.1}$$

In the same spirit, a connection on a line bundle is a set of one-forms $\{\alpha_i\}$ on O_i such that on double overlaps $O_{ij} = O_i \cap O_j$,

$$\alpha_i = \alpha_j + \lambda_{ij} d\lambda_{ij}^{-1} \ . \tag{2.1.2}$$

Actually, we are interested only in isomorphism classes of line bundles with connection. Indeed, all physical observables are obtained from Wilson loops, and these cannot distinguish between a bundle with connection (λ_{ij}, α_i) and an equivalent one $(\lambda'_{ij}, \alpha'_i)$, that by definition satisfies

$$\lambda'_{ij} = \tilde{\lambda}_i \lambda_{ij} \tilde{\lambda}_j^{-1} \, , \qquad \alpha'_i = \alpha_i + \tilde{\lambda}_i d\tilde{\lambda}_i^{-1} \ , \tag{2.1.3}$$

where $\tilde{\lambda}_i$ are $U(1)$ valued smooth functions on O_i. We are thus led to consider the class $[\lambda_{ij}, \alpha_i]$ of all couples (λ_{ij}, α_i) that satisfy (2.1.2), and where $(\lambda_{ij}, \alpha_i) \sim (\lambda'_{ij}, \alpha'_i)$ iff (2.1.3) holds. The space of all these classes (called **Deligne classes**) is the **first Deligne cohomology group** $H^1(M, \mathcal{D}^1)$.

Similarly, we can consider the **second Deligne class** $[\lambda_{ijk}, \alpha_{ij}, \beta_i] \in H^2(M, \mathcal{D}^2)$, where now $\lambda_{ijk} : O_{ijk} \to U(1)$ is, in the multiplicative sense, totally antisymmetric in its indices, $\lambda_{ijk} = \lambda_{jik}^{-1} = \lambda_{kij}$ etc., and satisfies the cocycle condition on quadruple overlaps $O_{ijkl} = O_i \cap O_j \cap O_k \cap O_l$

$$\lambda_{ijk} \lambda_{jkl}^{-1} \lambda_{ikl} \lambda_{ijl}^{-1} = 1 \ . \tag{2.1.4}$$

The connection one-form $\{\alpha_{ij}\}$ satisfies on O_{ijk}

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} + \lambda_{ijk} d\lambda_{ijk}^{-1} = 0 \tag{2.1.5}$$

and the curving two-form $\{\beta_i\}$ satisfies on O_{ij}

$$\beta_i - \beta_j + d\alpha_{ij} = 0 \ . \tag{2.1.6}$$

The triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$ gives the zero Deligne class if

$$(\lambda_{ijk}, \alpha_{ij}, \beta_i) = D(\tilde{\lambda}_{ij}, \tilde{\alpha}_i) \, , \tag{2.1.7}$$

where D is the Deligne coboundary operator, and $\tilde{\lambda}_{ij} : O_{ij} \to U(1)$ are smooth functions and $\tilde{\alpha}_i$ are smooth one-forms on O_i. Explicitly (2.1.7) reads[1]

$$\lambda_{ijk} = \tilde{\lambda}_{ik} \tilde{\lambda}_{jk}^{-1} \tilde{\lambda}_{ij}^{-1} \, , \tag{2.1.8}$$

$$\alpha_{ij} = -\tilde{\alpha}_i + \tilde{\alpha}_j + \tilde{\lambda}_{ij} d\tilde{\lambda}_{ij}^{-1} \, , \tag{2.1.9}$$

$$\beta_i = d\tilde{\alpha}_i \, . \tag{2.1.10}$$

There is also a geometric structure associated with the triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$, it is that of **(abelian) gerbe** [21] or **bundle gerbe** [64] (with a connection and curving).

Equivalence classes of gerbes with connection and curving are in 1-1 correspondence with Deligne classes.

With abuse of language we occasionally say that $[\mathcal{G}] = [\lambda_{ijk}, \alpha_{ij}, \beta_i]$ is the equivalence class of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$. As before, gauge invariant (physical) quantities can be obtained from the holonomy (Wilson surface), and this depends only on the equivalence class of the gerbe.

Gerbes are also called 1-gerbes in order to distinguish them from 2-gerbes and higher gerbes. In the same way as abelian 1-gerbes were described above, we can define abelian $n-1$-gerbes

[1]The Deligne coboundary operator is $D = \pm\delta + d$, the sign factor in front of the Čech coboundary operator depends on the degree of the form D acts on; it insures $D^2 = 0$.

with curvings using Deligne cohomology classes in $H^n(M, \mathcal{D}^n)$ [21]. Correspondingly, we have characteristic classes in $H^{n+1}(M, \mathbb{Z})$. The case $n = 1$ gives equivalence classes of line bundles with connections, and in this case the characteristic class is the Chern class of the line bundle.

An important example of a 1-gerbe is a torsion gerbe, i.e. a gerbe with a characteristic class being a torsion class (let say an n-torsion) in $H^3(M, \mathbb{Z})$. Such a torsion gerbe can be obtained form a lifting gerbe, i.e. from a gerbe that describes the obstruction of lifting a $PU(n)$ bundle to a $U(n)$ one. We now describe this lifting gerbe and the associated twisted $U(n)$-bundle. Let $P \to M$ be a $PU(n)$ bundle and consider the exact sequence $U(1) \to U(n) \xrightarrow{\pi} PU(n)$. Consider an open cover $\{U_\alpha\}$ of $PU(n)$ with sections $s^\alpha : U_\alpha \subset PU(n) \to U(n)$. We can always choose a good cover $\{O_i\}$ of M such that each transition function g_{ij} of $P \to M$ has image contained in some U_α. Let $G_{ij} = s^\alpha(g_{ij})$, these are $U(n)$-valued functions and satisfy:

$$G_{ik} G_{jk}^{-1} G_{ij}^{-1} = \lambda_{ijk} \, , \tag{2.1.11}$$

where λ_{ijk} is $U(1)$-valued as is easily seen by applying the projection π and using the cocycle relation for the g_{ij} transition functions. We say that G_{ij} are the transition functions for a twisted $U(n)$ bundle and that the lifting gerbe is defined by the twist λ_{ijk}. It is indeed easy to check that the λ_{ijk} satisfy the cocycle condition (2.1.4) on quadruple overlaps O_{ijkl}. A connection for a twisted bundle is a set of Lie$(U(n))$-valued one-forms A_i such that $\alpha_{ij} \equiv -A_i + G_{ij} A_j G_{ij}^{-1} + G_{ij} d G_{ij}^{-1}$ is a connection for the corresponding gerbe (in particular $\pi_* A$ is a connection on the initial $PU(n)$ bundle P).[2] We restate this construction this way: consider the couple (G_{ij}, A_i), and define

$$\mathbf{D}(G_{ij}, A_i) := (G_{ik} G_{jk}^{-1} G_{ij}^{-1}, -A_i + G_{ij} A_j G_{ij}^{-1} + G_{ij} d G_{ij}^{-1}, \tfrac{1}{n} Tr d A_i) \, . \tag{2.1.12}$$

If this triple has abelian entries then it defines a gerbe, and (G_{ij}, A_i) is called a **twisted bundle**. We also say that the twisted bundle (G_{ij}, A_i) is twisted by the gerbe $\mathbf{D}(G_{ij}, A_i)$. Notice that the nonabelian $\mathbf{D}$ operation becomes the abelian Deligne coboundary operator D if $n = 1$ in $U(n)$ [cf.(2.1.7)].

Following the above discussion of 1-gerbes, for the purposes of this thesis, we understand under an **abelian 2-gerbe with curvings** on M a quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$. Here $\lambda_{ijkl} : O_{ijkl} \equiv O_i \cap O_j \cap O_k \cap O_l \to U(1)$ is a Čech 3-cocycle

$$\lambda_{ijkl} \lambda_{ijlm} \lambda_{jklm} = \lambda_{iklm} \lambda_{ijkm} \qquad \text{on} \;\; O_{ijklm} \, , \tag{2.1.13}$$

and λ_{ijkl} is totally antisymmetric, $\lambda_{ijkl} = \lambda_{jikl}^{-1}$ etc. Next, $\alpha_{ijk} \in \Omega^1(O_{ijk})$, $\beta_{ij} \in \Omega^2(O_{ij})$ and $\gamma_i \in \Omega^3(O_i)$ are a collection of local one, two, and three-forms totally antisymmetric in their respective indices and subject to the following relations:

$$\alpha_{ijk} + \alpha_{ikl} - \alpha_{ijl} - \alpha_{jkl} = \lambda_{ijkl} d \lambda_{ijkl}^{-1} \qquad \text{on} \;\; O_{ijk} \, , \tag{2.1.14}$$

$$\beta_{ij} + \beta_{jk} - \beta_{ik} = d\alpha_{ijk} \qquad \text{on} \;\; O_{ijk} \, , \tag{2.1.15}$$

$$\gamma_i - \gamma_j = d\beta_{ij} \qquad \text{on} \;\; O_{ij} \, . \tag{2.1.16}$$

The equivalence class of the 2-gerbe with curvings $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ is given by the Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$, where the quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ represents the zero Deligne

[2] Here we use the notation Lie(G) for the Lie algebra g of a Lie group G.

class if it is of the form

$$\lambda_{ijkl} = \tilde{\lambda}_{ijl}^{-1}\tilde{\lambda}_{jkl}^{-1}\tilde{\lambda}_{ijk}\tilde{\lambda}_{ikl}\,, \tag{2.1.17}$$

$$\alpha_{ijk} = \tilde{\alpha}_{ij} + \tilde{\alpha}_{jk} + \tilde{\alpha}_{ki} + \tilde{\lambda}_{ijk} d\tilde{\lambda}_{ijk}^{-1}\,, \tag{2.1.18}$$

$$\beta_{ij} = \tilde{\beta}_i - \tilde{\beta}_j + d\tilde{\alpha}_{ij}\,, \tag{2.1.19}$$

$$\gamma_i = d\tilde{\beta}_i\,. \tag{2.1.20}$$

The above equations are summarized in the expression

$$(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i) = D(\tilde{\lambda}_{ijk}, \tilde{\alpha}_{ij}, \tilde{\beta}_i)\,, \tag{2.1.21}$$

where D is the Deligne coboundary operator, $\tilde{\lambda}_{ijk}$ are $U(1)$-valued functions on O_{ijk} and $\tilde{\alpha}_{ij}, \tilde{\beta}_i$ are respectively 1- and 2-forms on O_{ij} and on O_i.

The Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i] \in H^3(M, \mathcal{D}^3)$ (actually the cocycle $\{\lambda_{ijkl}\}$) defines an integral class $\xi \in H^4(M, \mathbb{Z})$; this is the characteristic class of the 2-gerbe. Moreover, $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$ defines a closed integral 4-form

$$\frac{1}{2\pi i} G = \frac{1}{2\pi i} d\gamma_i\,. \tag{2.1.22}$$

The 4-form G is a representative of $\xi_{\mathbb{R}}$: the image of the integral class ξ in real de Rham cohomology.

2.2. **Abelian bundle gerbes.** Here we describe the construction of M. Murray [64], which identifies the geometric objects realizing the classes in $H^3(X, \mathbb{Z})$ in a similar spirit as line bundles realize classes in $H^3(X, \mathbb{Z})$ (see also, e.g., [65, 42, 45]). Let Y be a manifold. Consider a surjective submersion $\wp : Y \to X$, which in particular admits local sections. Let $\{O_i\}$ be the corresponding covering of X with local sections $\sigma_i : O_i \to Y$, i.e., $\wp\sigma_i = id$. We also consider $Y^{[n]} = Y \times_X Y \times_X Y \ldots \times_X Y$, the n-fold fibre product of Y, i.e., $Y^{[n]} := \{(y_1, \ldots y_n) \in Y^n \mid \wp(y_1) = \wp(y_2) = \ldots \wp(y_n)\}$. Given a (complex) line bundle $\mathcal{L}$ over $Y^{[2]}$ we denote by $\mathcal{L}_{12} = p_{12}^*(\mathcal{L})$ the line bundle on $Y^{[3]}$ obtained as a pullback of $\mathcal{L}$ under $p_{12} : Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{L}_{13}$ and $\mathcal{L}_{23}$. Consider a quadruple $(\mathcal{L}, Y, X, \ell)$, where $\mathcal{L}$ is a line bundle, $Y \to X$ a surjective submersion and ℓ an isomorphism of line bundles $\ell : \mathcal{L}_{12}\mathcal{L}_{23} \to \mathcal{L}_{13}$. We now consider bundles $\mathcal{L}_{12}$, $\mathcal{L}_{23}$, $\mathcal{L}_{13}$, $\mathcal{L}_{24}$, $\mathcal{L}_{34}$, $\mathcal{L}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} : Y^{[4]} \to Y^{[2]}$ etc. and also the line bundle isomorphisms ℓ_{123}, ℓ_{124}, ℓ_{123}, ℓ_{234} induced by projections $p_{123} : Y^{[4]} \to Y^{[3]}$ etc.

The quadruple $\mathcal{G} = (\mathcal{L}, Y, X, \ell)$, where $Y \to X$ is a surjective submersion, $\mathcal{L}$ is a line bundle over $Y^{[2]}$, and $\ell : \mathcal{L}_{12}L_{23} \to \mathcal{L}_{13}$ an isomorphism of line bundles over $Y^{[3]}$, is called an **(abelian) bundle gerbe** if ℓ satisfies the cocycle condition (associativity) on $Y^{[4]}$

$$\begin{tikzcd} \mathcal{L}_{12}\mathcal{L}_{23}\mathcal{L}_{34} \arrow[r, "\ell_{234}"] \arrow[d, "\ell_{123}"'] & \mathcal{L}_{12}\mathcal{L}_{24} \arrow[d, "\ell_{124}"] \\ \mathcal{L}_{13}\mathcal{L}_{34} \arrow[r, "\ell_{134}"] & \mathcal{L}_{14}\,. \end{tikzcd} \tag{2.2.1}$$

Let us also mention that there exists a proper notion of an isomorphism (the so-called **stable isomorphism** [66]) for abelian bundle gerbes, such that the categories of abelian bundle gerbes and of Čech 2-cocycles (2.1.4) are equivalent.

Further, an abelian bundle 1-gerbe can be equipped with a connection and a curving, so that locally it becomes represented by the full Deligne 2-cocycle. Without going into details, we just notice that the connection on an abelian 1-gerbe $\mathcal{G}$ can be defined as a connection on the line bundle $\mathcal{L}$ fulfilling a more or less obvious compatibility condition on $Y^{[3]}$. Also, the above described twisted principal bundles with connections can be cast in the language of bundle gerbes (cf. bundle gerbe modules of ([12]).

Abelian bundle 2-gerbes have been introduced in [25] and discussed in detail in [80].

2.3. **Global worldsheet anomalies of D-branes.** Here we briefly describe the so-called **inflow mechanism**, as it applies to a stack of n D-branes and the corresponding Freed-Witten anomaly [34], [22]. The method described here will be applied later in the thesis to the case of M5-branes, in which case 1-gerbes will be replaced by 2-gerbes and principal bundles by nonabelian gerbes.

In string theory, the background B-field is naturally interpreted as a 1-gerbe with connection and curving on the spacetime manifold M [22, 49]. Let $[\lambda_{ijk}, \alpha_{ij}, \beta_i]$ be the corresponding Deligne class and H the associated 3-form. Further, let Q be be a cycle embedded in the spacetime manifold M, on which cycle open (super)strings can end (i.e., we have D-branes wrapping Q) and $[\omega_{ijk}, 0, 0]$ be the Deligne class associated with the second Stiefel-Whitney class $\omega_2 \in H^2(Q, \mathbb{Z}_2)$ of the normal bundle of Q (or, which is the same, with its image W_3 in $H^3_{tors}(Q, \mathbb{Z})$). It can be shown [22] that the general condition for a stack of n D-branes to be wrapping the cycle Q in M is the existence of a twisted bundle (G_{ij}, A_i) (2.1.12) on Q such that

$$[\lambda_{ijk}, \alpha_{ij}, \beta_i]|_Q - [\omega_{ijk}, 0, 0] = [\mathbf{D}(G_{ij}, A_i)] + [1, 0, B_Q] \,, \tag{2.3.1}$$

where B_Q is a 2-form on Q. In particular, for the characteristic classes of these gerbes we have (cf. [34]),

$$[H]|_Q - W_3 = \xi_{[\mathbf{D}(G_{ij}, A_i)]} \,, \tag{2.3.2}$$

where $[H]|_Q \equiv \xi_{\mathcal{G}|_Q}$ is the characteristic class of the restriction to Q of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$ associated with the 3-form H, and $W_3 = \beta(\omega_2)$ is the obstruction for having Spin^c structure on the normal bundle of Q Here β is the Bockstein homomorphism associated with the short exact sequence $\mathbb{Z} \stackrel{\times 2}{\to} \mathbb{Z} \to \mathbb{Z}_2$.

2.4. **Higher crossed modules and simplicial groups.** Hypercrossed complexes of groups or hypergroupoids, in particular crossed modules 2-crossed modules, will take over the role of groups in generalizing principal bundles to bundle gerbes and their higher analogues. They relate to maybe more familiar simplicial group (here we assume familiarity with the basic definitions of the theory of simplicial objects [58]) via a nonabelian version of the famous Dold-Kan correspondence. We briefly sketch the relation between simplicial groups and hypercrossed complexes of groups. The basic idea comes from [27] and is further developed and formalized in [26]. We follow [27, 67, 33].

Although the above mentioned references[3] work with simplicial sets, the constructions and statements relevant relevant for our purposes can be straightforwardly formulated in the context of simplicial manifolds. Let G be a simplicial Lie group. We denote the corresponding face and degeneracy mappings ∂_i and s_i, respectively.

[3]and also our basic reference regarding simplicial objects [58] as well as other useful references [29, 39]

The **Moore complex** NG of G is the Lie group chain complex (NG, δ) with

$$NG_n := \bigcap_{i=1}^{n} \ker \partial_i$$

and the differentials $\delta_n : NG_n \to NG_{n-1}$ induced from the respective 0th face maps ∂_0 by restriction. It is a normal complex, i.e. $\delta_n NG_n$ is a normal subgroup of NG_{n-1}.[4]. Of course, $NG_0 = G_0$ The Moore complex NG is said to be of length k if NG_n is trivial for $n > k$.[5]

The Moore complex NG carries a structure of a Lie **hypercrossed complex** structure, form which it can be reconstructed [27, 26]. For a simplicial Lie group G,

$$G_n \cong (\dots(NG_n \rtimes s_0NG_{n-1})) \rtimes \cdots \rtimes s_{n-1}\dots s_0NG_1) \tag{2.4.1}$$

The bracketing an ordering of the terms should be clear from the first few terms of the sequence:

$$\begin{aligned}
&G_1 \cong NG_1 \rtimes s_0NG_0\\
&G_2 \cong (NG_2 \rtimes s_0NG_1) \rtimes (s_1NG_1 \rtimes s_1s_0NG_0)\\
&G_3 \cong ((NG_3 \rtimes s_0NG_2) \rtimes (s_1NG_2 \rtimes s_1s_0NG_1))\rtimes\\
&\qquad((s_2NG_2 \rtimes s_2s_0NG_1) \rtimes (s_2s_1NG_1 \rtimes s_2s_1s_0NG_0)).
\end{aligned} \tag{2.4.2}$$

We are not going to spell out the rather complicated definition of a hypercrossed complex [26]. Instead, we give some examples.

A 1-hypercrossed complex of Lie groups is the same thing as a **Lie crossed module** or a Lie 2-group [5, 8]; Let H and D be two Lie groups. We say that H is a crossed D-module if there is a Lie group morphism $\delta_1 : H \to D$ and a smooth action of D on H $(d, h) \mapsto {}^dh$ such that

$${}^{\delta_1(h)}h' = hh'h^{-1} \text{ (Peiffer condition)}$$

for $h, h' \in H$, and

$$\delta_1({}^dh) = d\delta_1(h)d^{-1}$$

for $h \in H, d \in D$ hold true.

We will use the notation $H \overset{\delta_1}{\to} D$ or $H \to D$ for a crossed module.

Starting from a Lie crossed module $H \to D$ we can build up the corresponding simplicial Lie group. Explicitly, cf. Proposition 2.4.1,

$$G_0 = D, \quad G_1 = (H \rtimes D), \quad G_2 = (H \rtimes (H \rtimes D)), \quad \text{etc.}$$

The construction can be interpreted as the internal nerve of the associated internal category in the category of Lie groups (a strict Lie 2-group). A Lie 2-hypercrossed complex is the same thing as a **Lie 2-crossed module** [27]; A Lie 2-crossed module is a complex of Lie groups

$$H \overset{\delta_2}{\to} D \overset{\delta_1}{\to} K \tag{2.4.3}$$

together with smooth left actions by automorphisms of K on H and D (and on K by conjugation), and the **Peiffer pairing**, which is an smooth equivariant map $\{\,,\,\} : D \times D \to H$, i.e., ${}^k\{d_1, d_2\} = \{{}^kd_1, {}^kd_2\}$ such that:

[4] It is a normal subgroup of G_{n-1} too.

[5] The objects of the full subcategory of simplicial groups with Moore complex of length k are also called k-**hypergroupoids**[38].

i) (2.4.3) is a complex of K-modules, i.e., δ_2 and δ_1 are K-equivariant and $\delta_2\delta_1(h) = 1$ for $h \in H$,
ii) $d_1 d_2 d_1^{-1} = \delta_2\{d_1, d_2\}\,{}^{\delta_1(d_1)}d_2$, for $d_1, d_2 \in D$,
iii) $h_1 h_2 h_1^{-1} h_2^{-1} = \{\delta_2 h_1, \delta_2 h_2\}$, for $h_1, h_2 \in H$,
iv) $\{d_1 d_2, d_3\} = \{d_1, d_2 d_3 d_2^{-1}\}\,{}^{\delta_1(d_1)}\{d_2, d_3\}$, for $d_1, d_2, d_3 \in D$,
v) $\{d_1, d_2 d_3\} = {}^{d_1 d_2 d_1^{-1}}\{d_1, d_3\}\{d_1, d_2\}$, for $d_1, d_2, d_3 \in D$,
vi) $\{\delta_2(h), d\}\{d, \delta_2(h)\} = h^{\delta_1(d)}(h^{-1})$, for $d \in D, h \in H$,

wherein the notation ${}^k d$ and ${}^k h$ for left actions of the element $k \in K$ on elements $d \in D$ and $h \in H$ has been used.

The corresponding simplicial Lie group is given explicitly by, cf. 2.4.1,

$$G_0 = K, \quad G_1 = (D \rtimes K), \quad G_2 = (H \rtimes D) \rtimes (D \rtimes K)), \tag{2.4.4}$$

$$G_3 = (H \rtimes (H \rtimes D)) \rtimes ((H \rtimes D) \rtimes (D \rtimes K)), \quad \text{etc.} \tag{2.4.5}$$

Let us note, that there are obvious notions of a morphisms for Lie crossed modules and Lie 2-crossed modules.

We refer the interested reader to [26] for a thorough discussion of hypercrossed complexes of groups and their relation to simplicial groups.

Continuing on general discussion, at each level n, there is an lexicographically ordered set $S(n)$ of 2^n sets, which defines the compositions of the degeneracy maps appearing in the decomposition of G_n. Explicitly for $S(n)$ we have:

$$\{\emptyset < \{0\} < \{1\} < \{1,0\} < \{2\} < \{2,0\} < \{2,1\} < \{2,1,0\} < \ldots < \{n-1, \ldots, 1\} < \{n-1, \ldots, 0\}\}.$$

The important role in the theory of hypercrossed complexes is played by the actions $G_0 \times NG_n \to NG_n$ defined by

$$g_0 \times g_n \mapsto {}^{g_0}g_n : (s_{n-1} \ldots s_0 g_0) g_n (s_{n-1} \ldots s_0 g_0)^{-1}$$

and the so called **Peiffer pairings**. In order to define these, we will use the multi-indices like α and β from $\bigcup_n S(n)$ to write s_α for products of degeneracy maps

$$s_0,\ s_1,\ s_1 s_0,\ s_2,\ s_2 s_0,\ s_2 s_1,\ s_2 s_1 s_0, \ldots$$

In particular, for $g \in NG_{n-\sharp\alpha}$ we have $s_\alpha g \in G_n$. For each n consider the set $P(n)$ of pairs (α, β) such that $\emptyset < \alpha < \beta$ and $\alpha \cap \beta = \emptyset$, where $\alpha \cap \beta$ is the set of indices belonging to both α and β.

The following definition will be needed later when describing differential graded Li algebras arising as differentiation of simplicial Lie groups.

The Peiffer pairing (or lifting) $F_{\alpha,\beta}(g, h) \in NG_n$ for $g \in NG_{n-\sharp\alpha}$, $h \in NG_{n-\sharp\beta}$ and $(\alpha, \beta) \in P(n)$ is defined by

$$F_{\alpha,\beta}(g, h) = p_n(s_\alpha(g) s_\beta(h) s_\alpha(g)^{-1} s_\beta(h)^{-1}),$$

where p_n is the projection to NG_n. For the projector p_n, we have $p_n = p_n^1 \ldots p_n^n$ with $p_n^i(g) = g s_{i-1} \partial_i g^{-1}$.

For us, the relevant Peiffer pairings at each level n will be those defined for pairs $(\alpha, \beta) \in P(n)$ such that $\alpha \cup \beta = \{0, \ldots n\}$. We shall denote the set of such pairs $\bar{P}(n)$.

For a simplicial Lie algebra $\mathfrak{g}$, we have the corresponding Moore complex $N\mathfrak{g}$ of Lie algebras, which carries a structure of a hypercrossed complex of Lie algebras, cf. [1]. All the definitions and statements of this section have, of course, their infinitesimal counterparts. Since these are obvious, we shall not formulate them explicitly.

As shown by Quillen [70], there is an adjunction between simplicial Lie algebras and dg-Lie algebras. The part of the adjunction going from simplicial Lie algebras to dg-Lie algebras acts on the underlying simplicial vector spaces as the Moore complex functor N.

2.5. **Simplicial principal bundles.** Let G be a simplicial Lie group and X a simplicial manifold. In this paper we use the name *principal G-bundle* for a *twisted Cartesian product.* Therefore, we start with defining twisting functions. Again, we will denote by ∂_i and s_i the corresponding face and degeneracy maps. We follow [58].[6]

For a smooth function $\tau : X_n \to G_{n-1}$ to be a **twisting**, the following conditions should be fulfilled:

$$\begin{aligned} \partial_0\tau(x)\tau(\partial_0 x) &= \tau(\partial_1 x), \\ \partial_i\tau(x) &= \tau(\partial_{i+1}x) \quad \text{for} \quad i > 0, \\ s_i\tau(x) &= \tau(s_{i+1}x) \quad \text{for} \quad i \geq 0, \\ \tau(s_0 x) &= e_n \quad \text{for} \quad x \in X_n. \end{aligned}$$

The fundamental role is played by **twisted Cartesian products**: Let τ be a twisting function. A twisted Cartesian product $P(\tau) = G \times_\tau X$ (alternatively a **principal G-bundle**, or simply **G-bundle**, $P \to X$) is the simplicial manifold with simplices

$$P(\tau)_n = G_n \times X_n$$

and with the following face and degeneracy maps

$$\begin{aligned} \partial_i(g, x) &= (\partial_i g, \partial_i x) \quad \text{for} \quad i > 0, \\ \partial_0(g, x) &= (\partial_0 g.\tau(x), \partial_0 x), \\ s_i(g, x) &= (s_i g, s_i x) \quad \text{for} \quad i \geq 0. \end{aligned}$$

The principal (left) G-action

$$G_n \times P(\tau)_n \to P(\tau)_n, \quad g'_n \times (g_n, x_n) \mapsto (g'_n g_n, x_n)$$

and the projection

$$\pi_n : P_n \to X_n, \quad (g_n, x_n) \mapsto x_n$$

are smooth simplicial maps.

We call two twistings τ' and τ **equivalent** if there exists a smooth map $\psi : X \to G$ such that

$$\begin{aligned} \partial_0\psi(x).\tau'(x) &= \tau(x).\psi(\partial_0 x), \\ \partial_i\psi(x) &= \psi(\partial_i x) \quad \text{if} \quad i > 0, \\ s_i\psi(x) &= \psi(s_i x) \quad \text{if} \quad i \geq 0. \end{aligned}$$

In particular a twisting or the corresponding G-bundle $P(\tau)$ is trivial iff

$$\tau(x) = \partial_0\psi(x)^{-1}.\psi(\partial_0 x),$$

with ψ as above.

As with ordinary bundles, simplicial principal bundles can be pulled back and their structure groups can be changed using simplicial Lie group morphisms. Twistings transform under these operations in an obvious way.

---

[6] Again, passing from sets to manifolds is straightforward.

There is a canonical construction of the **classifying space** $\overline{W}G$ and of the **universal** G-**bundle** WG. The classifying space $\overline{W}G$ is defined as follows. $\overline{W}G_0$ has one element $*$ and $\overline{W}G_n = G_{n-1} \times G_{n-2} \times \ldots \times G_0$ for $n > 0$. Face and degeneracy maps are

$$s_0(*) = e_0, \qquad \partial_i(g_0) = * \quad \text{for} \quad i = 0 \text{ or } 1$$

and

$$\begin{aligned} \partial_0(g_n, \ldots g_0) &= (g_{n-1}, \ldots, g_0), \\ \partial_{i+1}(g_n, \ldots, g_0) &= (\partial_i g_n, \ldots, \partial_1 g_{n-i+1}, \partial_0 g_{n-i} \cdot g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \\ s_0(g_{n-1}, \ldots, g_0) &= (e_n, g_{n-1}, \ldots, g_0), \\ s_{i+1}(g_{n-1}, \ldots, g_0) &= (s_i g_{n-1}, \ldots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0), \end{aligned}$$

for $n > 0$. With the choice of a twisting given by

$$\tau(g_{n-1}, \ldots, g_0) = g_{n-1}$$

we have the universal G-principal bundle

$$WG = G \times_\tau \overline{W}G.$$

The relation between twistings and simplicial maps $X \to \overline{W}G$ is the following one.

The map $f_\tau : X \to \overline{W}G$ given by

$$x \mapsto * \quad for \quad x \in X_0$$

and

$$x \mapsto (\tau(x), \tau(\partial_0 x), \ldots, \tau(\partial_0^i x), \ldots, \tau(\partial_0^{n-1} x)) \quad for \quad x \in X_n,\, n > 0$$

is a smooth simplicial map.

Vice versa, a smooth simplicial map $f : X \to \overline{W}_G$, given by

$$x \mapsto * \quad for \quad x \in X_0$$

and

$$x \mapsto (g^{(n)}_{n-1}(x), \ldots, g^{(n)}_0(x)) \quad for \quad x \in X_n,\ n > 0$$

defines a twisting by

$$\tau_f(x) = g^{(n)}_{n-1}(x) \quad for \quad x \in X_n,\ n > 0.$$

We have $\tau_{f_\tau} = \tau$ and $f_{\tau_f} = f$.

The mane of the universal bundle is justified by the following result. The principal G-bundle $G \times_\tau X$ corresponding to the twisting τ is obtained from the universal bundle WG as a pullback under the simplicial map f_τ.

Let us finish this section with a very short discussion of differentiation of certain simplicial manifolds. In [78], Ševera describes the so called **1-jet** of a (truncated) **simplicial Kan (super)manifold**.[7] Let us note that a simplicial Lie group G, its classifying space $\overline{W}G$ and the corresponding universal bundle WG are such simplicial Kan manifolds. The rough idea is to look at simplicial maps from the simplicial supermanifold – given at the level n as the n-th Cartesian power of the superline $\mathbb{R}^{0|1}$ with obvious face and degeneracy maps[8] – to a given simplicial Kan manifold. If the simplicial Kan manifold has only one zero simplex, such simplicial maps can be described in terms of L_∞-algebras,[9] which are the resulting 1-jets. For

[7]The interested reader can find the corresponding rather technical definitions in Ševera's paper or in [P7].

[8]the nerve of the pair groupoid with object elements of $\mathbb{R}^{0|1}$ and morphisms elements of $\mathbb{R}^{0|1} \times \mathbb{R}^{0|1}$. We postpone the definition of a nerve of a category, in particular of a groupoid, until the next section.

[9]More generally by L_∞-algebroids [85]. For L_∞-algebras see, e.g., [57, 56]. For the opposite, the integration of L_∞-algebras, see [36, 41].

instance, if the simplicial Lie group is at each level given by the same Lie group with all face and degeneracy maps given by the identity map, then the 1-jet is the just the corresponding Lie algebra. If the simplicial manifold is the Atyiah groupoid of an ordinary Lie group principal bundle, then its 1-jet is the corresponding Atyiah algebroid, and its sections are connections on the principal bundle. Hence, understanding 1-jets of classifying spaces $\overline{W}G$ can be viewed as a fist step in understanding connections on simplicial principal bundles, which in view of results in Section 3.1 can be viewed as higher nonabelian bundle gerbes.[10]

2.6. **Nerves, linear orders.** Simplicial spaces [58] coming from (Duskin [32] and possibly other [55, 84, 79]) nerves of topological bicatgeories [11] (and spaces obtained by geometric realizations of these nerves) play a central role in the very last part of this thesis devoted to classifying spaces and topoi. Let us recall that the **nerve** of a (topological) category is defined as a simplicial space $N\mathbb{C}$ with space of n-simplices $N\mathbb{C}_n$ being the fibred product space $\mathbb{C}_1 \times_{\mathbb{C}_0} \times \ldots \times_{\mathbb{C}_0} \mathbb{C}_1$ of all composable strings of arrows of $\mathbb{C}$. The degeneracy maps $N\mathbb{C}_{n-1} \to N\mathbb{C}_n$ are given by insertions of identity arrows. The face maps $N\mathbb{C}_n \to N\mathbb{C}_{n-1}$ (except the 0th and nth, which are given by dropping the first and the last arrow, respectively) are given by compositions of arrows. In case of a small category, the nerve is just a simplicial set.

Let us recall that the **Duskin nerve** of a (topological) bicategory $\mathbb{B}$ is a 3-coskeletal simplicial space $N\mathbb{B}$ with 0-simplices the objects x_0 of $2\mathbb{C}$, 1-simplices the 1-arrows $x_0 \overset{x_{01}}{\to} x_1$ of $\mathbb{B}$ and 2-simplices are 2-cells which are triangles $x_{02} \overset{x_{012}}{\Longrightarrow} x_{01}x_{12}$ filled with a 2-arrow x_{012}.

For the future reference, let us collect few definitions and result, which we repeat almost verbatim from [63]. For a simplicial space Y the **geometric realization** $|Y|$ will always mean the thickened (fat) geometric realization. This is defined as a topological space obtained from the disjoint sum $\sum_{n\geq 0} X_n \times \Delta^n$ by the the equivalence relations

$$(\alpha^*(x), t) \sim (x, \alpha(t)) \tag{2.6.1}$$

for all injective (order-preserving) arrows $\alpha\colon [n] \to [m] \in \Delta$, any $x \in X_m$ and any $t \in \Delta^n$, where Δ^n is the standard topological n-simplex. In the case of a so-called good simplicial space [3] (e.g., all Y_n re CW-complexes), this geometric realization is homotopy equivalent to the geometric realization of the underlying simplicial set of Y, which is defined as above but allowing for all arrows in Δ.

Linear order over a topological space X is a sheaf L on X together with a subsheaf $O \subseteq L \times_X L$ such that for each point $x \in X$ the stalk L_x is nonempty and linearly ordered by the relation $y \leq z$ iff $(y, z) \in O_x$, for $y, z \in L_x$. A mapping $L \to L'$ between two linear orders over X is a mapping of sheaves restricting for each $x \in X$ to an order preserving map of stalks $L_x \to L'_x$. This defines a **category of linear orders on** X.

A linear order L on X defines an obvious topological category with L as space of objects and the order subsheaf $O \subseteq L \times_X L$ as space of arrows. Hence, we can speak of a nerve NL of the linear order L. This nerve is obviously a simplicial sheaf on X (a simplicial space with étale maps into X).

For any space X and any simplicial space Y write $Lin(X, Y)$ for the category of linear orders (L, aug) on X equipped with a simplicial map (**augmentation**) $\mathrm{aug} : NL \to Y$ from the nerve of L to Y. A morphism $(L, \mathrm{aug}) \to (L', \mathrm{aug}')$ in $Lin(X, Y)$ is a map of linear orders $L \to L'$ such that the induced map $NL \to NL'$ on the nerves respects the augmentations.

[10] See, e.g., [71] for discussion of L_∞-algebra valued connections.
[11] See, e.g., [11] for the definition of a bicategory.

We call two objects $E_0, E_1 \in Lin(X,Y)$ **concordant** if there exists an $E \in Lin(X\times[0,1],Y)$ such that we have $E_0 \cong i_0^*(E)$ and $E_0 \cong i_1^*(E)$ under the obvious inclusions $i_0, i_1 : X \hookrightarrow X \times [0,1]$. $Lin_c(X,Y)$ will denote the collection of concordance classes of objects from $Lin(X,Y)$.

Let Y be a simplicial space. For any space X there is a natural equivalence of categories

$$\mathrm{Hom}(Sh(X), Sh(Y)) \simeq Lin(X,Y). \tag{2.6.2}$$

Here $\mathrm{Hom}(Sh(X), Sh(Y))$ is the category of geometric morphisms of topoi $Sh(X)$ and $Sh(Y)$, with morphisms being natural transformations.

On homotopy classes of topos morphisms we have the natural bijection

$$[Sh(X), Sh(Y)] \cong Lin_c(X,Y). \tag{2.6.3}$$

Let X be a CW-complex and Y be a locally contractible simplicial space. There is a natural bijection between homotopy classes of maps $[X, |Y|]$ and concordance classes $Lin_c(X,Y)$.

2.7. **Classifying spaces, classifying topoi.** The above described constructions can be applied to the case of $\mathbb{C}$-principal bundles, where $\mathbb{C}$ is a topological category. Let us start with discussing the case of a topological group G. Principal G-bundles over a (topological) space X are classified by the first Čech cohomology $H^1(X,G)$ of X with coefficients in G. Under some mild conditions, these Čech cohomology classes are in 1-1 correspondence with homotopy classes of maps $[X, BG]$ from X to the classifying space BG (see, e.g., [74, 44]). For example, the elements of

$$H^1(X, U(1)) \cong H^2(X, \mathbb{Z}) \cong [X, BU(1)] \tag{2.7.1}$$

classify line bundles. Thus, characteristic classes for bundles can be obtained as pullbacks of cohomology classes on BG. One way to define the classifying space is to take the geometric realization $|NG|$ of the nerve NG of the group G.

The notion of a principal bundle and of the classifying space can be generalized from the case of a topological group G to the case of a topological category $\mathbb{C}$ [63]. Roughly speaking, a **$\mathbb{C}$-principal bundle** over X can be defined as a continuous functor from the topological category defined by an ordered open covering $\mathcal{U} = \{U_i\}$ (more generally from a linear order L over X [63]) to the category $\mathbb{C}$, i.e., as a $\mathbb{C}$-valued Čech 1-cocycle. Again, the **classifying space** $B\mathbb{C}$ is defined as the geometric realization $|N\mathbb{C}|$ of the nerve $N\mathbb{C}$ of the category $\mathbb{C}$. If X is a CW complex and $\mathbb{C}$ has contractible spaces of objects and arrows, concordance classes $Lin_c(X,\mathbb{C})$ of principal $\mathbb{C}$-bundles are in 1-1 correspondence with the homotopy classes of maps $[X, B\mathbb{C}]$:

$$Lin_c(X,\mathbb{C}) \cong [X, B\mathbb{C}]. \tag{2.7.2}$$

The above restrictions on X and $\mathbb{C}$ can be abandoned if one considers, instead of the classifying space $B\mathbb{C}$, the classifying topos $\mathcal{B}\mathbb{C}$ (still, all spaces have to be assumed to be sober, i.e., every closed subset which can not be written as a union of two smaller closed sets is a closure of a unique one point set). The classifying topos $\mathcal{B}\mathbb{C}$ is the so-called Deligne topos $Sh(N\mathbb{C})$ of sheaves on the nerve $N\mathbb{C}$ of the category $\mathbb{C}$. Let us recall that a **sheaf S on a simplicial space** Y is defined to be a system of sheaves S^n on Y_n, for $n \geq 0$, together with sheaf maps $S(\alpha)\colon Y(\alpha)^* S^n \to S^m$ for each $\alpha\colon [n] \to [m]$. These maps are required to satisfy the proper functoriality conditions [63]. Equipped with properly defined morphisms we have the category of sheaves $Sh(Y)$ on the simplicial space Y. The category $Sh(Y)$ of sheaves on a simplicial space is a topos, which is called the **Deligne topos**.

There is an equivalence [63] between the category $Lin(X,\mathbb{C})$ of $\mathbb{C}$-principal bundles and category of geometric morphisms $\mathrm{Hom}(Sh(X), Sh(N\mathbb{C}))$ between (Grothendieck) topoi $Sh(X)$ and $Sh(N\mathbb{C})$:

$$Lin(X,\mathbb{C}) \simeq \mathrm{Hom}(Sh(X), Sh(N\mathbb{C})). \tag{2.7.3}$$

Let us motivate the later discussion, for the case of one dimension higher, by considering the example of abelian 1-gerbes. We have

$$H^2(X, U(1)) \cong H^3(X,\mathbb{Z}) \cong [X, B^2U(1)]. \tag{2.7.4}$$

In this situation, $B^2U(1)$ can be given the following interpretation. Starting with $U(1)$ we can consider the strict Lie 2-group (see, e.g., [7]) with only one object, one 1-arrow and 2-arrows being the elements of $U(1)$, or equivalently, the corresponding crossed module [20]. Then the classifying space $B^2U(1)$ is (homotopy) equivalent to the geometric realization of the so-called Duskin nerve [32] of this strict 2-group. The "classifying properties" of geometric realizations of the so-called Duskin's nerves have been investigated in cases of strict Lie 2-groups, topological 2-groups and topological bicategories in [P6], [9] and [3], respectively. One of the results of the present thesis is the description of the classifying topos and its properties for any topological bicategory.

2.8. **Noncommutative line bundles .** Nocommutative line bundles are yet another generalization of line bundles. We will use them later to introduce noncommuative gerbes, which will turn out to be genuine nonabelian gerbes.[12]

Let (M,θ) be a general Poisson manifold, and let $\star$ be the Kontsevich's deformation quantization of the Poisson tensor θ. Further, let us consider a good covering $\{U^i\}$ of M. For the purposes of this thesis, a **noncommutative line bundle** $\mathcal{L}$ is defined by a collection of $\mathbb{C}[[\hbar]]$-valued local transition functions $G^{ij} \in C^\infty(U^i \cap U^j)[[\hbar]]$ (that can be thought valued in the enveloping algebra of $U(1)$, see [46]), and a collection of maps $\mathcal{D}^i : C^\infty(U^i)[[\hbar]] \to C^\infty(U^i)[[\hbar]]$, formal power series in $\hbar$, starting with the identity, and with coefficients being differential operators, such that

$$G^{ij} \star G^{jk} = G^{ik} \tag{2.8.1}$$

on $U^i \cap U^j \cap U^k$, $G^{ii} = 1$ on U^i, and

$$\mathrm{Ad}_\star G^{ij} = \mathcal{D}^i \circ (\mathcal{D}^j)^{-1} \tag{2.8.2}$$

on $U^i \cap U^j$ or, equivalently, $\mathcal{D}^i(f) \star G^{ij} = G^{ij} \star \mathcal{D}^j(f)$ for all $f \in C^\infty(U^i \cap U^j)[[\hbar]]$. Obviously, with this definition the local maps $\mathcal{D}^i$ can be used to define *globally* a new star product $\star'$ (because the inner automorphisms $\mathrm{Ad}_\star G^{ij}$ do not affect $\star'$)

$$\mathcal{D}^i(f \star' g) = \mathcal{D}^i f \star \mathcal{D}^i g\ . \tag{2.8.3}$$

We say that two line bundles $\mathcal{L}_1 = \{G_1^{ij}, \mathcal{D}_1^i, \star\}$ and $\mathcal{L}_2 = \{G_2^{ij}, \mathcal{D}_2^i, \star\}$ are **equivalent** if there exists a collection of invertible local functions $H^i \in C^\infty(U^i)[[\hbar]]$ such that

$$G_1^{ij} = H^i \star G_2^{ij} \star (H^j)^{-1} \tag{2.8.4}$$

[12] A noncommutative line bundle is a finite projective module. In the present context it can be understood as a quantization of a line bundle over a compact manifold in the sense of deformation quantization. Here we shall take the properties of quantized line bundles as derived in [48, 47] as a formal definition of a noncommutative line bundle.

and

$$\mathcal{D}^i_1 = \mathrm{Ad}_\star H^i \circ \mathcal{D}^i_2 \ . \tag{2.8.5}$$

The **tensor product** of two line bundles $\mathcal{L}_1 = \{G^{ij}_1, \mathcal{D}^i_1, \star_1\}$ and $\mathcal{L}_2 = \{G^{ij}_2, \mathcal{D}^i_2, \star_2\}$ is well defined if $\star_2 = \star'_1$ (or $\star_1 = \star'_2$.) Then the corresponding tensor product is a line bundle $\mathcal{L}_2 \otimes \mathcal{L}_1 = \mathcal{L}_{21} = \{G^{ij}_{12}, \mathcal{D}^{ij}_{12}, \star_1\}$ defined as

$$G^{ij}_{12} = \mathcal{D}^i_1(G^{ij}_2) \star_1 G^{ij}_1 = G^{ij}_1 \star_1 \mathcal{D}^j_1(G^{ij}_2) \tag{2.8.6}$$

and

$$\mathcal{D}^i_{12} = \mathcal{D}^i_1 \circ \mathcal{D}^i_2 \ . \tag{2.8.7}$$

The order of indices of $\mathcal{L}_{21}$ indicates the bimodule structure of the corresponding space of sections to be defined later, whereas the first index on the G_{12}'s and $\mathcal{D}_{12}$'s indicates the star product (here: $\star_1$) by which the objects multiply.

A **section** $\Psi = (\Psi^i)$ is a collection of functions $\Psi^i \in C^\infty_{\mathbb{C}}(U^i)[[\hbar]]$ satisfying consistency relations

$$\Psi^i = G^{ij} \star \Psi^i \tag{2.8.8}$$

on all intersections $U^i \cap U^j$. With this definition the space of sections $\mathcal{E}$ is a right $\mathcal{A}_x = (C^\infty(M)[[\hbar]], \star)$ module. We shall use the notation $\mathcal{E}_{\mathcal{A}_x}$ for it. The right action of the function $f \in \mathcal{A}_x$ is the regular one

$$\Psi . f = (\Psi^k \star f) \ . \tag{2.8.9}$$

Using the maps $\mathcal{D}^i$ it is easy to turn $\mathcal{E}$ also into a left $\mathcal{A}_x{}' = (C^\infty(M)[[\hbar]], \star')$ module ${}_{\mathcal{A}_x{}'}\mathcal{E}$. The left action of $\mathcal{A}_x{}'$ is given by

$$f.\Psi = (\mathcal{D}^i(f) \star \Psi^i) \ . \tag{2.8.10}$$

It is easy to check, using (2.8.2), that the left action (2.8.10) is compatible with (2.8.8). From the property (2.8.3) of the maps $\mathcal{D}^i$ we find

$$f.(g.\Psi) = (f \star' g).\Psi \ . \tag{2.8.11}$$

Together we have a **bimodule** structure ${}_{\mathcal{A}_x{}'}\mathcal{E}_{\mathcal{A}_x}$ on the space of sections.

There is an obvious way of tensoring sections. The section

$$\Psi^i_{12} = \mathcal{D}^i_1(\Psi^i_2) \star_1 \Psi^i_1 \tag{2.8.12}$$

is a section of the tensor product line bundle (2.8.6), (2.8.7). Tensoring of line bundles naturally corresponds to tensoring of bimodules.

Let us note that if we assume the base manifold M to be compact, then the space of sections $\mathcal{E}$ as a right $\mathcal{A}_x$-module is projective of finite type. Of course, the same holds if $\mathcal{E}$ is considered as a left $\mathcal{A}'_x$ module. Also let us note that the two algebras $\mathcal{A}_x$ and $\mathcal{A}'_x$ are **Morita equivalent**. Up to a global isomorphism they must be related by an action of the **Picard group** $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ as follows. Let $L \in \mathrm{Pic}(M)$ be a (complex) line bundle on M and c its Chern class. Let F be a curvature two form on M whose cohomology class $[F]$ is (the image in $\mathbb{R}$ of) the Chern cass c. Consider the formal Poisson structure θ' given by the geometric series

$$\theta' = \theta(1 + \hbar F \theta)^{-1}. \tag{2.8.13}$$

In this formula θ and F are understood as maps $\theta : T^*M \to TM$, $F : TM \to T^*M$ and θ' is the result of the indicated map compositions. Then $\star'$ must (up to a global isomorphism) be the deformation quantization of θ' corresponding to $c \in H^2(M,\mathbb{Z})$. This construction depends only on the integer cohomology class c, indeed if c is the trivial class then $F = da$ and the corresponding quantum line bundle is trivial, i.e.,

$$G^{ij} = (H^i)^{-1} \star H^j \ . \tag{2.8.14}$$

In this case the linear map

$$\mathcal{D} = \mathrm{Ad}_\star H^i \circ \mathcal{D}^i \tag{2.8.15}$$

defines a global equivalence (a stronger notion than Morita equivalence) of $\star$ and $\star'$.

Finally, using, e.g., the Hochschild complex we can introduce a natural differential calculus on the algebra $\mathcal{A}_x$ and consequently, in the spirit of noncommutative geometry [28], also the notion of a connection and curvature [48].

3. Some results

3.1. **Nonabelian bundle gerbes.** In paper [P2], the main goal was to generalize the theory of abelian bundle gerbes and their differential geometry, due to Murray [64], to the nonabelian case. Hence, in contrary to the previous approaches to nonabelian gerbes (e.g., [37, 14, 16, 17]), our study was from the differential geometry viewpoint. We believe that it is primarily in this context that nonabelian gerbes structures can appear and can be recognized in physics. It is for example in this context that one would like to have a formulation of Yang-Mills theory with higher forms. The idea followed in [P2] was to replace the "transition" line bundles $\mathcal{L}$ by G-principal bundles with additional structure, which would allow to multiply them. Let us mention [10, 9, 54] for different or independent approaches[13], e.g., [73] for discussion of the holonomy of gerbes.

Let $(G \overset{\partial}{\to} D)$ be a crossed module of Lie groups and X a manifold. Let $P \to X$ be a left principal G-bundle, such that the principal D-bundle $D \times_\partial P$ is trivial with a trivialization defined by a section (i.e., a left G-equivariant smooth map) $\mathfrak{d} : P \to D$. We call the couple $(P, \mathfrak{d})$ a $(G \to D)$-bundle. Two $(G \to D)$-bundles $(P, \mathfrak{d})$ and $(P', \mathfrak{d}')$ over X are isomorphic if they are isomorphic as left G-bundles by an isomorphism $\ell : P \to P'$ and $\mathfrak{d}'\ell = \mathfrak{d}$. Obviously, a pullback of a $(G \to D)$-bundle is again a $(G \to D)$-bundle.

The $(G \to D)$-bundle $(P, \mathfrak{d})$ is also a right principal G-bundle with the right action of G given by $p.l = {}^{\mathfrak{d}(p)}(g).p$ for $p \in P, l \in G$. The left and right actions commute, hence, P has naturally the structure if a principal G-bibundle [40, 37, 14]. The section $\mathfrak{d}$ is G-biequivariant. Let $(P, \mathfrak{d})$ and $(\tilde{P}, \tilde{\mathfrak{d}})$ are two $(G \to D)$-bundles over X. Let us define an equivalence relation on the Whitney sum $P \oplus \tilde{P} = P \times_X \tilde{P}$ by $(pl, \tilde{p}) \sim (p, g\tilde{p})$, for $(p, \tilde{p}) \in P \oplus \tilde{P}$ and $g \in G$. Then $(P\tilde{P} := (P \oplus \tilde{P})/\sim, \mathfrak{d}\tilde{\mathfrak{d}})$ with $\mathfrak{d}\tilde{\mathfrak{d}}([p, \tilde{p}]) := \mathfrak{d}(p)\tilde{\mathfrak{d}}(\tilde{p})$ is a $(G \to D)$-bundle.

Let Y be a manifold. Consider a surjective submersion $\wp : Y \to X$, which in particular admits local sections. Let $\{O_i\}$ be the corresponding covering of X with local sections $\sigma_i : O_i \to Y$, i.e., $\wp\sigma_i = id$. We also consider $Y^{[n]} = Y \times_X Y \times_X Y \ldots \times_X Y$, the n-fold fibre product of Y, i.e., $Y^{[n]} := \{(y_1, \ldots y_n) \in Y^n \mid \wp(y_1) = \wp(y_2) = \ldots \wp(y_n)\}$. Given a $(G \to D)$-bundle $\mathcal{P} = (P, \mathfrak{d})$ over $Y^{[2]}$ we denote by $\mathcal{P}_{12} = p_{12}^*(\mathcal{P})$ the crossed module bundle on $Y^{[3]}$ obtained as a pullback of $\mathcal{P}$ under $p_{12} : Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{P}_{13}$ and $\mathcal{P}_{23}$. Consider a quadruple $(\mathcal{P}, Y, X, \ell)$, where $\mathcal{P} = (P, \mathfrak{d})$ is a crossed module bundle, $Y \to X$ a surjective submersion and ℓ an isomorphism of crossed

[13]We have also to mention [4] for a different generalization of abelian gerbes, the 2-vector bundles – categorifications of vector bundles.

module bundles $\ell : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$. We now consider bundles $\mathcal{P}_{12}$, $\mathcal{P}_{23}$, $\mathcal{P}_{13}$, $\mathcal{P}_{24}$, $\mathcal{P}_{34}$, $\mathcal{P}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} : Y^{[4]} \to Y^{[2]}$ etc. and also the crossed module isomorphisms ℓ_{123}, ℓ_{124}, ℓ_{123}, ℓ_{234} induced by projections $p_{123} : Y^{[4]} \to Y^{[3]}$ etc. Now we can define a $(G \to D)$-bundle gerbe for a general crossed module of Lie groups.

Definition 3.1. *The quadruple $(\mathcal{P}, Y, X, \ell)$, where $Y \to X$ is a surjective submersion, $\mathcal{P}$ is a crossed module bundle over $Y^{[2]}$, and $\ell : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$ an isomorphism of crossed module bundles over $Y^{[3]}$, is called a crossed module bundle gerbe if ℓ satisfies the cocycle condition (associativity) on $Y^{[4]}$*

$$\begin{array}{ccc} \mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{34} & \xrightarrow{\ell_{234}} & \mathcal{P}_{12}\mathcal{P}_{24} \\ {\scriptstyle \ell_{123}}\big\downarrow & & \big\downarrow{\scriptstyle \ell_{124}} \\ \mathcal{P}_{13}\mathcal{P}_{34} & \xrightarrow{\ell_{134}} & \mathcal{P}_{14}\,. \end{array} \tag{3.1.1}$$

Abelian bundle gerbes as introduced in [64], [65] are $(U(1) \to 1)$-bundle gerbes. More generally, if $A \to 1$ is a crossed module then A is necessarily an abelian group and an abelian bundle gerbe can be identified as an $(A \to 1)$-bundle gerbe.

A $(1 \to G)$-bundle gerbe is the same thing as a G-valued function g on $Y^{[2]}$ satisfying on $Y^{[3]}$ the cocycle relation $g_{12}g_{23} = g_{23}$ and hence, a principal G-bundle on X (more precisely, a descent datum of a principal G-bundle).

The stable isomorphism of two $(G \to D)$-bundle gerbes is defined as follows.

Definition 3.2. *Two crossed module bundle gerbes $(\mathcal{P}, Y, X, \ell)$ and $(\mathcal{P}', Y', X, \ell')$ are stably isomorphic if there exists a crossed module bundle $\mathcal{Q} \to \bar{Y} = Y \times_X Y'$ such that over $\bar{Y}^{[2]}$ the crossed module bundles $q^*\mathcal{P}$ and $\mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ are isomorphic. The corresponding isomorphism $\tilde{\ell} : q^*\mathcal{P} \to \mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ should satisfy on $\bar{Y}^{[3]}$ (with an obvious abuse of notation) the condition*

$$\tilde{\ell}_{13}\ell = \ell'\tilde{\ell}_{23}\tilde{\ell}_{12}\,. \tag{3.1.2}$$

In the above definition, q and q' are projections onto first and second factor of $\bar{Y} = Y \times_X Y'$ and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the pullbacks of $\mathcal{Q} \to \bar{Y}$ to $\bar{Y}^{[2]}$ under respective projections form $\bar{Y}^{[2]}$ to $\bar{Y}$ etc.

Locally, bundle gerbes can be described in terms of 2-cocycles as follows. First, let us notice that the trivializing cover $\{O_i\}$ of the map $\wp : Y \to X$ defines a new surjective submersion $\wp' : Y' = \coprod O_i \to X$. The local sections of $Y \to X$ define a map $f : Y' \to Y$, which is compatible with the maps $\wp$ and $\wp'$, i.e., such that $\wp f = \wp'$. We know that crossed module bundle gerbes $\mathcal{G}_f$ and $\mathcal{G}$ are stably isomorphic. Hence, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one. We have the following proposition.

Proposition 3.3. *A crossed module bundle gerbe can be locally described by a 2-cocycle (d_{ij}, g_{ijk}), where the smooth maps $d_{ij} : O_{ij} \to D$ and $g_{ijk} : O_{ijk} \to G$ fulfill the following conditions:*

$$d_{ij}d_{jk} = \partial(g_{ijk})d_{ik} \quad \text{on} \quad O_{ijk} \tag{3.1.3}$$

and

$$g_{ijk}g_{ikl} = {}^{d_{ij}}g_{jkl}g_{ijl} \quad \text{on} \quad O_{ijkl}\,. \tag{3.1.4}$$

2-cocycles (d_{ij}, g_{ijk}) *and* (d'_{ij}, g'_{ijk}) *corresponding to stably isomorphic* $(G \to D)$*-bundle gerbes are related by*

$$d'_{ij} = d_i \partial(g_{ij}) d_{ij} d_j^{-1} \tag{3.1.5}$$

$${}^{d_i^{-1}}g'_{ijk} = g_{ij}\, {}^{d_{ij}}g_{jk} g_{ijk} g_{ik}^{-1}, \tag{3.1.6}$$

with smooth functions $g_{ij} : O_{ij} \to G$ *and* $d_i : O_i \to D$.

These are, of course, the well-known formulas from non-abelian cohomology theory, for nonabelian 2-cocycles (see, e.g. [14, 16, 62]). Also, as shown in [P2], the following theorem holds true.

Theorem 3.4. *Stable isomorphism classes of crossed module bundle gerbes are one to one with stable isomorphism classes of 2-cocycles denotes as* $H^1(M, H \to D)$.

Let us mention, that it is possible to develop the theory of connections and curvings on nonabelian bundle gerbes in the framework described in this section. This is also one of the results of paper [P2]. The construction is rather involved since, in contrary to the abelian case, it is not enough to consider ordinary connections of the special G-principal bundles P over $Y^{[2]}$, which enter the definition of the nonabelian bundle gerbe. However, as introduced and studied in [P2], nonabelian bundle gerbes, connections and curvings are very natural concepts also in classical differential geometry. We will give the corresponding local description in the next section. We just mention the following result of [P2], which can be proved using the partition of unity.[14]

Theorem 3.5. *On each crossed module bundle gerbe there exist a connection and a curving.*

In [P6], crossed module bundle gerbes were identified within simplicial principal bundles. We know from Sec.2.5 that a (Lie) Moore complex of length two posses a structure of a (Lie) crossed module and vice-versa, given a (Lie) crossed module $H \to D$ we reconstruct a simplicial Lie group with the Moore complex of length two. As readily seen from the description in Sec. 2.5, this simplicial Lie group is the nerve of the Lie groupoid with objects being group elements of H and morphisms being elements of D. Let us use notation $H_{(H\to D)}$ for this nerve.

Let us consider another 1-groupoid $\mathcal{C}_{\{O_\alpha\}}$, related to an open covering $\{O_\alpha\}$, described as follows. Objects are pairs (x, O_α) with $x \in O_\alpha$ and there is unique morphism $(x, O_\alpha) \to (y, O_\beta)$ iff $x = y \in O_\alpha \cap O_\beta$. Let $N\mathcal{C}_{\{O_\alpha\}}$ denote the nerve of this groupoid.

Consider a simplicial map $N\mathcal{C}_{\{O_\alpha\}} \to \overline{W}N\mathcal{C}_{(H\to D)}$. Then the maps between 1- 2- and 3-simplices give us the gerbe transition functions. We also see that the at the 0-level, the simplicial twisting τ_0 is identified with $d_{\alpha\beta}$. At the level one, τ_1 identifies with $d_{\alpha\gamma}d_{\beta\gamma}^{-1} \xrightarrow{h_{\alpha\beta\gamma}} d_{\alpha\beta}$. A similar identification can easily be done also for the equivalence data of twistings and the local stable equivalence data of bundle gebres. Hence, we can conclude that the following proposition holds.

Proposition 3.6. *Stable equivalence classes of crossed module bundle gerbes are described by homotopy classes of simplicial maps* $N\mathcal{C}_{\{O_\alpha\}} \to \overline{W}N\mathcal{C}_{(H\to D)}$*, i.e., they are the same things as equivalence classes of simplicial principal* $N\mathcal{C}_{(H\to D}$*-bundles over* $N\mathcal{C}_{\{O_\alpha\}}$)

[14]We will give a local description of the connection and curving for twisted nonabelian gerbes in the next section.

Associated to the simplicial group $N_{(H\to D)}$, we have its geometric realization $|N_{(H\to D)}|$, which is a topological group. Also associated to the classifying (simplicial) space $\overline{W}N_{(H\to D)}$, we have the topological space $|\overline{W}N_{(H\to D)}|$. Related to the above proposition, we have the following theorem P[6] (cf. also [9])

Theorem 3.7. *The space $|\overline{W}N_{(H\to D)}|$ is a classifying space of $(H\to D)$-bundle gerbes and at the same time is also a classifying space of $|N_{(H\to D)}|$-principal bundles. Hence, the stable equivalence classes of $(H\to D)$-bundle gerbes are in a one to one correspondence with $|N_{(H\to D)}|$-principal bundles.*

An example related to the string group and string structures is the crossed module $\tilde{\Omega}G\to PG$. Here, G is a (simply-connected, compact, simple) Lie group, $\tilde{\Omega}G$ is the centrally extended group of based smooth loops and PG is the based path group. The case relevant to string group (see, e.g., [21, 83, 6, 72] for models of string group) and string structures is $G = Spin(n)$ (see, e.g., [44], for a short discussion). The topological group $|NC_{\tilde{\Omega}G\to PG}|$ is a model of the String group and the corresponding principal bundles are string structures. In this case the above theorem gives a "gerby" classification of string structures.

Also, in [P6], an attempt has been made to interpret connections and curvings on bundle gerbes in simplicial terms.

3.2. **Twisted nonabelian gerbes.** In paper [P1], we developed, based on [P2], the theory of twisted nonabelian gerbes on the level of cocycles. Our aim was to generalize the twisted principal bundles and also to generalize the "inflow" mechanism, as described above for D-branes, to the case of M5-branes. The notion of a twisted 1-gerbe (2-gerbe module) can be introduced performing a similar construction as in the case of a twisted principal bundle (2.1.11). For concreteness, we assume the crossed module to be the one of the form $G\to Aut(G)$. While twisted nonabelian bundles are described by nonabelian transition functions $\{G_{ij}\}$, twisted nonabelian gerbes are described by transition functions $\{f_{ijk}, \varphi_{ij}\}$ that are respectively valued in G and in $Aut(G)$, $f_{ijk} : O_{ijk}\to G$, $\varphi_{ij} : O_{ij}\to Aut(G)$, and where the action of φ_{ij} on $U(1)$ is trivial: $\varphi_{ij}|_{U(1)} = id$. The twisted cocycle relations now read

$$\lambda_{ijkl} = f^{-1}_{ikl} f^{-1}_{ijk} \varphi_{ij}(f_{jkl}) f_{ijl}\,, \tag{3.2.1}$$

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}}\varphi_{ik}\ , \tag{3.2.2}$$

where $\{\lambda_{ijkl}\}$ is $U(1)$-valued. It is not difficult to check that $\{\lambda_{ijkl}\}$ is a Čech 3-cocycle and thus defines a 2-gerbe (without curvings). In the particular case $\lambda_{ijkl} = 1$ equations (3.2.1), (3.2.2) define a nonabelian 1-gerbe (without curvings).

One can also consider twisted gerbes with connection 1-forms: $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i)$ where $a_{ij}\in \mathrm{Lie}(G)\otimes\Omega^1(O_{ij})$, $\mathcal{A}_i \in \mathrm{Lie}(Aut(G))\otimes\Omega^1(O_i)$, and twisted gerbes with curvings:

$$(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i) \tag{3.2.3}$$

where B_i, d_{ij} are 2-forms and H_i 3-forms, all of them $\mathrm{Lie}(G)$-valued; $B_i \in \mathrm{Lie}(G)\otimes\Omega^2(O_i)$, $d_{ij}\in \mathrm{Lie}(G)\otimes\Omega^2(O_{ij})$, $H_i\in\mathrm{Lie}(G)\otimes\Omega^3(O_i)$. Before defining a twisted 1-gerbe we need to introduce some more notation. Given an element $X\in \mathrm{Lie}(Aut(G))$, we can construct a map (a 1-cocycle) $T_X : G\to \mathrm{Lie}(G)$ in the following way,

$$T_X(h) \equiv [he^{tX}(h^{-1})]\ , \tag{3.2.4}$$

where $[he^{tX}(h^{-1})]$ is the tangent vector to the curve $he^{tX}(h^{-1})$ at the point 1_G. Given a Lie($Aut(G)$)-valued form $\mathcal{A}$, we write $\mathcal{A} = \mathcal{A}^\rho X^\rho$ where $\{X^\rho\}$ is a basis of Lie($Aut(G)$). We then define $T_\mathcal{A}$ as

$$T_\mathcal{A} \equiv \mathcal{A}^\rho T_{X^\rho} \ . \tag{3.2.5}$$

We use the same notation $T_\mathcal{A}$ for the induced map on Lie(G). Now we extend this map to allow $T_\mathcal{A}$ to act on a Lie(G)-valued form $\eta = \eta^\alpha Y^\alpha$, where $\{Y^\alpha\}$ is a basis of Lie(G), by $T_\mathcal{A}(\eta) = \eta^\alpha \wedge T_\mathcal{A}(Y^\alpha)$. Also, we define

$$\mathcal{K}_i \equiv d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i \, , \tag{3.2.6}$$

$$k_{ij} \equiv da_{ij} + a_{ij} \wedge a_{ij} + T_{\mathcal{A}_i}(a_{ij}) \, . \tag{3.2.7}$$

Definition 3.8. *A twisted 1-gerbe is a set* $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$ *such that,* $\varphi_{ij}|_{U(1)} = id$, $T_{\mathcal{A}_i}|_{U(1)} = 0$,

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}}\varphi_{ik} \, , \tag{3.2.8}$$

$$\mathcal{A}_i + ad_{a_{ij}} = \varphi_{ij}\mathcal{A}_j\varphi_{ij}^{-1} + \varphi_{ij}d\varphi_{ij}^{-1} \, , \tag{3.2.9}$$

$$d_{ij} + \varphi_{ij}(d_{jk}) = f_{ijk}d_{ik}f_{ijk}^{-1} + T_{\mathcal{K}_i + ad_{B_i}}(f_{ijk}) \, , \tag{3.2.10}$$

$$\varphi_{ij}(H_j) = H_i + d\,d_{ij} + [a_{ij}, d_{ij}] + T_{\mathcal{K}_i + ad_{B_i}}(a_{ij}) - T_{\mathcal{A}_i}(d_{ij}) \, , \tag{3.2.11}$$

and such that $\mathbf{D}_H(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}) \equiv (\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ *has* $U(1)$*- and* Lie($U(1)$)*-valued elements, where*

$$\lambda_{ijkl} \equiv f_{ikl}^{-1}f_{ijk}^{-1}\varphi_{ij}(f_{jkl})f_{ijl} \, , \tag{3.2.12}$$

$$\alpha_{ijk} \equiv a_{ij} + \varphi_{ij}(a_{jk}) - f_{ijk}a_{ik}f_{ijk}^{-1} - f_{ijk}df_{ijk}^{-1} - T_{\mathcal{A}_i}(f_{ijk}) \, , \tag{3.2.13}$$

$$\beta_{ij} \equiv \varphi_{ij}(B_j) - B_i - d_{ij} + k_{ij} \, , \tag{3.2.14}$$

$$\gamma_i \equiv H_i - dB_i + T_{\mathcal{A}_i}(B_i) \, , \tag{3.2.15}$$

and where the the same notation φ_{ij} *has been used for the induced map* $\varphi_{ij} : O_{ij} \to Aut(Lie(G))$.

If there is zero on the LHS of equations (3.2.13), (3.2.14), (3.2.15) and 1 on the LHS of eq. (3.2.12), equations (3.2.8)-(3.2.15) define a nonabelian gerbe with connection and curving. A little algebra shows that in the less trivial situation we have the following proposition.

Proposition 3.9. *Assume that* λ_{ijkl} *is* $U(1)$*-valued and* α_{ijk}*,* β_{ij} *and* γ_i *are* Lie($U(1)$)*-valued, then the equations of the above definition guarantee that* $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ *is a 2-gerbe with connections and curvings; hence the name twisted 1-gerbe for the set* $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$.

We say that the nonabelian gerbe $(f_{ijk}, \varphi_{ij}, a_{ij}, d_{ij}, A_i, B_i, H_i)$ is twisted by the 2-gerbe $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$.

in the next section, twisted bundle gerbes are identified as bundle 2-gerbes with particular structure 2-crossed modules.

3.3. **Results on global worldsheet anomalies of M5-branes.** In order to describe the "inflow" mechanism for M5-branes, a slight generalization of twisted nonabelian gerbes to the case of the crossed module $\tilde{\Omega}G \to PG$ is needed. Here, G is a (simply-connected, compact, simple) Lie group, $\tilde{\Omega}G$ is the centrally extended group of based smooth loops and PG is the based path group of paths starting at the identity. The case relevant for a (six-dimensional, compact, oriented) M5-brane V embedded in an 11-dimensional spacetime spin manifold Y [87] is that of $G = E_8$ [31]. The case relevant to string group (see, e.g., [83, 6] for models of string group) and string structures is $G = Spin(n)$ (see, e.g., [45], for a short discussion).

Comparing with the case of D-branes living the 10-dimensional spacetime, there is now a "3-form" G replacing the "2-form" field B. Based on the discussion in [31], this field G together with the metric give rise to an abelian 2-gerbe with curvings, its restriction to V being referred as the Chern-Simons 2-gerbe CS. Let $[CS]$ be the corresponding Deligne class. Also, there exists an torsion element $\theta \in H^4(V, \mathbb{Z})$ [87], replacing it this situation the integral Stiefel-Whitney class W_3 from the D-brane case, with the corresponding Deligne class $[\vartheta_{ijkl}, 0, 0, 0]$. The following condition generalizing that of (2.3.1) has been proposed in paper [P1].

Conjecture 3.10. *In order for a "stack" of M5-branes to be wrapping the cycle V in Y, there should exist a twisted nonabelian gerbe $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij})$ satisfying [cf. (2.3.1)]*

$$[CS] - [\vartheta_{ijkl}, 0, 0, 0] = [\mathbf{D}_H(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij})] + [1, 0, 0, C_V]\,, \tag{3.3.1}$$

where $[1, 0, 0, C_V]$ is the trivial Deligne class associated with a global 3-form C_V.

Some arguments, based on homotopy properties of E_8, supporting this condition are given in [P1].

3.4. **Noanbelian bundle 2-gerbes.** Here we describe an approach nonbelian bundle gerbes based on 2-modules.[15] Before describing 2-crossed module bundle 2-gerbes, we need the following definition of a 2-crossed module bundle gerbe, which is a one level up generalization of a crossed module bundle. Let $(L \to M \to N)$ be a Lie 2-crossed module and $\mathcal{G}$ be an $(L \to M)$-bundle gerbe over X. From the definition of the 2-crossed module we see immediately that the maps $L \to 1$ and $\partial_2 : M \to N$ define a morphism of crossed modules $\mu : (L \overset{\partial_1}{\to} M) \to (1 \to N)$. Changing the structure crossed module using μ we obtain a $(1 \to N)$-bundle gerbe $\mu(\mathcal{G})$, which is the same thing an N-principal bundle. Hence, the following definition makes sense.

Definition 3.11. *Let $\mathcal{G}$ be an $L \to M$-bundle gerbe such that the principal bundle $\mu(\mathcal{G})$ over X is trivial with a section $\boldsymbol{n} : \mu(\mathcal{G}) \to N$. We call the pair $(\mathcal{G}, \boldsymbol{n})$ a 2-crossed module bundle gerbe.*

Let us note that $1 \to M \to N$-bundle gerbe is an $M \to N$-bundle. The following interpretation of the trivializing section $\boldsymbol{n}$ will be useful later. For the $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n}) = ((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ the trivializing section $\boldsymbol{n}$ of the left principal N-bundle $\mu(\mathcal{G})$ is the same thing as an N-valued function $\mathbf{n}$ on Y such that $\partial_2(\boldsymbol{m}) = \mathbf{n}_1 \mathbf{n}_2^{-1}$.

Obviously, we have a pullback, if $f : X \to X'$ then we put $f^*(\mathcal{G}, \boldsymbol{n}) = (f^*(\mathcal{G}), f^*\boldsymbol{n})$; this will again be a 2-crossed module bundle gerbe.

Definition 3.12. *We call two $(L \to M \to N)$-bundle gerbes $(\mathcal{G}, \boldsymbol{n})$ and $(\mathcal{G}', \boldsymbol{n}')$ over the same manifold X stably isomorphic if there exists a stable isomorphism $\boldsymbol{q} := (\mathcal{Q}, \tilde{\boldsymbol{\ell}}) : \mathcal{G} \to \mathcal{G}'$ of $(L \to M)$-bundle gerbes such that $\boldsymbol{n}'\mu(\boldsymbol{q}) = \boldsymbol{n}$ holds true for the induced isomorphism of trivial bundles $\mu(\boldsymbol{q}) : \mu(\mathcal{G}) \to \mu(\mathcal{G}')$.*

[15] For an alternative approach based on crossed squares, see [16].

Pullbacks preserve stable isomorphisms, in particular a pullback of a trivial 2-crossed module bundle gerbe is again a trivial 2-crossed module bundle gerbe.

W can go even further and introduce isomorphism of stable isomorphisms.[16]

Definition 3.13. *Let $((\mathcal{P}, Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ and $((\mathcal{P}', Y', X, \boldsymbol{\ell}'), \boldsymbol{n}')$ be two 2-crossed module bundle gerbes and $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ two stable isomorphisms between them. We call $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ isomorphic if there is an isomorphism $\underline{\boldsymbol{\ell}} : \mathcal{Q} \to \mathcal{R}$ of crossed module bundles on $\bar{Y} = Y \times_X Y'$ such that (with an obvious abuse of notation) the diagram*

$$\begin{array}{ccc} q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{Q}}} & \mathcal{Q}_1 q'^* \mathcal{P}' \mathcal{Q}_2^{-1} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow {\scriptstyle \underline{\boldsymbol{\ell}}_1 \underline{\boldsymbol{\ell}}_2^{-1}} \\ q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{R}}} & \mathcal{R}_1 q'^* \mathcal{P}' \mathcal{R}_2^{-1} \end{array} \tag{3.4.1}$$

is commutative.

Obviously, pullbacks preserve isomorphisms of stable isomorphisms.

To describe the relevant 2-cocyles, we again assume, without loss of generality, $Y = \coprod O_i$. with thw covering $\{O_i\}$ being a good one, in which case the $(L \to M \to N)$-bundle gerbe is characterized by transition functions (n_i, m_{ij}, l_{ijk}), $n_i : O_i \to N$, $m_{ij} : O_{ij} \to M$, $l_{ijk} : O_{ijk} \to L$ fulfilling 2-cocycle relations

$$n_i = \partial_2(m_{ij}) n_j$$

$$m_{ij} m_{jk} = \partial_1(l_{ijk}) m_{ik}$$

$$l_{ijk} l_{ikl} = {}^{m_{ij}} l_{jkl} l_{ijl}$$

on O_{ij}, O_{ijk} and O_{ijkl}, respectively.

In terms of 2-cocycles the stable isomorphism $(l_{ijk}, m_{ij}, n_i) \sim (l'_{ijk}, m'_{ij}, n'_i)$ is expressed by relations

$$n'_i = \partial_2(m_i) n_i \tag{3.4.2}$$

$$m'_{ij} = m_i \partial_1(l_{ij}) m_{ij} m_j^{-1} \tag{3.4.3}$$

$${}^{m_i^{-1}} l'_{ijk} = l_{ij} \, {}^{m_{ij}} l_{jk} l_{ijk} l_{ik}^{-1} \tag{3.4.4}$$

Two $(L \to M \to N)$ valued 2-cocycles related as above will be called equivalent. The corresponding set of equivalence classes will be denoted by $H^0(X, L \to M \to N)$.

Locally, two collections of stable isomorphism data (m_i, l_{ij}) and (m'_i, l'_{ij}) are isomorphic if

$$m'_i = \partial_1(l_i) m_i$$

$$l'_{ij} = l_i l_{ij} \, {}^{m_{ij}} l_j^{-1}$$

We have the following proposition proposition.

Proposition 3.14. *Stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are in a bijective correspondence with the set $H^0(X, L \to M \to N)$.*

[16] Of course could have done that already in case ordinary bundle gebres, but we really need only now.

The nice thing about $L \to M \to N$-gerbes is that they can be multiplied. Let $(\mathcal{G}, \boldsymbol{n}) = ((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ and $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = ((\tilde{P}, \tilde{\boldsymbol{m}}), Y, X, \tilde{\boldsymbol{\ell}}), \tilde{\boldsymbol{n}})$ be two 2-crossed module bundle gerbes. Let us again consider on $Y^{[2]}$ the Whitney sum $P \oplus \tilde{P}$ and introduce an equivalence relation on $P \oplus \tilde{P}$ by

$$(p._{\mathbf{n}}l, \tilde{p}) \sim_{\mathbf{n}} (p, l\tilde{p})$$

and define $\bar{P} = P._{\mathbf{n}}\tilde{P} = P \oplus \tilde{P}/\sim_{\mathbf{n}}$. We will denote an element of $P._{\mathbf{n}}\tilde{P}$ defined by the equivalence class of $(p, \tilde{p}) \in P \oplus \tilde{P}$ as $[p, \tilde{p}]_{\mathbf{n}}$ in order to distinguish it from equivalence class $[p, \tilde{p}] \in P\tilde{P}$ defined previously. Also, put

$$\bar{\boldsymbol{m}} = \boldsymbol{m}\, {}^{\mathbf{n}_2}\tilde{\boldsymbol{m}}$$

It is easy to see that $\bar{\mathcal{P}} := (\bar{P}, \bar{\boldsymbol{m}})$ is an $(L \to M)$-bundle on $Y^{[2]}$. Let us note that also $\partial_2(\bar{\boldsymbol{m}}) = \bar{\mathbf{n}}_1\bar{\mathbf{n}}_2$ on $Y^{[2]}$ with

$$\bar{\boldsymbol{n}} = \boldsymbol{n}\tilde{\boldsymbol{n}}$$

Now on $Y^{[3]}$ we do have the pullbacks $\mathcal{P}_{12}$, $\tilde{\mathcal{P}}_{12}$, $\bar{\mathcal{P}}_{12}$, etc. An element of $\bar{P}_{12}\bar{P}_{23}$ is then given by $((y_1, y_2, y_3), [[p, \tilde{p}]_{\mathbf{n}}, [p', \tilde{p}']_{\mathbf{n}}])$ with $(y_1, y_2, y_3) \in Y^{[3]}$, $p \in P$ and $\tilde{p} \in \tilde{P}$ in the respective fibres of P and $\tilde{P}$ over $(y_1, y_2) \in Y^{[2]}$, and $p' \in P$ and $\tilde{p}' \in \tilde{P}$ are in the respective fibres of P and $\tilde{P}$ over $(y_2, y_3) \in Y^{[2]}$. Finally, we define $\bar{\boldsymbol{\ell}} : \bar{P}_{12}\bar{P}_{23} \to \bar{P}_{13}$ as

$$\bar{\boldsymbol{\ell}}((y_1, y_2, y_3), [[p, \tilde{p}]_{\mathbf{n}}, [p', \tilde{p}']_{\mathbf{n}}]) := ((y_1, y_2, y_3), [\boldsymbol{\ell}([p, p'], \tilde{\boldsymbol{\ell}}[\tilde{p}, \tilde{p}']]_{\mathbf{n}})$$

Now it is a rather lengthy but straightforward check to establish the following proposition.

Proposition 3.15. $(\bar{\mathcal{G}}, \bar{\boldsymbol{n}}) := ((\bar{P}, \bar{\mathbf{m}}), Y, X, \bar{\boldsymbol{\ell}}), \bar{\boldsymbol{n}})$ *defines an $(L \to M \to N)$-bundle gerbe, the product of $(L \to M \to N)$-bundle gerbes $(\mathcal{G}, \boldsymbol{n}) = ((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ and $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = ((\tilde{P}, \tilde{\boldsymbol{m}}), Y, X, \tilde{\boldsymbol{\ell}}), \tilde{\boldsymbol{n}})$.*

The product formulas for the corresponding transition functions (2-cocycles) of the product $\bar{\mathcal{G}} = \mathcal{G}\tilde{\mathcal{G}}$ of two 2-crossed module bundles are given by

$$\bar{n}_i = n_i\tilde{n}_i$$

$$\bar{m}_{ij} = m_{ij}\, {}^{n_j}\tilde{m}_{ij}$$

$$\bar{l}_{ijk} = l_{ijk}\, {}^{m_{ik}}\{m_{jk}{}^{-1}, {}^{n_j}\tilde{m}_{ij}\}\, {}^{n_i}\tilde{l}_{ijk}$$

The inverse $(n_i, m_{ij}, l_{ijk})^{-1}$ is given by

$$(n_i^{-1}, {}^{n_j^{-1}}m_{ij}^{-1}, {}^{n_k^{-1}}\{m_{jk}^{-1}, m_{ij}^{-1}\}^{-1}\, {}^{n_i^{-1}}l_{ijk}^{-1})$$

Finally, we can define nonabelian bundle 2-gebres. For, consider again a surjective submersion $\pi : Y \to X$. Let, as before, $p_{ij} : Y^{[3]} \to Y^{[2]}$ denote the projection to the i-th and j-th component, and similarly for projections of higher fibred powers $Y^{[n]}$ of Y. Let $L \overset{\partial_1}{\to} M \overset{\partial_2}{\to} N$ be a 2-crossed module.

Definition 3.16. *A 2-crossed module bundle 2-gerbe is defined by a quintuple $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$, where $\mathfrak{G} = (\mathcal{G}, \boldsymbol{n})$ is a 2-crossed module bundle gerbe over $Y^{[2]}$,*

$$\boldsymbol{m} : \mathfrak{G}_{12}\mathfrak{G}_{23} \to \mathfrak{G}_{13}$$

is a stable isomorphism on $Y^{[3]}$ of the product $\mathfrak{G}_{12}\mathfrak{G}_{23}$ of the pullback 2-crossed module bundle gerbes $\mathfrak{G}_{12} = p_{12}^*\mathfrak{G}$ and $\mathfrak{G}_{23} = p_{23}^*\mathfrak{G}$ and the pullback 2-crossed module bundle gerbe $\mathfrak{G}_{13} = p_{13}^*\mathfrak{G}$, and

$$\boldsymbol{\ell} : \boldsymbol{m}_{124}\boldsymbol{m}_{234} \to \boldsymbol{m}_{134}\boldsymbol{m}_{123}$$

is an isomorphism of the composition of pullbacks of stable isomorphisms $p_{124}^*\boldsymbol{m}$ *and* $p_{234}^*\boldsymbol{m}$ *and the composition of pullbacks of stable isomorphisms* $p_{123}^*\boldsymbol{m}$ *and* $p_{134}^*\boldsymbol{m}$ *on* $Y^{[4]}$*. On* $Y^{[5]}$*, the isomorphism* $\boldsymbol{\ell}$ *should satisfy the obvious coherence relation*

$$\boldsymbol{\ell}_{1345}\boldsymbol{\ell}_{1235} = \boldsymbol{\ell}_{1234}\boldsymbol{\ell}_{1245}\boldsymbol{\ell}_{2345}.$$

Let us note, that Abelian bundle 2-gerbes as introduced in [80, 82, 25] are $(U(1) \to 1 \to 1)$-bundle 2-gerbes. A $(1 \to M \to M)$-bundle 2-gerbe is an $M \to N)$-bundle gerbe. Obviously, we have pullbacks. If $f : X \to X'$ is a map then we can pullback $Y \to X$ to $f^*(Y) \to X'$ with a map $\tilde{f} : f^*(Y) \to Y$ covering f. There are induced maps $\tilde{f}^{[n]} : f^*(Y)^{[n]} \to Y^{[n]}$. The pullback $f^*(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell}) := (\tilde{f}^{[2]*}\mathfrak{G}, f^*(Y), f(X), \tilde{f}^{[3]*}\boldsymbol{m}, \tilde{f}^{[4]*}\boldsymbol{\ell})$ is again an $(L \to M \to N)$-bundle 2-gerbe.

The notion of a stable isomorphism is defined as follows:

Definition 3.17. *Two 2-crossed module bundle 2-gerbes* $((\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ *and* $(\mathfrak{G}', Y', X, \boldsymbol{m}', \boldsymbol{\ell}')$ *are stably isomorphic if there exists a 2-crossed module bundle gerbe* $\mathfrak{Q} \to \bar{Y} = Y \times_X Y'$ *such that over* $\bar{Y}^{[2]}$ *the 2-crossed module bundle gerbes* $q^*\mathfrak{G}$ *and* $\mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1}$ *are stably isomorphic. Let* $\tilde{\boldsymbol{m}}$ *be the stable isomorphism* $\tilde{\boldsymbol{m}} : q^*\mathfrak{G} \to \mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1}$*. Then we ask on* $Y^{[3]}$ *(with an obvious abuse of notation) for the existence of an isomorphism* $\tilde{\boldsymbol{\ell}}$ *of stable isomorphisms*

$$\tilde{\boldsymbol{\ell}} : \boldsymbol{m}'\tilde{\boldsymbol{m}}_{23}\tilde{\boldsymbol{m}}_{12} \to \tilde{\boldsymbol{m}}_{13}\boldsymbol{m}$$

fulfilling on $Y^{[4]}$

$$\boldsymbol{\ell}_{1234}\tilde{\boldsymbol{\ell}}_{124}\tilde{\boldsymbol{\ell}}_{234} = \tilde{\boldsymbol{\ell}}_{134}\tilde{\boldsymbol{\ell}}_{123}\boldsymbol{\ell}'_{1234}$$

Here q *and* q' *are projections onto first and second factor of* $\bar{Y} = Y \times_X Y'$ *and* $\mathfrak{Q}_1$ *and* $\mathfrak{Q}_2$ *are the pullbacks of* $\mathfrak{Q}$ *to* $\bar{Y}^{[2]}$ *under respective projections* p_1*,* p_2 *form* $\bar{Y}^{[2]}$ *to* $\bar{Y}$*, etc.*

Correspondingly we have a stable isomorphism of stable isomorphisms.

Definition 3.18. *Let* $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ *and* $(\mathfrak{G}', Y', X, \boldsymbol{m}', \boldsymbol{\ell}')$ *be two 2-crossed module bundle 2-gerbes and* $(\mathfrak{Q}, \tilde{\boldsymbol{m}}_{\mathfrak{Q}}, \tilde{\boldsymbol{\ell}}_{\mathfrak{Q}})$ *and* $(\mathfrak{R}, \tilde{\boldsymbol{m}}_{\mathfrak{R}}, \tilde{\boldsymbol{\ell}}_{\mathfrak{R}})$ *two stable isomorphisms between them. We call these two stable isomorphisms stably isomorphic if there is a stable isomorphism* $\underline{\boldsymbol{m}} : \mathfrak{Q} \to \mathfrak{R}$ *of 2-crossed module bundles on* $\bar{Y} = Y \times_X Y'$ *such that (with an obvious abuse of notation) the diagram*

$$\begin{array}{ccc} q^*\mathfrak{G} & \xrightarrow{\tilde{\boldsymbol{m}}_{\mathfrak{Q}}} & \mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow {\scriptstyle \underline{\boldsymbol{m}}_1\underline{\boldsymbol{m}}_2{}^{-1}} \\ q^*\mathfrak{G} & \xrightarrow{\tilde{\boldsymbol{m}}_{\mathfrak{R}}} & \mathfrak{R}_1 q'^*\mathfrak{G}'\mathfrak{R}_2^{-1} \end{array}$$

commutes up to an isomorphism of stable isomorphisms

$$\underline{\boldsymbol{\ell}} : \tilde{\boldsymbol{m}}_{\mathfrak{Q}}\underline{\boldsymbol{m}}_1\underline{\boldsymbol{m}}_2{}^{-1} \to \tilde{\boldsymbol{m}}_{\mathfrak{R}}$$

on $\bar{Y}^{[2]}$*, fulfilling on* $\bar{Y}^{[3]}$

$$\tilde{\boldsymbol{\ell}}_{\mathfrak{Q}}\underline{\boldsymbol{\ell}}_{13} = \underline{\boldsymbol{\ell}}_{12}\underline{\boldsymbol{\ell}}_{23}\tilde{\boldsymbol{\ell}}_{\mathfrak{R}}$$

The local description follows below. Without loss of generality, we can again assume the case when $Y = \coprod O_i$ and the covering $\{O_i\}$ is a good one, in which case the $(L \to M \to N)$-bundle gerbe can be described by transition functions $(n_{ij}, m_{ijk}, l_{ijkl})$ $n_{ij} : O_{ij} \to N$, $m_{ijk} : O_{ijk} \to M$ and $l_{ijkl} : O_{ijkl} \to L$ satisfying

$$\begin{gathered} n_{ij}n_{jk} = \partial_2(m_{ijk})n_{ik} \\ m_{ijk}m_{ikl} = \partial_1(l_{ijkl})\, {}^{n_{ij}}m_{jkl}m_{ijl} \\ l_{ijkl}\ {}^{n_{ij}m_{jkl}}(l_{ijlm})\, {}^{n_{ij}}l_{jklm} = {}^{m_{ijk}}l_{iklm}\{m_{ijk},\, {}^{n_{ik}}m_{klm}\}\ {}^{n_{ij}n_{jk}m_{klm}}(l_{ijkm}) \end{gathered} \tag{3.4.5}$$

We shall not give explicit formulas relating transition functions (3-cocycles) of two stably isomorphic 2-crossed module bundle 2-gerbes. We introduce the notation $H^1(X, L \to M \to N)$ for the equivalence classes of 3-cocycles. We just give the formulas for transition functions $(n_{ij}, m_{ijk}, l_{ijkl})$ of a trivial 2-crossed module bundle 2-gerbe:

$$\begin{gathered} n_{ik} = n_i^{-1}\partial_2(m_{ij})n_j \\ {}^{n_i}m_{ijl} = \partial_1(l_{ijk}^{-1})\, m_{ij}m_{jk}m_{ik}^{-1} \\ {}^{n_i}l_{ijkl} = \ {}^{n_i m_{ijk}}(l_{ikl}^{-1})l_{ijk}^{-1}\, {}^{m_{ij}}l_{jkl}\{m_{ij},\, {}^{n_j}m_{jkl}\}\ {}^{n_i n_{ij} m_{jkl}}(l_{ijl}) \end{gathered} \tag{3.4.6}$$

We introduce the notation $H^1(X, L \to M \to N)$ for the corresponding equivalence classes of 3-cocycles.

Hence, we can summarize the discussion in the following proposition.

Theorem 3.19. *Stable isomorphism classes of $(L \to M \to N)$-bundle 2-gerbes are in a bijective correspondence with the set $H^1(X, L \to M \to N)$.*

We finish with the following observation concerning twisted bundle gerbes introduced earlier in previous section. These are 2-crossed module bundle 2-gerbes with structure 2-crossed module $U(1) \to L \xrightarrow{\delta} M$, associated with a central extension of M by $U(1)$.

3.5. **Differentiation of classifying spaces $\overline{W}G$.** Here, foolowing [P7], we state the result on 1-jets of classifying spaces of simplicial Lie groups in the language of differential graded Lie algebras (DGLA's). This allows us to make a relation to the Quillen's adjunctions between simplicial Lie algebras and DGLA's. Let G be a simplicial Lie group G with the Moore complex of length k. Let $\mathfrak{g}$ denote the corresponding simplicial Lie algebra. What we have is a k-term DGLA $L = \oplus_{n=0}^{k} L_{-n}$ with components in degrees $0, -1, \ldots -k$, given by $L_{-n} = N\mathfrak{g}_n$. The differentials $d_n : N\mathfrak{g}_n \to N\mathfrak{g}_{n+1}$ are given by the restrictions $d_n = \partial_0|_{N\mathfrak{g}_n}$ of the zeroth face maps, i.e by the differentials δ_n of the Moore complex $N\mathfrak{g}$, i.e, for $x_n \in N\mathfrak{g}_n$

$$d_n x_n = \delta_n x_n. \tag{3.5.1}$$

The only nonzero brackets are the binary brackets. The nonzero binary brackets are determined by the following prescription:

The bracket $N\mathfrak{g}_0 \times N\mathfrak{g}_0 \to N\mathfrak{g}_0$ is just the Lie bracket on $N\mathfrak{g}_0$, i.e for $x_0 \in N\mathfrak{g}_0$ and $y_0 \in N\mathfrak{g}_0$

$$[x_0, y_0]. \tag{3.5.2}$$

The brackets $N\mathfrak{g}_0 \times N\mathfrak{g}_n \to N\mathfrak{g}_n$: $(x, y) \mapsto [x_0, x_n] = -[x_n, x_0]$ are given by the action of $N\mathfrak{g}_0$ on $N\mathfrak{g}_n$

$$[x_0, x_n] = -[x_n, x_0] = {}^{x_0}x_n. \tag{3.5.3}$$

The bracket $N\mathfrak{g}_{n_1} \times N\mathfrak{g}_{n_2} \to N\mathfrak{g}_n$ with $n = n_1 + n_2$, for n_1 and n_2 nonzero, is described as follows: For $x_{n_1} \in N\mathfrak{g}_{n_1}$ and $x_{n_2} \in N\mathfrak{g}_{n_2}$

(3.5.4)
$$[x_{n_1}, x_{n_2}] = \sum_{(\alpha,\beta)\in\bar{P}(n_1,n_2)} \pm f_{\alpha,\beta}(x_{n_1}, x_{n_2}) + (-1)^{(n_1+1)(n_2+1)} \sum_{(\alpha,\beta)\in\bar{P}(n_2,n_1)} \pm f_{\alpha,\beta}(x_{n_2}, x_{n_1})$$

The $\pm$ sign is given by the product of parity of $n_1(n_2+1)$ and the parity of the shuffle defined by the pair $(\alpha, \beta) \in \bar{P}(n_1, n_2)$. Here $\bar{P}(n_1, n_2) \subset \bar{P}(n)$ denotes the subset of $P(n)$ consisting of those pairs $(\alpha, \beta) \in \bar{P}(n)$, for which $n - \sharp\alpha = n_1, n - \sharp\beta = n_2$.

Let us now consider an arbitrary simplicial Lie algebra $\mathfrak{g}$ with Moore complex of length k. Associated to $\mathfrak{g}$ we have the (unique) simplicial group G integrating it, such that all its components are simply connected. Therefore, starting with $\mathfrak{g}$, we can consider the 1-jet of G. Correspondingly, we have the following theorem:

Proposition 3.20. *Let $\mathfrak{g}$ be a simplicial Lie group with Moore complex $N\mathfrak{g}$ of length k. Then $N\mathfrak{g}$ or becomes a DGLA. The differential and the binary brackets are explicitly given by formulas (3.5.1-3.5.4). This DGLA structure on $N\mathfrak{g}$ is the same one as described by Quillen's construction in Proposition 4.4 of* [70].

3.6. **Classifying topoi of topological bicategories.** In paper [P3], some established results [63] on classifying spaces and topoi were put together in a new way, with consequences for bicategories.

Let $\mathbb{B}$ be a topological bicategory. In analogy with the case of a topological category we have the following definition.

Definition 3.21. *The classifying topos $\mathcal{B}\mathbb{B}$ of the topological bicategory $\mathbb{B}$ is defined as $Sh(N\mathbb{B})$, the topos of sheaves on the Duskin nerve $N\mathbb{B}$. Similarly, the classifying space $B\mathbb{B}$ of a topological bicategory $\mathbb{B}$ is the geometric realization $|N\mathbb{B}|$ of its nerve $N\mathbb{B}$.*

Also, for a topological bicategory $\mathbb{B}$ write $Lin(X, \mathbb{B})$ for the category of linear orders over X equipped with an augmentation $\mathrm{aug}\colon NL \to N\mathbb{B}$.

Definition 3.22. *An object E of $Lin(X, \mathbb{B})$ is called a Duskin principal $\mathbb{B}$- bundle. We call two Duskin principal $\mathbb{B}$-bundles E_0 and E_1 on X concordant if there exists a Duskin principal $\mathbb{B}$-bundle on $X \times [0, 1]$ such that we have the equivalences $E_0 \simeq i_0^*(E)$ and $E_0 \simeq i_1^*(E)$ under the obvious inclusions $i_0, i_1\colon X \hookrightarrow X \times [0, 1]$.*

We can consider a linear order L as a locally trivial bicategory (with only trivial 2-morphisms). In this case the Duskin nerve of L coincides with the ordinary nerve of L which justifies the same notation NL for both nerves. Therefore, an augmentation $NL \to N\mathbb{B}$ is the same, by the nerve construction, as a continuous normal lax functor $L \to \mathbb{B}$. Similarly to the case of topological category (2.7.3) we have the following "classifying" property of the classifying topos $\mathcal{B}\mathbb{B}$.

Theorem 3.23. *For a topological bicategory $\mathbb{B}$ and a topological space X there is a natural equivalence of categories*

(3.6.1)
$$\mathrm{Hom}(Sh(X), \mathcal{B}\mathbb{B}) \simeq Lin(X, \mathbb{B}).$$

On homotopy classes of topos morphisms we have the natural bijection

(3.6.2)
$$[Sh(X), \mathcal{B}\mathbb{B}] \cong Lin_c(X, \mathbb{B}).$$

Let us recall that the topological bicategory $\mathbb{B}$ is locally contractible if its spaces of objects, 1-arrows and 2-arrows are locally contractible. The "classifying" property of the classifying space $B\mathbb{B}$ now follows as a corollary.

Corollary 3.24. *For a locally contractible bicategory $\mathbb{B}$ and a CW-complex X there is a natural bijection*

$$[X, B\mathbb{B}] \cong Lin_c(X, \mathbb{B}). \tag{3.6.3}$$

If, in addition, the topological bicategory $\mathbb{B}$ is a so-called "good" one [3] then the above is true also if we use, instead of the thickened geometric realization of the nerve, the geometric realization of the underlying simplicial set. The case of a good topological bicategory, as well as the sufficient conditions for a bicategory being a good one, are discussed in [3]. Thus, as a corollary we have a slight generalization of the result of Baas, Bökstedt and Kro [3]. As shown in [P3], similar results apply also to other types of nerves of bicategories, such as the Lack-Paoli [55], Tansamani [84] and Simpson [79] nerve.

3.7. **Noncommutative gerbes and quantization of twisted Poisson structures.** Noncommutative gebres were introduced in [P4]. Let us consider any covering $\{U_\alpha\}$ (not necessarily a good one) of a manifold M. Here, comparing to Sec. 2.8, we switch from upper Latin to lower Greek indices to label the local patches. The reason for the different notation will become clear soon. Consider each local patch equipped with its own star product $\star_\alpha$ the deformation quantization of a local Poisson structure θ_α. We assume that on each double intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the local Poisson structures θ_α and θ_β are related similarly as in the previous section via some integral closed two form $F_{\beta\alpha}$, which is the curvature of a line bundle $L_{\beta\alpha} \in \mathrm{Pic}(U_{\alpha\beta})$

$$\theta_\alpha = \theta_\beta(1 + \hbar F_{\beta\alpha}\theta_\beta)^{-1}. \tag{3.7.1}$$

Let us now consider a good covering $U^i_{\alpha\beta}$ of each double intersection. $U_\alpha \cap U_\beta$ [17] with a noncommutative line bundle $\mathcal{L}_{\beta\alpha} = \{G^{ij}_{\alpha\beta}, \mathcal{D}^i_{\alpha\beta}, \star_\alpha\}$

$$G^{ij}_{\alpha\beta} \star_\alpha G^{jk}_{\alpha\beta} = G^{ik}_{\alpha\beta}\ , \qquad G^{ii}_{\alpha\beta} = 1\ , \tag{3.7.2}$$

$$\mathcal{D}^i_{\alpha\beta}(f) \star_\alpha G^{ij}_{\alpha\beta} = G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(f) \tag{3.7.3}$$

and

$$\mathcal{D}^i_{\alpha\beta}(f \star_\beta g) = \mathcal{D}^i_{\alpha\beta}(f) \star_\alpha \mathcal{D}^i_{\alpha\beta}(g)\ . \tag{3.7.4}$$

The opposite order of indices labeling the line bundles and the corresponding transition functions and equivalences simply reflects a choice of convention. As in the previous section the order of indices of $\mathcal{L}_{\alpha\beta}$ indicates the bimodule structure of the corresponding space of sections, whereas the order of Greek indices on G's and D's indicates the star product in which the objects multiply. The product always goes with the first index of the multiplied objects.

This and the following definitions (including their justification and consistency) are among the main results of [P4].

Definition 3.25. *A noncommutative gerbe is characterised by the following axioms:*

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[17] At this point we use so called hypercovers. Hypercovers are defined, e.g., in [2] A thorough treatment with non-abelian cohomology classes is given in [15]. The hypercovers are not necessary if M is paracompact.

Axiom 1: *$\mathcal{L}_{\alpha\beta} = \{G^{ij}_{\beta\alpha}, \mathcal{D}^i_{\beta\alpha}, \star_\beta\}$ and $\mathcal{L}_{\beta\alpha} = \{G^{ij}_{\alpha\beta}, \mathcal{D}^i_{\alpha\beta}, \star_\alpha\}$ are related as follows*

$$\{G^{ij}_{\beta\alpha}, \mathcal{D}^i_{\beta\alpha}, \star_\beta\} = \{(\mathcal{D}^j_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta}), (\mathcal{D}^i_{\alpha\beta})^{-1}, \star_\beta\} \tag{3.7.5}$$

i.e. $\mathcal{L}_{\alpha\beta} = \mathcal{L}^{-1}_{\beta\alpha}$. (Notice also that $(\mathcal{D}^j_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta}) = (\mathcal{D}^i_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta})$.)

Axiom 2: *On the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ the tensor product $\mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ is equivalent to the line bundle $\mathcal{L}_{\gamma\alpha}$. Explicitly*

$$G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(G^{ij}_{\beta\gamma}) = \Lambda^i_{\alpha\beta\gamma} \star_\alpha G^{ij}_{\alpha\gamma} \star_\alpha (\Lambda^j)^{-1}_{\alpha\beta\gamma} \ , \tag{3.7.6}$$

$$\mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma} = \mathrm{Ad}_{\star_\alpha} \Lambda^i_{\alpha\beta\gamma} \circ \mathcal{D}^i_{\alpha\gamma} \ . \tag{3.7.7}$$

Axiom 3: *On the quadruple intersection $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$*

$$\Lambda^i_{\alpha\beta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\delta} = \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta} \ , \tag{3.7.8}$$

$$\Lambda^i_{\alpha\beta\gamma} = (\Lambda^i_{\alpha\gamma\beta})^{-1} \qquad \text{and} \qquad \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\alpha}) = \Lambda^i_{\alpha\beta\gamma} \ . \tag{3.7.9}$$

With slight abuse of notation we have used Latin indices $\{i, j, ..\}$ to label both the good coverings of the intersection of the local patches U_α and the corresponding transition functions of the consistent restrictions of line bundles $\mathcal{L}_{\alpha\beta}$ to these intersections. A short comment on the consistency of Axiom 3 is in order. Let us define

$$\mathcal{D}^i_{\alpha\beta\gamma} = \mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma} \circ \mathcal{D}^i_{\gamma\alpha} \ . \tag{3.7.10}$$

Then it is easy to see that

$$\mathcal{D}^i_{\alpha\beta\gamma} \circ \mathcal{D}^i_{\alpha\gamma\delta} \circ \mathcal{D}^i_{\alpha\delta\beta} = \mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma\delta} \circ \mathcal{D}^i_{\beta\alpha} \ . \tag{3.7.11}$$

In view of (3.7.7) this implies that

$$\Lambda^i_{\alpha\beta\gamma\delta} \equiv \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta} \star_\alpha \Lambda^i_{\alpha\delta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\beta}$$

is central. Using this and the associativity of $\star_\alpha$ together with (3.7.6) applied to the triple tensor product $\mathcal{L}_{\delta\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ transition functions

$$G^{ij}_{\alpha\beta\gamma} \equiv G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(G^{ij}_{\beta\gamma}) \star_\alpha \mathcal{D}^j_{\alpha\beta}(\mathcal{D}^j_{\beta\gamma}(G^{ij}_{\gamma\delta})) \tag{3.7.12}$$

reveals that $\Lambda^i_{\alpha\beta\gamma\delta}$ is independent of i. It is therefore consistent to set $\Lambda^i_{\alpha\beta\gamma\delta}$ equal to 1. A similar consistency check works also for (3.7.9). If we replace all noncommutative line bundles $\mathcal{L}_{\alpha\beta}$ in Axioms 1-3 by equivalent ones, we get by definition an equivalent noncommutative gerbe.

Also, let us note that there is a natural (contravariant) connection on a noncommutative gerbe.

Using Axiom 2 one can show that the product bundle

$$\mathcal{L}_{\alpha\beta\gamma\delta} = \mathcal{L}_{\alpha\beta\gamma} \otimes \mathcal{L}_{\alpha\gamma\delta} \otimes \mathcal{L}_{\alpha\delta\beta} \otimes \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\delta\gamma} \otimes \mathcal{L}_{\beta\alpha} \tag{3.7.13}$$

is trivial: it has transition functions $G^{ij}_{\alpha\beta\gamma\delta} = 1$ and maps $\mathcal{D}^i_{\alpha\beta\gamma\delta} = \mathrm{id}$. The constant unit section is thus well defined on this bundle. On $\mathcal{L}_{\alpha\beta\gamma\delta}$ we also have the section $(\Lambda^i_{\alpha\beta\gamma\delta})$. Axiom 3 implies $(\Lambda^i_{\alpha\beta\gamma\delta})$ to be the unit section. If two of the indices α, β, γ, δ are equal, triviality of the bundle $\mathcal{L}_{\alpha\beta\gamma\delta}$ implies (3.7.9). Using for example the first relation in (3.7.9) one can show that (3.7.8) written in the form $\mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta} \star_\alpha \Lambda^i_{\alpha\delta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\beta} = 1$ is invariant under cyclic permutations of any three of the four factors appearing on the l.h.s..

If we now assume that all line bundles $\mathcal{L}_{\beta\alpha}$ are trivial (this is for example the case when $\{U_\alpha\}$ is a good covering) then $F_{\alpha\beta} = da_{\alpha\beta}$ for each $U_\alpha \cap U_\beta$ and

$$G^{ij}_{\alpha\beta} = (H^i_{\alpha\beta})^{-1} \star_\alpha H^j_{\alpha\beta}$$

$$\mathcal{D}_{\alpha\beta} = \mathrm{Ad}_{\star_\alpha} H^i_{\alpha\beta} \circ \mathcal{D}^i_{\alpha\beta} \ .$$

It then easily follows that

$$\Lambda_{\alpha\beta\gamma} \equiv H^i_{\alpha\beta} \star_\alpha \mathcal{D}^i_{\alpha\beta}(H^i_{\beta\gamma}) \star_\alpha \mathcal{D}^i_{\alpha\beta}\mathcal{D}^i_{\beta\gamma}(H^i_{\gamma\alpha}) \star_\alpha \Lambda^i_{\alpha\beta\gamma} \tag{3.7.14}$$

defines a global function on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$. $\Lambda_{\alpha\beta\gamma}$ is just the quotient of the two sections $\left(H^i_{\alpha\beta} \star_\alpha \mathcal{D}^i_{\alpha\beta}(H^i_{\beta\gamma}) \star_\alpha \mathcal{D}^i_{\alpha\beta}\mathcal{D}^i_{\beta\gamma}(H^i_{\gamma\alpha})\right)^{-1}$ and $\Lambda^i_{\alpha\beta\gamma}$ of the triple tensor product $\mathcal{L}_{\alpha\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$. On the quadruple overlap $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ it satisfies conditions analogous to (3.7.8) and (3.7.9)

$$\Lambda_{\alpha\beta\gamma} \star_\alpha \Lambda_{\alpha\gamma\delta} = \mathcal{D}_{\alpha\beta}(\Lambda_{\beta\gamma\delta}) \star_\alpha \Lambda_{\alpha\beta\delta} \ , \tag{3.7.15}$$

$$\Lambda_{\alpha\beta\gamma} = (\Lambda_{\alpha\gamma\beta})^{-1} \qquad \text{and} \qquad \mathcal{D}_{\alpha\beta}(\Lambda_{\beta\gamma\alpha}) = \Lambda_{\alpha\beta\gamma} \ . \tag{3.7.16}$$

Also

$$\mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = \mathrm{Ad}_{\star_\alpha} \Lambda_{\alpha\beta\gamma} \ . \tag{3.7.17}$$

Definition 3.26. *We say that formulas (3.7.15)-(3.7.17) define a noncommutative 2-cocycle, which we take as a definition of a noncommutative gerbe in the case of a good covering $\{U_\alpha\}$.*

From now on we shall consider only good coverings. A noncommutative gerbe defined by $\Lambda_{\alpha\beta\gamma}$ and $\mathcal{D}_{\alpha\beta}$ is said to be trivial if there exists a global star product $\star$ on M and a collection of “twisted” transition functions $G_{\alpha\beta}$ defined on each overlap $U_\alpha \cap U_\beta$ and a collection $\mathcal{D}_\alpha$ of local equivalences between the global product $\star$ and the local products $\star_\alpha$

$$\mathcal{D}_\alpha(f) \star \mathcal{D}_\alpha(g) = \mathcal{D}_\alpha(f \star_\alpha g)$$

satisfying the following two conditions:

$$G_{\alpha\beta} \star G_{\beta\gamma} = \mathcal{D}_\alpha(\Lambda_{\alpha\beta\gamma}) \star G_{\alpha\gamma} \tag{3.7.18}$$

and

$$\mathrm{Ad}_\star G_{\alpha\beta} \circ \mathcal{D}_\beta = \mathcal{D}_\alpha \circ \mathcal{D}_{\alpha\beta} \ . \tag{3.7.19}$$

Locally, every noncommutative gerbe is trivial as is easily seen from (3.7.15), (3.7.16) and (3.7.17) by fixing the index α.

We conclude this section with the following remark concerning the role of local functions $\Lambda_{\alpha\beta\gamma}$ and $\mathcal{D}_{\alpha\beta}$ satisfying relations (3.7.15)-(3.7.17). These represent a honest non-abelian 2-cocycle, as defined for example in [15]. It follows from the discussion of section 2, that each $\mathcal{D}_{\alpha\beta}$ defines an equivalence, in the sense of deformation quantization, of star products $\star_\alpha$ and $\star_\beta$ on $U_\alpha \cap U_\beta$. The non-triviality of the non-abelian 2-cocycle (3.7.15)-(3.7.17) can therefore be seen as an obstruction to gluing the collection of local star products $\{\star_\alpha\}$, i.e., the collection of local rings $C^\infty(U_\alpha)[[\hbar]]$, into a global one. We also mention that in [50] a 2-cocycle similar to that of (3.7.15)-(3.7.17) represents an obstruction to gluing together certain local rings appearing in quantization of contact manifolds.

Now we can turn our attention to deformation quntization of twisted Poisson structures. Let $H \in H^3(M, \mathbb{Z})$ be a closed integral three form on M. Such a form is known to define a gerbe on M. We can find a good covering $\{U_\alpha\}$ and local potentials B_α with $H = dB_\alpha$ for

H. On $U_\alpha \cap U_\beta$ the difference of the two local potentials $B_\alpha - B_\beta$ is closed and hence exact: $B_\alpha - B_\beta = da_{\alpha\beta}$. On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = -i\lambda_{\alpha\beta\gamma} d\lambda^{-1}_{\alpha\beta\gamma}. \tag{3.7.20}$$

The collection of local functions $\{\lambda_{\alpha\beta\gamma}\}$ represents a gerbe.

Let us also assume the existence of a formal antisymmetric bivector field $\theta = \theta^{(0)} + \hbar\theta^{(1)} + \dots$ on M such that

$$[\theta, \theta] = \hbar\, \theta^* H\ , \tag{3.7.21}$$

where $[\ ,\ ]$ is the Schouten-Nijenhuis bracket and θ^* denotes the natural map sending n-forms to n-vector fields by "using θ to raise indices". Explicitly, in local coordinates, $\theta^* H^{ijk} = \theta^{im}\theta^{jn}\theta^{ko}H_{mno}$. We call θ a Poisson structure twisted by H [76, 68, 51]. On each U_α we can introduce a local formal Poisson structure $\theta_\alpha = \theta(1 - \hbar B_\alpha \theta)^{-1}$, $[\theta_\alpha, \theta_\alpha] = 0$. The Poisson structures θ_α and θ_β are related on the intersection $U_\alpha \cap U_\beta$ as in (3.7.1)

$$\theta_\alpha = \theta_\beta(1 + \hbar F_{\beta\alpha}\theta_\beta)^{-1}\ , \tag{3.7.22}$$

with an exact $F_{\beta\alpha} = da_{\beta\alpha}$. Now we can use Kontsevich's formality [52] to obtain local star products $\star_\alpha$ and to construct for each intersection $U_\alpha \cap U_\beta$ the corresponding equivalence maps $\mathcal{D}_{\alpha\beta}$. See [47, 48] for an explicit formula for the equivalence maps. According to our discussion in the previous section these $\mathcal{D}_{\alpha\beta}$, supplemented by trivial transition functions, define a collection of trivial line bundles $\mathcal{L}_{\beta\alpha}$. On each triple intersection we then have

$$\mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = \mathrm{Ad}_{\star_\alpha} \Lambda_{\alpha\beta\gamma}\ . \tag{3.7.23}$$

It follows from the discussion after formula (3.7.9) that the collection of local functions $\{\Lambda_{\alpha\beta\gamma}\}$ represents a noncommutative gerbe (a deformation quantization of the classical gerbe represented by $\{\lambda_{\alpha\beta\gamma}\}$) if each of the central functions $\Lambda_{\alpha\beta\gamma\delta}$ introduced there can be chosen to be equal to 1. The following proposition relies on results contained in [77, section 5][18] and [53] that this is really the case.

Proposition 3.27. *Local functions $\{\Lambda_{\alpha\beta\gamma}\}$ and local maps $\mathcal{D}_{\alpha\beta}$ indeed represent a noncommutative gerbe, a deformation quantization of the classical gerbe represented by $\{\lambda_{\alpha\beta\gamma}\}$.*

As mentioned at the end of the previous section, the non-triviality of the non-abelian 2-cocycle (3.7.15)-(3.7.17) can be seen as an obstruction to gluing the collection of local star products $\{\star_\alpha\}$, i.e., the collection of local rings $C^\infty(U_\alpha)[[\hbar]]$, into a global one. Hence, in the context of this section, this obstruction comes as a deformation quantization of the classical obstruction to gluing together local formal Poisson structures $\{\ ,\ \}_\alpha$ into a global one.

Finally, let us mention [60, 18, 19] for some closely related work.

4. Conclusions

The author of the thesis believes that the results described in the thesis fit well into the recent trends of exploring possible generalizations of the symmetry principles underlying our present understanding of quantum field theory. Since the higher and noncommutative structures discussed in this thesis arose naturally in both mathematics (algebraic geometry, homological algebra, category theory) and physics (quantum field theory, string theory), there is a good chance that their study will help us in our attempts of identifying and understanding the fundamental mathematical structure of quantum field and string theory.

[18]Here, quantization of twisted Poisson structures is discussed independently.

Concerning more specific questions, the author believes that the results on higher gauge theories comply with the recent interest in categorification of various mathematics structures. Indeed, gerbes can be thought as categorifications of bundles, their structure crossed modules are 2-group categorifiction of ordinary groups. Abelian gerbes already proved to be relevant to interrelated problems in mathematics (string structures, twisted K-theory, elliptic cohomology) and physics (global worldsheet anomalies and holonomy of D-branes). One can speculate, that further extended objects in string theory and/or their charges can be described using non-abelian higher gauge theories, in analogy with D-branes. Similarly, the results on nocommuatative gebres fit well into recent interest in noncommutative geometry in quantum field theory and string theory (e.g., D-branes) [75]. Moreover, they are honest non-abelian gerbes.

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5. Appendix; Papers [P1] - [P7]

Gerbes, M5-brane anomalies and **E_8** gauge theory

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Gerbes, M5-brane anomalies and E_8 gauge theory

Paolo Aschieri[abc] **and Branislav Jurčo**[bc]

[a] *Dipartimento di Scienze e Tecnologie Avanzate, Universitá del Piemonte Orientale and INFN*
Via Bellini 25/G, 15100 Alessandria, Italy
[b] *Max-Planck-Institut für Physik*
Föhringer Ring 6, D-80805, München, Germany
[c] *Sektion Physik, Universität München*
Theresienstr. 37, D-80333 München, Germany
E-mail: aschieri@theorie.physik.uni-muenchen.de, jurco@theorie.physik.uni-muenchen.de

Abstract: Abelian gerbes and twisted bundles describe the topology of the NS 3-form gauge field strength H. We review how they have been usefully applied to study and resolve global anomalies in open string theory. Abelian 2-gerbes and twisted nonabelian gerbes describe the topology of the 4-form field strength G of M-theory. We show that twisted nonabelian gerbes are relevant in the study and resolution of global anomalies of multiple coinciding M5-branes. Global anomalies for one M5-brane have been studied by Witten and by Diaconescu, Freed and Moore. The structure and the differential geometry of twisted nonabelian gerbes (i.e. modules for 2-gerbes) is defined and studied. The nonabelian 2-form gauge potential living on multiple coinciding M5-branes arises as curving (curvature) of twisted nonabelian gerbes. The nonabelian group is in general $\tilde{\Omega}E_8$, the central extension of the E_8 loop group. The twist is in general necessary to cancel global anomalies due to the nontriviality of the 11-dimensional 4-form field strength G and due to the possible torsion present in the cycles the M5-branes wrap. Our description of M5-branes global anomalies leads to the D4-branes one upon compactification of M-theory to Type IIA theory.

Keywords: Chern-Simons Theories, M-Theory, Anomalies in Field and String Theories, Differential and Algebraic Geometry.

Contents

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1. Introduction

The topology of gauge theories with 2-form gauge potentials is a fascinating subject both from the physics and the mathematics perspectives. Consider for example in string theory a stack of n coinciding D-branes, they usually form a $\mathrm{U}(n)$ vector bundle, however when there is a topologically nontrivial NS B-field background, this is generally no more the case. Cancellation of global anomalies requires the $\mathrm{U}(n)$ bundle to be *twisted* in order to accommodate for the nontrivial topology of the B-field. Thus the study of D-brane charges in the presence of nontrivial backgrounds leads to generalize the usual notion of fibre bundle. A twisted $\mathrm{U}(n)$ bundle has transition functions G_{ij} that satisfy the twisted cocycle relations $G_{ij}G_{jk}G_{ki} = \lambda_{ijk}$, where λ_{ijk} are $\mathrm{U}(1)$ valued functions. It follows that λ_{ijk} satisfy the cocycle relations $\lambda_{ijk}\lambda_{jkl}^{-1}\lambda_{ikl}\lambda_{ijl}^{-1} = 1$, this is the characteristic property of the transition functions of a bundle gerbe. In short, bundle gerbes, or simply gerbes, are a higher version of line bundles, and the gauge potential for these structures is the 2-form B in the same way as the connection 1-form A is the gauge potential associated with line bundles. As we have sketched, associated with a gerbe we have a twisted bundle (also called gerbe module). The fact that a stack of D-branes in a nontrivial background forms a twisted bundle was studied in [1], confirmed using worldsheet global anomalies in [2] and further generalized using twisted K-theory in [3]–[5].

The structure of gauge theories with 3-form gauge potentials is similarly fascinating and rich, the corresponding geometrical structure is that of a 2-gerbe (if the 4-form field-strength is integral). A main motivation for studying these structures is provided by the 3-form C-field of 11-dimensional supergravity. In particular it is interesting to study which M5-brane configurations are compatible with a topologically nontrivial C-field. By requiring the vanishing of global anomalies, topological aspects of the partition function of a single M5-brane have been studied in [6, 7], and in the presence of a nontrivial background in [7] and in [8, 9]. We refer to [10] for the underlying mathematical structures.

In this note we define twisted nonabelian gerbes, these are a higher version of twisted bundles, we study their properties and show that they are associated with abelian 2-gerbes (they are 2-gerbe modules). Using global anomalies cancellation arguments we then see that the geometrical structure underlying a stack of M5-branes is indeed that of a twisted nonabelian gerbe. The associated 2-gerbe is constructed from the C-field data. The twist is necessary in order to accommodate for the nontrivial topology of the C-field. A twisted nonabelian gerbe is (partly) characterized by a nonabelian 2-form gauge potential, in the case of a single M5-brane this becomes the abelian chiral gauge potential of the self-dual 3-form on the M5-brane. Moreover, an M5-brane becomes a D4-brane upon the appropriate compactification of M-theory to Type IIA string theory, and correspondingly the 2-gerbe becomes a gerbe, and the twisted gerbe becomes a twisted bundle. A particular case is when the 2-gerbe is trivial, then the stack of M5-branes gives a nonabelian gerbe. This corresponds in Type IIA to a stack of D-branes forming a bundle. The differential geometry of nonabelian gerbes has been studied at length in [11] (using algebraic geometry) and in [12] (using differential geometry).

It may sound strange to discuss the physical (string theory) relevance of twisted nonabelian gerbes before studying that of the easier case of nonabelian gerbes. This route is however dictated by anomaly cancellation arguments and by the strong analogy between M5-branes in M-theory (with open M2-branes ending on them) and D-branes in Type IIA (with open strings ending on them). Indeed, as we emphasise in section 3, the study of global open string anomalies in the presence of a closed string NS B-field background is enough to conclude that there must be a U(1) gauge potential on a D-brane, and that therefore a D-brane configuration is associated with a line bundle. Even more, if the NS 3-form field H is torsion class (i.e. it is trivial in real De Rham cohomology but not in integer cohomology) then we are obliged to consider coinciding branes forming twisted $\mathrm{U}(n)$ bundles, and this implies that $\mathrm{U}(N)$ bundles also arise for coinciding branes in the previous case where B is torsionless. Similarly, nontrivial backgrounds in M-theory, giving rise to torsion classes, force us to describe the configuration of a stack of M5-branes via twisted nonabelian gerbes. Nonabelian gerbes are then recovered as a special case.

The knowledge of the topology of coinciding M5-branes is a first step toward the formulation of the dynamics of these nonabelian gauge fields. Indeed the full structure of a (twisted) nonabelian gerbe is considerably richer than just a local nonabelian 2-form gauge potential, for example we also have a local 1-form gauge potential and its corresponding

2-form field strength. It is using all these gauge potentials and their gauge transformations (analyzed in section 4) that one can attack the problem of constructing an action describing the dynamics of a stack of M5-branes.

A prominent role in nonabelian gerbes in M-theory is played by the E_8 group. Indeed, for topological considerations, the 2-form gauge potential can be always thought to be valued in ΩE_8, the E_8 loop group, and for twisted nonabelian gerbes in $\tilde{\Omega} E_8$, the central extension of the E_8 loop group. This is so because of the simple homotopy structure of E_8. This corresponds to the fact, exploited in [15], and recalled at the end of this paper, that in Type IIA theory a stack of D-branes gives in general a twisted $\tilde{\Omega} E_8$ bundle, so that at least for topological considerations we can consider the gauge potential to be $\tilde{\Omega} E_8$ valued. This adds to the growing evidence that E_8 plays a main role in M-theory. For example the subtle topology of the 3-form C-field is conveniently described considering it as a composite field, obtained via E_8 valued 1-form gauge potentials, roughly we have $C \sim CS(A_i) = \mathrm{Tr}(A_i dA_i) + \frac{2}{3}\,\mathrm{Tr}(A_i^3)$. Gauge theory with E_8 gauge group has been used in [13], and then, for manifolds with boundary, in [8] in order to globally define the Chern-Simons topological term $\Phi(C) \sim \int \frac{1}{6} C \wedge G \wedge G$. It has been shown in [14] to nicely confirm the K-theory formalism in Type IIA theory upon compactification of M-theory. For further work in this direction see for example [15, 16]. Another instance where E_8 gauge theory appears in M-theory is in Hořava-Witten [17]. Finally it is well known that exceptional groups duality symmetries appear after compactification of supergravity theories, and it has been proposed that these symmetries follow from a hidden E_{11} symmetry of 11-dimensional supergravity [18].

It is interesting to notice that the E_8 formulation of the C-field is not the only one, in particular in [19] another formulation related to $OSp(1|32)$ gauge theory was studied, and is currently investigated, see for example [20]. It might well be that a relation between these two different descriptions can lead to a further understanding of the possibly dynamical role of the E_8 gauge theory.

This paper is organized as follows. Section 2 is a review of abelian gerbes. There are many ways of introducing these structures (see [21] for a recent and nice introduction to the subject), we choose a minimal approach, mainly focusing on Deligne cohomology classes [22], these are a refinement of integral cohomology. Gerbes are then a geometric realizations of Deligne classes. They are equivalent to differential characters, also called Cheeger-Simons characters [23], in this case it is the holonomy of these higher order bundles that is emphasized.

Global worldsheet anomalies of open strings ending on D-branes where studied in [24]; in section 3 we use gerbes in order to construct anomaly free worldsheet actions of strings ending on multiple coinciding D-branes. We mainly follow [2] and [5], but also uncover some details (especially about gauge transformations), simplify the presentation when torsion is present, and emphasize that the gauge fields on the branes can be inferred just from the NS B-field in the bulk.

Section 4 defines and studies twisted nonabelian gerbes. We then give an explicit construction using the loop group of E_8; we also see that any twisted nonabelian gerbe can be realized by lifting an E_8 bundle.

Section 5 uses twisted nonabelian gerbes in order to describe a stack of M5-branes.

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2. Gerbes

2.1 Abelian 1-Gerbes

Line bundles can be described using transition functions. Consider a cover $\{O_i\}$ of the base space M, then a line bundle is given by a set of U(1) valued smooth transition functions $\{\lambda_{ij}\}$ that satisfy $\lambda_{ij} = \lambda_{ji}^{-1}$ and that on triple overlaps $O_{ijk} = O_i \cap O_j \cap O_k$ satisfy the cocycle condition

$$\lambda_{ij}\lambda_{jk} = \lambda_{ik}\,. \tag{2.1}$$

In the same spirit, a connection on a line bundle is a set of one-forms $\{\alpha_i\}$ on O_i such that on double overlaps $O_{ij} = O_i \cap O_j$,

$$\alpha_i = \alpha_j + \lambda_{ij} d\lambda_{ij}^{-1}\,. \tag{2.2}$$

Actually we are interested only in isomorphic classes of line bundles with connection, indeed all physical observables are obtained from Wilson loops, and these cannot distinguish between a bundle with connection (λ_{ij}, α_i) and an equivalent one $(\lambda'_{ij}, \alpha'_i)$, that by definition satisfies

$$\lambda'_{ij} = \tilde{\lambda}_i \lambda_{ij} \tilde{\lambda}_j^{-1}\,, \qquad \alpha'_i = \alpha_i + \tilde{\lambda}_i d\tilde{\lambda}_i^{-1}\,, \tag{2.3}$$

where $\tilde{\lambda}_i$ are U(1) valued smooth functions on O_i. We are thus led to consider the class $[\lambda_{ij}, \alpha_i]$, of all couples (λ_{ij}, α_i) that satisfy (2.2), and where $(\lambda_{ij}, \alpha_i) \sim (\lambda'_{ij}, \alpha'_i)$ iff (2.3) holds. The space of all these classes (called Deligne classes) is the Deligne cohomology group $H^1(M, \mathcal{D}^1)$. Wilson loops for the Deligne class $[\lambda_{ij}, \alpha_i]$ are given in subsection 2.4.

Similarly we can consider the Deligne class $[\lambda_{ijk}, \alpha_{ij}, \beta_i] \in H^2(M, \mathcal{D}^2)$ where now $\lambda_{ijk} : O_{ijk} \to \mathrm{U}(1)$ is totally antisymmetric in its indices, $\lambda_{ijk} = \lambda_{jik}^{-1} = \lambda_{kij}$ etc., and it satisfies the cocycle condition on triple overlaps

$$\lambda_{ijk}\lambda_{jkl}^{-1}\lambda_{ikl}\lambda_{ijl}^{-1} = 1\,, \tag{2.4}$$

while the connection one-form $\{\alpha_{ij}\}$ satisfies on O_{ijk}

$$\alpha_{ij} + \alpha_{jk} + \alpha_{ki} + \lambda_{ijk} d\lambda_{ijk}^{-1} = 0 \tag{2.5}$$

and the curving two-form $\{\beta_i\}$ satisfies on O_{ij}

$$\beta_i - \beta_j + d\alpha_{ij} = 0\,. \tag{2.6}$$

The triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$ gives the zero Deligne class if

$$(\lambda_{ijk}, \alpha_{ij}, \beta_i) = D(\tilde{\lambda}_{ij}, \tilde{\alpha}_i)\,, \tag{2.7}$$

where D is the Deligne coboundary operator, and $\tilde{\lambda}_{ij} : O_{ij} \to \mathrm{U}(1)$ are smooth functions and $\tilde{\alpha}_i$ are smooth one-forms on O_i. Explicitly (2.7) reads[1]

$$\begin{aligned} \lambda_{ijk} &= \tilde{\lambda}_{ik}\tilde{\lambda}_{jk}^{-1}\tilde{\lambda}_{ij}^{-1} & (2.8)\\ \alpha_{ij} &= -\tilde{\alpha}_i + \tilde{\alpha}_j + \tilde{\lambda}_{ij} d\tilde{\lambda}_{ij}^{-1}\,, & (2.9)\\ \beta_i &= d\tilde{\alpha}_i\,. & (2.10) \end{aligned}$$

[1]The Deligne coboundary operator is $D = \pm\delta + d$, the sign factor in front of the Čech coboundary operator depends on the degree of the form D acts on; it insures $D^2 = 0$.

There is also a geometric structure associated with the triple $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$, it is that of (abelian) gerbe [22] or bundle gerbe [33]. Equivalence classes of gerbes with connection and curving are in 1-1 correspondence with Deligne classes, and with abuse of language we say that $[\mathcal{G}] = [\lambda_{ijk}, \alpha_{ij}, \beta_i]$ is the equivalence class of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$. The holonomy of an abelian gerbe is given in subsection 2.4. As before, gauge invariant (physical) quantities can be obtained from the holonomy (Wilson surface), and this depends only on the equivalence class of the gerbe.

Gerbes are also called 1-gerbes in order to distinguish them from 2-gerbes.

2.2 Abelian 2-Gerbes

Following the previous section, for the purposes of this paper, we understand under an abelian 2-gerbe with curvings on M a quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$. Here $\lambda_{ijkl} : O_{ijkl} \equiv O_i \cap O_j \cap O_k \cap O_l \to \mathrm{U}(1)$ is a 2-Čech cocycle

$$\lambda_{ijkl}\lambda_{ijlm}\lambda_{jklm} = \lambda_{iklm}\lambda_{ijkm} \qquad \text{on} \quad O_{ijklm}\,, \tag{2.11}$$

and λ_{ijkl} is totally antisymmetric, $\lambda_{ijkl} = \lambda_{jikl}^{-1}$ etc.. Next $\alpha_{ijk} \in \Omega^1(O_{ijk})$, $\beta_{ij} \in \Omega^2(O_{ij})$ and $\gamma_i \in \Omega^3(O_i)$ are a collection of local one, two, and three-forms totally antisymmetric in their respective indices and subject to the following relations:

$$\alpha_{ijk} + \alpha_{ikl} - \alpha_{ijl} - \alpha_{jkl} = \lambda_{ijkl} d\lambda_{ijkl}^{-1} \qquad \text{on} \quad O_{ijk}\,, \tag{2.12}$$

$$\beta_{ij} + \beta_{jk} - \beta_{ik} = d\alpha_{ijk} \qquad \text{on} \quad O_{ijk}\,, \tag{2.13}$$

$$\gamma_i - \gamma_j = d\beta_{ij} \qquad \text{on} \quad O_{ij}\,. \tag{2.14}$$

The equivalence class of the 2-gerbe with curvings $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ is given by the Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$, where the quadruple $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ represents the zero Deligne class if it is of the form

$$\lambda_{ijkl} = \tilde{\lambda}_{ijl}^{-1}\tilde{\lambda}_{jkl}^{-1}\tilde{\lambda}_{ijk}\tilde{\lambda}_{ikl}\,, \tag{2.15}$$

$$\alpha_{ijk} = \tilde{\alpha}_{ij} + \tilde{\alpha}_{jk} + \tilde{\alpha}_{ki} + \tilde{\lambda}_{ijk} d\tilde{\lambda}_{ijk}^{-1}\,, \tag{2.16}$$

$$\beta_{ij} = \tilde{\beta}_i - \tilde{\beta}_j + d\tilde{\alpha}_{ij}\,, \tag{2.17}$$

$$\gamma_i = d\tilde{\beta}_i\,. \tag{2.18}$$

The above equations are summarized in the expression

$$(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i) = D\left(\tilde{\lambda}_{ijk}, \tilde{\alpha}_{ij}, \tilde{\beta}_i\right), \tag{2.19}$$

where D is the Deligne coboundary operator, $\tilde{\lambda}_{ijk}$ are $\mathrm{U}(1)$ valued functions on O_{ijk} and $\tilde{\alpha}_{ij}, \tilde{\beta}_i$ are respectively 1- and 2-forms on O_{ij} and on O_i.

The Deligne class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i] \in H^3(M, \mathcal{D}^3)$ (actually the cocycle $\{\lambda_{ijkl}\}$) defines an integral class $\xi \in H^4(M, \mathbb{Z})$; this is the characteristic class of the 2-gerbe. Moreover $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$ defines the closed integral 4-form

$$\frac{1}{2\pi i} G = \frac{1}{2\pi i} d\gamma_i\,. \tag{2.20}$$

The 4-form G is a representative of $\xi_{\mathbb{R}}$: the image of the integral class ξ in real de Rham cohomology.

In the same way as abelian 2-gerbes were described above we can define abelian $n-1$-gerbes with curvings using Deligne cohomology classes in $H^n(M, \mathcal{D}^n)$. Correspondingly we have characteristic classes in $H^{n+1}(M, \mathbb{Z})$. The case $n = 1$ gives equivalence classes of line bundles with connections, and in this case the characteristic class is the Chern class of the line bundle.

The relation between a Deligne class and its characteristic class leads to the following exact sequence ([22], see [25] for an elementary proof)

$$0 \to \Omega^n_{\mathbb{Z}}(M) \to \Omega^n(M) \to H^n(M, \mathcal{D}^n) \to H^{n+1}(M, \mathbb{Z}) \to 0\,, \tag{2.21}$$

where $\Omega^n_{\mathbb{Z}}(M)$ is the space of closed integral (i.e. whose integration on n-cycles is an integer) n-forms on M. We also have the exact sequence (see for example [26])

$$0 \to H^n(M, \mathrm{U}(1)) \to H^n(M, \mathcal{D}^n) \to \Omega^{n+1}_{\mathbb{Z}}(M) \to 0\,, \tag{2.22}$$

where, as in (2.20), $G \in \Omega^{n+1}_{\mathbb{Z}}(M)$ is the curvature of the $n-1$-gerbe $(\lambda_{i_1,\ldots i_{n+1}}, \alpha_{i_1\ldots i_n}, \ldots,$ $\gamma_i)$.

It is a result of [23], that $H^n(M, \mathcal{D}^n)$ is isomorphic to the space of differential characters $\check{H}^{n+1}(M)$ (Cheeger-Simons characters). An element of $\check{H}^{n+1}(M)$ is a pair (h, F) where h is a homomorphism from the group of n-cycles $\mathbb{Z}_n(M)$ to U(1) and F is an $(n+1)$-form. The pair (h, F) is such that for any $(n+1)$-chain $\mu \in C_{n+1}(M)$ with boundary $\partial\mu$ the following relation holds

$$h(\partial\mu) = \exp\left(\int_\mu F\right). \tag{2.23}$$

The isomorphism with Deligne cohomology groups is given essentially via the holonomy of an $n-1$-gerbe, and $F = G$.

2.3 Special cases

An important example of a 2-gerbe is obtained from an element θ belonging to the torsion subgroup $H^4_{tors}(M, \mathbb{Z})$ of $H^4(M, \mathbb{Z})$. Every torsion element θ is the image of an element $\vartheta \in H^3(M, \mathbb{Q}/\mathbb{Z})$ via the Bockstein homomorphism $\beta : H^3(M, \mathbb{Q}/\mathbb{Z}) \to H^4(M, \mathbb{Z})$ associated with the exact sequence $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. As a Čech cocycle ϑ can be represented as a $\mathbb{Q}/\mathbb{Z}$ valued cocycle $\{\vartheta_{ijkl}\}$. Now $\{\vartheta_{ijkl}\}$ can be thought of as a Čech cocycle with values in U(1) valued functions on O_{ijkl}, we have of course $d\vartheta_{ijkl} = 0$ and we can thus consider the 2-gerbe $(\vartheta_{ijkl}, 0, 0, 0)$. The equivalence class of this 2-gerbe is the Deligne class

$$[\vartheta_{ijkl}, 0, 0, 0]\,; \tag{2.24}$$

it depends only on $\theta = \beta(\vartheta)$, the characteristic class of this Deligne class.

Given a globally defined 3-form $C \in \Omega^3(M)$ we can construct the Deligne class

$$[1, 0, 0, C|_{O_i}]\,. \tag{2.25}$$

Accordingly with (2.21) it has trivial characteristic class and it is the zero Deligne class iff $C \in \Omega^3_{\mathbb{Z}}(M)$. Indeed in this case we can write $(1, 0, 0, C|_{O_i}) = D(\lambda_{ijk}, \alpha_{ij}, \beta_i)$ where $(\lambda_{ijk}, \alpha_{ij}, \beta_i)$ is a 1-gerbe with curvature C. Following [5], Deligne classes like $[1, 0, 0, C|_{O_i}]$

will be called *trivial*. Notice that a trivial characteristic class is the same as a zero characteristic class while a trivial Deligne class is usually not a zero Deligne class.

These two constructions obviously also apply to n-gerbes. In particular we have the torsion 1-gerbe class

$$[\vartheta_{ijk}, 0, 0]\,, \tag{2.26}$$

associated with the element $\theta \in H^3_{tors}(M, \mathbb{Z})$. Similarly we have the trivial 1-gerbe class $[1, 0, B|_{O_i}]$ associated with a globally defined 2-form $B \in \Omega^2(M)$.

Another family of 2-gerbes, the so-called Chern-Simons 2-gerbes, comes from a principal G-bundle $P_G \to M$. Its characteristic class is the first Pontryagin class of P_G, $p_1 \in H^4(M, \mathbb{Z})$. If G is connected, simply connected and simple and if A is a connection on P_G, given locally by a collection of Lie(G)-valued one-forms A_i, then the image of p_1 in real cohomology equals the cohomology class of $\mathrm{Tr}\, F^2$, and we can identify the local three-forms γ_i with the Chern-Simons forms $CS(A_i)$,

$$\gamma_i = \mathrm{Tr}(A_i dA_i) + \frac{2}{3}\,\mathrm{Tr}(A_i^3)\,.$$

The two-forms β_{ij} and α_{ijkl} and the Čech cocycle λ_{ijkl} can in principle be obtained by solving descent equations [27] (see also [28]). We will denote the 2-gerbe obtained this way as $CS(p_1)$.

Notice that if $\dim M \leq 15$, then there is a one to one correspondence between $H^4(M, \mathbb{Z})$ and isomorphism classes of principal E_8 bundles on M, see [29] for an elementary proof. This follows from the fact that the first nontrivial homotopy group of E_8, except $\pi_3(E_8) = \mathbb{Z}$, is $\pi_{15}(E_8)$. We then have that up to the 14th-skeleton E_8 is homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}, 3)$ (defined as the space whose only nontrivial homotopy group is $\pi_3(K(\mathbb{Z}, 3)) = \mathbb{Z}$). Similarly up to the 15th-skeleton we have $BE_8 \sim K(\mathbb{Z}, 4)$, where BE_8 is the classifying space of E_8 principal bundles. For the homotopy classes of maps from M to E_8 it then follows that $[M, E_8] = [M, K(\mathbb{Z}, 3)] = H^3(M, \mathbb{Z})$ if $\dim M \leq 14$, and similarly $\{$Equivalence classes of E_8 bundles on $M\} = [M, BE_8] = [M, K(\mathbb{Z}, 4)] = H^4(M, \mathbb{Z})$ if $\dim M \leq 15$. Therefore, corresponding to an element $a \in H^4(M, \mathbb{Z})$ we have an E_8 principal bundle $P(a) \to M$ with $p_1(P(a)) = a$ and picking a connection A on $P(a)$ we have a Deligne class, the Chern-Simons 2-gerbe $CS(a)$, with a being its characteristic class.

As in this paper we are mainly concerned with 2-gerbes associated with 5-branes embedded in 11-dimesional spacetime it is worth to recall also the homotopy groups of the groups G_2, Spin_n, F_4, E_6 and E_7. Except π_3 which is of course $\mathbb{Z}$ in each case, the first nonzero ones are $\pi_6(G_2)$, $\pi_7(\mathrm{Spin}_n)$ where $n \geq 7$, $\pi_8(F_4)$, $\pi_8(E_6)$ and $\pi_{11}(E_7)$. So in the case of a 5-brane, with 6-dimensional worldsheet M, we can replace E_8 bundles with G_2, Spin_n where $n \geq 7$, F_4, E_6 or E_7 bundles in the above discussion.

2.4 Holonomy of line bundles, 1-gerbes and 2-gerbes

The holonomy can be associated with any Deligne class. It gives the corresponding differential character for cycles that arise as images of triangulated manifolds. Here we just collect formulas in the case of 0-, 1- and 2-gerbes [30], see also [5].

Line bundles. The holonomy of $[\lambda_{ij}, \alpha_i]$ around a loop $\varsigma : S \to M$ can be calculated splitting S in sufficiently small arches b and corresponding vertices v, such that each $\varsigma(b)$ is completely contained in an open O_i of the cover $\{O_i\}$ of M. The index i depends on the arch e, we thus call it $\rho(e)$, and write $\varsigma(e) \subset O_{\rho(e)}$; we also associate an index $\rho(v)$ with every vertex v and write $\varsigma(v) \subset O_{\rho(v)}$. We then have

$$\mathrm{hol}(\varsigma) = \prod_e \exp \int_e \varsigma^* \alpha_{\rho(e)} \prod_{v \subset e} \lambda^{\sigma_{e,v}}_{\rho(e)\rho(v)}(\varsigma(v)) \,, \tag{2.27}$$

where $\sigma_{e,v} = 1$ if v is the final point of the oriented arch e, and -1 if it is the initial point. Note that the holonomy depends only on the class $[\lambda_{ij}, \alpha_i]$ and not on the representative (λ_{ij}, α_i) or on the splitting of S or the choice of the index map ρ. Of course if the loop is the boundary of a disk, i.e., if $\zeta : D \to M$ is such that $\zeta|_{\partial D} = \varsigma$, then $\mathrm{hol}(\varsigma) = e^{\int_D \zeta^* F}$.

1-gerbes. We now consider the map $\zeta : \Sigma \to M$ where Σ is a 2-cycle that we triangulate with faces, edges and vertices, denoted f, e and v. The faces f inherit the orientation of Σ, we also choose an orientation for the edges e. It is always possible to choose a triangulation subordinate to the open cover O_i of M and define an index map ρ which maps faces, edges and vertices, into the index set of the covering of M in a way that $\zeta(f) \in O_{\rho(f)}$, etc.. The holonomy of the class $[\lambda_{ijk}, \alpha_{ij}, \beta_i]$ is then

$$\mathrm{hol}(\zeta) = \prod_f \exp \int_f \zeta^* \beta_{\rho(f)} \prod_{e \subset f} \exp \int_e \zeta^* \alpha_{\rho(f)\rho(e)} \prod_{v \subset e \subset f} \lambda_{\rho(f)\rho(e)\rho(v)}(\zeta(v)) \,, \tag{2.28}$$

where it is understood that $\alpha_{\rho(f)\rho(e)}$ appears with the opposite sign if f and e have incompatible orientations. Similarly the inverse of $\lambda_{\rho(f)\rho(e)\rho(v)}$ appears if f and e have incompatible orientations or if v is not the final vertex of e. As before the holonomy depends only on the equivalence class of the gerbe and not on the chosen representative gerbe. It is also independent from the choice of triangulation, of index map ρ and of orientation of the edges.

2-gerbes. We now consider the map $\xi : \Gamma \to M$ where Γ is a 3-cycle. We triangulate it with tetrahedrons, faces, edges and vertices, denoted t, f, e and v. The triangulation is chosen to be subordinate to the open cover $\{O_i\}$ of M. The index map ρ now maps tetrahedrons, faces etc. into the index set of the covering $\{O_i\}$. The formula for the holonomy of the class $[\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i]$ is

$$\mathrm{hol}(\xi) = \prod_t \exp \int_t \xi^* \gamma_{\rho(t)} \prod_{f \subset t} \exp \int_f \xi^* \beta_{\rho(t)\rho(f)} \times \tag{2.29}$$

$$\times \prod_{e \subset f \subset t} \exp \int_e \xi^* \alpha_{\rho(t)\rho(f)\rho(e)} \prod_{v \subset e \subset f \subset t} \lambda_{\rho(t)\rho(f)\rho(e)\rho(v)}(\xi(v)) \,. \tag{2.30}$$

3. Open strings worldsheet anomalies, 1-gerbes and twisted bundles

It is commonly said that the low energy effective action of a stack of n branes is a $\mathrm{U}(n)$ Yang-Mills theory. Therefore n coinciding branes are associated with a $\mathrm{U}(n)$ bundle. More in general, in the presence of a nontrivial H field we do not have a $\mathrm{U}(n)$ bundle, rather a twisted one, i.e. we have a PU(n) bundle that cannot be lifted to a U(n) one, i.e. the PU(n) transition functions g_{ij} cannot be lifted to U(n) transition functions G_{ij} such that under the projection $\mathrm{U}(n) \to \mathrm{PU}(n)$ we have $G_{ij} \to g_{ij}$ and such that the cocycle condition $G_{ij}G_{jk}G_{ki} = 1$ holds. The twisting is necessary in order to cancel global worldsheet anomalies for open strings ending on D-branes. In this section we study this mechanism. Consider for simplicity the path integral for open bosonic string theory in the presence of D-branes wrapping a cycle Q inside spacetime M and with a given closed string background metric g and NS three form H. We have

$$\int \mathcal{D}\zeta \; e^{i\int_\Sigma L_{\rm NG}} \, e^{\int_\Sigma \zeta^* d^{-1}H} \, \mathrm{Tr}\,\mathrm{hol}^{-1}_{\partial\Sigma}(\zeta^* A) \tag{3.1}$$

here $\zeta : \Sigma \to M$ are maps from the open string worldsheet Σ to the target spacetime M such that the image of the boundary $\partial\Sigma$ lives on Q, we denote by $\Sigma_Q(M)$ this space, $L_{\rm NG}$ is the Nambu-Goto lagrangian, $\int_\Sigma \zeta^* d^{-1}H$ is locally given by $\int_\Sigma \zeta^* B = \int_\Sigma \varepsilon^{\alpha\beta} B_{MN}\partial_\alpha X^M \partial_\beta X^N$ and is the topological coupling of the open string to the NS field, and $\mathrm{Tr}\ \mathrm{hol}_\gamma(\zeta^* A)$ is the trace of the holonomy (Wilson loop) around the boundary $\partial\Sigma$ of the nonabelian gauge field A that lives on the n coincident D-branes wrapping Q. Now, while the exponential of the Nambu-Goto action is a well defined function from $\Sigma_Q(M)$ to the circle U(1), the other U(1) factor $e^{\int_\Sigma \zeta^* B}$ is more problematic because only $H = dB$ is globally defined, while $B =$ "$d^{-1}H$" is defined only locally. In order to define this term we need to know not only the integral cohomology of H but the full Deligne class $[\mathcal{G}] = [\lambda_{ijk}, \alpha_{ij}, \beta_i]$ whose curvature is H. We call the gerbe $\mathcal{G}|_Q$ trivial if its class $[\mathcal{G}|_Q]$ is trivial i.e. if (cf. (2.25)): 1) H restricted to Q is cohomologically trivial, that is it exists a B_Q globally defined on Q such that

$$H|_Q = dB_Q \,, \tag{3.2}$$

and 2) the characteristic class ξ of the gerbe is trivial ($H|_Q$ is trivial also in integer cohomology). It turns out that if $\mathcal{G}|_Q$ is trivial, then defining

$$e^{\int_\Sigma \zeta^* d^{-1}H} \equiv \mathrm{hol}(\Sigma\sharp D) e^{\int_D \tilde\zeta^* B_Q} \tag{3.3}$$

we have a well defined function on $\Sigma_Q(M)$. Here D is the disk and $\tilde\zeta : D \to Q$ is such that the boundary of $\tilde\zeta(D)$ coincides with the boundary of $\zeta(\Sigma)$ (we have assumed Q simply connected and $\partial\Sigma$ a single loop). Moreover $\mathrm{hol}(\Sigma\sharp D) \equiv \mathrm{hol}(\zeta\sharp\tilde\zeta)$ is the holonomy of the closed surface $\zeta(\Sigma)\sharp\tilde\zeta(D)$ obtained by gluing together $\zeta(\Sigma)$ and $\tilde\zeta(D)$ (and thus in particular it is obtained by changing the orientation of D).

The two terms $\mathrm{hol}(\Sigma\sharp D)$ and $e^{\int_D \tilde\zeta^* B}$ depend on $\tilde\zeta : D \to Q$ and are not functions on $\Sigma_Q(M)$ but respectively sections of a U(1) (or line) bundle $\partial^{-1}\mathcal{L}^{-1}_{[\mathcal{G}|_Q]}$ on $\Sigma_Q(M)$ and of the opposite bundle $\partial^{-1}\mathcal{L}_{[\mathcal{G}|_Q]}$ on $\Sigma_Q(M)$ so that indeed their product is a well defined function on $\Sigma_Q(M)$.[2] The bundle $\partial^{-1}\mathcal{L}_{[\mathcal{G}|_Q]} \to \Sigma_Q(M)$ is constructed from the 1-gerbe

[2] A section of a canonically trivial bundle such as $\mathcal{L}^{-1}\mathcal{L} \to \Sigma_Q(M)$ is automatically a global function on $\Sigma_Q(M)$ because $\mathcal{L}^{-1}\mathcal{L} \to \Sigma_Q(M)$ has the canonical section 1 (locally 1 is the product of an arbitrary section s^{-1} of $\mathcal{L}^{-1}$ and of the corresponding section s of $\mathcal{L}$) and two global sections define a U(1) function on the base space

class. Without entering this construction (described after eq. (3.7)) we can directly see that expression (3.3) is a well defined function on $\Sigma_Q(M)$ by showing its independence from the choice of the map $\tilde{\zeta}$. Given another map $\tilde{\zeta}'$ we have

$$\mathrm{hol}(\zeta\sharp\tilde{\zeta})/\mathrm{hol}(\zeta\sharp\tilde{\zeta}') = \mathrm{hol}(\tilde{\zeta}'\sharp\tilde{\zeta}) = e^{\int_D \tilde{\zeta}'^* B_Q - \int_D \tilde{\zeta}^* B_Q}\,, \tag{3.4}$$

where the first equality is the holonomy gluing property and the last equality holds because the integral of B_Q on $\tilde{\zeta}'\sharp\tilde{\zeta}$ equals the holonomy of the gerbe since B_Q gives a gerbe $(1, 0, B_Q)$ on Q equivalent to $\mathcal{G}|_Q$: $[1, 0, B_Q] = [\mathcal{G}|_Q]$.

Expression (3.3) depends on the equivalence class of the initial gerbe $\mathcal{G}$ and also on B_Q, not just on $[1, 0, B_Q]$. Had we chosen a different 2-form B'_Q such that $[1, 0, B'_Q] = [\mathcal{G}|_Q] = [1, 0, B_Q]$, then the result would have differed by the phase

$$e^{\int_D \tilde{\zeta}^*(B'_Q - B_Q)}\,, \tag{3.5}$$

where $\frac{1}{2\pi i}\omega \equiv \frac{1}{2\pi i}(B'_Q - B_Q)$ is a closed integral 2-form, recall (2.22). In order to absorb this extra phase (this gauge transformation) we have to consider the last term in (3.1): $\mathrm{Tr}\ \mathrm{hol}_{\partial\Sigma}(\zeta^* A)$. This is a well defined U(1)-valued function on $\Sigma_Q(M)$ and A is a true $\mathrm{U}(n)$ connection on a nonabelian bundle on Q, with Tr the trace in the fundamental of $\mathrm{U}(n)$. Under the gauge transformation $B_Q \to B'_Q = B_Q + \omega$ we have to transform accordingly the $\mathrm{U}(n)$ bundle in order to compensate for the phase factor (3.5). This is obtained considering the new $\mathrm{U}(n)$ bundle with curvature $F' = F + \omega$ obtained by tensoring the initial $\mathrm{U}(N)$ bundle on Q with the U(1) bundle on Q defined by the closed 2-form ω (the definition of this U(1) bundle is unique since we have considered Q simply connected). If we consider just one D-brane we recover the gauge invariance of the total U(1) field $B_Q - F$; the gauge transformations locally read $B_Q \to B_Q + d\Lambda$ and $A \to A + \Lambda$.

In conclusion using anomaly cancellation we have seen that if the open strings couple to the B field, then their ends must couple to a U(1) gauge field A. So far there is no requirement for nonabelian gauge fields.

The situation is more involved if $\mathcal{G}|_Q$ has torsion, i.e. if the three form H restricted to Q is cohomologically trivial, but the characteristic class of the gerbe is nontrivial. In this case (3.3) is not well defined because $[1, 0, B_Q] \neq [\mathcal{G}|_Q]$. However any torsion gerbe can be obtained form a lifting gerbe, i.e. from a gerbe that describes the obstruction of lifting a $\mathrm{PU}(n)$ bundle to a $\mathrm{U}(n)$ one (with appropriate n). We now describe this lifting gerbe and the associated twisted $\mathrm{U}(n)$ bundle. Let $P \to M$ be a $\mathrm{PU}(n)$ bundle and consider the exact sequence $\mathrm{U}(1) \to \mathrm{U}(n) \xrightarrow{\pi} \mathrm{PU}(n)$. Consider an open cover $\{U_\alpha\}$ of $\mathrm{PU}(n)$ with sections $s^\alpha : U_\alpha \subset \mathrm{PU}(n) \to \mathrm{U}(n)$. Consider also a good cover $\{O_i\}$ of M such that each transition function g_{ij} of $P \to M$ has image contained in a U_i (this is always doable, we also fix a map from the couples of indices (i, j) to the α indices). Let $G_{ij} = s^\alpha(g_{ij})$, these are $\mathrm{U}(n)$ valued functions and satisfy:

$$G_{ik} G_{jk}^{-1} G_{ij}^{-1} = \lambda_{ijk}\,,$$

where λ_{ijk} is U(1) valued as is easily seen applying the projection π and using the cocycle relation for the g_{ij} transition functions. We say that G_{ij} are the transition functions for a $\mathrm{U}(n)$ twisted bundle and that the lifting gerbe is defined by the twist λ_{ijk}. It is indeed easy to check that the λ_{ijk} satisfy the cocycle condition on quadruple overlaps O_{ijkl}. A connection for a twisted bundle is a set of 1-forms A_i such that $\alpha_{ij} \equiv -A_i + G_{ij}A_jG_{ij}^{-1} + G_{ij}dG_{ij}^{-1}$ is a connection for the corresponding gerbe (in particular $\pi_* A$ is a connection on the initial $\mathrm{PU}(n)$ bundle P). We restate this construction this way: consider the couple (G_{ij}, A_i), and define

$$\begin{aligned}\mathbf{D}(G_{ij}, A_i) &\equiv \left((\delta G)_{ijk}, (\delta A)_{ij} + G_{ij}dG_{ij}^{-1}, \frac{1}{n}\operatorname{Tr} dA_i \right) \\ &= \left(G_{ik}G_{jk}^{-1}G_{ij}^{-1}, -A_i + G_{ij}A_jG_{ij}^{-1} + G_{ij}dG_{ij}^{-1}, \frac{1}{n}\operatorname{Tr} dA_i \right). \end{aligned} \tag{3.6}$$

If this triple has abelian entries then it defines a gerbe, and (G_{ij}, A_i) is called a twisted bundle. We also say that the twisted bundle (G_{ij}, A_i) is twisted by the gerbe $\mathbf{D}(G_{ij}, A_i)$. Notice that the nonabelian $\mathbf{D}$ operation becomes the abelian Deligne coboundary operator D if $n = 1$ in $\mathrm{U}(n)$ (cf. (2.7)).

More in general, if $\mathcal{G}|_Q = (\lambda_{ijk}, \alpha_{ij}, \beta_i)|_Q$, is torsion then it follows from the results in [31] that one can always find a twisted bundle (G_{ij}, A_i) such that

$$(\lambda_{ijk}, \alpha_{ij}, \beta_i)|_Q = \mathbf{D}(G_{ij}, A_i) + (1, 0, B_Q)\,, \tag{3.7}$$

where B_Q is a globally defined abelian 2-form.

We can now correctly define the path integral (3.1). We proceed is three steps.

i) Using the holonomy gluing property it is easy to see that $\mathrm{hol}(\Sigma \sharp D) \equiv \mathrm{hol}(\zeta \sharp \tilde{\zeta})$ is a section of the line bundle $\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]} \to \Sigma_Q(M)$ at the point $\zeta \in \Sigma_Q(M)$. The line bundle $\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]} \to \Sigma_Q(M)$ is the pull back to $\Sigma_Q(M)$ of the line bundle on loop space $\mathcal{L}_{[-\mathcal{G}|_Q]} \to L(Q)$. We characterize $\mathcal{L}_{[-\mathcal{G}|_Q]} \to L(Q)$ (here $-\mathcal{G}|_Q$ is a generic gerbe over Q) by realizing its sections $L(Q) \to \mathcal{L}_{[-\mathcal{G}|_Q]}$ through functions $s : D(Q) \to \mathbb{C}$ where $D(Q)$ is the space of maps from the disk D into Q; the boundaries of these maps are loops in $L(Q)$. The function s is a section of $\mathcal{L}_{[-\mathcal{G}|_Q]} \to L(Q)$ if $s(\tilde{\zeta}) = \mathrm{hol}(\tilde{\zeta}\sharp\tilde{\zeta}')s(\tilde{\zeta}')$ for all $\tilde{\zeta}, \tilde{\zeta}' \in D(Q)$ that are equal on the boundary: $\tilde{\zeta}|_{\partial D} = \tilde{\zeta}'|_{\partial D}$. Expression $\mathrm{hol}(\tilde{\zeta}\sharp\tilde{\zeta}')$ above is the holonomy of $[-\mathcal{G}|_Q]$ on the closed surface $\tilde{\zeta}(D)\sharp\tilde{\zeta}'(D)$ obtained by gluing together $\tilde{\zeta}(D)$ and $\tilde{\zeta}'(D)$.

ii) If we define $\mathcal{T} = \mathbf{D}(G_{ij}, A_i)$, then (3.7) reads $\mathcal{G}|_Q - \mathcal{T} = (1, 0, B_Q)$ and we see that $e^{\int_D \tilde{\zeta}^* B_Q}$ is a section of the line bundle $\mathcal{L}_{[\mathcal{G}|_Q - \mathcal{T}]} \to L(Q)$. From i) and ii) we see that we need a section of the line bundle $\mathcal{L}_{[\mathcal{T}]} \to L(Q)$.

iii) A section of the line bundle $\mathcal{L}_{[\mathcal{T}]} \to L(Q)$ is given by the inverse of the trace of the holonomy of the twisted $\mathrm{U}(n)$ bundle (G_{ij}, A_i). The definition is as follows. The pull back of $\mathcal{T}$ on the disk D via $\tilde{\zeta} : D \to Q$ is trivial since D is two dimensional. We can thus write

$$\mathbf{D}(\tilde{\zeta}^* G_{ij}, \tilde{\zeta}^* A_i) = D(\tilde{\lambda}_{ij}, \tilde{\alpha}_i) + (1, 0, b) \tag{3.8}$$

so that $(\tilde{\zeta}^* G_{ij}\, \lambda_{ij}^{-1}, \tilde{\zeta}^* A_i - \tilde{\alpha}_i)$ is a true $\mathrm{U}(n)$ bundle. We then define

$$\operatorname{Tr} \mathrm{hol}_{\partial\Sigma}(\zeta^* A) \equiv \operatorname{Tr} \mathrm{hol}_{\partial\Sigma}(\tilde{\zeta}^* A - \tilde{\alpha}) e^{-\int_D b}\,, \tag{3.9}$$

where $\mathrm{Tr\,hol}_{\partial\Sigma}(\tilde\zeta^*A-\tilde\alpha)$ is the trace of the holonomy of the $\mathrm{U}(N)$ bundle $(\tilde\zeta^*G_{ij}\,\tilde\lambda_{ij}^{-1},\tilde\zeta^*A_i-\tilde\alpha_i)$. Note that if we consider the couple (G_{ij},A_i) in the adjoint representation, then it defines a true $\mathrm{SU}(n)$ bundle. Consistently, if in (3.9) we consider the trace in the adjoint representation instead of the trace in the fundamental, we then obtain the holonomy of this $\mathrm{SU}(n)$ bundle. It is easy to check that definition (3.9) is independent from the choice of the trivialization $\tilde\lambda,\tilde\alpha$, b and of the map $\tilde\zeta:D\to Q$.

We conclude that expression (3.1) is well defined because we have the product of the three sections

$$\mathrm{hol}(\Sigma\sharp D)\; e^{\int_D\tilde\zeta^*B_Q}\;\mathrm{Tr\,hol}^{-1}_{\partial\Sigma}(\tilde\zeta^*A)\,, \tag{3.10}$$

respectively sections of the bundles $\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]}$, $\partial^{-1}\mathcal{L}_{[\mathcal{G}|_Q-\mathcal{T}]}$ and $\partial^{-1}\mathcal{L}_{[\mathcal{T}]}$ on the base space $\Sigma_Q(M)$ obtained by pulling back the corresponding bundles on the loop space $L(Q)$ via the map $\Sigma_Q(M)\overset{\partial}{\to}L(Q)$. The product of these three bundles is canonically trivial. Expression (3.10) depends on the Deligne classes $[\mathcal{G}]$, $[\mathcal{T}]$ and on the potential B_Q; it is easily seen that it does not depend on the choice of the map $\tilde\zeta:D\to Q$. In order to obtain a gauge invariant action the gauge transformation $B_Q\to B'_Q=B_Q+\omega$ comes always together with the transformation of the twisted $\mathrm{U}(N)$ bundle (G_{ik},A_i) obtained by tensoring (G_{ik},A_i) with the $\mathrm{U}(1)$ bundle on Q defined by the closed 2-form ω.

In conclusion using anomaly cancellation we have seen that if the open strings couple to the B field — more precisely to the gerbe class $[\mathcal{G}]$ — then their ends must couple to a twisted $\mathrm{U}(n)$ gauge field A if on the boundary $\mathcal{G}$ is torsion.

For sake of simplicity, up to now we have omitted spinor fields. In superstring theory, due to the determinant of the spinor fields, we have an extra term entering the functional integral: Pfaff. This is a section of the bundle $\partial(\mathcal{L}_{[\omega_{ijk},0,0]})\to\Sigma_Q(M)$ where $[\omega_{ijk},0,0]$ is the Deligne class associated with the second Stiefel-Whitney class $\omega_2\in H^2(Q,\mathbb{Z}_2)$ of the normal bundle of Q [or, which is the same, with its image W_3 in $H^3_{tors}(Q,\mathbb{Z})$]. In this case we consider a $\mathrm{PU}(n)$ bundle $P\to Q$ with curvature two form such that instead of (3.7) the following equation holds, $(\lambda_{ijk},\alpha_{ij},\beta_i)|_Q-(\omega_{ijk},0,0)=\mathbf{D}(G_{ij},A_i)+(1,0,B_Q)$. Correspondingly, the product

$$\mathrm{Pfaff}\;\mathrm{hol}(\Sigma\sharp D)\; e^{\int_D\tilde\zeta^*B_Q}\;\mathrm{Tr\,hol}^{-1}_{\partial\Sigma}(\tilde\zeta^*A)\,, \tag{3.11}$$

is a well defined function on $\Sigma_Q(M)$ because $\partial^{-1}\mathcal{L}_{[\omega_{ijk},0,0]}\,\partial^{-1}\mathcal{L}_{[-\mathcal{G}|_Q]}\partial^{-1}\mathcal{L}_{[1,0,B|_Q]}\,\partial^{-1}\mathcal{L}_{[\mathcal{T}]}$ is the trivial bundle. We thus arrive at the general condition for a stack of D-branes to be wrapping a cycle Q in M. It is

$$[\lambda_{ijk},\alpha_{ij},\beta_i]|_Q-[\omega_{ijk},0,0]=[\mathbf{D}(G_{ij},A_i)]+[1,0,B_Q]\,, \tag{3.12}$$

i.e. the stack of D-branes must form a twisted bundle, the twist being given by a gerbe that up to a trivial gerbe is equal to the initial gerbe associated with the 3-form H minus the gerbe obtained from the second Stiefel-Whitney class of the normal bundle of Q. In particular, for the characteristic classes of these gerbes we have,

$$[H]|_Q-W_3=\xi_{[\mathbf{D}(G_{ij},A_i)]}\,, \tag{3.13}$$

where $[H]|_Q \equiv \xi_{\mathcal{G}|_Q}$ is the characteristic class of the restriction to Q of the gerbe $\mathcal{G} = (\lambda_{ijk}, \alpha_{ij}, \beta_i)$ associated with the 3-form H, and $W_3 = \beta(\omega_2)$ is the obstruction for having Spin^c structure on the normal bundle of Q, in fact β is the Bockstein homomorphism associated with the short exact sequence $\mathbb{Z} \stackrel{\times 2}{\rightarrow} \mathbb{Z} \rightarrow \mathbb{Z}_2$.

4. Twisted nonabelian gerbes (2-gerbe modules)

In this section we slightly generalize the notion of twisted bundle (1-gerbe module) and then consider the one degree higher case. In (3.6) twisted U(n) bundles where defined. More generally, consider the central extension:

$$\mathrm{U}(1) \rightarrow G \stackrel{\pi}{\rightarrow} G/\,\mathrm{U}(1) \tag{4.1}$$

i.e. where U(1) is mapped into the center $Z(G)$ of G. (In the following we will not distinguish between U(1) and its image $\ker\pi \subset G$). A twisted G bundle with connection A and curvature F is a triple (G_{ij}, A_i, F_i) such that

$$F_i = G_{ij} F_j G_{ij}^{-1}\,, \tag{4.2}$$

and such that

$$\begin{aligned} \mathbf{D}_F(G_{ij}, A_i) &\equiv \Big((\delta G)_{ijk}, (\delta A)_{ij} + G_{ij} dG_{ij}^{-1}, dA_i + A_i \wedge A_i - F_i\Big) \\ &= (G_{ik} G_{jk}^{-1} G_{ij}^{-1}, -A_i + G_{ij} A_j G_{ij}^{-1} + G_{ij} dG_{ij}^{-1}, dA_i + A_i \wedge A_i - F_i) \end{aligned} \tag{4.3}$$

has U(1)- and Lie(U(1))-valued entries.

It is not difficult to check that the triple (4.3) defines a gerbe (hint: since the group extension is central, $d((\delta A)_{ij} + G_{ij} dG_{ij}^{-1}) = -dA_i - A_i \wedge A_i + G_{ij}(dA_j + A_j \wedge A_j) G_{ij})$. In the U($n$) case there was no need to introduce the extra data of the curvature F because at the Lie algebra level Lie(U(n)) $= Lie(\mathrm{U}(n)/\,\mathrm{U}(1)) \otimes$ Lie(U(1)), so that F was canonically constructed from A.

The notion of twisted 1-gerbe (2-gerbe module) can be introduced performing a similar construction. While twisted nonabelian bundles are described by nonabelian transition functions $\{G_{ij}\}$, twisted nonabelian gerbes are described by transition functions $\{f_{ijk}, \varphi_{ij}\}$ that are respectively valued in G and in $Aut(G)$, $f_{ijk} : O_{ijk} \rightarrow G$, $\varphi_{ij} : O_{ij} \rightarrow Aut(G)$, and where the action of φ_{ij} on U(1) is trivial: $\varphi_{ij}|_{\mathrm{U}(1)} = id$. The twisted cocycle relations now read

$$\lambda_{ijkl} = f_{ikl}^{-1} f_{ijk}^{-1} \varphi_{ij}(f_{jkl}) f_{ijl}\,, \tag{4.4}$$

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}} \varphi_{ik}\;, \tag{4.5}$$

where $\{\lambda_{ijkl}\}$ is U(1)-valued. It is not difficult to check that $\{\lambda_{ijkl}\}$ is a Čech 3-cocycle and thus defines a 2-gerbe (without curvings). This cocycle may not satisfy the antisymmetry property in its indices, this however can always be achieved by a gauge transformation with a trivial cocycle. In the particular case $\lambda_{ijkl} = 1$ equations (4.4), (4.5) define a nonabelian 1-gerbe (without curvings).

One can also consider twisted gerbes with connections 1-forms: $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i)$ where $a_{ij} \in \mathrm{Lie}(G) \otimes \Omega^1(O_{ij})$, $\mathcal{A}_i \in \mathrm{Lie}(Aut(G)) \otimes \Omega^1(O_i)$, and twisted gerbes with curvings:

$$(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)\,, \tag{4.6}$$

where B_i, d_{ij} are 2-forms and H_i 3-forms, all of them $\mathrm{Lie}(G)$-valued; $B_i \in \mathrm{Lie}(G) \otimes \Omega^2(O_i)$, $d_{ij} \in \mathrm{Lie}(G) \otimes \Omega^2(O_{ij})$, $H_i \in \mathrm{Lie}(G) \otimes \Omega^3(O_i)$. Before defining a twisted 1-gerbe we need to introduce some more notation. Given an element $X \in \mathrm{Lie}(Aut(G))$, we can construct a map (a 1-cocycle) $T_X : G \to \mathrm{Lie}(G)$ in the following way,

$$T_X(h) \equiv [he^{tX}(h^{-1})]\,,$$

where $[he^{tX}(h^{-1})]$ is the tangent vector to the curve $he^{tX}(h^{-1})$ at the point 1_G; if X is inner, i.e. $X = ad_Y$ with $Y \in \mathrm{Lie}(G)$, then $e^{tX}(h^{-1}) = e^{tY}h^{-1}e^{-tY}$ and we simply have $T_X(h) = T_{ad_Y}(h) = hYh^{-1} - Y$. Given a $\mathrm{Lie}(Aut(G))$-valued form $\mathcal{A}$, we write $\mathcal{A} = \mathcal{A}^\rho X^\rho$ where $\{X^\rho\}$ is a basis of $\mathrm{Lie}(Aut(G))$. We then define $T_\mathcal{A}$ as

$$T_\mathcal{A} \equiv \mathcal{A}^\rho T_{X^\rho}\,. \tag{4.7}$$

We use the same notation $T_\mathcal{A}$ for the induced map on $\mathrm{Lie}(G)$. Now we extend this map to allow $T_\mathcal{A}$ to act on a $\mathrm{Lie}(G)$-valued form $\eta = \eta^\alpha Y^\alpha$, where $\{Y^\alpha\}$ is a basis of $\mathrm{Lie}(G)$, by $T_\mathcal{A}(\eta) = \eta^\alpha \wedge T_\mathcal{A}(Y^\alpha)$. Also we define

$$\mathcal{K}_i \equiv d\mathcal{A}_i + \mathcal{A}_i \wedge \mathcal{A}_i\,, \tag{4.8}$$

$$k_{ij} \equiv da_{ij} + a_{ij} \wedge a_{ij} + T_{\mathcal{A}_i}(a_{ij})\,. \tag{4.9}$$

A twisted 1-gerbe is a set $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$ such that, $\varphi_{ij}|_{\mathrm{U}(1)} = id$, $T_{\mathcal{A}_i}|_{\mathrm{U}(1)} = 0$,

$$\varphi_{ij}\varphi_{jk} = Ad_{f_{ijk}}\varphi_{ik}\,, \tag{4.10}$$

$$\mathcal{A}_i + ad_{a_{ij}} = \varphi_{ij}\mathcal{A}_j\varphi_{ij}^{-1} + \varphi_{ij}d\varphi_{ij}^{-1}\,, \tag{4.11}$$

$$d_{ij} + \varphi_{ij}(d_{jk}) = f_{ijk}d_{ik}f_{ijk}^{-1} + T_{\mathcal{K}_i + ad_{B_i}}(f_{ijk})\,, \tag{4.12}$$

$$\varphi_{ij}(H_j) = H_i + d\,d_{ij} + [a_{ij}, d_{ij}] + T_{\mathcal{K}_i + ad_{B_i}}(a_{ij}) - T_{\mathcal{A}_i}(d_{ij})\,, \tag{4.13}$$

and such that $\mathbf{D}_H(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}) \equiv (\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ has $\mathrm{U}(1)$- and $\mathrm{Lie}(\mathrm{U}(1))$-valued elements, where

$$\lambda_{ijkl} \equiv f_{ikl}^{-1}f_{ijk}^{-1}\varphi_{ij}(f_{jkl})f_{ijl}\,, \tag{4.14}$$

$$\alpha_{ijk} \equiv a_{ij} + \varphi_{ij}(a_{jk}) - f_{ijk}a_{ik}f_{ijk}^{-1} - f_{ijk}df_{ijk}^{-1} - T_{\mathcal{A}_i}(f_{ijk})\,, \tag{4.15}$$

$$\beta_{ij} \equiv \varphi_{ij}(B_j) - B_i - d_{ij} + k_{ij}\,, \tag{4.16}$$

$$\gamma_i \equiv H_i - dB_i + T_{\mathcal{A}_i}(B_i)\,, \tag{4.17}$$

and we have used the same notation φ_{ij} for the induced map $\varphi_{ij} : O_{ij} \to Aut(\mathrm{Lie}(G))$.[3]

[3]In the special case $G = \mathrm{U}(1)$ the $\mathbf{D}_H$ operation is equivalent to the usual Deligne coboundary operator, provided we change $f \to f^{-1}, B \to -B$, $(d \to -d,\ H \to -H)$ and set $H = 0$.

If there is zero on the l.h.s. of equations (4.15)–(4.17) and 1 on the l.h.s. of eq. (4.14), equations (4.10)–(4.17) define a nonabelian gerbe[4]. A little algebra, see the appendix, shows that in the less trivial situation, when we assume that λ_{ijkl} is U(1)-valued and α_{ijk}, β_{ij} and γ_i are Lie(U(1))-valued, the above equations guarantee that $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$ is a honest 2-gerbe; hence the name twisted 1-gerbe for the set $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$.

The 2-gerbe may not satisfy the antisymmetry property in its indices. This however can always be achieved by a gauge transformation with a trivial Deligne class.

We say that the nonabelian gerbe $(f_{ijk}, \varphi_{ij}, a_{ij}, d_{ij}, A_i, B_i, H_i)$ is twisted by the 2-gerbe $(\lambda_{ijkl}, \alpha_{ijk}, \beta_{ij}, \gamma_i)$. We can also say (compare to the one degree lower situation) that we have a 2-gerbe module, or that we have a lifting 2-gerbe. The name "lifting 2-gerbe" comes from the following observation: under the projection π, that enters the group extension $\mathrm{U}(1) \to G \xrightarrow{\pi} G/\,\mathrm{U}(1)$, the twisting 2-gerbe disappears and we are left with an ordinary $G/\,\mathrm{U}(1)$-nonabelian gerbe (for example the map φ_{ij} is now given by $\pi(\varphi(\hat{g}))$ and is independent from the lifting $\hat{g}$ of the element $g \in G/\,\mathrm{U}(1)$). The twisting 2-gerbe is the obstruction to lift the nonabelian $G/\,\mathrm{U}(1)$-gerbe to a G-gerbe.

4.1 Twisted $\tilde{\Omega} E_8$ gerbes

Consider the exact sequence of groups,

$$1 \to \Omega E_8 \to PE_8 \xrightarrow{\pi} E_8 \to 1\,, \tag{4.18}$$

where the loop group ΩE_8 is the space of loops based at the identity 1_{E_8}, and the based path group PE_8 is the space of paths starting at the identity 1_{E_8}.[5] Expression (4.18) states that ΩE_8 is a normal subgroup of PE_8, the quotient being E_8. Consider now the problem of lifting an E_8 bundle to a PE_8 bundle. Since every path can be homotopically deformed to the identity path, we have that PE_8 is contractible, and therefore every PE_8 bundle is the trivial bundle. This implies that only the trivial E_8 bundle can be lifted. Any nontrivial E_8 bundle cannot be lifted and we thus obtain a nontrivial ΩE_8 1-gerbe. If $\mathrm{dim} M \leq 15$ (equivalence classes of) E_8 bundles are in 1-1 correspondence with elements $a \in H^4(M, \mathbb{Z})$ and we can say that a is the obstruction to lift the E_8 bundle, i.e. that a characterizes the gerbe. More explicitly the ΩE_8 gerbe has $Aut(\Omega E_8)$ valued maps φ_{ij} coming from the conjugation action by some $G_{ij} \in PE_8$ (these are the transition functions of the twisted PE_8 bundle associated with the ΩE_8 gerbe). Since ΩE_8 is normal in PE_8 also the actions $T_{\mathcal{A}_i}$ of the Lie($Aut(\Omega E_8)$) valued one forms $\mathcal{A}_i$ on $h \in \Omega E_8$ and $X \in \mathrm{Lie}(\Omega E_8)$ can be understood as $T_{\mathcal{A}_i}(h) = hA_ih^{-1} - A_i$ and $T_{\mathcal{A}_i}(X) = [X, A_i]$ with some Lie(PE_8)-valued forms A_i that are a lift of the connection on the E_8 bundle. (See [12] for more details on gerbes from an exact sequence $1 \to H \to G \xrightarrow{\pi} G/H \to 1$).

Finally consider the (universal) central extension of ΩE_8,

$$1 \to \mathrm{U}(1) \to \tilde{\Omega} E_8 \xrightarrow{\pi} \Omega E_8 \to 1\,, \tag{4.19}$$

[4]A general nonabelian 1-gerbe is defined by equations (4.10)-(4.17), where now the group G is an arbitrary group (not necessarily a central extension).

[5]More precisely we should use smooth loops and paths with sitting instant [32].

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and try to lift the ΩE_8 gerbe to an $\tilde{\Omega} E_8$ gerbe, this is in general not possible and the obstruction gives rise to a twisted $\tilde{\Omega} E_8$ gerbe, the twist being described by a 2-gerbe. Actually one has always an obstruction in lifting the ΩE_8 gerbe if at least $M \leq 14$, and therefore the lifting 2-gerbe thus obtained has characteristic class a (characterizing the initial E_8 bundle). The twisted $\tilde{\Omega} E_8$ gerbe has $Aut(\tilde{\Omega} E_8)$ valued maps φ_{ij}, obtained extending the previous φ_{ij} maps in such a way that they act trivially on the center U(1) of $\tilde{\Omega} E_8$, also the $T_{\mathcal{A}_i}$ map is similarly extended.

A similar statement holds for E_8 replaced by G_2, Spin_n where $n \geq 7$, F_4, E_6, E_7, when one correspondingly lowers the dimension of M.

5. M5-brane anomaly, 2-gerbes and twisted nonabelian 1-gerbes

In section 3 cancellation of global anomalies appearing in the open string worldsheet with strings ending on a stack of D-branes led to condition (3.13) for the D-brane configuration (charges). Here one could in principle follow a similar approach and study global anomalies of the path integral of open M2-branes ending on M5 branes. An alternative approach is to study anomalies of 11-dimensional supergravity in the presence of M5-branes. The relevant mechanism is the cancellation between anomalies of the M5 brane quantum effective action and anomaly inflow from the 11-dimensional bulk through a non invariance of the Chern-Simons plus Green-Schwarz topological term $\Phi(C) \sim \int \frac{1}{6} C \wedge G \wedge G - C I_8(g)$ where C is the 3-form potential of 11-dimensional supergravity, $G = dC$ and $I_8 \sim (\mathrm{Tr}\, R^2)^2 - \mathrm{Tr}\, R^4$, with R being the curvature. We are interested in the global aspects of this mechanism, where we cannot assume that C is globally defined and that G is topologically trivial. This problem has been studied in [13, 6, 7]; and in the more general case where the 11-dimensional space has boundaries in [8]. Let Y be the 11-dimensional spacetime: a spin manifold. Let also V be the six dimensional M5-brane worldvolume embedded in Y $\iota : V \hookrightarrow Y$, we assume it compact and oriented. It turns out [7] that if the field strength G is cohomologically trivial on V and V is the product space $V = S \times Q$, with S a circle with supersymmetric spin structure and Q a five manifold, then the M5-brane can wrap V iff Q is a Spin^c manifold. If this is not the case the M5-brane has a global anomaly: one detects it from the vanishing of the M5-brane partition function. The partition function is zero every time that there is a torsion element $\theta \in H^3_{tors}(Q, \mathbb{Z}) \subset H^3(Q, \mathbb{Z})$ different from zero. More in general, without assuming that $V = S \times Q$, we have a global anomaly if there exists an element $\theta \in H^4_{tors}(V, \mathbb{Z})$ different from zero. As suggested in [7] a way to cancel this anomaly is to turn on a background field G such that, essentially,

$$[G]|_V = \theta \,, \tag{5.1}$$

where $[G]|_V$ is the integral class associated with G restricted to V. This condition should be compared to (3.13) when the l.h.s. of (3.13) is zero: $[H]|_Q = W_3$. In the case $V = S \times Q$, dimensional reduction of the M5-brane on the circle S leads to a Type IIA D4-brane wrapping Q and satisfying $[H]|_Q = W_3$.

In [8] condition (5.1) is sharpened. First a mathematically precise definition of C and of $\Phi(C)$ is given, it is in terms of connections on E_8 bundles. Associated with the field strength G on spacetime Y with metric g, there is an integral cohomology class $a \in H^4(Y, \mathbb{Z})$. This determines an E_8 bundle $P(a) \to Y$ (cf. section 2.3). The field C can then be described by a couple (A, c) where A is an E_8 connection on $P(a)$ and c is a globally defined $Lie(\mathrm{U}(1))$-valued 3-form on Y. We denote by $\check{C} = (A, c)$ this E_8 description of the C-field. In particular the holonomy of $\check{C}$ around a 3-cycle Σ is given as

$$\mathrm{hol}_\Sigma(\check{C}) = \exp\left[\left(\int_\Sigma CS(A) - \frac{1}{2}CS(\omega) + c\right)\right],$$

with properly normalized Chern-Simons terms corresponding to the gauge field A and the spin connection ω such that $\exp[(\int_\Sigma CS(A)]$ is well defined and $\exp[\frac{1}{2}(\int_\Sigma CS(\omega)]$ has a sign ambiguity. To be more precise these should be the holonomy of the E_8 Chern-Simons 2-gerbe and the proper square root of the holonomy of the Chern-Simons 2-gerbe associated with the metric.

Subsequently in [8] the electric charge associated with the C field is studied. From the C field equation of motion that are nonlinear, $d \star G = \frac{1}{2}G^2 - I_8$, we have that the C field and the background metric induce an electric charge that is given by the cohomology class

$$[\frac{1}{2}G^2 - I_8]_{DR} \in H^8(Y, \mathbb{R})\,. \tag{5.2}$$

However the electric charge is an integer cohomology class (because of Dirac quantization, due to the existence of fundamental electric M2-branes and magnetic M5-membranes). In [8] the integral lift of (5.2) is studied and denoted $\Theta_Y(\check{C})$ (and also $\Theta_Y(a)$).

In order to study the anomaly inflow, we consider a tubular neighbourhood of V in Y. Since this is diffeomorphic to the total space of the normal bundle $N \to V$, we identify these two spaces. Let $X = S_r(N)$ be the 10-dimensional sphere bundle of radius r; the fibres of $X \xrightarrow{\pi} V$ are then 4-speres. An 11-dimensional manifold Y_r with boundary X is then constructed by removing from Y the disc bundle $D_r(N)$ of radius r; $Y_r = Y - D_r(N)$ (we can also say that Y_r is the complement of the tubular neighbourhood $D_r(N)$). We call Y_r the bulk manifold. Then one has the bulk C field path integral $\Psi_{bulk}(\check{C}_X) \sim \int exp[G \wedge \star G]\Phi(\check{C}_{Y_r})$ where the integral is over all equivalence classes of $\check{C}_{Y_r}$ fields that on the boundary assume the fixed value $\check{C}_X$. The wavefunction $\Psi_{bulk}(\check{C}_X)$ is section of a line bundle $\mathcal{L}$ on the space of $\check{C}_X$ fields. This wavefunction appears together with the M5-brane partition function $\Psi_{M5}(\check{C}_V)$ that depends on the $\check{C}$ field on the M5-brane, or better, on an infinitesimally small ($r \to 0$) tubular neighbourhood of the M5-brane. Anomaly cancellation requires $\Psi_{bulk}\Psi_{M5}$ to be gauge invariant and therefore Ψ_{M5} has to be a section of the opposite bundle of $\mathcal{L}$ ($\check{C}$ fields on V and $\check{C}$ fields on X can be related according to the exact sequence (5.4)). Let's study the various cases.

I) We can have Ψ_{bulk} gauge invariant, and this is shown to imply $\Theta_Y(\check{C}_X) = 0$. This last condition is the decoupling condition, indeed if $\Theta_Y(\check{C}_X) \neq 0$ then charge conservation requires that M2-branes end on the M5-brane and the M5-brane is thus not decoupled from the bulk. If Ψ_{bulk} is gauge invariant also Ψ_{M5} needs to be, and this holds if $\theta = 0$.

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II) More generally we can have $\theta \neq 0$ but then invariance of $\Psi_{bulk}\Psi_{M5}$ can be shown to imply

$$\pi_*(\Theta_X) = \theta\,, \tag{5.3}$$

where π_* is integration over the fibre. The map π_* enters the exact sequence

$$0 \to H^k(V,\mathbb{Z}) \xrightarrow{\pi^*} H^k(X,\mathbb{Z}) \xrightarrow{\pi_*} H^{k-4}(V,\mathbb{Z}) \to 0\,, \tag{5.4}$$

where π^* is just pull back associated with the bundle $X \xrightarrow{\pi} V$. The exactness of this sequence (obtained from the Gysin sequence) follows from X being oriented compact and spin, and V oriented and compact. Condition (5.3) is the precise version of condition (5.1).

We now compare this situation to that in 10 dimensional Type IIA theory, described at the end of section 3, and therefore we are led to consider the following more general case.

III) Here $\Psi_{bulk}\Psi_{M5}$ is not gauge invariant (therefore it is a section of a line bundle) but we can consider a new partition function Ψ'_{M5} that is obtained from a "stack" of M5-branes instead of just a single brane. This stack gives rise to a twisted gerbe $(f_{ijk}, \varphi_{ij}, a_{ij}, \mathcal{A}_i, B_i, d_{ij}, H_i)$ on V so that in particular Ψ'_{M5} depends also from the nonabelian gauge fields B_i and H_i.[6] In order for $\Psi_{bulk}\Psi'_{M5}$ to be well defined, the twisted gerbe has to satisfy [cf. (3.12)],

$$[CS(\pi_*(\Theta_X))] - [\vartheta_{ijkl},0,0,0] = [\mathbf{D}_H(f_{ijk},\varphi_{ij},a_{ij},\mathcal{A}_i,B_i,d_{ij})] + [1,0,0,C_V]\,, \tag{5.5}$$

where, as constructed in subsection 2.3, $CS(\pi_*(\Theta_X))$ is the Chern-Simons 2-gerbe associated with $\pi_*(\Theta_X)$ and a choice of connection on the E_8 bundle with first Pontryagin class $\pi_*(\Theta_X)$ (all other 2-gerbes differ by a global 3-form, see (2.21)), while $[\vartheta_{ijkl},0,0,0]$ is the 2-gerbe class associated with the torsion class θ (i.e. $\beta(\vartheta) = \theta$, cf (2.24)), and $[1,0,0,C_V]$ is the trivial Deligne class associated with the global 3-form C_V.

In particular (5.5) implies

$$\pi_*(\Theta_X) - \theta = \xi_{\mathbf{D}_H(G_{ijk},\varphi_{ij},a_{ij},\mathcal{A}_i,B_i,d_{ij})}\,, \tag{5.6}$$

where on the r.h.s. we have the characteristic class of the lifting 2-gerbe.

The correspondence of this construction with that described in section 3, is strengthened by slightly generalizing the results of section 3. In fact there we always considered $[H]|_Q - W_3$ trivial in De Rham cohomology. This implied that the torsion class $[H]|_Q - W_3$ was interpreted as the characteristic class of a gerbe associated with a twisted U(n) bundle for some $n \in \mathbb{Z}$. However (at least mathematically) one can consider the more general case

[6] Of course we have the special case when a stack of M5-branes gives a nonabelian gerbe. Then H_i is the curvature of B_i. These two fields should not be confused, and have nothing to do with the NS B field and its curvature H.

where $[H]|_Q - W_3 \neq 0$ also in De Rham cohomology. Here too we have a twisted bundle, but with structure group U($\mathcal{H}$), the group of unitary operators on the complex, separable and infinite dimensional Hilbert space $\mathcal{H}$. This case corresponds to an infinite number of D-branes wrapping the cycle Q, and the relevant central extension is $\mathrm{U}(1) \to \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$. When dim$Q \leq 13$ (which is always the case in superstring theory), we can replace, for homotopy purposes, PU($\mathcal{H}$) with ΩE_8 and U($\mathcal{H}$) with $\tilde{\Omega} E_8$, so that the group extension $\mathrm{U}(1) \to \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ is replaced with $\mathrm{U}(1) \to \tilde{\Omega} E_8 \to \Omega E_8$. Now consider a stack of M5-branes wrapping a cycle $V = S \times Q$ and dimensionally reduce M-theory to Type IIA along the circle S. Then the M5-branes become D4-branes and the twisted ΩE_8 1-gerbe becomes a twisted ΩE_8 bundle.

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A. Proof that a twisted 1-gerbe defines a 2-gerbe

The cocycle condition for λ_{ijkl} is straightforward. In order to show that α_{ijk} as defined in (4.15) satisfies the 2-gerbe condition

$$\alpha_{ijk} + \alpha_{ikl} - \alpha_{ijl} - \alpha_{jkl} = \lambda_{ijkl} d\lambda_{ijkl}^{-1},$$

we rewrite the l.h.s. as $\alpha_{ijk} + Ad_{f_{ijk}}\alpha_{ikl} - Ad_{\varphi_{ij}(f_{jkl})}\alpha_{ijl} - \alpha_{jkl}$, we then use the definition of α_{ijk} and the following properties of the map $T_{\mathcal{A}}$,

$$\begin{aligned} T_{\mathcal{A}}(hk) &= T_{\mathcal{A}}(h) + kT_{\mathcal{A}}(h)k^{-1}, \qquad \text{(cocycle property)} && \text{(A.1)} \\ \varphi_{ij}(T_{\mathcal{A}}(h)) &= T_{\varphi_{ij}\mathcal{A}\varphi_{ij}^{-1}}(\varphi_{ij}(h)), && \text{(A.2)} \\ T_{-\varphi_{ij}d\varphi_{ij}^{-1}}(\varphi_{ij}(h)) &= \varphi_{ij}(h)d\varphi_{ij}(h^{-1}) - \varphi_{ij}(hdh^{-1}), && \text{(A.3)} \end{aligned}$$

where h, k are elements of G, and more in general functions from some open neighbourhood of M into G. Finally $T_{\mathcal{A}_i}(\varphi_{ij}(f_{jkl})f_{ijl}) = T_{\mathcal{A}_i}(f_{ijk}f_{ikl})$ since $T_{\mathcal{A}_i}(\lambda_{ijkl}) = 0$.

Similarly in order to show that

$$\beta_{ij} + \beta_{jk} + \beta_{ki} = d\alpha_{ijk}$$

we rewrite the l.h.s. as $\beta_{ij} + \varphi_{ij}(\beta_{jk}) + f_{ijk}\beta_{ki}f_{ijk}^{-1}$ and then use the following equality

$$k_{ij} + \varphi_{ij}(k_{jk}) = f_{ijk}k_{ik}f_{ijk}^{-1} + T_{\mathcal{K}_i}(f_{ijk}) + d\alpha_{ijk}, \tag{A.4}$$

that follows from (4.8)–(4.11), the algebra here is the same as for usual gerbes. We also have $\mathcal{K}_i + ad_{k_{ij}} = \varphi_{ij}\mathcal{K}_j\varphi_{ij}^{-1}$, and the Bianchi identity

$$dk_{ij} + [a_{ij}, k_{ij}] + T_{\mathcal{K}_i}(a_{ij}) - T_{\mathcal{A}_i}(k_{ij}). \tag{A.5}$$

Relation

$$\gamma_i - \gamma_j = d\beta_{ij}$$

that we rewrite as $\gamma_i - \varphi_{ij}(\gamma_j) = d\beta_{ij}$ follows from (A.5), (4.13) and $T_{-\varphi_{ij}d\varphi_{ij}^{-1}}(\varphi_{ij}(B_j)) = -d(\varphi_{ij}(B_j)) + \varphi_{ij}(dB_j)$.

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Nonabelian Bundle Gerbes, Their Differential Geometry and Gauge Theory

Paolo Aschieri[1,2,3], **Luigi Cantini**[4,5], **Branislav Jurčo**[2,3]

[1] Dipartimento di Scienze e Tecnologie Avanzate, Universitá del Piemonte Orientale, and INFN, Corso Borsalino 54, 15100 Alessandria, Italy
[2] Max-Planck-Institut für Physik, Föhringer Ring 6, 80805 München, Germany
[3] Sektion Physik, Universität München, Theresienstr. 37, 80333 München, Germany.
E-mail: aschieri@theorie.physik.uni-muenchen.de; jurco@theorie.physik.uni-muenchen.de
[4] Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa & INFN sezione di Pisa, Italy
[5] Department of Physics, Queen Mary, University of London, Mile End Road, London E1 4NS, UK.
E-mail: l.cantini@qmul.ac.uk

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Abstract: Bundle gerbes are a higher version of line bundles, we present nonabelian bundle gerbes as a higher version of principal bundles. Connection, curving, curvature and gauge transformations are studied both in a global coordinate independent formalism and in local coordinates. These are the gauge fields needed for the construction of Yang-Mills theories with 2-form gauge potential.

1. Introduction

Fibre bundles, besides being a central subject in geometry and topology, provide the mathematical framework for describing global aspects of Yang-Mills theories. Higher abelian gauge theories, i.e. gauge theories with abelian 2-form gauge potential appear naturally in string theory and field theory, and here too we have a corresponding mathematical structure, that of the abelian gerbe (in algebraic geometry) and of the abelian bundle gerbe (in differential geometry). Thus abelian bundle gerbes are a higher version of line bundles. Complex line bundles are geometric realizations of the integral 2$^{\text{nd}}$ cohomology classes $H^2(M, \mathbb{Z})$ on a manifold, i.e. the first Chern classes (whose de Rham representative is the field strength). Similarly, abelian (bundle) gerbes are the next level in realizing integral cohomology classes on a manifold; they are geometric realizations of the 3$^{\text{rd}}$ cohomology classes $H^3(M, \mathbb{Z})$. Thus the curvature 3-form of a 2-form gauge potential is the de Rham representative of a class in $H^3(M, \mathbb{Z})$. This class is called the Dixmier-Douady class [1, 2]; it topologically characterizes the abelian bundle gerbe in the same way that the first Chern class characterizes complex line bundles.

One way of thinking about abelian gerbes is in terms of their local transition functions [3, 4]. Local "transition functions" of an abelian gerbe are complex line bundles on double overlaps of open sets satisfying cocycle conditions for tensor products over quadruple overlaps of open sets. The nice notion of abelian bundle gerbe [5] is related to this picture. Abelian gerbes and bundle gerbes can be equipped with additional structures, that of a connection 1-form and of curving (the 2-form gauge potential), and that of (bundle) gerbe modules (with or without connection and curving). Their holonomy can be introduced and studied [3, 4, 6–9]. The equivalence class of an abelian gerbe with connection and curving is the Deligne class on the base manifold. The top part of the Deligne class is the class of the curvature, the Dixmier-Douady class.

Abelian gerbes arise in a natural way in quantum field theory [10–12], where their appearance is due to the fact that one has to deal with abelian extensions of the group of gauge transformations; this is related to chiral anomalies. Gerbes and gerbe modules appear also very naturally in TQFT [13], in the WZW model [14] and in the description of D-brane anomalies in the nontrivial background 3-form H-field (identified with the Dixmier-Douday class) [15–17]. Coinciding (possibly infinitely many) D-branes are submanifolds "supporting" bundle gerbe modules [6] and can be classified by their (twisted) K-theory. The relation to the boundary conformal field theory description of D-branes is due to the identification of equivariant twisted K-theory with the Verlinde algebra [18, 19]. For the role of K-theory in D-brane physics see e.g. [20–22].

In this paper we study the nonabelian generalization of abelian bundle gerbes and their differential geometry, in other words we study higher Yang-Mills fields. Nonabelian gerbes arose in the context of nonabelian cohomology [23, 1] (see [24] for a concise introduction), see also ([25]). Their differential geometry –from the algebraic geometry point of view– is discussed thoroughly in the recent work of Breen and Messing [26] (and their combinatorics in [27]). Our study on the other hand is from the differential geometry viewpoint. We show that nonabelian bundle gerbes connections and curvings are very natural concepts also in classical differential geometry. We believe that it is primarily in this context that these structures can appear and can be recognized in physics. It is for example in this context that one would like to have a formulation of Yang-Mills theory with higher forms. These theories should be relevant in order to describe coinciding NS5-branes with D2-branes ending on them. They should be also relevant in the study of M5-brane anomaly. We refer to [28–30] for some attempts in constructing higher gauge fields.

Abelian bundle gerbes are constructed using line bundles and their products. One can also study $U(1)$ bundle gerbes; here line bundles are replaced by their corresponding principal $U(1)$ bundles. In the study of nonabelian bundle gerbes it is more convenient to work with nonabelian principal bundles than with vector bundles. Actually principal bundles with additional structures are needed. We call these objects (principal) bibundles and D-H bundles (D and H being Lie groups). Bibundles are fibre bundles (with fiber H) which are at the same time left and right principal bundles (in a compatible way). They are the basic objects for constructing (principal) nonabelian bundle gerbes. The first part of this paper is therefore devoted to their description. In Sect. 2 we introduce bibundles, D-H bundles (i.e. principal D bundles with extra H structure) and study their products. In Sect. 3 we study the differential geometry of bibundles, in particular we define connections, covariant exterior derivatives and curvatures. These structures are generalizations of the corresponding structures on usual principal bundles. We thus describe them using a language very close to that of the classical reference books [31] or [32]. In particular a connection on a bibundle needs to satisfy a relaxed equivariance property, this is the price to be paid in order to incorporate nontrivially the additional bibundle structure. We are thus lead to introduce the notion of a 2-connection $(\boldsymbol{a}, A)$ on a bibundle. Products of bibundles with connections give a bibundle with connection only if the initial connections were compatible. We call this compatibility the summability conditions for 2-connections; a similar summability condition is established also for horizontal forms (e.g. 2-curvatures).

In Sect. 4, using the product between bibundles we finally introduce (principal) bundle gerbes. Here too we first describe their structure (including stable equivalence) and then only later in Sect. 7 we describe their differential geometry. We start with the proper generalization of abelian bundle gerbes in the sense of Murray [5]; we then describe the relation to the Hitchin type presentation [3, 4], where similarly to the abelian case, nonabelian gerbes are described in terms of their "local transition functions" which are bibundles on double overlaps of open sets. The properties of the products of these bibundles over triple and quadruple overlaps define the gerbe and its nonabelian Čech 2-cocycle.

Section 5 is devoted to the example of the lifting bundle gerbe associated with the group extension $H \to E \to G$. In this case the bundle gerbe with structure group H appears as an obstruction to lift to E a G-principal bundle P.

Again by generalizing the abelian case, bundle gerbe modules are introduced in Sect. 6. Since we consider principal bibundles we obtain modules that are D-H bundles (compatible with the bundle gerbe structure). With each bundle gerbe there is canonically associated an $Aut(H)$-H bundle. In the lifting bundle gerbe example a module is given by the trivial E-H bundle.

In Sect. 7 we introduce the notion of a bundle gerbe connection and prove that on a bundle gerbe a connection always exists. Bundle gerbe connections are then equivalently described as collections of local 2-connections on local bibundles (the "local transition functions of the bundle gerbe") satisfying a nonabelian cocycle condition on triple overlaps of open sets. Given a bundle gerbe connection we immediately have a connection on the canonical bundle gerbe module can. We describe also the case of a bundle gerbe connection associated with an arbitrary bundle gerbe module. In particular we describe the bundle gerbe connection in the case of a lifting bundle gerbe.

Finally in Sect. 8 we introduce the nonabelian curving $\boldsymbol{b}$ (the 2-form gauge potential) and the corresponding nonabelian curvature 3-form $\boldsymbol{h}$. These forms are the nonabelian generalizations of the string theory B and H fields.

2. Principal Bibundles and Their Products

Bibundles (bitorsors) were first studied by Grothendieck [33] and Giraud [1], their cohomology was studied in [34]. We here study these structures using the language of differential geometry.

Given two $U(1)$ principal bundles E, $\tilde{E}$, on the same base space M, one can consider the fiber product bundle $E\tilde{E}$, defined as the $U(1)$ principal bundle on M whose fibers are the product of the E and $\tilde{E}$, fibers. If we introduce a local description of E and $\tilde{E}$, with transition functions h^{ij} and $\tilde{h}^{ij}$ (relative to the covering $\{U^i\}$ of M), then $E\tilde{E}$ has transition functions $h^{ij}\tilde{h}^{ij}$.

In general, in order to multiply principal nonabelian bundles one needs extra structure. Let E and $\tilde{E}$ be H-principal bundles, we use the convention that H is acting on the bundles from the left. Then in order to define the H principal left bundle $E\tilde{E}$ we need also a right action of H on E. We thus arrive at the following

Definition 1. *An H principal bibundle E on the base space M is a bundle on M that is both a* left *H principal bundle and a* right *H principal bundle and where left and right H actions commute*

$$\forall\, h,k \in H\ ,\ \forall e \in E,\ (k\,e) \triangleleft h = k(e \triangleleft h)\ ; \tag{1}$$

we denote with $p\ : E \to M$ the projection to the base space.

Before introducing the product between principal bibundles we briefly study their structure. A morphism W between two principal bibundles E and $\tilde{E}$ is a morphism between the bundles E and $\tilde{E}$ compatible with both the left and the right action of H:

$$W(k\, e \triangleleft h) = k\, W(e) \tilde{\triangleleft} h; \tag{2}$$

here $\tilde{\triangleleft}$ is the right action of H on $\tilde{E}$. As for morphisms between principal bundles on the same base space M, we have that morphisms between principal bibundles on M are isomorphisms.

Trivial bibundles. Since we consider only principal bibundles we will frequently write bibundle for principal bibundle. The product bundle $M \times H$, where left and right actions are the trivial ones on H [i.e. $k\,(x,h) \triangleleft h' = (x, khh')$] is a bibundle. We say that a bibundle T is trivial if T is isomorphic to $M \times H$.

Proposition 2. *We have that T is trivial as a bibundle iff it has a global central section, i.e. a global section $\boldsymbol{\sigma}$ that intertwines the left and the right action of H on T:*

$$\forall h \in H\,,\ \forall x \in M,\ \ h\,\boldsymbol{\sigma}(x) = \boldsymbol{\sigma}(x) \triangleleft h. \tag{3}$$

Proof. Let $\boldsymbol{\sigma}$ be a global section of T, define $W_{\boldsymbol{\sigma}} : M \times H \to T$ as $W_{\boldsymbol{\sigma}}(x,h) = h\,\boldsymbol{\sigma}(x)$, then T and $M \times H$ are isomorphic as left principal bundles. The isomorphism $W_{\boldsymbol{\sigma}}$ is also a right principal bundle isomorphism iff (3) holds. □

Note also that the section $\boldsymbol{\sigma}$ is unique if H has trivial centre. An example of nontrivial bibundle is given by the trivial left bundle $M \times H$ equipped with the nontrivial right action $(x,h) \triangleleft h' = (x, h\chi(h'))$, where χ is an outer automorphism of H. We thus see that bibundles are in general *not* locally trivial. Short exact sequences of groups provide examples of bibundles that are in general nontrivial as left bundles [cf. (112), (113)].

The $\boldsymbol{\varphi}$ map. We now further characterize the relation between left and right actions. Given a bibundle E, the map $\boldsymbol{\varphi} : E \times H \to H$ defined by

$$\forall e \in E\,,\ \forall h \in H\,,\ \boldsymbol{\varphi}_e(h)\, e = e \triangleleft h \tag{4}$$

is well defined because the left action is free, and transitive on the fibers. For fixed $e \in E$ it is also one-to-one since the right action is transitive and left and right actions are free. Using the compatibility between left and right actions it is not difficult to show that $\boldsymbol{\varphi}$ is equivariant w.r.t. the left action and that for fixed $e \in E$ it is an automorphism of H:

$$\boldsymbol{\varphi}_{he}(h') = h\boldsymbol{\varphi}_e(h')h^{-1}, \tag{5}$$

$$\boldsymbol{\varphi}_e(hh') = \boldsymbol{\varphi}_e(h)\boldsymbol{\varphi}_e(h'), \tag{6}$$

we also have

$$\boldsymbol{\varphi}_{e \triangleleft h}(h') = \boldsymbol{\varphi}_e(hh'h^{-1}). \tag{7}$$

Vice versa, given a left bundle E with an equivariant map $\boldsymbol{\varphi} : E \times H \to H$ that restricts to an H automorphism $\boldsymbol{\varphi}_e$, we have that E is a bibundle with right action defined by (4).

Using the $\boldsymbol{\varphi}$ map we have that a global section $\boldsymbol{\sigma}$ is a global central section (i.e. that a trivial left principal bundle is trivial as a bibundle) iff [cf. (3)], $\forall x \in M$ and $\forall h \in H$,

$$\varphi_{\sigma(x)}(h) = h. \tag{8}$$

In particular, since $e \in E$ can be always written as $e = h'\sigma$, we see that φ_e is always an *inner* automorphism,

$$\varphi_e(h) = \varphi_{h'\sigma}(h) = Ad_{h'}(h)\ . \tag{9}$$

Vice versa, we have that

Proposition 3. *If H has trivial centre then an H bibundle E is trivial iff φ_e is an inner automorphism for all $e \in E$.*

Proof. Consider the local sections $t^i : U^i \to E$, since H has trivial centre the map $k(t) : U^i \to H$ is uniquely defined by $\varphi_{t^i}(h') = Ad_{k(t^i)}h'$. From (5), $\varphi_{ht^i}(h') = Ad_h\, Ad_{k(t^i)}h'$, and therefore the sections $k(t^i)^{-1}t^i$ are central because they satisfy $\varphi_{k(t^i)^{-1}t^i}(h') = h'$. In the intersections $U^{ij} = U^i \cap U^j$ we have $t^i = h^{ij}t^j$ and therefore $k(t^i)^{-1}t^i = k(t^j)^{-1}t^j$. We can thus construct a global central section. □

Any principal bundle with H abelian is a principal bibundle in a trivial way, the map φ is given simply by $\varphi_e(h) = h$.

Now let us recall that a global section $\sigma : M \to E$ on a principal H-bundle $E \to M$ can be identified with an H-equivariant map $\overline{\sigma} : E \to H$. With our (left) conventions, $\forall E \in E$,

$$e = \overline{\sigma}(e)\sigma(x)\ .$$

Notice, by the way, that if E is a trivial bibundle with a global section σ, then $\overline{\sigma}$ is bi-equivariant, i.e.: $\overline{\sigma}(heh') = h\overline{\sigma}(e)h'$ iff σ is central. We apply this description of a global section of a left principal bundle to the following situation: Consider an H-bibundle E. Let us form $Aut(H) \times_H E$ with the help of the canonical homomorphism $Ad : H \to Aut(H)$. Then it is straightforward to check that $\overline{\sigma} : [\eta, e] \mapsto \eta \circ \varphi_e$ with $\eta \in Aut(H)$ is a global section of the left $Aut(H)$-bundle $Aut(H) \times_H E$. So $Aut(H) \times_H E$ is trivial as a left $Aut(H)$-bundle. On the other hand if E is a left principal H-bundle such that $Aut(H) \times_H E$ is a trivial left $Aut(H)$-bundle then it has a global section $\overline{\sigma} : Aut(H) \times_H E \to Aut(H)$ and the structure of an H-bibundle on E is given by $\varphi_e \equiv \overline{\sigma}([id, e])$. We can thus characterize H-bibundles without mentioning their right H structure,

Proposition 4. *A left H-bundle E is an H-bibundle if and only if the (left) $Aut(H)$-bundle $Aut(H) \times_H E$ is trivial.*

Any trivial left H-bundle T can be given a trivial H-bibundle structure. We consider a trivialization of T, i.e. an isomorphism $T \to M \times H$ and pull back the trivial right H-action on $M \times H$ to T. This just means that the global section of the left H-bundle T associated with the trivialization $T \to M \times H$, is by definition promoted to a global central section.

Product of bibundles. In order to define the product bundle $E\tilde{E}$ we first consider the fiber product (Withney sum) bundle

$$E \oplus \tilde{E} \equiv \{(e, \tilde{e}) \mid p(e) = \tilde{p}(\tilde{e})\} \tag{10}$$

with projection $\rho : E \oplus \tilde{E} \to M$ given by $\rho(e, \tilde{e}) = p(e) = \tilde{p}(\tilde{e})$. We now can define the product bundle $E\tilde{E}$ with base space M via the equivalence relation

$$\forall h \in H \; (e, h\tilde{e}) \sim (e \triangleleft h\,, \tilde{e}); \tag{11}$$

we write $[e, \tilde{e}]$ for the equivalence class and

$$E\tilde{E} \equiv E \oplus_H \tilde{E} \equiv \{\,[e, \tilde{e}]\,\}. \tag{12}$$

The projection $p\tilde{p} \;:\; E\tilde{E} \to M$ is given by $p\tilde{p}[e, \tilde{e}] = p(e) = \tilde{p}(\tilde{e})$. One can show that $E\tilde{E}$ is an H principal bundle; the action of H on $E\tilde{E}$ is inherited from that on E: $h[e, \tilde{e}] = [he, \tilde{e}]$. Concerning the product of sections we have that if $\boldsymbol{s} \;:\; U \to E$ is a section of E (with $U \subseteq M$), and $\tilde{\boldsymbol{s}} \;:\; U \to \tilde{E}$ is a section of $\tilde{E}$, then

$$\boldsymbol{s}\tilde{\boldsymbol{s}} \equiv [\,\boldsymbol{s}, \tilde{\boldsymbol{s}}\,] \;:\; U \to E\tilde{E} \tag{13}$$

is the corresponding section of $E\tilde{E}$.

When also $\tilde{E}$ is an H principal bibundle, with right action $\tilde{\triangleleft}$, then $E\tilde{E}$ is again an H principal bibundle with right action $\triangleleft\tilde{\triangleleft}$ given by

$$[e, \tilde{e}] \triangleleft\tilde{\triangleleft} h \;=\; [e, \tilde{e} \,\tilde{\triangleleft}\, h]. \tag{14}$$

It is easy to prove that the product between H principal bibundles is associative.

Inverse bibundle. The inverse bibundle E^{-1} of E has by definition the same total space and base space of E but the left action and the right actions $\triangleleft^{-1}$ are defined by

$$h\, e^{-1} = (e \triangleleft h^{-1})^{-1}\,, e^{-1} \triangleleft^{-1} h = (h^{-1} e)^{-1}; \tag{15}$$

here e^{-1} and e are the same point of the total space, we write e^{-1} when the total space is endowed with the E^{-1} principal bibundle structure, we write e when the total space is endowed with the E principal bibundle structure. From Definition (15) it follows that $he^{-1} = e^{-1} \triangleleft^{-1} \varphi_e(h)$. Given the sections $\boldsymbol{t}^i \;:\; U^i \to E$ of E we canonically have the sections $\boldsymbol{t}^{i^{-1}} \;:\; U^i \to E^{-1}$ of E^{-1} (here again $\boldsymbol{t}^i(x)$ and $\boldsymbol{t}^{i^{-1}}(x)$ are the same point of the total space). The section $\boldsymbol{t}^{i^{-1}}\boldsymbol{t}^i$ of $E^{-1}E$ is central, i.e. it satisfies (3). We also have $\boldsymbol{t}^{i^{-1}}\boldsymbol{t}^i = \boldsymbol{t}^{j^{-1}}\boldsymbol{t}^j$ in U^{ij}; we can thus define a canonical (natural) global central section $\mathcal{I}$ of $E^{-1}E$, thus showing that $E^{-1}E$ is canonically trivial. Explicitly we have $\overline{I}[e^{'-1}, e] = h$ with $e' \triangleleft h = e$. Similarly for EE^{-1}. The space of isomorphism classes of H-bibundles on M [cf. (2)] can now be endowed with a group structure. The unit is the isomorphism class of the trivial product bundle $M \times H$. The inverse of the class represented by E is the class represented by E^{-1}.

Consider two isomorphic bibundles E and E' on M. The choice of a specific isomorphism between E and E' is equivalent to the choice of a global central section of the bibundle $EE^{'-1}$, i.e. a global section that satisfies (3). Indeed, let $\boldsymbol{f}$ be a global section of $EE^{'-1}$, given an element $e \in E$ with base point $x \in M$, there is a unique element $e^{'-1} \in E^{'-1}$ with base point $x \in M$ such that $[e, e^{'-1}] = \boldsymbol{f}(x)$. Then the isomorphism $E \sim E'$ is given by $e \mapsto e'$; it is trivially compatible with the right H-action, and it is compatible with the left H-action and because of the centrality of $\boldsymbol{f}$.

More generally let us consider two isomorphic left H-bundles $E \overset{W}{\sim} E'$ which are not necessarily bibundles. Let us write a generic element $(e, e') \in E \oplus E'$ in the form $(e, hW(e))$ with a properly chosen $h \in H$. We introduce an equivalence relation on

$E \oplus E'$ by $(e, hW(e)) \sim (h'e, hW(h'e))$. Then $T = E \oplus E'/\sim$ is a trivial left H-bundle with global section $\bar{\sigma}([e, hW(e)]) = h^{-1}$ (the left H-action is inherited from E). Recalling the comments after Proposition 4, we equip T with trivial H-bibundle structure and global central section $\bar{\sigma}$. Next we consider the product TE' and observe that any element $[[e, e'_1], e''_2] \in TE'$ can be written as $[[\tilde{e}, W(\tilde{e})], W(\tilde{e})]$ with a unique $\tilde{e} \in E$. We thus have a canonical isomorphism between E and TE' and therefore we write $E = TE'$. Vice versa if T is a trivial bibundle with global central section $\bar{\sigma} : T \to H$ and E,E' are left H-bundles and $E = TE'$, i.e E is canonically isomorphic to TE', then we can consider the isomorphism $E \overset{W}{\sim} E'$ defined by $W([t, e']) = \bar{\sigma}(t)e'$ (here $[t, e']$ is thought of as an element of E because of the identification $E = TE'$). It is then easy to see that the trivial bibundle with section given by this isomorphism W is canonically isomorphic to the initial bibundle T.

We thus conclude that the choice of an isomorphism between two left H-bundles E and E' is equivalent to the choice of a trivialization (the choice of a global central section) of the bibundle T, in formulae

$$E \overset{W}{\sim} E' \Longleftrightarrow E = TE', \tag{16}$$

where T has a given global central section.

Local coordinates description. We recall that an atlas of charts for an H principal left bundle E with base space M is given by a covering $\{U^i\}$ of M, together with sections $\boldsymbol{t}^i : U^i \to E$ (the sections $\boldsymbol{t}^i$ determine isomorphisms between the restrictions of E to U^i and the trivial bundles $U^i \times H$). The transition functions $h^{ij} : U^{ij} \to H$ are defined by $\boldsymbol{t}^i = h^{ij}\boldsymbol{t}^j$. They satisfy on U^{ijk} the cocycle condition

$$h^{ij}h^{jk} = h^{ik}\,.$$

On U^{ij} we have $h^{ij} = {h^{ji}}^{-1}$. A section $\boldsymbol{s} : U \to E$ has local representatives $\{s^i\}$ where $s^i : U \cap U^i \to H$ and in U^{ij} we have

$$s^i h^{ij} = s^j\ . \tag{17}$$

If E is also a bibundle we set

$$\varphi^i \equiv \boldsymbol{\varphi}_{\boldsymbol{t}^i} : U^i \to Aut(H), \tag{18}$$

and we then have $\forall\, h \in H\ ,\ \ \varphi^i(h)h^{ij} = h^{ij}\varphi^j(h)\ ,$ i.e.

$$Ad_{h^{ij}} = \varphi^i \circ {\varphi^j}^{-1}\ . \tag{19}$$

We call the set $\{h^{ij}, \varphi^i\}$ of transition functions and φ^i maps satisfying (19) a set of local data of E. A different atlas of E, i.e. a different choice of sections $\boldsymbol{t}'^i = r^i\boldsymbol{t}^i$ where $r^i : U^i \to H$ (we can always refine the two atlases and thus choose a common covering $\{U^i\}$ of M), gives local data

$$h'^{ij} = r^i h^{ij} {r^j}^{-1}\ \ , \tag{20}$$

$$\varphi'^i = Ad_{r^i} \circ \varphi^i\ \ . \tag{21}$$

We thus define two sets of local data $\{h^{ij}, \varphi^i\}$ and $\{h^{ij}, \varphi^i\}$ to be equivalent if they are related by (20), (21).

One can reconstruct an H-bibundle E from a given set of local data $\{h^{ij}, \varphi^i\}$ relative to a covering $\{U^i\}$ of M. For short we write $E = \{h^{ij}, \varphi^i\}$. The total space of this bundle is the set of triples (x, h, i) where $x \in U^i$, $h \in H$, modulo the equivalence relation $(x, h, i) \sim (x', h', j)$ iff $x = x'$ and $hh^{ij} = h'$. We denote the equivalence class by $[x, h, i]$. The left H action is $h'[x, h, i] = [x, h'h, i]$. The right action, given by $[x, h, i] \triangleleft h' = [x, h\varphi^i(h'), i]$ is well defined because of (19). The h^{ij}'s are transition functions of the atlas given by the sections $\boldsymbol{t}^i : U^i \to E$, $\boldsymbol{t}^i(x) = [x, 1, i]$, and we have $\boldsymbol{\varphi}_{\boldsymbol{t}^i} = \varphi^i$. It is now not difficult to prove that equivalence classes of local data are in one-to-one correspondence with isomorphism classes of bibundles. [Hint: $\boldsymbol{t}'^{i^{-1}}(r^i\boldsymbol{t}^i)$ is central and i independent.]

Given two H bibundles $E = \{h^{ij}, \varphi^i\}$ and $\tilde{E} = \{\tilde{h}^{ij}, \tilde{\varphi}^i\}$ on the same base space M, the product bundle $E\tilde{E}$ has transition functions and left H-actions given by (we can always choose a covering $\{U^i\}$ of M common to E and $\tilde{E}$)

$$E\tilde{E} = \{h^{ij}\varphi^j(\tilde{h}^{ij}), \varphi^i \circ \tilde{\varphi}^i\}. \tag{22}$$

If $\tilde{E}$ is not a bibundle the product $E\tilde{E}$ is still a well defined bundle with transition functions $h^{ij}\varphi^j(\tilde{h^{ij}})$. Associativity of the product (22) is easily verified. One also shows that if $s^i, \tilde{s}^i : U \cap U^i \to H$ are local representatives for the sections $\boldsymbol{s} : U \to E$ and $\tilde{\boldsymbol{s}} : U \to \tilde{E}$ then the local representative for the product section $\boldsymbol{s}\tilde{\boldsymbol{s}} : U \to E\tilde{E}$ is given by

$$s^i\varphi^i(\tilde{s}^i)\,. \tag{23}$$

The inverse bundle of $E = \{h^{ij}, \varphi^i\}$ is

$$E^{-1} = \{\varphi^{j^{-1}}(h^{ij^{-1}}), \varphi^{i^{-1}}\} \tag{24}$$

(we also have $\varphi^{j^{-1}}(h^{ij})^{-1} = \varphi^{i^{-1}}(h^{ij^{-1}})$). If $\boldsymbol{s} : U \to E$ is a section of E with representatives $\{s^i\}$ then $\boldsymbol{s}^{-1} : U \to E^{-1}$, has representatives $\{\varphi^{i^{-1}}(s^{i^{-1}})\}$.

A trivial bundle T with global central section $\boldsymbol{t}$, in an atlas of charts subordinate to a cover U^i of the base space M, reads

$$T = \{f^i f^{j^{-1}},\ Ad_{f^i}\}\,, \tag{25}$$

where the section $\boldsymbol{t} \equiv \boldsymbol{f}^{-1}$ has local representatives $\{f^{i^{-1}}\}$. For future reference notice that $T^{-1} = \{f^{i^{-1}}f^j,\ Ad_{f^{i^{-1}}}\}$ has global central section $\boldsymbol{f} = \{f^i\}$, and that $E\,T^{-1}E^{-1}$ is trivial,

$$E\,T^{-1}E^{-1} = \{\varphi^i(f^{i^{-1}})\varphi^j(f^j),\ Ad_{\varphi^i(f^{i^{-1}})}\}\,. \tag{26}$$

We denote by $\boldsymbol{\varphi}(\boldsymbol{f})$ the global central section $\{\varphi^i(f^i)\}$ of $E\,T^{-1}E^{-1}$. Given an *arbitrary* section $\boldsymbol{s} : U \to E$, we have, in U

$$\boldsymbol{\varphi}(\boldsymbol{f}) = \boldsymbol{s}\,\boldsymbol{f}\,\boldsymbol{s}^{-1}. \tag{27}$$

Proof. $\boldsymbol{f}\boldsymbol{s}^{-1} = \{f^i Ad_{f^{i^{-1}}}(\varphi^{i^{-1}}(s^{i^{-1}}))\} = \{\varphi^{i^{-1}}(s^{i^{-1}})f^i\}$ and therefore $\boldsymbol{s}\,\boldsymbol{f}\,\boldsymbol{s}^{-1} = \{\varphi^i(f^i)\} = \boldsymbol{\varphi}(\boldsymbol{f})$. Property (27) is actually the defining property of $\boldsymbol{\varphi}(\boldsymbol{f})$. Without using an atlas of charts, we define the global section $\boldsymbol{\varphi}(\boldsymbol{f})$ of $ET^{-1}E^{-1}$ to be that section that locally satisfies (27). The definition is well given because centrality of $\boldsymbol{f}$ implies that

$\boldsymbol{\varphi}(\boldsymbol{f})$ is independent from $\boldsymbol{s}$. Centrality of the global section $\boldsymbol{f}$ also implies that $\boldsymbol{\varphi}(\boldsymbol{f})$ is a global central section. If $\overline{\sigma}$ is the global central section of T, the corresponding global section $\overline{\sigma}'$of $E\,T^{-1}E^{-1}$ is $\overline{\sigma}'[e, t, e^{'-1}] = \varphi_e(\overline{\sigma}(t))h$ with $e = he'$. □

The pull-back of a bi-principal bundle is again a bi-principal bundle. It is also easy to verify that the pull-back commutes with the product.

***D*-*H* bundles.** We can generalize the notion of a bibundle by introducing the concept of a crossed module.

We say that H is a crossed D-module [35] if there is a group homomorphism $\alpha : H \to D$ and an action of D on H denoted as $(d, h) \mapsto {}^d h$ such that

$$\forall h,\ h' \in H\ ,\ {}^{\alpha(h)}h' = hh'h^{-1} \tag{28}$$

and for all $h \in H, d \in D$,

$$\alpha({}^d h) = d\alpha(h)d^{-1} \tag{29}$$

holds true.

Notice in particular that $\alpha(H)$ is normal in D. The canonical homomorphism $Ad : H \to Aut(H)$ and the canonical action of $Aut(H)$ on H define on H the structure of a crossed $Aut(H)$-module. Given a D-bundle Q we can use the homomorphism $t : D \to Aut(H),\ t \circ \alpha = Ad$ to form $Aut(H) \times_D Q$.

Definition 5. *Consider a left D-bundle Q on M such that the $Aut(H)$-bundle $Aut(H)\times_D Q$ is trivial. Let σ be a global section of $Aut(H) \times_D Q$. We call the couple (Q, σ) a D-H bundle.*

Notice that if $\overline{\sigma} : Aut(H) \times_D Q \ni [\eta, q] \mapsto \overline{\sigma}([\eta, q]) \in Aut(H)$ is a global section of $Aut(H) \times_D Q$ then
i) the automorphism $\boldsymbol{\psi}_q \in Aut(H)$ defined by

$$\boldsymbol{\psi}_q \equiv \overline{\sigma}([id, q]) \tag{30}$$

is D-equivariant,

$$\boldsymbol{\psi}_{dq}(h) = {}^d\boldsymbol{\psi}_q(h), \tag{31}$$

ii) the homomorphism $\boldsymbol{\xi}_q : H \to D$ defined by

$$\boldsymbol{\xi}_q(h) \equiv \alpha \circ \boldsymbol{\psi}_q(h) \tag{32}$$

gives a fiber preserving action $q \triangleleft h \equiv \boldsymbol{\xi}_q(h)q$ of H on the right, commuting with the left D-action, i.e.

$$\forall\ h \in H, d \in D, q \in Q,\quad (d\,q) \triangleleft h = d(q \triangleleft h)\ . \tag{33}$$

Vice versa we easily have

Proposition 6. *Let H be a crossed D-module. If Q is a left D bundle admitting a right fiber preserving H action commuting with the left D action, and the homomorphism $\boldsymbol{\xi}_q : H \to D$, defined by $q \triangleleft h = \boldsymbol{\xi}_q(h)q$ is of the form (32) with a D-equivariant $\boldsymbol{\psi}_q \in Aut(H)$ [cf. (31)], then Q is a D-H bundle.*

There is an obvious notion of an isomorphism between two D-H bundles (Q, σ) and $(\tilde{Q}, \tilde{\sigma})$; it is an isomorphism between D-bundles Q and $\tilde{Q}$ intertwining between σ and $\tilde{\sigma}$. In the following we denote a D-H bundle (Q, σ) simply as Q without spelling out explicitly the choice of a global section σ of $Aut(H) \times_D Q$. As in the previous section out of a given isomorphism we can construct a trivial D-bibundle Z with a global central section z^{-1} such that $\tilde{Q}$ and ZQ are canonically identified and we again write this as $\tilde{Q} = ZQ$. The $\boldsymbol{\psi}$ map of Z is given by $Ad_{\overline{z}^{-1}}$.

Note that the product of a trivial D-bibundle Z and a D-H bundle Q is well-defined and gives again a D-H bundle.

The trivial bundle $M \times D \to M$, with right H-action given by $(x, d) \triangleleft h = (x, d\alpha(h))$, is a D-H bundle, we have $\boldsymbol{\psi}_{(x,d)}(h) = {}^d h$. A D-H bundle Q is trivial if it is isomorphic to $M \times D$. Similarly to the case of a bibundle we have that a D-H bundle is trivial iff it has a global section $\boldsymbol{\sigma}$ which is central with respect to the left and the right actions of H on Q,

$$\boldsymbol{\sigma}(x) \triangleleft h = \alpha(h)\boldsymbol{\sigma}(x) \; . \tag{34}$$

The corresponding map $\overline{\sigma} : Q \to D$ is then bi-equivariant

$$\overline{\sigma}(dq \triangleleft h) = d\overline{\sigma}(q)\alpha(h). \tag{35}$$

The pull-back of a D-H bundle is again a D-H bundle.

The trivial bundle $Aut(H) \times_H E$ (cf. Proposition 4) is an $Aut(H)$-H bundle.

Proof. The left $Aut(H)$ and the right H actions commute, and they are related by $[\eta, e]h = Ad_{\eta(\boldsymbol{\varphi}_e(h))}[\eta, e]$; we thus have $\boldsymbol{\psi}_{[\eta,e]} = \eta \circ \boldsymbol{\varphi}_e$, which structures $Aut(H) \times_H E$ into an $Aut(H)$-H bundle. Moreover $\overline{\sigma}([\eta, e]) = \eta \circ \boldsymbol{\varphi}_e$ is bi-equivariant, hence $Aut(H) \times_H E$ is isomorphic to $M \times Aut(H)$ as an $Aut(H)$-H bundle.

More generally, we can use the left H-action on D given by the homomorphism $\alpha : H \to D$ to associate to a bibundle E the bundle $D \times_H E$. The H-automorphism $\boldsymbol{\psi}_{[d,e]}$ defined by $\boldsymbol{\psi}_{[d,e]} = {}^d\boldsymbol{\varphi}_e(h)$ endows $D \times_H E$ with a D-H bundle structure. □

There is the following canonical construction associated with a D-H module. We use the D-action on H to form the associated bundle $H \times_D Q$. Using the equivariance property (31) of $\boldsymbol{\psi}_q$ we easily get the following proposition.

Proposition 7. *The associated bundle $H \times_D Q$ is a trivial H-bibundle with actions $h'[h, q] = [\boldsymbol{\psi}_q(h')h, q]$ and $[h, q] \triangleleft h' = [h\boldsymbol{\psi}_q(h'), q]$, and with global central section given by $\bar{\sigma}([h, q]) = \boldsymbol{\psi}_q^{-1}(h)$.*

The local coordinate description of a D-H bundle Q is similar to that of a bibundle. We thus omit the details. We denote by d^{ij} the transition functions of the left principal D-bundle Q. Instead of local maps (18) we now have local maps $\psi^i : U^i \to Aut(H)$, such that (compare to (19))

$${}^{d_{ij}}h = \psi^i \circ {\psi^j}^{-1}(h). \tag{36}$$

The product QE of a D-H bundle Q with a H-bibundle E can be defined as in (11), (12). The result is again a D-H bundle. If Q is locally given by $\{d^{ij}, \psi^i\}$ and H is locally given by $\{h^{ij}, \varphi^i\}$ then QE is locally given by $\{d^{ij}\xi^j(h^{ij}), \psi^i \circ \varphi^i\}$. Moreover if $Z = \{z^i z^{j^{-1}}, Ad_{z^i}\}$ is a trivial D-bibundle with section $z^{-1} = \{z^{i^{-1}}\}$, then the well-defined D-H bundle ZQ is locally given by $\{z^i d^{ij} z^{j^{-1}}, {}^{z^i} \circ \psi^i\}$. We have the following associativity property:

$$(ZQ)E = Z(QE) \,, \tag{37}$$

and the above products commute with pull-backs.

Given a D-H bundle Q and a trivial H-bibundle T with section $\boldsymbol{f}^{-1}$ there exists a unique trivial D-bibundle $\boldsymbol{\xi}(T)$ with section $\boldsymbol{\xi}(\boldsymbol{f}^{-1})$ such that

$$QT = \boldsymbol{\xi}(T)Q \,, \tag{38}$$

i.e. such that for any local section $\boldsymbol{s}$ of Q one has $\boldsymbol{s}\boldsymbol{f}^{-1} = \boldsymbol{\xi}(\boldsymbol{f}^{-1})\boldsymbol{s}$. The notations $\boldsymbol{\xi}(T)$, $\boldsymbol{\xi}(\boldsymbol{f}^{-1})$ are inferred from the local expressions of these formulae. Indeed, if locally $T = \{f^i f^{j^{-1}}, Ad_{f^i}\}$ and $\boldsymbol{f} = \{f^i\}$, then $\boldsymbol{\xi}(T) = \{\xi^i(f^i)\xi^j(f^j)^{-1}, Ad_{\xi^i(f^i)}\}$ and $\boldsymbol{\xi}(\boldsymbol{f}) = \{\xi^i(f^i)\}$.

Finally, as was the case for bibundles, we can reconstruct a D-H bundle Q from a given set of local data $\{d^{ij}, \psi^i\}$ relative to a covering $\{U^i\}$ of M. Equivalence of local data for D-H bundles is defined in such a way that isomorphic (equivalent) D-H bundles have equivalent local data, and vice versa.

3. Connection and Curvature on Principal Bibundles

Since a bibundle E on M is a bundle on M that is both a left principal H-bundle and a right principal H-bundle, one could then define a connection on a bibundle to be a one-form $\boldsymbol{a}$ on E that is both a left and a right principal H-bundle connection. This definition [more precisely the requirement $\mathcal{A}^r = 0$ in (49)] preserves the left-right symmetry property of the bibundle structure, but it turns out to be too restrictive, indeed not always a bibundle can be endowed with such a connection, and furthermore the corresponding curvature is valued in the center of H. If we insist in preserving the left-right symmetry structure we are thus led to generalize (relax) equivariance property of a connection and thus define the notion of connection. In this section we will see that a connection on a bibundle is a couple $(\boldsymbol{a}, A)$, where $\boldsymbol{a}$ is a one-form on E with values in Lie(H) while A is a Lie$(Aut(H))$ valued one-form on M. In particular we see that if $A = 0$ then $\boldsymbol{a}$ is a left connection on E where E is considered just as a left principal bundle. We recall that a connection $\boldsymbol{a}$ on a left principal bundle E satisfies [31].

i) The pull-back of $\boldsymbol{a}$ on the fibers of E is the right invariant Maurer-Cartan one-form. Explicitly, let $e \in E$, let $g(t)$ be a curve from some open interval $(-\varepsilon, \varepsilon)$ of the real line into the group H with $g(0) = 1_H$, and let $[g(t)]$ denote the corresponding tangent vector in 1_H and $[g(t)e]$ the vertical vector based in $e \in E$. Then

$$\boldsymbol{a}[g(t)e] = -[g(t)] \,. \tag{39}$$

Equivalently $\boldsymbol{a}[g(t)e] = \zeta_{[g(t)]}$, where $\zeta_{[g(t)]}$ is the right-invariant vector field associated with $[g(t)] \in$ Lie(H), i.e. $\zeta_{[g(t)]}|_h = -[g(t)h]$.

ii) Under the left H-action we have the equivariance property

$$l^{h*}\boldsymbol{a} = Ad_h \boldsymbol{a}, \tag{40}$$

where l^h denotes left multiplication by $h \in H$.

Now property *i)* is compatible with the bibundle structure on E in the following sense, if $\boldsymbol{a}$ satisfies *i)* then $-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})$ pulled back on the fibers is the left invariant Maurer-Cartan one-form

$$-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})[eg(t)] = [g(t)] \,, \tag{41}$$

here with abuse of notation we use the same symbol $\boldsymbol{\varphi}^{-1}$ for the map $\boldsymbol{\varphi}^{-1} \,:\, E \times H \to H$ and its differential map $\boldsymbol{\varphi}_*^{-1} \,:\, E \times \mathrm{Lie}(H) \to \mathrm{Lie}(H)$. Property (41) is equivalent to $\boldsymbol{a}[g(t)e] = \xi_{[g(t)]}$, where $\xi_{[g(t)]}$ is the left-invariant vectorfield associated with $[g(t)] \in \mathrm{Lie}(H)$, i.e. $\xi_{[g(t)]}|_h = [hg(t)]$. Property (41) is easily proven,

$$-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})[eg(t)] = -\boldsymbol{\varphi}_e^{-1}(\boldsymbol{a}[\boldsymbol{\varphi}_e(g(t))e]) = \boldsymbol{\varphi}_e^{-1}[\boldsymbol{\varphi}_e(g(t))] = [g(t)] \,.$$

Similarly, on the vertical vectors v_V of E we have $\left(r^{h^*}\boldsymbol{a} - \boldsymbol{a}\right)(v_V) = 0 \,,\ \left(l^{h^*}\varphi^{-1}(\boldsymbol{a}) - \varphi^{-1}(\boldsymbol{a})\right)(v_V) = 0$ and

$$\left(l^{h^*}\boldsymbol{a} - Ad_h\boldsymbol{a}\right)(v_V) = 0 \quad , \tag{42}$$

$$\left(r^{h^*}\varphi^{-1}(\boldsymbol{a}) - Ad_{h^{-1}}\varphi^{-1}(\boldsymbol{a})\right)(v_V) = 0 \quad . \tag{43}$$

On the other hand property *ii)* is not compatible with the bibundle structure, indeed if $\boldsymbol{a}$ satisfies (40) then it can be shown (see later) that $-\boldsymbol{\varphi}^{-1}(\boldsymbol{a})$ satisfies

$$r^{h^*}\boldsymbol{\varphi}^{-1}(\boldsymbol{a}) = Ad_{h^{-1}}\boldsymbol{\varphi}^{-1}(\boldsymbol{a}) - p^*T'(h^{-1}), \tag{44}$$

where $T'(h)$ is a given one-form on the base space M, and $p \,:\, E \to M$. In order to preserve the left-right symmetry structure we are thus led to generalize (relax) the equivariance property *ii)* of a connection. Accordingly with (42) and (44) we thus require

$$l^{h^*}\boldsymbol{a} = Ad_h\boldsymbol{a} + p^*T(h), \tag{45}$$

where $T(h)$ is a one-form on M. From (45) it follows

$$T(hk) = T(h) + Ad_hT(k) \,, \tag{46}$$

i.e., T is a 1-cocycle in the group cohomology of H with values in $\mathrm{Lie}(H) \otimes \Omega^1(M)$. Of course if T is a coboundary, i.e. $T(h) = h\boldsymbol{\chi}h^{-1} - \boldsymbol{\chi}$ with $\boldsymbol{\chi} \in \mathrm{Lie}(H) \otimes \Omega^1(M)$, then $\boldsymbol{a} + \boldsymbol{\chi}$ is a connection. We thus see that Eq. (45) is a nontrivial generalization of the equivariance property only if the cohomology class of T is nontrivial.

Given an element $X \in \mathrm{Lie}(Aut(H))$, we can construct a corresponding 1-cocycle T_X in the following way,

$$T_X(h) \equiv [he^{tX}(h^{-1})] \,,$$

where $[he^{tX}(h^{-1})]$ is the tangent vector to the curve $he^{tX}(h^{-1})$ at the point 1_H; if H is normal in $Aut(H)$ then $e^{tX}(h^{-1}) = e^{tX}h^{-1}e^{-tX}$ and we simply have $T_X(h) = hXh^{-1} - X$. Given a $\mathrm{Lie}(Aut(H))$-valued one-form A on M, we write $A = A^\rho X^\rho$, where $\{X^\rho\}$ is a basis of $\mathrm{Lie}(Aut(H))$. We then define T_A as

$$T_A \equiv A^\rho T_{X^\rho} \,. \tag{47}$$

Obviously, $p^*T_A = T_{p^*A}$. Following these considerations we define

Definition 8. *A* 2*-connection on* E *is a couple* $(\boldsymbol{a}, A)$ *where:*

i) $\boldsymbol{a}$ *is a* $\mathrm{Lie}(H)$ *valued one-form on* E *such that its pull-back on the fibers of* E *is the right invariant Maurer-Cartan one-form, i.e.* $\boldsymbol{a}$ *satisfies (39),*
ii) A *is a* $\mathrm{Lie}(Aut(H))$ *valued one-form on* M*,*
iii) the couple $(\boldsymbol{a}, A)$ *satisfies*

$$l^{h^*}\boldsymbol{a} = Ad_h\boldsymbol{a} + p^*T_A(h)\ . \tag{48}$$

This definition seems to break the left-right bibundle symmetry since, for example, only the left H action has been used. This is indeed *not* the case

Theorem 9. *If* $(\boldsymbol{a}, A)$ *is a* 2*-connection on* E *then* $(\boldsymbol{a}^r, A^r)$*, where* $\boldsymbol{a}^r \equiv -\boldsymbol{\varphi}^{-1}(\boldsymbol{a})$*, satisfies (39) and (48) with the left* H *action replaced by the right* H *action (and right-invariant vectorfields replaced by left-invariant vectorfields), i.e. it satisfies (41) and*

$$r^{h^*}\boldsymbol{a}^r = Ad_{h^{-1}}\boldsymbol{a}^r + p^*T_{A^r}(h^{-1})\ , \tag{49}$$

here A^r *is the one-form on* M *uniquely defined by the property*

$$p^*A^r = \boldsymbol{\varphi}^{-1}(p^*A + ad_{\boldsymbol{a}})\boldsymbol{\varphi} + \boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}\ . \tag{50}$$

Proof. First we observe that from (39) and (48) we have

$$l^{h'^*}\boldsymbol{a} = Ad_{h'}\boldsymbol{a} + p^*T_A(h') + h'dh'^{-1}, \tag{51}$$

where now $h' = h'(e)$, i.e. h' is an H-valued function on the total space E. Setting $h' = \boldsymbol{\varphi}(h)$, with $h \in H$ we have

$$\begin{aligned} r^{h^*}\boldsymbol{a} = l^{\boldsymbol{\varphi}(h)^*}\boldsymbol{a} &= Ad_{\boldsymbol{\varphi}(h)}\boldsymbol{a} + p^*T_A(\boldsymbol{\varphi}(h)) + \boldsymbol{\varphi}(h)d\boldsymbol{\varphi}(h^{-1}) \\ &= \boldsymbol{a} + \boldsymbol{\varphi}(T_{\mathcal{A}^r}(h)); \end{aligned} \tag{52}$$

in equality (52) we have defined

$$\mathcal{A}^r \equiv \boldsymbol{\varphi}^{-1}(p^*A + ad_{\boldsymbol{a}})\boldsymbol{\varphi} + \boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}\,. \tag{53}$$

Equality (52) holds because of the following properties of T,

$$T_{\boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}}(h) = \boldsymbol{\varphi}^{-1}\left(\boldsymbol{\varphi}(h)d\boldsymbol{\varphi}(h^{-1})\right)\ , \tag{54}$$

$$T_{\boldsymbol{\varphi}^{-1}p^*A\boldsymbol{\varphi}}(h) = \boldsymbol{\varphi}^{-1}\left(T_{p^*A}(\boldsymbol{\varphi}(h))\right)\ , \tag{55}$$

$$T_{ad_{\boldsymbol{a}}}(h) = Ad_h\boldsymbol{a} - \boldsymbol{a}\ . \tag{56}$$

From (52), applying $\boldsymbol{\varphi}^{-1}$ and then using (7) one obtains

$$r^{h^*}\boldsymbol{a}^r = Ad_{h^{-1}}\boldsymbol{a}^r + T_{\mathcal{A}^r}(h^{-1})\ . \tag{57}$$

Finally, comparing (43) with (57) we deduce that for all $h \in H$, $T_{\mathcal{A}^r}(h)(v_V) = 0$, and this relation is equivalent to $\mathcal{A}^r(v_V) = 0$. In order to prove that $\mathcal{A}^r = p^*A^r$, where A^r is a one-form on M, we then just need to show that $\mathcal{A}^r$ is invariant under the H action, $l^{h^*}\mathcal{A}^r = \mathcal{A}^r$. This is indeed the case because $l^{h^*}(\boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}) = \boldsymbol{\varphi}^{-1}Ad_{h^{-1}}dAd_h\boldsymbol{\varphi} = \boldsymbol{\varphi}^{-1}d\boldsymbol{\varphi}$, and because

$$l^{h*}\left(\boldsymbol{\varphi}^{-1}(p^*A+ad_{\boldsymbol{a}})\boldsymbol{\varphi}\right)=\boldsymbol{\varphi}^{-1}Ad_{h^{-1}}(p^*A+l^{h^*}ad_{\boldsymbol{a}})Ad_h\boldsymbol{\varphi}$$
$$=\boldsymbol{\varphi}^{-1}\left(Ad_{h^{-1}}p^*AAd_h+ad_{\boldsymbol{a}}+ad_{Ad_{h^{-1}}T_{p^*A}(h)}\right)\boldsymbol{\varphi}$$
$$=\boldsymbol{\varphi}^{-1}(p^*A+ad_{\boldsymbol{a}})\boldsymbol{\varphi}\ . \qquad \square$$

Notice that if $(\boldsymbol{a}, A)$ and $(\boldsymbol{a'}, A')$ are 2-connections on E then so is the affine sum

$$\left(p^*(\lambda)\boldsymbol{a}+(1-p^*\lambda)\boldsymbol{a'}\ ,\ \lambda A+(1-\lambda)A'\right) \tag{58}$$

for any (smooth) function λ on M.

As in the case of principal bundles we define a vector $v\in T_eE$ to be horizontal if $\boldsymbol{a}(v)=0$. The tangent space T_eE is then decomposed in the direct sum of its horizontal and vertical subspaces; for all $v\in T_eE$, we write $v=\mathrm{H}v+\mathrm{V}v$, where $Vv=[e^{-t\boldsymbol{a}(v)}e]$. The space of horizontal vectors is however not invariant under the usual left H-action, indeed

$$\boldsymbol{a}(l^h{}_*(\mathrm{H}v))=T_A(h)(v)\ ,$$

in this formula, as well as in the sequel, with abuse of notation T_A stands for T_{p^*A}.

Remark 10. It is possible to construct a new left H-action $\mathcal{L}_*$ on T_*E, that is compatible with the direct sum decomposition $T_*E=\mathrm{H}T_*E+\mathrm{V}T_*E$. We first define, for all $h\in H$,

$$\begin{aligned}&L_A^h\ :\ T_*E\to \mathrm{V}T_*E\ ,\\ &T_eE\ni v\mapsto [e^{tT_A(h)(v)}he]\in \mathrm{V}T_{he}E\ ,\end{aligned} \tag{59}$$

and notice that L_A^h on vertical vectors is zero, therefore $L_A^h\circ L_A^h=0$. We then consider the tangent space map,

$$\mathcal{L}_*^h\equiv l_*^h+L_A^h\ . \tag{60}$$

It is easy to see that $\mathcal{L}_*^{hk}=\mathcal{L}_*^h\circ\mathcal{L}_*^k$ and therefore that $\mathcal{L}_*$ defines an action of H on T_*H. We also have

$$\mathcal{L}^{h^*}\boldsymbol{a}=Ad_h\boldsymbol{a}\ . \tag{61}$$

Finally the action $\mathcal{L}_*^h$ preserves the horizontal and vertical decomposition $T_*E=\mathrm{H}T_*E+\mathrm{V}T_*E$, indeed

$$\mathrm{H}\mathcal{L}_*^hv=\mathcal{L}_*^h\mathrm{H}v\ ,\ \mathrm{V}\mathcal{L}_*^hv=\mathcal{L}_*^h\mathrm{V}v. \tag{62}$$

Proof. Let $v=[\gamma(t)]$. Then $\mathrm{H}\mathcal{L}_*^hv=\mathrm{H}l_*^hv=[h\gamma(t)]-[e^{-t\boldsymbol{a}[h\gamma(t)]}e]=[h\gamma(t)]+[e^{t(l^{h^*}\boldsymbol{a})(v)}he]=[h\gamma(t)]+[he^{t\boldsymbol{a}(v)}e]+[e^{tT_A(h)(v)}he]=\mathcal{L}_*^h(v+[e^{t\boldsymbol{a}(v)}e])=\mathcal{L}_*^h\mathrm{H}v$. $\square$

Curvature. An n-form $\boldsymbol{\vartheta}$ is said to be horizontal if $\boldsymbol{\vartheta}(u_1,u_2,\dots u_n)=0$ whenever at least one of the vectors $u_i\in T_eE$ is vertical. The exterior covariant derivative $D\boldsymbol{\omega}$ of an n-form $\boldsymbol{\omega}$ is the $(n+1)$-horizontal form defined by

$$D\boldsymbol{\omega}(v_1, v_2, \dots, v_{n+1}) \equiv d\boldsymbol{\omega}(\mathrm{H}v_1, \mathrm{H}v_2, \dots, \mathrm{H}v_{n+1}) \\ -(-1)^n T_A(\boldsymbol{\omega})(\mathrm{H}v_1, \mathrm{H}v_2, \dots, \mathrm{H}v_{n+1}) \tag{63}$$

for all $v_i \in T_e E$ and $e \in E$. In the above formula $T_A(\boldsymbol{\omega})$ is defined by

$$T_A(\boldsymbol{\omega}) \equiv \boldsymbol{\omega}^\alpha \wedge T_{A*}(X^\alpha) , \tag{64}$$

where $T_{A*} : \mathrm{Lie}(H) \to \mathrm{Lie}(H) \otimes \Omega^1(E)$ is the differential of $T_A : H \to \mathrm{Lie}(H) \otimes \Omega^1(E)$. If H is normal in $Aut(H)$ we simply have $T_A(\boldsymbol{\omega}) = \boldsymbol{\omega}^\rho \wedge p^* A^\sigma [X^\rho, X^\sigma] = [\boldsymbol{\omega}, p^* A]$, where now X^ρ are generators of $\mathrm{Lie}(Aut(H))$.

The 2-curvature of the 2-connection $(\boldsymbol{a}, A)$ is given by the couple

$$(\boldsymbol{k}, K) \equiv \big(D\boldsymbol{a} \, , \, dA + A \wedge A\big). \tag{65}$$

We have the Cartan structural equation

$$\boldsymbol{k} = d\boldsymbol{a} + \frac{1}{2}[\boldsymbol{a}, \boldsymbol{a}] + T_A(\boldsymbol{a}) , \tag{66}$$

where $\frac{1}{2}[\boldsymbol{a}, \boldsymbol{a}] = \frac{1}{2}\boldsymbol{a}^\alpha \wedge \boldsymbol{a}^\beta [X^\alpha, X^\beta] = \boldsymbol{a} \wedge \boldsymbol{a}$ with $X^\alpha \in \mathrm{Lie}(H)$.

The proof of Eq. (66) is very similar to the usual proof of the Cartan structural equation for principal bundles. One has just to notice that the extra term $T_A(\boldsymbol{a})$ is necessary since $d\boldsymbol{a}(\mathrm{V}v, \mathrm{H}u) = -\boldsymbol{a}([\mathrm{V}v, \mathrm{H}u]) = T_{A*}(\boldsymbol{a}(\mathrm{V}v))(\mathrm{H}u) = -T_A(\boldsymbol{a})(\mathrm{V}v, \mathrm{H}u)$.

The 2-curvature $(\boldsymbol{k}, K)$ satisfies the following generalized equivariance property:

$$l^{h^*}\boldsymbol{k} = Ad_h \boldsymbol{k} + T_K(h) , \tag{67}$$

where with abuse of notation we have written $T_K(h)$ instead of $T_{p^*K}(h)$. We also have the Bianchi identities, $dK + A \wedge K = 0$ and

$$D\boldsymbol{k} = 0 . \tag{68}$$

Given an horizontal n-form $\boldsymbol{\vartheta}$ on E that is Θ-equivariant, i.e. that satisfies $l^{h^*}\boldsymbol{\vartheta} = Ad_h \boldsymbol{\vartheta} + T_\Theta(h)$, where Θ is an n-form on M, we have the structural equation

$$D\boldsymbol{\vartheta} = d\boldsymbol{\vartheta} + [\boldsymbol{a}, \boldsymbol{\vartheta}] + T_\Theta(\boldsymbol{a}) - (-1)^n T_A(\boldsymbol{\vartheta}), \tag{69}$$

where $[\boldsymbol{a}, \boldsymbol{\vartheta}] = \boldsymbol{a}^\alpha \wedge \boldsymbol{\vartheta}^\beta [X^\alpha, X^\beta] = \boldsymbol{a} \wedge \boldsymbol{\vartheta} - (-1)^n \boldsymbol{\vartheta} \wedge \boldsymbol{a}$. The proof is again similar to the usual one (where $\Theta = 0$) and is left to the reader. We also have that $D\boldsymbol{\vartheta}$ is $(d\Theta + [A, \Theta])$-equivariant,

$$l^{h^*} D\boldsymbol{\vartheta} = Ad_h \boldsymbol{\vartheta} + T_{d\Theta + [A, \Theta]}(h) . \tag{70}$$

Combining (69) and (68) we obtain the explicit expression of the Bianchi identity

$$d\boldsymbol{k} + [\boldsymbol{a}, \boldsymbol{k}] + T_K(\boldsymbol{a}) - T_A(\boldsymbol{k}) = 0. \tag{71}$$

We also have

$$D^2 \boldsymbol{\vartheta} = [\boldsymbol{k}, \boldsymbol{\vartheta}] + T_\Theta(\boldsymbol{k}) - (-1)^n T_K(\boldsymbol{\vartheta}) . \tag{72}$$

As was the case for the 2-connection $(\boldsymbol{a}, A)$, also for the 2-curvature $(\boldsymbol{k}, K)$ we can have a formulation using the right H action instead of the left one. Indeed one can prove that if $(\boldsymbol{k}, K)$ is a 2-curvature then $(\boldsymbol{k}^r, K^r)$ where

$$\boldsymbol{k}^r = -\boldsymbol{\varphi}^{-1}(\boldsymbol{k}) \,,\ K^r = \boldsymbol{\varphi}^{-1}(K + ad_{\boldsymbol{k}})\boldsymbol{\varphi}$$

is the right 2-curvature associated with the right 2-connection $(\boldsymbol{a}^r, A^r)$. In other words we have that $\boldsymbol{k}^r$ is horizontal and that

$$\boldsymbol{k}^r = \boldsymbol{k}_{\boldsymbol{a}^r} \,,\ K^r = K_{A^r}$$

(for the proof we used $T_{A^r}(\boldsymbol{\varphi}^{-1}(X)) = \boldsymbol{\varphi}^{-1}([X, \boldsymbol{a}] + T_A(X)) + d\boldsymbol{\varphi}^{-1}(X)$, $X \in \mathrm{Lie}(H)$). We also have

$$r^{h*}\boldsymbol{k}^r = Ad_{h^{-1}}\boldsymbol{k}^r + T_{K^r}(h^{-1}) \,. \tag{73}$$

More in general consider the couple $(\boldsymbol{\vartheta}, \Theta)$ where $\boldsymbol{\vartheta}$, is an horizontal n-form on E that is Θ-equivariant. Then we have the couple $(\boldsymbol{\vartheta}^r, \Theta^r)$, where $\boldsymbol{\vartheta}^r = -\boldsymbol{\varphi}^{-1}(\boldsymbol{\vartheta})$ is an horizontal n-form on E that is right Θ^r-equivariant,

$$r^{h*}\boldsymbol{\vartheta}^r = Ad_{h^{-1}}\boldsymbol{\vartheta}^r + T_{\Theta^r}(h^{-1}), \tag{74}$$

with $\Theta^r = \boldsymbol{\varphi}^{-1}(\Theta + ad_{\boldsymbol{\vartheta}})\boldsymbol{\varphi}$.

The pull-back of a 2-connection (or of a horizontal form) on a principal H-bibundle is a 2-connection (horizontal form) on the pulled back principal H-bibundle, moreover the exterior covariant derivative -and in particular the definition of 2-curvature- commutes with the pull-back operation.

Local coordinates description.. Let's consider the sections $\boldsymbol{t}^i : U^i \to E$ subordinate to the covering $\{U^i\}$ of M. Let $\iota : H \times U^i \to p^{-1}(U^i) \subset E$ be the local trivialization of E induced by $\boldsymbol{t}^i$ according to $\iota(x, h) = h\boldsymbol{t}^i(x)$, where $x \in M$. We define the one-forms on $U^i \subset M$,

$$a^i = \boldsymbol{t}^{i*}\boldsymbol{a} \,, \tag{75}$$

then, the local expression of $\boldsymbol{a}$ is $ha^ih^{-1} + T_A(h) + hdh^{-1}$, more precisely,

$$\iota^*(\boldsymbol{a})_{(x,h)}(v_M, v_H) = ha^i(x)h^{-1}(v_M) + T_{A(x)}(h)(v_M) + hdh^{-1}(v_H) \,, \tag{76}$$

where v_M ,v_H are respectively tangent vectors of $U^i \subset M$ at x, and of H at h, and where $-hdh^{-1}$ denotes the Maurer—Cartan one-form on H evaluated at $h \in H$. Similarly the local expression for $\boldsymbol{k}$ is $hk^ih^{-1} + T_K(h)$, where $k^i = \boldsymbol{t}^{i*}\boldsymbol{k}$.

Using the sections $\{\boldsymbol{t}^i\}$ we also obtain an explicit expression for A^r,

$$A^r = \boldsymbol{t}^{i*}\mathcal{A}^r = \varphi_i^{-1}(A + ad_{a^i})\varphi_i + \varphi_i^{-1}d\varphi_i \,. \tag{77}$$

Of course in U^{ij} we have $\boldsymbol{t}^{i*}\mathcal{A}^r = \boldsymbol{t}^{j*}\mathcal{A}^r$, so that A^r is defined on all M. In U^{ij} we also have $a^i = h^{ij}a^jh^{ij^{-1}} + h^{ij}dh^{ij^{-1}} + T_A(h^{ij})$ and $k^i = h^{ij}k^jh^{ij^{-1}} + T_K(h^{ij})$.

Sum of 2-connections. If the group H is abelian, on the product bundle E_1E_2 there is the natural connection $\boldsymbol{a}_1+\boldsymbol{a}_2$ obtained from the connections $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ on E_1 and E_2. In this subsection we generalize to the nonabelian case the sum of connections. Consider the following diagram:

$$\begin{tikzcd} E_1 \oplus E_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_\oplus"] & E_2 \\ E_1 & E_1E_2 \end{tikzcd} \tag{78}$$

and let $(\boldsymbol{a}_1, A_2)$ be a 2-connection on E_1 and $(\boldsymbol{a}_2, A_2)$ a 2-connection on E_2. Recalling the definition of the product E_1E_2, we see that the one-form on $E_1 \oplus E_2$

$$\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2) \tag{79}$$

is the pull-back of a one-form on E_1E_2 iff, for all $v_1 \in T_{e_1}E$, $v_2 \in T_{e_2}E$ and $h \in H$,

$$\begin{aligned} &\big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big)_{(e_1,e_2)}(v_1, v_2) \\ &\quad = \big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big)_{(e_1h^{-1},he_2)}(r_*^h v_1, l_*^h v_2) \\ &\qquad + \big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big)_{(e_1h^{-1},he_2)}([e_1h^{-1}(t)], [h(t)e_2]), \end{aligned}$$

where $h(t)$ is an arbitrary curve in H with $h(0) = 1_H$. Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ satisfy the Cartan-Maurer condition (39) the last addend vanishes identically and therefore the expression is equivalent to

$$\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2) = rl^{h*}\big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big), \tag{80}$$

where

$$\begin{aligned} rl^h \;:\; E_1 \oplus E_2 &\to E_1 \oplus E_2 \;, \\ (e_1, e_2) &\mapsto (e_1h^{-1}, he_2) \;. \end{aligned}$$

Now, using (7), and then (52) we have

$$\begin{aligned} rl^{h*}\big(\pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2)\big) &= \pi_1^* r^{h^{-1}*}\boldsymbol{a}_1 + \boldsymbol{\varphi}_1 Ad_{h^{-1}}(\pi_2^* l^{h*}\boldsymbol{a}_2) \\ &= \pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2) \\ &\quad + \boldsymbol{\varphi}_1\big(\pi_1^* T_{A_1^r}(h^{-1}) + \pi_2^* Ad_{h^{-1}} T_{A_2}(h)\big) \end{aligned}$$

and the last addend vanishes iff

$$A_2 = {A_1}^r \;. \tag{81}$$

In conclusion, when (81) holds, there exists a one-form on E_1E_2, denoted by $\boldsymbol{a}_1+\boldsymbol{a}_2$, such that

$$\pi_\oplus^*(\boldsymbol{a}_1+\boldsymbol{a}_2) = \pi_1^*\boldsymbol{a}_1 + \boldsymbol{\varphi}_1(\pi_2^*\boldsymbol{a}_2). \tag{82}$$

From this expression it is easy to see that $(\boldsymbol{a}_1+\boldsymbol{a}_2, A_1)$ is a 2-connection on E_1E_2. We then say that $(\boldsymbol{a}_1, A_1)$ and $(\boldsymbol{a}_2, A_2)$ (or simply that $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$) are *summable* and we write

$$(\boldsymbol{a}_1, A_1) + (\boldsymbol{a}_2, A_2) = (\boldsymbol{a}_1+\boldsymbol{a}_2, A_1) \;. \tag{83}$$

Notice that the sum operation $+$ thus defined is associative (and noncommutative). In other words, if $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are summable, and if $\boldsymbol{a}_2$ and $\boldsymbol{a}_3$ are summable then $\boldsymbol{a}_1 \dot{+} (\boldsymbol{a}_2 \dot{+} \boldsymbol{a}_3) = (\boldsymbol{a}_1 \dot{+} \boldsymbol{a}_2) \dot{+} \boldsymbol{a}_3$ and $(\boldsymbol{a}_1 \dot{+} \boldsymbol{a}_2 \dot{+} \boldsymbol{a}_3, A_1)$ is a 2-connection on $E_1 E_2 E_3$.

We also have a summability criterion for the couples $(\boldsymbol{\vartheta}_1, \Theta_1)$ and $(\boldsymbol{\vartheta}_2, \Theta_2)$ where $\boldsymbol{\vartheta}_i, i = 1, 2$ is an horizontal n-form on E_i that is Θ_i-equivariant. We have that $(\boldsymbol{\vartheta}_1, \Theta_1) + (\boldsymbol{\vartheta}_2, \Theta_2) = (\boldsymbol{\vartheta}_1 \dot{+} \boldsymbol{\vartheta}_2, \Theta_1)$ where

$$\pi_\oplus^*(\boldsymbol{\vartheta}_1 \dot{+} \boldsymbol{\vartheta}_2) = \pi_1^* \boldsymbol{\vartheta}_1 + \boldsymbol{\varphi}_1(\pi_2^* \boldsymbol{\vartheta}_2) \tag{84}$$

is a well defined horizontal Θ_1-equivariant n-form on $E_1 E_2$ iff

$$\Theta_2 = {\Theta_1}^r \; . \tag{85}$$

We have

$$(D_{\boldsymbol{a}_1} \boldsymbol{\vartheta}_1, D_{A_1} \Theta_1) \dot{+} (D_{\boldsymbol{a}_2} \boldsymbol{\vartheta}_2, D_{A_2}) \Theta_2 = (D_{\boldsymbol{a}_1 \dot{+} \boldsymbol{a}_2} (\boldsymbol{\vartheta}_1 \dot{+} \boldsymbol{\vartheta}_2), D_{A_1} \Theta_1), \tag{86}$$

with obvious notation: $D_{\boldsymbol{a}} \boldsymbol{\vartheta} = d\boldsymbol{\vartheta} + [\boldsymbol{a}, \boldsymbol{\vartheta}] + T_\Theta(\boldsymbol{a}) - (-1)^n T_A(\boldsymbol{\vartheta})$ and $D_A \Theta = d\Theta + [A, \Theta]$. Also the summability of curvatures is a direct consequence of the summability of their corresponding connections. If $(\boldsymbol{a}_1, A_1) + (\boldsymbol{a}_2, A_2) = (\boldsymbol{a}_1 \dot{+} \boldsymbol{a}_2, A_1)$ then

$$(\boldsymbol{k}_1, K_1) + (\boldsymbol{k}_2, K_2) = (\boldsymbol{k}_1 \dot{+} \boldsymbol{k}_2, K_1) \; , \tag{87}$$

and we also have

$$\boldsymbol{k}_{a_1 \dot{+} a_2} = \boldsymbol{k}_1 \dot{+} \boldsymbol{k}_2 \; . \tag{88}$$

Summability is preserved under isomorphism, i.e. if $\boldsymbol{a}_i$ are summable connections on E_i $(i = 1, 2)$ and we have isomorphisms $\sigma_i : E'_i \to E_i$, then $\sigma_i^*(\boldsymbol{a}_i)$ are summable and $\sigma_1^*(\boldsymbol{a}_2) + \sigma_2^*(\boldsymbol{a}_2) = \sigma_{12}^*(\boldsymbol{a}_1 \dot{+} \boldsymbol{a}_2)$, where we have considered the induced isomorphism $\sigma_{12} \equiv \sigma_1 \sigma_2 \; : \; E'_1 E'_2 \to E_1 E_2$. The same property holds for horizontal forms.

4. Nonabelian Bundle Gerbes

Now that we have the notion of product of principal bibundles we can define nonabelian bundle gerbes generalizing the construction studied by Murray [5] (see also Hitchin [3] and [4]) in the abelian case.

Consider a submersion $\wp : Y \to M$ (i.e. a map onto with differential onto) we can always find a covering $\{O_\alpha\}$ of M with local sections $\sigma_\alpha \; : \; O_\alpha \to Y$, i.e. $\wp \circ \sigma_\alpha = id$. The manifold Y will always be equipped with the submersion $\wp \; : \; Y \to M$. We also consider $Y^{[n]} = Y \times_M Y \times_M Y \ldots \times_M Y$ the n-fold fiber product of Y, i.e. $Y^{[n]} \equiv \{(y_1, \ldots y_n) \in Y^n \mid \wp(y_1) = \wp(y_2) = \ldots \wp(y_n)\}$.

Given a H principal bibundle $\mathcal{E}$ over $Y^{[2]}$ we denote by $\mathcal{E}_{12} = p_{12}^*(\mathcal{E})$ the H principal bibundle on $Y^{[3]}$ obtained as the pull-back of $p_{12} \; : \; Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{E}_{13}$ and $\mathcal{E}_{23}$.

Consider the quadruple $(\mathcal{E}, Y, M, \boldsymbol{f})$, where the H principal bibundle on $Y^{[3]}$, $\mathcal{E}_{12} \mathcal{E}_{23} \mathcal{E}_{13}^{-1}$ is trivial, and $\boldsymbol{f}$ is a global central section of $\left(\mathcal{E}_{12} \mathcal{E}_{23} \mathcal{E}_{13}^{-1}\right)^{-1}$ [i.e. $\boldsymbol{f}$ satisfies (3)]. Recalling the paragraph after formula (15) we can equivalently say that $\mathcal{E}_{12} \mathcal{E}_{23}$ and $\mathcal{E}_{13}$ are isomorphic, the isomorphism being given by the global central section $\boldsymbol{f}^{-1}$ of

$$\mathcal{T} \equiv \mathcal{E}_{12} \mathcal{E}_{23} \mathcal{E}_{13}^{-1} \; . \tag{89}$$

We now consider $Y^{[4]}$ and the bundles $\mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{13}, \mathcal{E}_{24}, \mathcal{E}_{34}, \mathcal{E}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} : Y^{[4]} \to Y^{[2]}$ etc., and $\mathcal{T}_{123}^{-1}, \mathcal{T}_{124}^{-1}, \mathcal{T}_{134}^{-1}$ relative to $p_{123} : Y^{[4]} \to Y^{[3]}$, etc. Since the product of bundles commutes with the pull-back of bundles, we then have

$$\mathcal{T}_{124}^{-1}\mathcal{E}_{12}(\mathcal{T}_{234}^{-1}\mathcal{E}_{23}\mathcal{E}_{34}) = \mathcal{T}_{134}^{-1}(\mathcal{T}_{123}^{-1}\mathcal{E}_{12}\mathcal{E}_{23})\mathcal{E}_{34} = \mathcal{E}_{14} \tag{90}$$

as bundles on $Y^{[4]}$. The first identity in (90) is equivalent to

$$\mathcal{T}_{124}^{-1}\,\mathcal{E}_{12}\mathcal{T}_{234}^{-1}\mathcal{E}_{12}^{-1} = \mathcal{T}_{134}^{-1}\,\mathcal{T}_{123}^{-1}. \tag{91}$$

Let us now consider the global central section $\boldsymbol{f}$ of $\mathcal{T}^{-1} = \mathcal{E}_{13}\mathcal{E}_{23}^{-1}\mathcal{E}_{12}^{-1}$ and denote by $\boldsymbol{f}_{124}$ ($\boldsymbol{f}_{234}$, etc.) the global central section of $\mathcal{T}_{124}^{-1}$ ($\mathcal{T}_{234}^{-1}$, etc.) obtained as the pull-back of $\boldsymbol{f}$. Consistently with (91) we can require the condition

$$\boldsymbol{f}_{124}\,\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234}) = \boldsymbol{f}_{134}\,\boldsymbol{f}_{123} \tag{92}$$

where, following the notation of (27), $\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234})$ is the section of $\mathcal{T}_{234}^{-1}$ that in any open $\mathcal{U} \subset Y^{[4]}$ equals $\boldsymbol{s}_{12}\boldsymbol{f}_{234}\boldsymbol{s}_{12}^{-1}$, where $\boldsymbol{s}_{12} : \mathcal{U} \to \mathcal{E}_{12}$ is any section of $\mathcal{E}_{12}$, in particular we can choose $\boldsymbol{s}_{12}$ to be the pull-back of a section $\boldsymbol{s}$ of $\mathcal{E}$.

Definition 11. *A Bundle gerbe $\mathcal{G}$ is the quadruple $(\mathcal{E}, Y, M, \boldsymbol{f})$ where the H principal bibundle on $Y^{[3]}$, $\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}$ is trivial and $\boldsymbol{f}$ is a global central section of $\left(\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}\right)^{-1}$ that satisfies (92).*

Recall that when H has trivial centre then the section $\boldsymbol{f}$ of $\mathcal{T}^{-1}$ is unique; it then follows that relation (92) is automatically satisfied because the bundle on the l.h.s. and the bundle on the r.h.s. of (91) admit just one global central section, respectively $\boldsymbol{f}_{124}\,\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234})$ and $\boldsymbol{f}_{134}\,\boldsymbol{f}_{123}$. Therefore, if H has trivial centre, a bundle gerbe $\mathcal{G}$ is simply the triple $(\mathcal{E}, Y, M)$, where $\mathcal{E}_{12}\mathcal{E}_{23}\,\mathcal{E}_{13}^{-1}$ is trivial.

Consider an H principal bibundle N over Z and let $\mathcal{N}_1 = p_1^*(N)$, $\mathcal{N}_2 = p_2^*(N)$, be the pull-back of N obtained respectively from $p_1 : Z^{[2]} \to Z$ and $p_2 : Z^{[2]} \to Z$ (p_1 projects on the first component, p_2 on the second). If $(\mathcal{E}, Z, M, \boldsymbol{f})$ is a bundle gerbe also $\left(\mathcal{N}_1\mathcal{E}\mathcal{N}_2^{-1}, Z, M, \boldsymbol{\varphi}_1(\boldsymbol{f})\right)$ is a bundle gerbe. Here $\boldsymbol{\varphi}_1(\boldsymbol{f})$ is the canonical global central section of the bibundle $\mathcal{N}_1\mathcal{T}^{-1}\mathcal{N}_1^{-1}$ and now $\mathcal{N}_1$ is the pull-back of N via $p_1 : Z^{[3]} \to Z$; locally $\boldsymbol{\varphi}_1(\boldsymbol{f}) = \boldsymbol{s}_1\boldsymbol{f}\boldsymbol{s}_1^{-1}$, where $\boldsymbol{s}_1$ is the pull-back of any local section $\boldsymbol{s}$ of N. Similarly also $\left(\eta\mathcal{E}, Z, M, \boldsymbol{\ell}_{13}^{-1}\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12}\right)$ is a bundle gerbe if η^{-1} is a trivial bundle on $Z^{[2]}$ with global central section $\boldsymbol{\ell}$ (as usual $\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})$ denotes the canonical section of $\mathcal{E}_{12}\eta_{23}^{-1}\mathcal{E}_{12}^{-1}$). These observations lead to the following definition [36]

Definition 12. *Two bundle gerbes $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ and $\mathcal{G}' = (\mathcal{E}', Y', M, \boldsymbol{f}')$ are stably isomorphic if there exists a bibundle $\mathcal{N}$ over $Z = Y \times_M Y'$ and a trivial bibundle η^{-1} over $Z^{[2]}$ with section $\boldsymbol{\ell}$ such that*

$$\mathcal{N}_1\,q'^*\mathcal{E}'\,\mathcal{N}_2^{-1} = \eta\,q^*\mathcal{E} \tag{93}$$

and

$$\boldsymbol{\varphi}_1(q'^*\boldsymbol{f}') = \boldsymbol{\ell}_{13}^{-1}q^*\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12}, \tag{94}$$

where $q^*\mathcal{E}$ and $q'^*\mathcal{E}'$ are the pull-back bundles relative to the projections $q : Z^{[2]} \to Y^{[2]}$ and $q' : Z^{[2]} \to Y'^{[2]}$. Similarly $q'^*\boldsymbol{f}'$ and $q^*\boldsymbol{f}$ are the pull-back sections relative to the projections $q : Z^{[3]} \to Y^{[3]}$ and $q' : Z^{[3]} \to Y'^{[3]}$.

The relation of stable isomorphism is an equivalence relation.

The bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ is called trivial if it is stably isomorphic to the trivial bundle gerbe $(Y \times H, Y, M, \mathbf{1})$; we thus have that $\mathcal{E}$ and $\mathcal{N}_1^{-1}\mathcal{N}_2$ are isomorphic as H-bibundles, i.e.

$$\mathcal{E} \sim \mathcal{N}_1^{-1}\mathcal{N}_2 \tag{95}$$

and that $\boldsymbol{f} = \boldsymbol{\varphi}_1^{-1}(\boldsymbol{\ell}_{13}^{-1}\boldsymbol{\ell}_{23}\boldsymbol{\ell}_{12})$, where $\boldsymbol{\ell}$ is the global central section of $\eta^{-1} \equiv \mathcal{N}_2\mathcal{E}^{-1}\mathcal{N}_1^{-1}$.

Proposition 13. *Consider a bundle gerbe $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ with submersion $\wp : Y \to M$; a new submersion $\wp' : Y' \to M$ and a (smooth) map $\sigma : Y' \to Y$ compatible with $\wp$ and $\wp'$ (i.e. $\wp \circ \sigma = \wp'$). The pull-back bundle gerbe $\sigma^*\mathcal{G}$ (with obvious abuse of notation) is given by $(\sigma^*\mathcal{E}, Y', M, \sigma^*\boldsymbol{f})$. We have that the bundle gerbes $\mathcal{G}$ and $\sigma^*\mathcal{G}$ are stably equivalent.*

Proof. Consider the following identity on $Y^{[4]}$:

$$\mathcal{E}_{11'}\mathcal{E}_{1'2'}\mathcal{E}_{22'}^{-1} = \eta_{12}\mathcal{E}_{12}, \tag{96}$$

where $\eta_{12} = \mathcal{T}_{11'2'}\mathcal{T}_{122'}^{-1}$ so that η_{12}^{-1} has section $\boldsymbol{\ell}_{12} = \boldsymbol{f}_{122'}^{-1}\boldsymbol{f}_{11'2'}$; the labelling $1, 1', 2, 2'$ instead of $1, 2, 3, 4$ is just a convention. Multiplying three times (96) we obtain the following identity between trivial bundles on $Y^{[6]}$ $\mathcal{E}_{11'}\,\mathcal{T}_{1'2'3'}\,\mathcal{E}_{11'}^{-1} = \eta_{12}\,\mathcal{E}_{12}\eta_{23}\mathcal{E}_{12}^{-1}\,\mathcal{T}_{123}\,\eta_{13}^{-1}$. The sections of (the inverses of) these bundles satisfy

$$\boldsymbol{\varphi}_{11'}(\boldsymbol{f}') = \boldsymbol{\ell}_{13}^{-1}\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12}\ , \tag{97}$$

thus $\mathcal{E}_{1'2'}$ and $\mathcal{E}_{12}$ give stably equivalent bundle gerbes. Next we pull-back the bundles in (96) using $(id, \sigma, id, \sigma) : Z^{[2]} \to Y^{[4]}$ where $Z = Y \times_M Y'$; recalling that the product commutes with the pull-back we obtain relation (93) with $\eta = (id, \sigma, id, \sigma)^*\eta_{12}$ and $N = (id, \sigma)^*\mathcal{E}$. We also pull-back (97) with $(id, \sigma, id, \sigma, id, \sigma) : Z^{[3]} \to Y^{[6]}$ and obtain formula (94). □

Theorem 14. *Locally a bundle gerbe is always trivial: $\forall x \in M$ there is an open O of x such that the bundle gerbe restricted to O is stably isomorphic to the trivial bundle gerbe $(Y|_O^{[2]} \times H, Y|_O, O, \mathbf{1})$. Here $Y|_O$ is Y restricted to O: $Y|_O = \{y \in Y \mid \wp(y) \in O \subset M\}$. Moreover in any sufficiently small open $\mathcal{U}$ of $Y|_O^{[3]}$ one has*

$$\boldsymbol{f} = \boldsymbol{s}''_{13}\boldsymbol{s}'^{-1}_{23}\boldsymbol{s}_{12}^{-1} \tag{98}$$

with $\boldsymbol{s}_{12}^{-1}$, $\boldsymbol{s}'^{-1}_{23}$ and $\boldsymbol{s}''_{13}$ respectively sections of $\mathcal{E}_{12}^{-1}$, $\mathcal{E}_{23}^{-1}$ and $\mathcal{E}_{13}$ that are pull-backs of sections of $\mathcal{E}$.

Proof. Choose $O \subset M$ such that there exists a section $\sigma : O \to Y|_O$. Define the maps

$$\begin{aligned} r_{[n]} \ &: \ Y|_O^{[n]} \to Y|_O^{[n+1]}, \\ (y_1, \ldots y_n) &\mapsto (y_1, \ldots y_n, \sigma(\wp(y_n))); \end{aligned}$$

notice that $\sigma(\wp(y_1)) = \sigma(\wp(y_2)) \ldots = \sigma(\wp(y_n))$. It is easy to check the following equalities between maps on $Y|_O^{[2]}$, $p_{12} \circ r_{[2]} = id$, $p_{13} \circ r_{[2]} = r_{[1]} \circ p_1$, $p_{23} \circ r_{[2]} = r_{[1]} \circ p_2$, and between maps on $Y|_O^{[3]}$,

$$\begin{aligned} &p_{123} \circ r_{[3]} = id\ , \quad p_{124} \circ r_{[3]} = r_{[2]} \circ p_{12}\ , \\ &p_{234} \circ r_{[3]} = r_{[2]} \circ p_{23}\ , p_{134} \circ r_{[3]} = r_{[2]} \circ p_{13}\ . \end{aligned} \tag{99}$$

We now pull back with $r_{[2]}$ the identity $\mathcal{E}_{12} = \mathcal{T}\mathcal{E}_{13}\mathcal{E}_{23}^{-1}$ and obtain the following local trivialization of $\mathcal{E}$

$$\mathcal{E} = r_{[2]}^*(\mathcal{T})\,\mathcal{N}_1\mathcal{N}_2^{-1},$$

where $\mathcal{N}_1 = p_1^*(N)$, $\mathcal{N}_2 = p_2^*(N)$ and $N = r_{[1]}^*(\mathcal{E})$. Let $\mathcal{U} = U \times_O U' \times_O U'' \subset Y|_O^{[3]}$, where U, U', U'' are opens of $Y|_O$ that respectively admit the sections $\boldsymbol{n} : U \to N$, $\boldsymbol{n'} : U' \to N$, $\boldsymbol{n''} : U'' \to N$. Consider the local sections $\boldsymbol{s} = r_{[2]}^*(\boldsymbol{f}^{-1})\boldsymbol{n}_1\boldsymbol{n'}_2^{-1}$: $U \times_O U' \to \mathcal{E}$, $\boldsymbol{s'} = r_{[2]}^*(\boldsymbol{f}^{-1})\boldsymbol{n'}_2\boldsymbol{n''}_3^{-1}$: $U' \times_O U'' \to \mathcal{E}$, $\boldsymbol{s''} = r_{[2]}^*(\boldsymbol{f}^{-1})\boldsymbol{n}_1\boldsymbol{n''}_3^{-1}$: $U \times_O U'' \to \mathcal{E}$ and pull them back to local sections $\boldsymbol{s}_{12}$ of $\mathcal{E}_{12}$, $\boldsymbol{s'}_{23}$ of $\mathcal{E}_{23}$ and $\boldsymbol{s''}_{13}$ of $\mathcal{E}_{13}$. Then (98) holds because, using (99), the product $\boldsymbol{s''}_{13}\boldsymbol{s'}_{23}^{-1}\boldsymbol{s}_{12}^{-1}$ equals the pull-back with $r_{[3]}$ of the section $\boldsymbol{f}_{134}^{-1}\boldsymbol{f}_{124}\boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234}) = \boldsymbol{f}_{123}$ [cf. (92)]. □

Local description. Locally we have the following description of a bundle gerbe; we choose an atlas of charts for the bundle $\mathcal{E}$ on $Y^{[2]}$, i.e. sections $\boldsymbol{t}^i : \mathcal{U}^i \to \mathcal{E}$ relative to a trivializing covering $\{\mathcal{U}^i\}$ of $Y^{[2]}$. We write $\mathcal{E} = \{h^{ij}, \varphi^i\}$. We choose also atlases for the pull-back bundles $\mathcal{E}_{12}, \mathcal{E}_{23}, \mathcal{E}_{13}$; we write $\mathcal{E}_{12} = \{h_{12}^{ij}, \varphi_{12}^i\}$, $\mathcal{E}_{23} = \{h_{23}^{ij}, \varphi_{23}^i\}$, $\mathcal{E}_{13} = \{h_{13}^{ij}, \varphi_{13}^i\}$, where these atlases are relative to a common trivializing covering $\{\mathcal{U}^i\}$ of $Y^{[3]}$. It then follows that $\mathcal{T} = \{f^i {f^j}^{-1}, Ad_{f^i}\}$, where $\{{f^i}^{-1}\}$ are the local representatives for the section $\boldsymbol{f}^{-1}$ of $\mathcal{T}$. We also consider atlases for the bundles on $Y^{[4]}$ that are relative to a common trivializing covering $\{\mathcal{U}^i\}$ of $Y^{[4]}$ (with abuse of notation we denote with the same index i all these different coverings[1]). Then (89), that we rewrite as $\mathcal{E}_{12}\mathcal{E}_{23} = \mathcal{T}\mathcal{E}_{13}$, reads

$$h_{12}^{ij}\varphi_{12}^j(h_{23}^{ij}) = f^i h_{13}^{ij} {f^j}^{-1} \quad , \quad \varphi_{12}^i \circ \varphi_{23}^i = Ad_{f^i} \circ \varphi_{13}^i \tag{100}$$

and relation (92) reads

$$\varphi_{12}^i(f_{234}^i) f_{124}^i = f_{123}^i f_{134}^i\ . \tag{101}$$

[1] An explicit construction is for example obtained pulling back the atlas of $\mathcal{E}$ to the pull-back bundles on $Y^{[3]}$ and on $Y^{[4]}$. The sections $\boldsymbol{t}^i : \mathcal{U}^i \to \mathcal{E}$ induce the associated sections $\boldsymbol{t}_{12}^i \equiv p_{12}^*(\boldsymbol{t}^i)$: $\mathcal{U}_{12}^i \to \mathcal{E}_{12}$ where $p_{12} : Y^{[3]} \to Y^{[2]}$ and $\mathcal{U}_{12}^i \equiv p_{12}^{-1}(\mathcal{U}^i)$. We then have $\mathcal{E}_{12} = \{h_{12}^{ij}, \varphi_{12}^i\}$ with $h_{12}^{ij} = p_{12}^*(h^{ij})$, $\varphi_{12}^i = p_{12}^*(\varphi^i)$. Similarly for $\mathcal{E}_{13}$, $\mathcal{E}_{23}$. The $Y^{[3]}$ covering given by the opens $\mathcal{U}^I \equiv \mathcal{U}^{ii'i''} \equiv \mathcal{U}_{12}^i \cap \mathcal{U}_{23}^{i'} \cap \mathcal{U}_{13}^{i''}$ can then be used for a common trivialization of the $\mathcal{E}_{12}$, $\mathcal{E}_{13}$ and $\mathcal{E}_{23}$ bundles; the respective sections are $\boldsymbol{t}_{12}^I = \boldsymbol{t}_{12}^i|_{\mathcal{U}^I}$, $\boldsymbol{t}_{23}^I = \boldsymbol{t}_{23}^{i'}|_{\mathcal{U}^I}$, $\boldsymbol{t}_{13}^I = \boldsymbol{t}_{13}^{i''}|_{\mathcal{U}^I}$; similarly for the transition functions $h_{12}^I, h_{23}^I, h_{13}^I$ and for $\varphi_{12}^I, \varphi_{23}^I, \varphi_{13}^I$. In $\mathcal{U}^I$ we then have $\boldsymbol{f}^{-1} = {f^I}^{-1}\boldsymbol{t}_{12}^I\boldsymbol{t}_{23}^I{\boldsymbol{t}_{13}^I}^{-1}$.

Bundles and local data on $\boldsymbol{M}$. Up to equivalence under stable isomorphisms, there is an alternative geometric description of bundle gerbes, in terms of bundles on M. Consider the sections $\sigma_\alpha : O_\alpha \to Y$, relative to a covering $\{O_\alpha\}$ of M and consider also the induced sections $(\sigma_\alpha, \sigma_\beta) : O_{\alpha\beta} \to Y^{[2]}$, $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma) : O_{\alpha\beta\gamma} \to Y^{[3]}$. Denote by $\mathcal{E}_{\alpha\beta}$, $\mathcal{T}_{\alpha\beta\gamma}$ the pull-back of the H-bibundles $\mathcal{E}$ and $\mathcal{T}$ via $(\sigma_\alpha, \sigma_\beta)$ and $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma)$. Denote also by $\boldsymbol{f}_{\alpha\beta\gamma}$ the pull-back of the section $\boldsymbol{f}$. Then, following the Hitchin description of abelian gerbes,

Definition 15. *A gerbe is a collection* $\{\mathcal{E}_{\alpha\beta}\}$ *of* H *principal bibundles* $\mathcal{E}_{\alpha\beta}$ *on each* $O_{\alpha\beta}$ *such that on the triple intersections* $O_{\alpha\beta\gamma}$ *the product bundles* $\mathcal{E}_{\alpha\beta}\mathcal{E}_{\beta\gamma}\mathcal{E}_{\alpha\gamma}^{-1}$ *are trivial, and such that on the quadruple intersections* $O_{\alpha\beta\gamma\delta}$ *we have* $\boldsymbol{f}_{\alpha\beta\delta}\boldsymbol{\varphi}_{\alpha\beta}(\boldsymbol{f}_{\beta\gamma\delta}) = \boldsymbol{f}_{\alpha\gamma\delta}\boldsymbol{f}_{\alpha\beta\gamma}$ *.*

We also define two gerbes, given respectively by $\{\mathcal{E}'_{\alpha\beta}\}$ and $\{\mathcal{E}_{\alpha\beta}\}$ (we can always consider a common covering $\{O_\alpha\}$ of M), to be stably equivalent if there exist bibundles $\mathcal{N}_\alpha$ and trivial bibundles $\eta_{\alpha\beta}$ with (global central) sections $\boldsymbol{\ell}_{\alpha\beta}^{-1}$ such that

$$\mathcal{N}_\alpha \mathcal{E}'_{\alpha\beta} \mathcal{N}_\beta^{-1} = \eta_{\alpha\beta}\mathcal{E}_{\alpha\beta} \ , \tag{102}$$

$$\boldsymbol{\varphi}_\alpha(\boldsymbol{f}'_{\alpha\beta\gamma}) = \boldsymbol{\ell}_{\alpha\gamma}^{-1}\boldsymbol{f}_{\alpha\beta\gamma}\boldsymbol{\varphi}_{\alpha\beta}(\boldsymbol{\ell}_{\beta\gamma})\boldsymbol{\ell}_{\alpha\beta} \ . \tag{103}$$

A local description of the $\mathcal{E}_{\alpha\beta}$ bundles in terms of the local data (100), (101) can be given considering the refinement $\{O^i_\alpha\}$ of the $\{O_\alpha\}$ cover of M such that $(\sigma_\alpha, \sigma_\beta)(O^{ij}_{\alpha\beta}) \subset \mathcal{U}^{ij} \subset Y^{[2]}$, the refinement $\{O^i_\alpha\}$ such that $(\sigma_\alpha, \sigma_\beta, \sigma_\gamma)(O^{ijk}_{\alpha\beta\gamma}) \subset \mathcal{U}^{ijk} \subset Y^{[3]}$, and similarly for $Y^{[4]}$. We can then define the local data on M

$$\begin{aligned} &h^{ij}_{\alpha\beta} \ : \ O^{ij}_{\alpha\beta} \to H \qquad && \varphi^i_{\alpha\beta} \ : \ O^i_{\alpha\beta} \to Aut(H),\\ &h^{ij}_{\alpha\beta} = h^{ij}_{12} \circ (\sigma_\alpha, \sigma_\beta) && \varphi^i_{\alpha\beta} = \varphi^i_{12} \circ (\sigma_\alpha, \sigma_\beta), \end{aligned} \tag{104}$$

and

$$\begin{aligned} &f^i_{\alpha\beta\gamma} \ : \ O^i_{\alpha\beta\gamma} \to H,\\ &f^i_{\alpha\beta\gamma} = f^i \circ (\sigma_\alpha, \sigma_\beta, \sigma_\gamma) \ . \end{aligned} \tag{105}$$

It follows that $\mathcal{E}_{\alpha\beta} = \{h^{ij}_{\alpha\beta}, \varphi^i_{\alpha\beta}\}$ and $\mathcal{T}_{\alpha\beta\gamma} = \{f^i_{\alpha\beta\gamma} f^{j\,-1}_{\alpha\beta\gamma}, Ad_{f^i_{\alpha\beta\gamma}}\}$. Moreover relations (100), (101) imply the relations between local data on M,

$$h^{ij}_{\alpha\beta}\varphi^j_{\alpha\beta}(h^{ij}_{\beta\gamma}) = f^i_{\alpha\beta\gamma} h^{ij}_{\alpha\gamma} f^{j\,-1}_{\alpha\beta\gamma} \ , \tag{106}$$

$$\varphi^i_{\alpha\beta} \circ \varphi^i_{\beta\gamma} = Ad_{f^i_{\alpha\beta\gamma}} \circ \varphi^i_{\alpha\gamma} \ , \qquad \varphi^i_{\alpha\beta}(f^i_{\beta\gamma\delta}) f^i_{\alpha\beta\delta} = f^i_{\alpha\beta\gamma} f^i_{\alpha\gamma\delta} \ . \tag{107}$$

We say that (107) define a nonabelian Čech 2-cocycle. From (102), (103) we see that two sets $\{h^{ij}_{\alpha\beta}, \varphi^i_{\alpha\beta}, f^i_{\alpha\beta\gamma}\}$, $\{h'^{ij}_{\alpha\beta}, \varphi'^i_{\alpha\beta}, f'^i_{\alpha\beta\gamma}\}$ of local data on M are stably isomorphic if

$$h^{ij}_\alpha \varphi^j_\alpha(h'^{ij}_{\alpha\beta})\, \varphi^j_\alpha \varphi'^j_{\alpha\beta} \varphi^{j-1}_\beta(h^{ij}_\beta) = \ell^i_{\alpha\beta} h^{ij}_{\alpha\beta} \ell^{j\,-1}_{\alpha\beta} \ , \tag{108}$$

$$\varphi^i_\alpha \circ \varphi'^i_{\alpha\beta} \circ \varphi^{i\,-1}_\beta = Ad_{\ell^i_{\alpha\beta}} \circ \varphi^i_{\alpha\beta} \ , \tag{109}$$

$$\varphi^i_\alpha(f'^i_{\alpha\beta\gamma}) = \ell^i_{\alpha\beta}\, \varphi^i_{\alpha\beta}(\ell^i_{\beta\gamma}) f^i_{\alpha\beta\gamma} \ell^{i\,-1}_{\alpha\gamma} \ , \tag{110}$$

here $\mathcal{N}_\alpha = \{h^{ij}_\alpha, \varphi_\alpha\}$, $\mathcal{E}'_{\alpha\beta} = \{h'^{ij}_{\alpha\beta}, \varphi'_{\alpha\beta}\}$ and $\eta_{\alpha\beta} = \{\ell^i_{\alpha\beta}\ell^{j\,-1}_{\alpha\beta}, Ad_{\ell^i_{\alpha\beta}}\}$.

We now compare the gerbe $\{\mathcal{E}_{\alpha\beta}\}$ obtained from a bundle gerbe $\mathcal{G}$ using the sections $\sigma_\alpha : O_\alpha \to Y$ to the gerbe $\{\mathcal{E}'_{\alpha\beta}\}$ obtained from $\mathcal{G}$ using a different choice of sections $\sigma'_\alpha : O_\alpha \to Y$. We first pull back the bundles in (96) using $(\sigma_\alpha, \sigma'_\alpha, \sigma_\beta, \sigma'_\beta) : O_{\alpha\beta} \to Y^{[4]}$; recalling that the product commutes with the pull-back we obtain the following relation between bundles respectively on $O_\alpha,\ O_{\alpha\beta},\ O_\beta$ and on $O_{\alpha\beta},\ O_{\alpha\beta}$,

$$\mathcal{N}_\alpha \mathcal{E}'_{\alpha\beta} \mathcal{N}_\beta^{-1} = \eta_{\alpha\beta} \mathcal{E}_{\alpha\beta};$$

here $\mathcal{N}_\alpha$ equals the pull-back of $\mathcal{E}_{11'}$ with $(\sigma_\alpha, \sigma'_\alpha) : O_\alpha \to Y^{[2]}$. We then pull back (97) with $(\sigma_\alpha, \sigma'_\alpha, \sigma_\beta, \sigma'_\beta, \sigma_\gamma, \sigma'_\gamma) : O_{\alpha\beta\gamma} \to Y^{[6]}$ and obtain formula (103). Thus $\{\mathcal{E}'_{\alpha\beta}\}$ and $\{\mathcal{E}_{\alpha\beta}\}$ are stably equivalent gerbes. We have therefore shown that the equivalence class of a gerbe (defined as a collection of bundles on $O_{\alpha\beta} \subset M$) is independent from the choice of sections $\sigma_\alpha : O_\alpha \to Y$ used to obtain it as pull-back from a bundle gerbe.

It is now easy to prove that equivalence classes of bundle gerbes are in one to one correspondence with equivalence classes of gerbes $\{\mathcal{E}_{\alpha\beta}\}$, and therefore with equivalence classes of local data on M. First of all we observe that a bundle gerbe $\mathcal{G}$ and its pull-back $\sigma^*\mathcal{G} = (\sigma^*\mathcal{E}, Y', M, \sigma^*\boldsymbol{f})$ (cf. Theorem 13) give the same gerbe $\{\mathcal{E}_{\alpha\beta}\}$ if we use the sections $\sigma'_\alpha : O_\alpha \to Y'$ for $\sigma^*\mathcal{G}$ and the sections $\sigma \circ \sigma'_\alpha : O_\alpha \to Y$ for $\mathcal{G}$. It then follows that two stably equivalent bundle gerbes give two stably equivalent gerbes. In order to prove the converse we associate to each gerbe $\{\mathcal{E}_{\alpha\beta}\}$ a bundle gerbe and then we prove that on equivalence classes this operation is the inverse of the operation $\mathcal{G} \to \{\mathcal{E}_{\alpha\beta}\}$. Given $\{\mathcal{E}_{\alpha\beta}\}$ we consider $Y = \sqcup O_\alpha$, the disjoint union of the opens $O_\alpha \subset M$, with projection $\wp(x, \alpha) = x$. Then $Y^{[2]}$ is the disjoint union of the opens $O_{\alpha\beta}$, i.e. $Y^{[2]} = \sqcup O_{\alpha\beta} = \cup O_{\alpha,\beta}$, where $O_{\alpha,\beta} = \{(\alpha, \beta, x) \,/\, x \in O_{\alpha\beta}\}$, similarly $Y^{[3]} = \sqcup O_{\alpha\beta\gamma} = \cup O_{\alpha,\beta,\gamma}$ etc.. We define $\mathcal{E}$ such that $\mathcal{E}|_{O_{\alpha,\beta}} = \mathcal{E}_{\alpha\beta}$ and we define the section $\boldsymbol{f}^{-1}$ of $\mathcal{T} = \mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1}$ to be given by $\boldsymbol{f}^{-1}|_{O_{\alpha,\beta,\gamma}} = \boldsymbol{f}_{\alpha\beta\gamma}^{-1}$, thus (92) holds. We write $(\sqcup\mathcal{E}_{\alpha\beta}, \sqcup O_\alpha, M, \sqcup \boldsymbol{f}_{\alpha\beta\gamma})$ for this bundle gerbe. If we pull it back with $\sigma_\alpha : O_\alpha \to Y$, $\sigma_\alpha(x) = (x, \alpha)$ we obtain the initial gerbe $\{\mathcal{E}_{\alpha\beta}\}$. In order to conclude the proof we have to show that $(\sqcup\mathcal{E}_{\alpha\beta}, \sqcup O_\alpha, M, \sqcup \boldsymbol{f}_{\alpha\beta\gamma})$ is stably isomorphic to the bundle gerbe $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ if $\{\mathcal{E}_{\alpha\beta}\}$ is obtained from $\mathcal{G} = (\mathcal{E}, Y, M, \boldsymbol{f})$ and sections $\sigma_\alpha : O_\alpha \to Y$. This holds because $(\sqcup\mathcal{E}_{\alpha\beta}, \sqcup O_\alpha, M, \sqcup \boldsymbol{f}_{\alpha\beta\gamma}) = \sigma^*\mathcal{G}$ with $\sigma : \sqcup O_\alpha \to Y$ given by $\sigma|_{O_\alpha} = \sigma_\alpha$.

We end this section with a comment on normalization. There is no loss in generality if we consider for all α, β and for all i,

$$\varphi^i_{\alpha\alpha} = id\,,\ \ f^i_{\alpha\alpha\beta} = 1\,,\ \ f^i_{\alpha\beta\beta} = 1. \tag{111}$$

Indeed first notice from (106) and (107) that $\varphi^i_{\alpha\alpha} = Ad_{f^i_{\alpha\alpha\alpha}}$ and $\varphi^i_{\alpha\alpha}(f^i_{\alpha\alpha\beta}) = f^i_{\alpha\alpha\alpha}$ so that $f^i_{\alpha\alpha\beta} = f^i_{\alpha\alpha\alpha}|_{O_{\alpha\beta}}$. Now, if $f^i_{\alpha\alpha\alpha} \neq 1$ consider the stably equivalent local data obtained from $\mathcal{E}'_{\alpha\beta} \equiv \eta_{\alpha\beta}\mathcal{E}_{\alpha\beta}$, where $\eta_{\alpha\beta} = \{\ell^i_{\alpha\beta}\ell^{j\,-1}_{\alpha\beta}, Ad_{\ell^i_{\alpha\beta}}\}$ with $\ell^i_{\alpha\beta} = f^{i\,-1}_{\alpha\alpha\alpha}|_{O_{\alpha\beta}}$. From (109) we have $\varphi'^i_{\alpha\alpha} = id$; from (110) we have $f'^i_{\alpha\alpha\beta} = 1$, it then also follows $f'^i_{\alpha\beta\beta} = 1$.

5. Nonabelian Gerbes from Groups Extensions

We here associate a bundle gerbe on the manifold M to every group extension

$$1 \to H \to E \stackrel{\pi}{\to} G \to 1 \tag{112}$$

and left principal G bundle P over M. We identify G with the coset ${}_H\backslash E$ so that E is a left H principal bundle. E is naturally a bibundle, the right action too is given by the group law in E

$$e \triangleleft h = eh = (ehe^{-1})e, \tag{113}$$

thus $\varphi_e(h) = ehe^{-1}$. We denote by $\tau : P^{[2]} \to G$, $\tau(p_1, p_2) = g_{12}$ the map that associates to any two points p_1, p_2 of P that live on the same fiber the unique element $g_{12} \in G$ such that $p_1 = g_{12}p_2$. Let $\mathcal{E} \equiv \tau^*(E)$ be the pull-back of E on $P^{[2]}$, explicitly $\mathcal{E} = \{(p_1, p_2; e) \mid \pi(e) = \tau(p_1, p_2) = g_{12}\}$. Similarly $\mathcal{E}_{12} = \{(p_1, p_2, p_3; e) \mid \pi(e) = \tau(p_1, p_2) = g_{12}\}$, for brevity of notations we set $e_{12} \equiv (p_1, p_2, p_3; e)$. Similarly with $\mathcal{E}_{23}$ and $\mathcal{E}_{13}$, while e_{13}^{-1} is a symbolic notation for a point of $\mathcal{E}_{13}^{-1}$. Recalling (15) we have

$$(he)_{13}^{-1} = (ek)_{13}^{-1} = k^{-1}\, e_{13}^{-1}, e_{13}^{-1} \triangleleft^{-1} h = k\, e_{13}^{-1}, \tag{114}$$

where $k = e^{-1}he$. We now consider the point

$$\boldsymbol{f}^{-1}(p_1, p_2, p_3) \equiv [e_{12}, e'_{23}, (ee')_{13}^{-1}] \in \mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1}, \tag{115}$$

where the square bracket denotes, as in (12), the equivalence class under the H action[2]. Expression (115) is well defined because $\pi(ee') = \pi(e)\pi(e') = g_{12}g_{23} = g_{13}$ the last equality following from $p_1 = g_{12}p_2$, $p_2 = g_{23}p_3$, $p_1 = g_{13}p_3$. Moreover $\boldsymbol{f}(p_1, p_2, p_3)$ is independent from e and e', indeed let $\hat{e}$ and $\hat{e}'$ be two other elements of E such that $\pi(\hat{e}) = \pi(e)$, $\pi(\hat{e}') = \pi(e')$; then $\hat{e} = he$, $\hat{e}' = h'e'$ with $h, h' \in H$ and $[\hat{e}_{12}, \hat{e}'_{23}, (\hat{e}\hat{e}')_{13}^{-1}] = [h\, e_{12}, h'\, e'_{23}, e'^{-1}h'^{-1}e^{-1}h^{-1}ee'\, (ee')_{13}^{-1}] = [e_{12}, e'_{23}, (ee')_{13}^{-1}]$. This shows that (115) defines a global section $\boldsymbol{f}^{-1}$ of $\mathcal{T} \equiv \mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1}$. Using the second relation in (114) we also have that $\boldsymbol{f}^{-1}$ is central so that $\mathcal{T}$ is a trivial bibundle. Finally (the inverse of) condition (92) is easily seen to hold and we conclude that $(\mathcal{E}, P, M, \boldsymbol{f})$ is a bundle gerbe. It is the so-called *lifting bundle gerbe*.

6. Bundle Gerbes Modules

The definition of a module for a nonabelian bundle gerbe is inspired by the abelian case [6].

Definition 16. *Given an H-bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$, an $\mathcal{E}$-module consists of a triple $(\mathcal{Q}, \mathcal{Z}, z)$, where $\mathcal{Q} \to Y$ is a D-H bundle, $\mathcal{Z} \to Y^{[2]}$ is a trivial D-bibundle and z is a global central section of $\mathcal{Z}^{-1}$ such that:*

i) On $Y^{[2]}$

$$\mathcal{Q}_1\mathcal{E} = \mathcal{Z}\mathcal{Q}_2 \tag{116}$$

and moreover

$$\boldsymbol{\varphi}_{12} = \boldsymbol{\psi}_1^{-1} \circ {}^{\bar{z}_{12}^{-1}} \circ \boldsymbol{\psi}_2. \tag{117}$$

ii) Equation (116) is compatible with the bundle gerbe structure of $\mathcal{E}$, i.e. from (116) we have $\mathcal{Q}_1\mathcal{T} = \mathcal{Z}_{12}\mathcal{Z}_{23}\mathcal{Z}_{13}^{-1}\mathcal{Q}_1$ on $Y^{[3]}$ and we require that

$$z_{23}z_{12} = z_{13}\boldsymbol{\xi}_1(\boldsymbol{f}) \tag{118}$$

holds true.

[2] It can be shown that a realization of the equivalence class $[e_{12}, e'_{23}] \in \mathcal{E}_{12}\mathcal{E}_{23}$ is given by $(p_1, p_2, p_3; ee')$, where ee' is just the product in E. (We won't use this property).

Remark 17. Let us note that the pair $(\mathcal{Z}, z^{-1})$ and the pair $(\mathcal{T}, \boldsymbol{f}^{-1})$ in the above definition give the isomorphisms

$$z : \mathcal{Q}_1\mathcal{E} \to \mathcal{Q}_2\ ,\ \ f : \mathcal{E}_{12}\mathcal{E}_{23} \to \mathcal{E}_{13} \tag{119}$$

respectively of D-H bundles on $Y^{[2]}$ and of bibundles on $Y^{[3]}$. Condition *ii)* in Definition 16 is then equivalent to the commutativity of the following diagram

$$\begin{tikzcd} \mathcal{Q}_1\mathcal{E}_{12}\mathcal{E}_{23} \arrow[r, "id_1 f"] \arrow[d, "z_{12}\, id_3"'] & \mathcal{Q}_1\mathcal{E}_{13} \arrow[d, "z_{13}"] \\ \mathcal{Q}_2\mathcal{E}_{23} \arrow[r, "z_{23}"] & \mathcal{Q}_3 \end{tikzcd} \tag{120}$$

Definition 18. *We call two bundle gerbe modules $(\mathcal{Q}, \mathcal{Z}, \boldsymbol{z})$ and $(\mathcal{Q}', \mathcal{Z}', \boldsymbol{z}')$ (with the same crossed module structure) equivalent if:*
i) $\mathcal{Q}$ and $\mathcal{Q}'$ are isomorphic as D-H bundles; we write $\mathcal{Q} = \mathcal{I}\mathcal{Q}'$ where the D-bibundle $\mathcal{I}$ has global central section $\boldsymbol{i}^{-1}$ and $\boldsymbol{\psi} = \bar{i}^{-1} \circ \boldsymbol{\psi}'$,
ii) the global central sections $\boldsymbol{z}$, $\boldsymbol{z}'$ and $\boldsymbol{i}^{-1}$ satisfy the condition $\boldsymbol{z}'_{12} = \boldsymbol{i}_2^{-1}\boldsymbol{z}_{12}\boldsymbol{i}_1$.

Let us now assume that we have two stably equivalent bundle gerbes $(\mathcal{E}, Y, M, \boldsymbol{f})$ and $(\mathcal{E}', Y', M, \boldsymbol{f}')$ with $Y' = Y$. We have [cf. (93), (94)] $\eta_{12}\mathcal{E}_{12} = \mathcal{N}_1\mathcal{E}'_{12}\mathcal{N}_2^{-1}$ and $\boldsymbol{\varphi}_1(\boldsymbol{f}') = \boldsymbol{\ell}_{13}^{-1}\boldsymbol{f}\boldsymbol{\varphi}_{12}(\boldsymbol{\ell}_{23})\boldsymbol{\ell}_{12}$. Let $\mathcal{Q}$ be an $\mathcal{E}$-module and $\mathcal{I}$ a trivial D-bibundle with a global central section $\boldsymbol{i}^{-1}$. It is trivial to check that $\mathcal{I}\mathcal{Q}\mathcal{N}$ is an $\mathcal{E}'$-module with $\mathcal{Z}'_{12} = \mathcal{I}_1\boldsymbol{\xi}_1(\eta_{12})\mathcal{Z}_{12}\mathcal{I}_2^{-1}$ and $\boldsymbol{z}'_{12} = \boldsymbol{i}_2^{-1}\boldsymbol{z}_{12}\boldsymbol{\xi}(\eta_{12})\boldsymbol{i}_1$. It is now easy to compare modules of stably equivalent gerbes that in general have $Y \neq Y'$.

Proposition 19. *Stably equivalent gerbes have the same equivalence classes of modules.*

Now we give the description of bundle gerbes modules in terms of local data on M. Let $\{EE_{\alpha\beta}\}$ be a gerbe in the sense of Definition 15.

Definition 20. *A module for the gerbe $\{\mathcal{E}_{\alpha\beta}\}$ is given by a collection $\{\mathcal{Q}_\alpha\}$ of D-H bundles such that on double intersections $O_{\alpha\beta}$ there exist trivial D-bibundles $\mathcal{Z}_{\alpha\beta}$, $\mathcal{Q}_\alpha\mathcal{E}_{\alpha\beta} = \mathcal{Z}_{\alpha\beta}\mathcal{Q}_\beta$, with global central sections $\boldsymbol{z}_{\alpha\beta}$ of $\mathcal{Z}_{\alpha\beta}^{-1}$ such that on triple intersections $O_{\alpha\beta\gamma}$,*

$$\boldsymbol{z}_{\beta\gamma}\boldsymbol{z}_{\alpha\beta} = \boldsymbol{z}_{\alpha\gamma}\boldsymbol{\xi}_\alpha(\boldsymbol{f}_{\alpha\beta\gamma}) \tag{121}$$

and on double intersections $O_{\alpha\beta}$

$$\boldsymbol{\varphi}_{\alpha\beta} = \boldsymbol{\psi}_\alpha^{-1} \circ \bar{z}_{\alpha\beta}^{-1} \circ \boldsymbol{\psi}_\beta. \tag{122}$$

Canonical module. For each H-bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ we have a canonical $\mathcal{E}$-module associated with it; it is constructed as follows. As a left $Aut(H)$-bundle the canonical module is simply the trivial bundle over Y. The right action of H is given by the canonical homomorphism $Ad : H \to Aut(H)$. For $(y, \eta) \in Y \times Aut(H)$ we have $\boldsymbol{\xi}_{(y,\eta)}(h) = \eta \circ Ad_h \circ \eta^{-1} = Ad_{\eta(h)}$ and $\boldsymbol{\psi}_{(y,\eta)}(h) = \eta(h)$. The $Aut(H)$-H bundle morphism $z : (Y \times Aut(H))_1\mathcal{E} \to (Y \times Aut(H))_2$ is given in the following way. A generic element of $(Y \times Aut(H))_1\mathcal{E}$ is of the form $[(y, y', (y, \eta)), e]$, where $\eta \in Aut(H)$, $(y, y') \in Y^{[2]}$and $e \in \mathcal{E}$ such that $p_1 \circ p(e) = y$ and $p_2 \circ p(e) = y'$. Here p is the projection $p : \mathcal{E} \to Y^{[2]}$. We set

$$z([(y, y', (y, \eta)), e]) = (y, y', (y', \eta \circ \boldsymbol{\varphi}_e)).$$

The commutativity of diagram (120) is equivalent to the following statement:

$$\eta \circ \boldsymbol{\varphi}_{f[e_1,e_2]} = \eta \circ \boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2},$$

and this is a consequence of the isomorphism of H-bibundles

$$f : \mathcal{E}_{12}\mathcal{E}_{23} \to \mathcal{E}_{13}.$$

We have

$$f([e_1, e_2]h) = (f[e_1, e_2])h\,,$$

but we also have

$$f([e_1, e_2)]h) = f(\boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2}(h)[e_1, e_2]) = \boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2}(h) f([e_1, e_2]).$$

On the other hand we can write

$$(f[e_1, e_2])h = \boldsymbol{\varphi}_{f[e_1,e_2]}(h) f[e_1, e_2].$$

Hence

$$\boldsymbol{\varphi}_{f[e_1,e_2]}(h) = \boldsymbol{\varphi}_{e_1} \circ \boldsymbol{\varphi}_{e_2}(h)$$

and the commutativity of diagram (120) follows. We denote the canonical module as *can* in the following.

In the case of a bundle gerbe $\mathcal{E}$ associated with the lifting of a G-principal bundle P, as described in Sect. 5, we have another natural module. We follow the notation of Sect. 5. In the exact sequence of groups (112),

$$1 \to H \to E \xrightarrow{\pi} G \to 1\,,$$

H is a normal subgroup. This gives the group H the structure of a crossed E-module.

The $\mathcal{E}$-module $\mathcal{Q}$ is simply the trivial E-H bundle $P \times E \to P$. The D-H bundle morphims $z' : \mathcal{Q}_1\mathcal{E} \to \mathcal{Q}_2$ is given by (recall $p_1 = \pi(\tilde{e})p_2$)

$$z'[(p_1, p_2, (p_1, e), (p_1, p_2, \tilde{e})] = (p_1, p_2, (p_2, e\tilde{e})),$$

which of course is compatible with the bundle gerbe structure of $\mathcal{E}$. Due to the exact sequence (112) we do have a homomorphism $E \to Aut(\mathrm{H})$ and hence we have a map

$$t : Y \times E \to Y \times Aut(H),$$

which is a morphism between the modules compatible with the module structures, i.e. the following diagram is commutative:

$$\begin{tikzcd} \mathcal{Q}_1\mathcal{E}_{12} \arrow[r, "z_{12}"] \arrow[d, "t"'] & \mathcal{Q}_2 \arrow[d, "t"] \\ can_1\mathcal{E}_{12} \arrow[r, "z'_{12}"] & can_2 \end{tikzcd} \tag{123}$$

More generally given any bundle gerbe $\mathcal{E}$ and an $\mathcal{E}$-module $\mathcal{Q}$ we have the trivial $Aut(H)$-H bundle $Aut(H) \times_D \mathcal{Q}$ (see Sect. 2). This gives a morphism $t\ :\ \mathcal{Q} \mapsto can$.

Now suppose that the bundle gerbe $\mathcal{E}$ is trivialized (stably equivalent to a trivial bundle gerbe) by $\mathcal{E}_{12} \sim \mathcal{N}_1^{-1}\mathcal{N}_2$ with $\mathcal{N}$ an H-bibundle on Y, hence a $\mathcal{E}$-module satisfies

$$\mathcal{Q}_2 \sim \mathcal{Q}_1\mathcal{E}_{12} \sim \mathcal{Q}_1\mathcal{N}_1^{-1}\mathcal{N}_2,$$

hence

$$\mathcal{Q}_2\mathcal{N}_2^{-1} \sim \mathcal{Q}_1\mathcal{N}_1^{-1}. \tag{124}$$

It easily follows from (124) that $\mathcal{Q}\mathcal{N}^{-1} \to Y$ gives descent data for a D-H bundle $\widetilde{Q}$ over M. Conversely given a D-H bundle $\widetilde{Q} \to M$ the bundle $p^*(\widetilde{Q})\mathcal{N}$ is a $\mathcal{E}$-module. This proves the following

Proposition 21. *For a trivial bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ the category of $\mathcal{E}$-modules is equivalent to the category of D-H bundles over the base space M.*

7. Bundle Gerbe Connections

Definition 22. *A bundle gerbe connection on a bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ is a 2-connection $(\boldsymbol{a}, A)$ on $\mathcal{E} \to Y^{[2]}$ such that*

$$\boldsymbol{a}_{12} \dot{+} \boldsymbol{a}_{23} = f^*\boldsymbol{a}_{13}\,, \tag{125}$$

or which is the same

$$\boldsymbol{a}_{12} \dot{+} \boldsymbol{a}_{23} \dot{+} \boldsymbol{a}^r_{13} = \overline{f}^{-1} d\overline{f} + T_{A_1}(\overline{f}^{-1}) \tag{126}$$

holds true.

In the last equation $\overline{f}^{-1}$ is the bi-equivariant map $\overline{f}^{-1} : \mathcal{T} \to H$ associated with the global central section $\boldsymbol{f}^{-1}$ of $\mathcal{T}$. Moreover we used that $(\boldsymbol{a}^r, A^r)$ is a right 2-connection on $\mathcal{E}$ and a left 2-connection on $\mathcal{E}^{-1}$ [cf. (15)].

Remark 23. It follows from (125) that for a bundle gerbe connection $A_{12} = A_{13}$ must be satisfied, hence A is a pull-back via p_1 on $Y^{[2]}$ of a one form defined on Y. We can set $A_1 \equiv A_{12} = A_{13}$. Definition 21 contains implicitly the requirement that $(\boldsymbol{a}_{12}, \boldsymbol{a}_{23})$ are summable, which means that $A_1^r = A_2$. More explicitly (see (53)):

$$\mathcal{A}_1 + ad_{\boldsymbol{a}_{12}} = \varphi_{12}\mathcal{A}_2\varphi_{12}^{-1} + \varphi_{12}d\varphi_{12}^{-1}. \tag{127}$$

The affine sum of bundle gerbe connections is again a bundle gerbe connection. This is a consequence of the following affine property for sums of the 2-connection. If on the bibundles E_1 and E_2 we have two couples of summable connections $(\boldsymbol{a}_1, \boldsymbol{a}_2)$, $(\boldsymbol{a}'_1, \boldsymbol{a}'_2)$, then $\lambda\boldsymbol{a}_1 + (1-\lambda)\boldsymbol{a}'_1$ is summable to $\lambda\boldsymbol{a}_2 + (1-\lambda)\boldsymbol{a}'_2$ and the sum is given by

$$(\lambda\boldsymbol{a}_1 + (1-\lambda)\boldsymbol{a}'_1) \dot{+} (\lambda\boldsymbol{a}_2 + (1-\lambda)\boldsymbol{a}'_2) = \lambda(\boldsymbol{a}_1 \dot{+} \boldsymbol{a}_2) + (1-\lambda)(\boldsymbol{a}'_1 \dot{+} \boldsymbol{a}'_2)\,. \tag{128}$$

We have the following theorem:

Theorem 24. *There exists a bundle gerbe connection $(\boldsymbol{a}, A)$ on each bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$.*

Proof. Let us assume for the moment the bundle gerbe to be trivial, $\mathcal{E} = \mathcal{N}_1^{-1}\mathcal{Z}\mathcal{N}_2$ with a bibundle $\mathcal{N} \to Y$ and a trivial bibundle $\mathcal{Z} \to Y^{[2]}$ with global central section z^{-1}. Consider on $\mathcal{Z}$ the 2-connection $(\boldsymbol{\alpha}, \tilde{A})$, where the Lie$(Aut(H))$-valued one-form $\tilde{A}$ on Y is the pull-back of a one-form on M. Here $\boldsymbol{\alpha}$ is canonically determined by $\tilde{A}$ and z^{-1}, we have $\boldsymbol{\alpha} = \overline{z}^{-1}d\overline{z} + T_A(\overline{z}^{-1})$. Next consider on $\mathcal{N}$ an arbitrary 2-connection $(\tilde{\boldsymbol{a}}, \tilde{A})$. Since $\tilde{A}$ is the pull-back of a one form on M we have that the sum $\boldsymbol{a} = \tilde{\boldsymbol{a}}_1^r\dot{+}\boldsymbol{\alpha}\dot{+}\tilde{\boldsymbol{a}}_2$ is well defined and that $(\boldsymbol{a}, A \equiv \tilde{A}^r)$ is a 2-connection on $\mathcal{E}$. Notice that under the canonical identification $\mathcal{Z}_{12}\mathcal{N}_2\mathcal{N}_2^{-1}\mathcal{Z}_{23} = \mathcal{Z}_{12}\mathcal{Z}_{23}$ we have the canonical identification $\boldsymbol{\alpha}_{12}\dot{+}\tilde{\boldsymbol{a}}_2\dot{+}\tilde{\boldsymbol{a}}_2^r\dot{+}\boldsymbol{\alpha}_{23} = \boldsymbol{\alpha}_{12}\dot{+}\boldsymbol{\alpha}_{23}$. The point here is that $\mathcal{N}_2\mathcal{N}_2^{-1}$ has the canonical section $\mathbf{1} = [n, n^{-1}]$, $n \in \mathcal{N}_2$, and that $\tilde{\boldsymbol{a}}_2\dot{+}\tilde{\boldsymbol{a}}_2^r = \bar{\mathbf{1}}d\bar{\mathbf{1}}^{-1} + T_A(\bar{\mathbf{1}})$ independently from $\tilde{\boldsymbol{a}}_2$. Then from $\mathcal{E} = \mathcal{N}_1^{-1}\mathcal{Z}\mathcal{N}_2$ we have $\mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{13}^{-1} = \mathcal{N}_1^{-1}\mathcal{Z}_{12}\mathcal{Z}_{23}\mathcal{Z}_{13}^{-1}\mathcal{N}_1$ and for the connections we have

$$\boldsymbol{a}_{12}\dot{+}\boldsymbol{a}_{23}\dot{+}\boldsymbol{a}_{13}^r = \tilde{\boldsymbol{a}}_1^r\dot{+}\boldsymbol{\alpha}_{12}\dot{+}\boldsymbol{\alpha}_{23}\dot{+}\boldsymbol{\alpha}_{13}^r\dot{+}\tilde{\boldsymbol{a}}_1 \,. \tag{129}$$

We want to prove that the r.h.s. of this equation equals the canonical 2-connection $\overline{f}^{-1}d\overline{f} + T_{A_1}(\overline{f}^{-1})$ associated with the trivial bundle $\mathcal{T}$ with section $\boldsymbol{f}^{-1}$. We first observe that a similar property holds for the sections of $\mathcal{Z}_{12}\mathcal{Z}_{23}\mathcal{Z}_{13}^{-1}$ and of $\mathcal{T}$: $\boldsymbol{f}^{-1} = \boldsymbol{\varphi}_1^{-1}(z_{12}^{-1}z_{23}^{-1}z_{13}) \equiv \boldsymbol{n}_1^{-1}z_{12}^{-1}z_{23}^{-1}z_{13}\boldsymbol{n}_1$ independently from the local section $\boldsymbol{n}_1$ of $\mathcal{N}_1$. Then one can explicitly check that this relation implies the relation $\tilde{\boldsymbol{a}}_1^r\dot{+}\boldsymbol{\alpha}_{12}\dot{+}\boldsymbol{\alpha}_{23}$ $\dot{+}\boldsymbol{\alpha}_{13}^r\dot{+}\tilde{\boldsymbol{a}}_1 = \overline{f}^{-1}d\overline{f} + T_{A_1}(\overline{f}^{-1})$. This proves the validity of the theorem in the case of a trivial bundle gerbe. According to Theorem 14 any gerbe is locally trivial, so we can use the affine property of bundle gerbe connections and a partition of unity subordinate to the covering $\{O_\alpha\}$ of M to extend the proof to arbitrary bundle gerbes. □

A natural question arises: can we construct a connection on the bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ starting with:

- its nonabelian Čech cocycle $\overline{f}^{-1} : \mathcal{T} \to H$, $\boldsymbol{\varphi} : \mathcal{E} \times H \to H$
- sections $\sigma_\alpha : O_\alpha \to Y$
- a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\{O_\alpha\}$ of M ?

The answer is positive. Let us describe the construction. First we use the local sections σ_α to map $Y|_{O_\alpha}^{[2]}$ to $Y^{[3]}$ via the map $r_\alpha^{[2]} : [y, y'] \mapsto [\sigma_\alpha(x), y, y']$, where $\wp(y) = \wp(y') = x$, similarly $r_\alpha^{[1]} : Y|_{O_\alpha}^{[1]} \to Y^{[2]}$. Next let us introduce the following H-valued one form $\boldsymbol{a}$

$$\boldsymbol{a} = \sum_\alpha \rho_\alpha r_\alpha^{[2]*}\boldsymbol{\varphi}_{12}^{-1}\left(\overline{f}\, d\,\overline{f}^{-1}\right) . \tag{130}$$

We easily find that

$$l^{h*}\boldsymbol{a} = Ad_h\boldsymbol{a} + p_1^*\left(h\sum_\alpha \rho_\alpha r_\alpha^{[1]*}\boldsymbol{\varphi}^{-1}(d\,\boldsymbol{\varphi}(h^{-1})\right) .$$

The Lie$(Aut(H))$-valued 1-form $\sum_\alpha \rho_\alpha r_\alpha^{[1]*}\boldsymbol{\varphi}^{-1}\,d\,\boldsymbol{\varphi}$ is, due to (5), well defined on Y. We set

$$A = \sum_\alpha \rho_\alpha r_\alpha^{[1]*}\boldsymbol{\varphi}^{-1}\,d\,\boldsymbol{\varphi} - d \tag{131}$$

for the sought Lie($Aut(H)$)-valued 1-form on Y. Using the cocycle property of $\bar{f}$ and $\boldsymbol{\varphi}$ we easily have

Proposition 25. *Formulas (130) and (131) give a bundle gerbe connection.*

Using (88) we obtain that the 2-curvature $(\boldsymbol{k}, K)$ of the bundle gerbe 2-connection $(\boldsymbol{a}, A)$ satisfies

$$\boldsymbol{k}_{12} + \boldsymbol{k}_{23} + \boldsymbol{k}^r_{13} = T_{K_1}(\overline{f}^{-1}) \,. \tag{132}$$

Connection on a lifting bundle gerbe. Let us now consider the example of a lifting bundle gerbe associated with an exact sequence of groups (112) and a G-principal bundle $P \to M$ on M. In this case, for any given connection $\bar{A}$ on P we can construct a connection on the lifting bundle gerbe. Let us choose a section $s :$ Lie(G) $\to$Lie(E); i.e a linear map such that $\pi \circ s = id$. We first define $A = s(\bar{A})$ and then consider the Lie(E) valued one-forms on $P^{[2]}$ given by $A_1 = p_1^* s(\bar{A})$ and $A_2 = p_2^* s(\bar{A})$, where p_1 and p_2 are respectively the projections onto the first and second factor of $P^{[2]}$. We next consider the one-form $\boldsymbol{a}$ on $\mathcal{E}$ that on $(p_1, p_2; e) \in \mathcal{E}$ is given by

$$\boldsymbol{a} \equiv e\mathcal{A}_2 e^{-1} + ede^{-1} - \mathcal{A}_1 \,, \tag{133}$$

here $\mathcal{A}_1 = p^*(A_1)$ and $\mathcal{A}_2 = p^*(A_2)$, with $p : \mathcal{E} \to P^{[2]}$. It is easy to see that $\pi^* \boldsymbol{a} = 0$ and that therefore $\boldsymbol{a}$ is Lie(H) valued; moreover $(\boldsymbol{a}, ad_A)$ is a 2-connection on $\mathcal{E}$. Recalling that on $\mathcal{E}$ we have $\boldsymbol{\varphi}_{(p_1, p_2; e)} = Ad_e$, it is now a straightforward check left to the reader to show that $(\boldsymbol{a}, ad_A)$ is a connection on the lifting bundle gerbe.

Connection on a module. Let us start discussing the case of the canonical module $can = Aut(H) \times Y$ (see Sect. 6). Let $(\boldsymbol{a}, A)$ be a connection on our bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$. The Lie($Aut(H)$)-valued one-form A on Y lifts canonically to the connection $\tilde{\mathcal{A}}$ on can defined, for all $(\eta, y) \in can$, by $\tilde{\mathcal{A}} = \eta \mathcal{A} \eta^{-1} + \eta d\eta^{-1}$. Let us consider the following diagram:

$$\begin{tikzcd} can_1 \oplus \mathcal{E} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[drr, "\pi_\oplus"] & \mathcal{E} & & \\ can_1 & & can_1\mathcal{E} \arrow[r, "z"] & can_2 \,. \end{tikzcd} \tag{134}$$

As in the case of the bundle gerbe connection we can consider whether the Lie($Aut(H)$)-valued one-form $\tilde{\mathcal{A}}_1 + \boldsymbol{\xi}(\boldsymbol{a})$ that lives on $can_1 \oplus \mathcal{E}$ is the pull-back under $\pi_\oplus$ of a one-form connection on $can_1\mathcal{E}$. If this is the case then we say that $\tilde{\mathcal{A}}_1$ and $\boldsymbol{a}$ are summable and we denote by $\tilde{\mathcal{A}}_1 + ad_{\boldsymbol{a}}$ the resulting connection on $can_1\mathcal{E}$. Let us recall that on can we have $\boldsymbol{\xi}_{(\eta,y)} = Ad \circ \boldsymbol{\psi}_{(\eta,y)}$ with $\boldsymbol{\psi}_{(\eta,y)}(h) = \eta(h)$. It is now easy to check that $\tilde{\mathcal{A}}_1$ and $\boldsymbol{a}$ are summable and that their sum equals the pull-back under z of the connection $\tilde{\mathcal{A}}_2$; in formulae

$$\tilde{\mathcal{A}}_1 + ad_{\boldsymbol{a}} = z^* \tilde{\mathcal{A}}_2 \,. \tag{135}$$

We also have that equality (135) is equivalent to the summability condition (127) for the bundle gerbe connection $\boldsymbol{a}$. Thus (135) is a new interpretation of the summability condition (127).

We now discuss connections on an arbitrary module $(\mathcal{Q}, \mathcal{Z}, \boldsymbol{z})$ associated with a bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ with connection $(\boldsymbol{a}, A)$. There are two natural requirements that a left connection $\mathcal{A}^D$ on the left D-bundle $\mathcal{Q}$ has to satisfy in order to be a module connection. The first one is that the induced connection $\widehat{\mathcal{A}^D}$ on $Aut(H) \times_D \mathcal{Q}$ has to be equal (under the isomorphism σ) to the connection $\tilde{A}$ of can. This condition reads

$$\mathcal{A}^D = \boldsymbol{\psi}\mathcal{A}\boldsymbol{\psi}^{-1} + \boldsymbol{\psi} d\boldsymbol{\psi}^{-1} , \tag{136}$$

where in the l.h.s. $\mathcal{A}^D$ is thought to be Lie$(Aut(H))$ valued. In other words on Y we require $\sigma^*\widehat{\mathcal{A}^D} = A$, where σ is the global section of $Aut(H) \times_D \mathcal{Q}$.

Next consider the diagram

$$\begin{tikzcd} \mathcal{Q}_1 \oplus \mathcal{E} \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, "\pi_\oplus"] & \mathcal{E} & \\ \mathcal{Q}_1 & \mathcal{Q}_1\mathcal{E} \arrow[r, "z"] & \mathcal{Q}_2 \ . \end{tikzcd} \tag{137}$$

We denote by $\mathcal{A}_1^D {+}\alpha(\boldsymbol{a})$ the well defined D-connection on $\mathcal{Q}_1\mathcal{E}$ that pulled back on $\mathcal{Q}_1 \oplus \mathcal{E}$ equals $\pi_1^*\mathcal{A}_1^D + \boldsymbol{\xi}(\pi_2^*\boldsymbol{a})$. It is not difficult to see that $\mathcal{A}_1^D$ is indeed summable to $\boldsymbol{a}$ if for all $h \in H$, $\alpha(T_{\mathcal{A}^D}(h)) = \alpha(T_{\boldsymbol{\psi}\mathcal{A}\boldsymbol{\psi}^{-1} + \boldsymbol{\psi} d\boldsymbol{\psi}^{-1}}(h))$. This summability condition is thus implied by (136). The second requirement that $\mathcal{A}^D$ has to satisfy in order to be a module connection is

$$\mathcal{A}_1^D {+}\alpha(\boldsymbol{a}) = z^*\mathcal{A}_2^D \, . \tag{138}$$

These conditions imply the summability condition (127) for the bundle gerbe connection $\boldsymbol{a}$.

Concerning the D-valued curvature $\mathcal{K}^D = d\mathcal{A}^D + \mathcal{A}^D \wedge \mathcal{A}^D$ we have

$$\mathcal{K}_1^D {+}\alpha(\boldsymbol{k_a}) = z_{12}^*\mathcal{K}_2^D \, . \tag{139}$$

In terms of local data a gerbe connection consists of a collection of local 2-connections $(\boldsymbol{a}_{\alpha\beta}, A_\alpha)$ on the local bibundles $\mathcal{E}_{\alpha\beta} \to O_{\alpha\beta}$. For simplicity we assume the covering $\{O_\alpha\}$ to be a good one. The explicit relations that the local maps $f_{\alpha\beta\gamma} : O_{\alpha\beta\gamma} \to H$, $\varphi_{\alpha\beta} : O_{\alpha\beta} \to Aut(H)$ and the local representatives A_α, K_α, $a_{\alpha\beta}$ and $k_{\alpha\beta}$ (forms on O_α, $O_{\alpha\beta}$, etc.) satisfy are

$$f_{\alpha\beta\gamma} f_{\alpha\gamma\delta} = \varphi_{\alpha\beta}(f_{\beta\gamma\delta}) f_{\alpha\beta\delta} \, , \tag{140}$$

$$\varphi_{\alpha\beta}\varphi_{\beta\gamma} = Ad_{f_{\alpha\beta\gamma}}\varphi_{\alpha\gamma} \, , \tag{141}$$

$$a_{\alpha\beta} + \varphi_{\alpha\beta}(a_{\beta\gamma}) = f_{\alpha\beta\gamma} a_{\alpha\gamma} f_{\alpha\beta\gamma}^{-1} + f_{\alpha\beta\gamma}\, d\, f_{\alpha\beta\gamma}^{-1} + T_{A_\alpha}(f_{\alpha\beta\gamma}) \, , \tag{142}$$

$$A_\alpha + ad_{a_{\alpha\beta}} = \varphi_{\alpha\beta} A_\beta \varphi_{\alpha\beta}^{-1} + \varphi_{\alpha\beta}\, d\, \varphi_{\alpha\beta}^{-1} \, , \tag{143}$$

$$k_{\alpha\beta} + \varphi_{\alpha\beta}(k_{\beta\gamma}) = f_{\alpha\beta\gamma} k_{\alpha\gamma} f_{\alpha\beta\gamma}^{-1} + T_{K_\alpha}(f_{\alpha\beta\gamma}), \tag{144}$$

and

$$K_\alpha + ad_{k_{\alpha\beta}} = \varphi_{\alpha\beta} K_\beta \varphi_{\alpha\beta}^{-1} \, . \tag{145}$$

8. Curving

In this section we introduce the curving two form $\boldsymbol{b}$. This is achieved considering a gerbe stably equivalent to $(\mathcal{E}, Y, M, \boldsymbol{f})$. The resulting equivariant H-valued 3-form $\boldsymbol{h}$ is then shown to be given in terms of a form on Y. This description applies equally well to the abelian case; there one can however impose an extra condition [namely the vanishing of (147)]. We also give an explicit general construction of the curving $\boldsymbol{b}$ in terms of a partition of unity. This construction depends only on the partition of unity, and in the abelian case it naturally reduces to the usual one that automatically encodes the vanishing of (147).

Consider a bundle gerbe $(\mathcal{E}, Y, M, \boldsymbol{f})$ with connection $(\boldsymbol{a}, A)$ and curvature $(\boldsymbol{k_a}, K_A)$ and an H-bibundle $\mathcal{N} \to Y$ with a 2-connection $(\boldsymbol{c}, A)$. Then we have a stably equivalent gerbe $(\mathcal{N}_1^{-1}\mathcal{E}\mathcal{N}_2, Y, M, \boldsymbol{\varphi}_1^{-1}(\boldsymbol{f}))$ with connection $(\boldsymbol{\theta}, A^{r_1})$ given by

$$\boldsymbol{\theta} = \boldsymbol{c}_1^{r_1} + \boldsymbol{a} + \boldsymbol{c}_2\,. \tag{146}$$

Also we can consider a K_A-equivariant horizontal 2-form $\boldsymbol{b}$ on $\mathcal{N}$. Again on the bibundle $\mathcal{N}_1^{-1}\mathcal{E}\mathcal{N}_2 \to Y^{[2]}$ we get a well defined $K_A^{r_1}$-equivariant horizontal 2-form

$$\tilde{\boldsymbol{\delta}} = \boldsymbol{b}_1^{r_1} + \boldsymbol{k_a} + \boldsymbol{b}_2\,. \tag{147}$$

Contrary to the abelian case we cannot achieve $\tilde{\boldsymbol{\delta}} = 0$, unless K_A is inner (remember $\tilde{\boldsymbol{\delta}}$ is always $K_A^{r_1}$-equivariant). Next we consider the equivariant horizontal H-valued 3-form $\boldsymbol{h}$ on $\mathcal{N}$ given by

$$\boldsymbol{h} = D_{\boldsymbol{c}}\boldsymbol{b}\,. \tag{148}$$

Because of the Bianchi identity $dK_A + [A, K_A] = 0$ this is indeed an equivariant form on $\mathcal{N}$. Obviously the horizontal form $\boldsymbol{\varphi}^{-1}(\boldsymbol{h})$ is invariant under the left H-action

$$l^{h*}\boldsymbol{\varphi}^{-1}(\boldsymbol{h}) = \boldsymbol{\varphi}^{-1}(\boldsymbol{h}), \tag{149}$$

and therefore it projects to a well defined form on Y.

Using now the property of the covariant derivative (86) and the Bianchi identity (68) we can write

$$\boldsymbol{h}_1^{r} + \boldsymbol{h}_2 = D_{\boldsymbol{\theta}}\tilde{\boldsymbol{\delta}}\,. \tag{150}$$

Finally from (72) we get the Bianchi identity for $\boldsymbol{h}$,

$$D_{\boldsymbol{c}}\boldsymbol{h} = [\boldsymbol{k_c}, \boldsymbol{b}] + T_{K_A}(\boldsymbol{k_c}) - T_{K_A}(\boldsymbol{b})\,. \tag{151}$$

For the rest of this section we consider the special case where $\mathcal{N}$ is a trivial bibundle with global central section $\bar{\sigma}$ and with 2-connection given by $(\boldsymbol{c}, A)$, where $\boldsymbol{c}$ is canonically given by $\bar{\sigma}$,

$$\boldsymbol{c} = \bar{\sigma}d\bar{\sigma}^{-1} + T_A(\bar{\sigma})\,.$$

Since the only H-bibundle $\mathcal{N} \to Y$ that we can canonically associate to a generic bundle gerbe is the trivial one (see Proposition 7), the special case where $\mathcal{N}$ is trivial seems quite a natural case.

In terms of local data curving is a collection $\{\boldsymbol{b}_\alpha\}$ of K_α-equivariant horizontal two forms on trivial H-bibundles $O_\alpha \times H \to O_\alpha$. Again we assume the covering O_α to be a good one and write out explicitly the relations to which the local representatives of b_α and h_α (forms on O_α) are subject:

$$k_{\alpha\beta} + \varphi_{\alpha\beta}(b_\beta) = b_\alpha + \delta_{\alpha\beta}\,, \tag{152}$$

$$\delta_{\alpha\beta} + \varphi_{\alpha\beta}(\delta_{\beta\gamma}) = f_{\alpha\beta\gamma}\delta_{\alpha\gamma} f^{-1}_{\alpha\beta\gamma} + T_{\nu_\alpha}(f_{\alpha\beta\gamma})\,, \tag{153}$$

$$\nu_\alpha \equiv K_\alpha + ad_{b_\alpha}\,, \tag{154}$$

$$h_\alpha = d\,b_\alpha - T_{A_\alpha}(b_\alpha)\,, \tag{155}$$

$$\varphi_{\alpha\beta}(h_\beta) = h_\alpha + d\delta_{\alpha\beta} + [a_{\alpha\beta}, \delta_{\alpha\beta}] + T_{\nu_\alpha}(a_{\alpha\beta}) - T_{A_\alpha}(\delta_{\alpha\beta}), \tag{156}$$

and the Bianchi identity

$$d\,h_\alpha + T_{K_A}(b_\alpha) = 0\,. \tag{157}$$

Here we introduced $\delta_{\alpha\beta} = \varphi_\alpha(\tilde{\delta}_{\alpha\beta})$. Equations (140)–(145) and (152)–(157) are the same as those listed after Theorem 6.4 in [26].

We now consider the case $Y = \sqcup O_\alpha$; this up to stable equivalence is always doable. Given a partition of unity $\{\rho_\alpha\}$ subordinate to the covering $\{O_\alpha\}$ of M, we have a natural choice for the H-valued curving 2-form $\boldsymbol{b}$ on $\sqcup O_\alpha \times H$. It is the pull-back under the projection $\sqcup O_\alpha \times H \to \sqcup O_\alpha$ of the 2-form

$$\sqcup \sum_\beta \rho_\beta k_{\alpha\beta} \tag{158}$$

on $Y = \sqcup O_\alpha$. In this case we have for the local H-valued 2-forms $\delta_{\alpha\beta}$ the following expression:

$$\begin{aligned}\delta_{\alpha\beta} &= \sum_\gamma \rho_\gamma (f_{\alpha\beta\gamma} k_{\alpha\gamma} f^{-1}_{\alpha\beta\gamma} - k_{\alpha\gamma} + T_{K_\alpha}(f_{\alpha\beta\gamma}))\\ &= \sum_\gamma \rho_\gamma (k_{\alpha\beta} + \varphi_{\alpha\beta}(k_{\beta\gamma}) - k_{\alpha\gamma})\,.\end{aligned} \tag{159}$$

We can now use Proposition 25 together with (158) in order to explicitly construct from the Čech cocycle $(\boldsymbol{f}, \boldsymbol{\varphi})$ an H-valued 3-form $\boldsymbol{h}$.

We conclude this final section by grouping together the global cocycle formulae that imply all the local expressions (140)–(145) and (152)–(157),

$$\boldsymbol{f}_{124}\, \boldsymbol{\varphi}_{12}(\boldsymbol{f}_{234}) = \boldsymbol{f}_{134}\, \boldsymbol{f}_{123}\,, \tag{92}$$

$$\boldsymbol{a}_{12} + \boldsymbol{a}_{23} = f^* \boldsymbol{a}_{13}\,, \tag{125}$$

$$\tilde{\boldsymbol{\delta}} = \boldsymbol{b}_1^{r_1} + \boldsymbol{k}_{\boldsymbol{a}} + \boldsymbol{b}_2\,, \tag{147}$$

$$\boldsymbol{h} = D_{\boldsymbol{c}} \boldsymbol{b}\,. \tag{148}$$

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THE CLASSIFYING TOPOS OF A TOPOLOGICAL BICATEGORY

IGOR BAKOVIĆ AND BRANISLAV JURČO

(*communicated by Ronald Brown*)

Abstract

For any topological bicategory $\mathbb{B}$, the Duskin nerve $N\mathbb{B}$ of $\mathbb{B}$ is a simplicial space. We introduce the classifying topos $\mathcal{B}\mathbb{B}$ of $\mathbb{B}$ as the Deligne topos of sheaves $\mathrm{Sh}(N\mathbb{B})$ on the simplicial space $N\mathbb{B}$. It is shown that the category of geometric morphisms $\mathrm{Hom}(\mathrm{Sh}(X), \mathcal{B}\mathbb{B})$ from the topos of sheaves $\mathrm{Sh}(X)$ on a topological space X to the Deligne classifying topos is naturally equivalent to the category of principal $\mathbb{B}$-bundles. As a simple consequence, the geometric realization $|N\mathbb{B}|$ of the nerve $N\mathbb{B}$ of a locally contractible topological bicategory $\mathbb{B}$ is the classifying space of principal $\mathbb{B}$-bundles, giving a variant of the result of Baas, Bökstedt and Kro derived in the context of bicategorical K-theory. We also define classifying topoi of a topological bicategory $\mathbb{B}$ using sheaves on other types of nerves of a bicategory given by Lack and Paoli, Simpson and Tamsamani by means of bisimplicial spaces, and we examine their properties.

1. Introduction

In a recent paper by Baas, Bökstedt and Kro [**1**], it was shown that the geometric realization $|N\mathbb{B}|$ of the Duskin nerve $N\mathbb{B}$ [**11**] of a good topological bicategory $\mathbb{B}$ is the classifying space of charted $\mathbb{B}$-bundles. The bicategory is called good if its Duskin nerve $N\mathbb{B}$ is a good simplicial space, i.e., all degeneracy maps are closed cofibrations. Special cases of topological 2-groups and Lie 2-groups were discussed in [**2**] and in [**15**], respectively.

The result of [**1**] generalizes the well-known fact that the geometric realization $|N\mathbb{C}|$ of the nerve $N\mathbb{C}$ of a locally contractible topological category $\mathbb{C}$ is the classifying space of principal $\mathbb{C}$-bundles (on CW-complexes). This is very nicely described by Moerdijk in [**21**]. The classifying topos $\mathcal{B}\mathbb{C}$ of a topological category $\mathbb{C}$ is also described there as the Deligne topos of sheaves $\mathrm{Sh}(N\mathbb{C})$ on the nerve $N\mathbb{C}$, and it is shown that the category of geometric morphisms $\mathrm{Hom}(\mathrm{Sh}(X), \mathcal{B}\mathbb{C})$ from the topos of sheaves $\mathrm{Sh}(X)$ on a topological space X to the Deligne topos is naturally equivalent

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to the category of principal $\mathbb{C}$-bundles. As a simple consequence, it is shown that the geometric realization $|N\mathbb{C}|$ of the nerve $N\mathbb{C}$ of a locally contractible topological category $\mathbb{C}$ is the classifying space of principal $\mathbb{C}$-bundles.

One purpose of this note is to introduce the classifying topos $\mathcal{B}\mathbb{B}$ of a topological bicategory $\mathbb{B}$ as the topos of sheaves $\mathrm{Sh}(N\mathbb{B})$ on the Duskin nerve $N\mathbb{B}$ of the bicategory $\mathbb{B}$, which is a simplicial space. The category of geometric morphisms $\mathrm{Hom}(\mathrm{Sh}(X), \mathcal{B}\mathbb{B})$ from the topos of sheaves $\mathrm{Sh}(X)$ on a topological space X to the classifying topos is naturally equivalent to the category of (suitably defined) principal $\mathbb{B}$-bundles. As a simple consequence, the geometric realization $|N\mathbb{B}|$ of the nerve $N\mathbb{B}$ of a locally contractible topological bicategory $\mathbb{B}$ is the classifying space of principal $\mathbb{B}$-bundles. Hence, we have a variant of the result of Baas, Bökstedt and Kro.

Another purpose of this note is to define classifying topoi of a topological bicategory $\mathbb{B}$ using sheaves on other types of nerves of the bicategory $\mathbb{B}$, the nerves according to Lack and Paoli [**17**] (or Simpson [**23**] and Tamsamani [**25**]), which can be viewed as bisimplicial spaces. Again, the category of topos morphisms from the topos of sheaves $\mathrm{Sh}(X)$ on a topological space X to the corresponding classifying topos is naturally equivalent to the respective category of (suitably defined) principal $\mathbb{B}$-bundles. As a simple consequence, the geometric realization of any of these nerves of a locally contractible topological bicategory $\mathbb{B}$ is the classifying space of the respective principal $\mathbb{B}$-bundles.

In Section 2, we recall some prerequisites from [**21**] regarding sheaves on a simplicial space and augmented linear orders over topological spaces. In Section 3, we recall, again from [**21**], the known facts about classifying spaces and topoi of topological categories (and the corresponding principal bundles). We describe a generalization to the case of bicategories, based on the Duskin nerve, in Section 4. Further preliminaries needed for the subsequent discussion of alternative definitions of classifying spaces and topoi of bicategories are given in Section 5. Finally, in Section 6, we describe a modification of the classifying topos of a topological bicategory (and the corresponding principal bundles) based on alternative definitions of the nerves according to Lack and Paoli, Simpson and Tamsamani.

This article is meant to be the first one in a sequence within a program, initiated by the authors, of classifying topoi of higher order structures in topology. It is a vast generalization of the program initiated by Moerdijk in [**21**] on the relation between classifying spaces and classifying topoi. Moerdijk's lecture notes arose out of an important question: What does the classifying space of a small category classify? In the article titled by the same question [**30**], Weiss proved the classifying property of the classifying space for slightly different geometric objects than those of Moerdijk, showing that the answer may not be unique.

Therefore, this article may be seen as an (one possible) answer to the following question: What does the classifying space of a topological bicategory classify? Bicategories are the weakest possible generalization of ordinary categories to the immediate next level of dimension. Like categories, bicategories do have a genuine simplicial set associated with them, their Duskin (geometric) nerve [**11**]. Based on unpublished work of Roberts on the characterization of the nerve of a strict n-category, Street postulated in [**24**] an equivalence between the category of *strict ω-categories* and a category of certain types of simplicial sets which are called *complicial sets*. The *Street-Roberts conjecture* was proved by Verity in [**27**], and in his subsequent papers, [**28**] and [**29**], he gave a characterization of *weak ω-categories*. Under this characterization, one should be able to capture classifying spaces and topoi of bicategories and other higher-dimensional categories, at least in so far as these concepts have found satisfactory definitions. Following such reasoning, we may define the classifying space of a weak ω-category as the geometric realization of the complicial set, which is its nerve, and the classifying topos of a weak ω-category would be the topos of sheaves on the same complicial set.

It would be interesting to compare this approach to classifying spaces of weak ω-categories with classifying spaces of crossed complexes defined by Brown and Higgins in [**8**], since there is a well-known equivalence between *strict ω-groupoids* and *crossed complexes* proved in [**7**] by the same authors. In particular, it would be interesting to see whether the methods we developed would allow one to define a classifying space of a weak ω-category by taking a fundamental crossed complex of its coherent simplicial nerve.

However, this article is not so cosmological in its scope, and its main contribution is to put together some established results on classifying spaces and classifying topoi in a new way, with consequences for the theory of bicategories. Since we are following Moerdijk's approach to classifying spaces and classifying topoi, we will omit all proofs, which can be found in Moerdijk's lecture notes. Many of the definitions and theorems in Sections 2, 3 and 5 are taken almost verbatim from [**21**].

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2. Simplicial spaces and linear orders over topological spaces

In this section, we recall some prerequisites regarding sheaves on a simplicial space and augmented linear orders over topological spaces. Almost all of the definitions and theorems are taken verbatim from [**21**], where the proofs of all statements of this section can be found.

2.1. Topological spaces

Let us recall that a closed set in a (topological) space X is *irreducible* if it can not be written as a union of two smaller closed sets. The space X is *sober* if every irreducible set is the closure $\overline{\{x\}}$ of the one point set $\{x\}$ of a unique $x \in X$. Every Hausdorff space is sober. In this note all spaces will be sober by assumption.

A space X is *locally equiconnected* (LEC) if the diagonal map $X \to X \times X$ is a closed cofibration. For example, CW-complexes are LEC.

A space X is *locally contractible* if it has a basis of contractible sets. Examples of locally contractible spaces are locally equiconnected spaces and, in particular, CW-complexes. For a locally contractible space the étale homotopy groups $\pi_n(\mathrm{Sh}(X), x_0)$ are naturally isomorphic to the ordinary homotopy groups $\pi_n(X, x_0)$ for each n.

2.2. Sheaves as étale spaces

Throughout this article, we will consider sheaves as sheaves of cross-sections of étale spaces. Recall that a bundle $p\colon E \to X$ over X is said to be an *étale space* over X if for each $e \in E$ there exists an open set $V \subset E$, with $e \in V$, such that $p(V) \subset X$ is open in X and the restriction $p|_V\colon V \to p(V)$ over V is a homeomorphism. There is a well-known equivalence

$$\mathrm{Etale}(X) \underset{\Lambda}{\overset{\Gamma}{\rightleftarrows}} \mathrm{Sh}(X),$$

where $\Gamma\colon \mathrm{Etale}(X) \to \mathrm{Sh}(X)$ is a functor which assigns to each étale space $p\colon E \to X$ over X the sheaf of all cross-sections of E. The functor $\Lambda\colon \mathrm{Sh}(X) \to \mathrm{Etale}(X)$ assigns to each sheaf S the étale space of *germes* of S, where the germ at the point $x \in X$ is an equivalence class $\mathrm{germ}_x\, s$ represented by $s \in S(U)$ under the equivalence relation, which relates two elements $s \in S(U)$ and $t \in S(V)$, if there is some open set $W \subset U \cap V$ such that $x \in W$ and $s|_W = t|_W$. The stalk of the sheaf S at the point $x \in X$ is the set $S_x = \{\mathrm{germ}_x\, s\colon s \in S(U), x \in U\}$ of all germs at x, which is formally a filtered colimit

$$S_x = \varinjlim_{x \in U} S(U)$$

of the restriction $S^{(x)}\colon \mathcal{O}_x(X)^{\mathrm{op}} \to \mathrm{Set}$ of the sheaf S to the filtered category $\mathcal{O}_x(X)^{\mathrm{op}}$ of open neighborhoods of the point $x \in X$. Then ΛS is an étale space $p\colon \coprod_{x \in X} S_x \to X$ whose sheaf of cross sections is canonically isomorphic to S. Therefore, we will simultaneously use the terms sheaves and étales spaces in the rest of this article.

2.3. Topoi

In the following, a *topos* will always mean a Grothendieck topos. $\mathrm{Sh}(X)$ will denote topos of sheaves on a (topological) space X. A sober space X can be recovered from the topos $\mathrm{Sh}(X)$, which is the faithful image of the space X in the world of topoi.

Further, $\mathrm{Hom}(\mathrm{Sh}(X), \mathrm{Sh}(Y))$ will denote the category of geometric morphisms from $\mathrm{Sh}(X)$ to $\mathrm{Sh}(Y)$. We will also use the same notation $\mathrm{Hom}(\mathcal{F}, \mathcal{E})$ in the more general case of any two topoi $\mathcal{F}$ and $\mathcal{E}$. By definition, a *geometric morphism* $f \in \mathrm{Hom}(\mathcal{F}, \mathcal{E})$ is a pair of functors $f^*\colon \mathcal{E} \to \mathcal{F}$ and $f_*\colon \mathcal{F} \to \mathcal{E}$, f^* being left adjoint to f_*, and also f^* being left exact, i.e., preserving finite limits.

Let us recall that a geometric morphism $f\colon \mathcal{F} \to \mathcal{E}$ between locally connected topoi is a *weak homotopy equivalence* if it induces isomorphisms on étale homotopy (pro)groups $\pi_0(\mathcal{F}) \cong \pi_0(\mathcal{E})$ and $\pi_n(\mathcal{F}, p) \cong \pi_n(\mathcal{E}, fq)$, for $n \geqslant 1$ for any basepoint $q \in \mathcal{F}$.

For the collection of homotopy classes of geometric morphism from $\mathcal{F}$ to $\mathcal{E}$ the usual notation $[\mathcal{F}, \mathcal{E}]$ will be used.

2.4. The singular functor

The following construction of a singular functor is taken from [**16**], where Kelly described it in the context of enriched $\mathcal{V}$-categories for any symmetric monoidal closed category $\mathcal{V}$, which is complete and cocomplete. Let

$$F\colon \mathbb{A} \to \mathbb{E}$$

be a functor from the small category $\mathbb{A}$. The *singular functor* of F is the functor

$$\mathbb{E}(F,1)\colon \mathbb{E} \to [\mathbb{A}^{\mathrm{op}}, \mathcal{V}],$$

which is obtained as the composite of the Yoneda embedding

$$\mathrm{Yon}\colon \mathbb{E} \to [\mathbb{E}^{i\,\mathrm{op}}, \mathcal{V}]$$

followed by the functor $[F^{\mathrm{op}}, \mathcal{V}]\colon [\mathbb{E}^{\mathrm{op}}, \mathcal{V}] \to [\mathbb{A}^{\mathrm{op}}, \mathcal{V}]$ given by restriction along a functor F. More explicitly, the singular functor $\mathbb{E}(F,1)$ sends any object E in $\mathbb{E}$ to the functor

$$\mathbb{E}(F(-),E)\colon \mathbb{A}^{\mathrm{op}} \to \mathcal{V},$$

which takes an object A in $\mathbb{A}$ to the hom-object $\mathbb{E}(F(A),E)$ in $\mathcal{V}$. If the category $\mathbb{E}$ is cocomplete, then the singular functor has a left adjoint

$$L\colon [\mathbb{A}^{\mathrm{op}}, \mathcal{V}] \to \mathbb{E}$$

defined for each presheaf $P\colon \mathbb{A}^{\mathrm{op}} \to \mathcal{V}$ as the colimit

$$L(P) = \varinjlim\big(\textstyle\int_{\mathbb{A}} P \xrightarrow{\pi_P} \mathbb{A} \xrightarrow{F} \mathbb{E}\big),$$

where $\int_{\mathbb{A}} P$ is the so-called Grothendieck construction [**20**] on a presheaf $P\colon \mathbb{A}^{\mathrm{op}} \to \mathcal{V}$.

2.5. Grothendieck nerve as a singular functor

Each ordinal $[n] = \{0 < 1 < \cdots < n\}$ can be seen as a category with the objects $0, 1, \ldots, n$ and a unique arrow $i \to j$ for each $0 \leqslant i \leqslant j \leqslant n$. Also, any monotone map between two ordinals may be seen as a functor. In this way, Δ becomes a full subcategory of Cat_1 with a fully faithful inclusion functor

$$J\colon \Delta \to \mathrm{Cat}_1.$$

For any small category $\mathbb{B}$, we see that the composite of the Yoneda embedding $\mathrm{Yon}\colon \mathbb{B} \to [\mathbb{B}^{\mathrm{op}}, \mathrm{Set}]$, followed by the restriction functor $[\mathbb{B}^{\mathrm{op}}, \mathrm{Set}] \to [\Delta^{\mathrm{op}}, \mathrm{Set}]$ along J, gives a singular functor of J. Specifically, the singular functor of J defines the Grothendieck nerve functor

$$N\colon \mathrm{Cat}_1 \to [\Delta^{\mathrm{op}}, \mathrm{Set}],$$

which sends any category $\mathbb{C}$ to the simplicial set $N\mathbb{C}$ which is the *nerve* of $\mathbb{C}$ whose n-simplices are defined by the set

$$N\mathbb{C}_n = [J([n]), \mathbb{C}],$$

where the right side denotes the set of functors from an ordinal $[n]$ to the category $\mathbb{C}$. The nerve functor is fully faithful, which means that the simplicial skeletal category Δ is an adequate subcategory of the category Cat_1 in the sense of Isbell [**13, 14**]. We also say that the corresponding embedding is dense, in the sense of Kelly [**16**].

2.6. Simplicial spaces

Let Δ be the simplicial model category having as objects the nonempty finite sets (ordinals) $[n] = \{0, 1, \dots, n\}$, for $n \geqslant 0$, and as arrows the order-preserving functions $\alpha\colon [n] \to [m]$. A *simplicial space (set)* is a contravariant functor from Δ into the category of spaces (sets). Its value at $[n]$ is denoted Y_n, and its action on the arrow $\alpha\colon [n] \to [m]$ is denoted as $Y(\alpha)\colon Y_m \to Y_n$. A simplicial space Y is called *locally contractible* if each Y_n has a basis of contractible sets.

For a simplicial space Y, the *geometric realization* $|Y|$ will always mean the thickened (fat) geometric realization. This is defined as a topological space obtained from the disjoint sum $\sum_{n\geqslant 0} X_n \times \Delta^n$ by the the equivalence relations

$$(\alpha^*(x), t) \sim (x, \alpha(t))$$

for all injective (order-preserving) arrows $\alpha\colon [n] \to [m] \in \Delta$, any $x \in X_m$ and any $t \in \Delta^n$, where Δ^n is the standard topological n-simplex. If all degeneracies are closed cofibrations, i.e., the simplicial space is a *good* simplicial space, then this geometric realization is homotopy equivalent to the geometric realization of the underlying simplicial set of Y, which is defined as above but allowing for all arrows in Δ. In particular, Y is good if all spaces Y_n are locally equiconnected [**1**]. Geometric realization of a locally contractible simplicial space is a locally contractible space.

Definition 2.1. A **sheaf** S **on a simplicial space** Y is defined to be a system of sheaves S^n on Y_n, for $n \geqslant 0$, together with sheaf maps $S(\alpha)\colon Y(\alpha)^* S^n \to S^m$ for each $\alpha\colon [n] \to [m]$. These maps are required to satisfy the following functoriality conditions:

(i) (normalization). $S(\mathrm{id}_{[n]}) = \mathrm{id}_{S_n}$, and

(ii) for any $\alpha\colon [n] \to [m]$, $\beta\colon [m] \to [k]$ the following diagram:

$$\begin{tikzcd}
Y(\beta)^*Y(\alpha)^*S^n \arrow[r, "Y(\beta)^*S(\alpha)"] \arrow[d, "\cong"'] & Y(\beta)^*S^m \arrow[d, "S(\beta)"] \\
Y(\beta\alpha)^*S^n \arrow[r, "S(\beta\alpha)"'] & S^k
\end{tikzcd}$$

is commutative. A morphism $f\colon S \to T$ of sheaves on Y consists of the maps $f_n\colon S^n \to T^n$ of sheaves on Y_n for each $n \geqslant 0$, which are compatible with the structure maps $S(\alpha)$ and $T(\alpha)$. This defines the category $\mathrm{Sh}(Y)$ of sheaves on the simplicial space Y.

Proposition 2.2. *The category* $\mathrm{Sh}(Y)$ *of sheaves on a simplicial space is a topos.*

Theorem 2.3. *For any simplicial space* Y*, the topoi* $\mathrm{Sh}(Y)$ *and* $\mathrm{Sh}(|Y|)$ *have the same weak homotopy type.*

Definition 2.4. A **linear order over a topological space** X is a sheaf $p\colon L \to X$ on X together with a subsheaf $O \subseteq L \times_X L$ such that for each point $x \in X$ the stalk

L_x is nonempty and is linearly ordered by the relation

$$y \leqslant z \qquad \text{if and only if} \qquad (y,z) \in O_x,$$

for $y, z \in L_x$. A mapping $L \to L'$ between two linear orders over X is a mapping of sheaves restricting for each $x \in X$ to an order-preserving map of stalks $L_x \to L'_x$. This defines a category of linear orders on X.

Example 2.5. An *open ordered covering* $\mathcal{U} = \{U_i\}_{i \in I}$ of a topological space X is a covering indexed over a partially ordered set I, which restricts to a total ordering on every finite subset $\{i_0, \ldots, i_n\}$ of I whenever the finite intersection $U_{i_0,\ldots,i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$ is nonempty. When a sheaf $p\colon L \to X$ is given by the projection $p\colon \coprod_{i \in I} U_i \to X$ from the disjoint union of open sets in the open ordered covering $\mathcal{U}$, the subsheaf $p^{[2]}\colon L \times_X L \to X$ is given by the induced projection $p^{[2]}\colon \coprod_{i,j \in I} U_{ij} \to X$ from the family $\{U_{ij}\}_{i,j \in I}$ of double intersections of open sets $\mathcal{U}$. The family of inclusions $i_{ij}\colon U_{ij} \hookrightarrow \coprod_{i,j \in I} U_{ij}$, for each $U_{ij} \neq \emptyset$ such that $i < j$, defines a subsheaf $O = \coprod_{i<j} U_{ij}$ of $L = \coprod_{i,j \in I} U_{ij}$ whose stalks O_x are linearly ordered for any $x \in X$. Therefore, open ordered coverings used by Baas, Bökstedt and Kro in [**1**] are examples of linear orders over X.

Remark 2.6. A linear order L over X defines an obvious topological category with L as the space of objects and the order subsheaf $O \subseteq L \times_X L$ as the space of arrows. Hence, we can speak of a nerve NL of the linear order L. This nerve is obviously a simplicial sheaf on X (a simplicial space with étale maps into X).

Recall that any open covering of a topological space X can be assembled into a simplicial sheaf over X with distinguished properties. Therefore, by the construction in Example 2.5 and following Remark 2.6, we may regard linear orders as generalizations of coverings of topological spaces.

Definition 2.7. For any space X and any simplicial space Y, write $\mathrm{Lin}(X,Y)$ for the category of linear orders (L, aug) over X equipped with a simplicial map (**augmentation**) $\mathrm{aug}\colon NL \to Y$ from the nerve of L to Y. A morphism $(L, \mathrm{aug}) \to (L', \mathrm{aug}')$ in $\mathrm{Lin}(X,Y)$ are maps of linear orders $L \to L'$ such that the induced map $NL \to NL'$ on the nerves respects the augmentations.

If we regard linear orders as generalizations of coverings of topological spaces, then augmentations of linear orders may be seen as cocycles on such coverings.

Example 2.8. Let $N\mathbb{C}$ be the nerve of a topological category $\mathbb{C}$. An augmentation $\mathrm{aug}\colon NL \to N\mathbb{C}$ of a linear order L defined by an open ordered covering $\mathcal{U} = \{U_i\}_{i \in I}$ of a topological space X, as in Example 2.5, is a Čech cocycle on the covering $\mathcal{U}$ with values in the category $\mathbb{C}$.

Definition 2.9. We call two objects $E_0, E_1 \in \mathrm{Lin}(X,Y)$ **concordant** if there exists an $E \in \mathrm{Lin}(X \times [0,1], Y)$ such that we have $E_0 \cong i_0^*(E)$ and $E_0 \cong i_1^*(E)$ under the obvious inclusions $i_0, i_1\colon X \hookrightarrow X \times [0,1]$. $\mathrm{Lin}_c(X,Y)$ will denote the collection of concordance classes of objects from $\mathrm{Lin}(X,Y)$.

Theorem 2.10. *Let Y be a simplicial space. For any space X there is a natural equivalence of categories*

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathrm{Sh}(Y)) \simeq \mathrm{Lin}(X, Y).$$

On homotopy classes of topos morphisms we have the natural bijection

$$[\mathrm{Sh}(X), \mathrm{Sh}(Y)] \cong \mathrm{Lin}_c(X, Y).$$

Corollary 2.11. *Let X be a* CW*-complex and Y be a locally contractible simplicial space. There is a natural bijection between homotopy classes of maps $[X, |Y|]$ and concordance classes* $\mathrm{Lin}_c(X, Y)$.

Remark 2.12. If, in addition, the simplicial space Y is a good one, then the above is also true if we use the geometric realization of the underlying simplicial set of Y instead of its thickened geometric realization. In particular, it does not matter which geometric realization we use if each of Y_n is LEC or a CW-complex.

3. Classifying spaces and classifying topoi of topological categories

In this section we specify the known results described in Section 1 to the case when the simplicial space Y is the nerve of a topological category $\mathbb{C}$. The reader who is interested in more details is referred to [**21**], which we again follow almost verbatim.

Definition 3.1. Let $\mathbb{C}$ be a topological category. The **classifying topos** $\mathcal{B}\mathbb{C}$ of a topological category is defined as the topos $\mathrm{Sh}(N\mathbb{C})$.

Definition 3.2. The **classifying space** $B\mathbb{C}$ of a topological category $\mathbb{C}$ is the geometric realization $|N\mathbb{C}|$ of its nerve $N\mathbb{C}$.

With these definitions we have the following corollary of Theorem 2.3.

Corollary 3.3. *For any topological category $\mathbb{C}$, the topos of sheaves* $\mathrm{Sh}(B\mathbb{C})$ *on the classifying space $B\mathbb{C}$ has the same weak homotopy type as the classifying topos $\mathcal{B}\mathbb{C}$.*

Definition 3.4. For any topological category $\mathbb{C}$, write $\mathrm{Lin}(X, \mathbb{C})$ for the category of linear orders over X equipped with an augmentation $NL \to N\mathbb{C}$. An object E of this category will be called a **principal $\mathbb{C}$-bundle**. We call two principal $\mathbb{C}$-bundles E_0 and E_1 on X **concordant** if there exists a principal $\mathbb{C}$-bundle on $X \times [0, 1]$ such that we have isomorphisms $E_0 \cong i_0^*(E)$ and $E_0 \cong i_1^*(E)$ under the obvious inclusions $i_0, i_1 \colon X \hookrightarrow X \times [0, 1]$.

Remark 3.5. The nerve construction leads to a bijection between principal $\mathbb{C}$-bundles and linear orders L equipped with a continuous functor $L \to \mathbb{C}$.

The fact that the classifying topos $\mathcal{B}\mathbb{C}$ classifies principal $\mathbb{C}$-bundles now follows immediately from Theorem 2.10.

Theorem 3.6. *For a topological category $\mathbb{C}$ and a topological space X, there is a natural equivalence of categories*

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathcal{B}\mathbb{C}) \simeq \mathrm{Lin}(X, \mathbb{C}).$$

On homotopy classes of topos morphisms, we have the natural bijection

$$[\mathrm{Sh}(X), \mathcal{B}\mathbb{C}] \cong \mathrm{Lin}_c(X, \mathbb{C}).$$

Similarly, the fact that the classification space $B\mathbb{C}$ classifies principal $\mathbb{C}$-bundles now follows from Corollary 2.11.

Definition 3.7. We say that a topological category $\mathbb{C}$ is **locally contractible** if its space of objects C_0 and its space of arrows C_1 are locally contractible. A topological category $\mathbb{C}$ is a **good topological category**, if its nerve $N\mathbb{C}$ is a good simplicial space.

Corollary 3.8. *For a locally contractible category $\mathbb{C}$ and a* CW*-complex X, there is a natural bijection*

$$[X, B\mathbb{C}] \cong \mathrm{Lin}_c(X, \mathbb{C}).$$

Remark 3.9. If, in addition, the topological category $\mathbb{C}$ is a good one, then the above is also true if we use the geometric realization of the underlying simplicial set instead of the thickened geometric realization of the nerve. In particular, it does not matter which geometric realization we use if all $N\mathbb{C}_n$ are LEC.

4. Classifying spaces and classifying topoi of topological bicategories I

In this section we specify the known results described in Section 1 to the case when the simplicial space Y is the nerve of a topological bicategory $\mathbb{B}$.

4.1. Duskin nerve as a singular functor

The Duskin nerve [**11**] can also be obtained as a singular functor when we take $\mathcal{V} = \mathrm{Set}$. Every category (in particular the category defined above by the ordinal $[n]$) can be seen as a locally discrete bicategory (the only 2-cells are identities) which gives a fully faithful inclusion

$$H \colon \Delta \to \mathrm{Bicat}_1,$$

where Bicat_1 denotes the category of bicategories and normal lax functors or normal morphisms of bicategories defined by Bénabou in [**6**]. The singular functor of the inclusion H is the Duskin nerve functor

$$N \colon \mathrm{Bicat}_1 \to [\Delta^{\mathrm{op}}, \mathrm{Set}],$$

which is fully faithful and sends a (small) bicategory $\mathbb{B}$ to its nerve $N\mathbb{B}$ which is a simplicial set whose n-simplices are defined by the set

$$N\mathbb{B}_n = [H([n]), \mathbb{B}].$$

The right side is a set of normal lax functors from an ordinal $[n]$ to the bicategory $\mathbb{B}$.

Definition 4.1. For an ordinal $[n]$ and a bicategory $\mathbb{B}$, a **normal lax functor** $(B, f, \beta)\colon [n] \to \mathbb{B}$ consists of the following data in $\mathbb{B}$:

(i) an object B_i for each $i \in [n]$,
(ii) a morphism $f_{ij}\colon B_i \to B_j$ for each $i, j \in [n]$ with $i \leqslant j$,
(iii) a 2-cell $\beta_{ijk}\colon f_{ij} \circ f_{jk} \Rightarrow f_{ik}$ for each $i, j, k \in [n]$ with $i \leqslant j \leqslant k$

$$\begin{tikzcd}
B_k \arrow[r, "f_{jk}"] \arrow[dr, "f_{ik}"'] & B_j \arrow[d, "f_{ij}"] \arrow[dl, phantom, "\Swarrow\beta_{ijk}"] \\
& B_i
\end{tikzcd}$$

such that the following conditions are satisfied:

- (*normalization*). For any $i \in [n]$ we have $f_{ii} = i_{B_i}\colon B_i \to B_i$, and for any $i, j \in [n]$, such that $i \leqslant j$, the corresponding 2-cells $\beta_{iij}\colon f_{ii} \circ f_{ij} \Rightarrow f_{ij}$ and $\beta_{ijj}\colon f_{ij} \circ f_{jj} \Rightarrow f_{ij}$ are given by the two 2-simplices

$$\begin{tikzcd}
B_j \arrow[r, "f_{ij}"] \arrow[dr, "f_{ij}"'] & B_i \arrow[d, "i_{B_i}"] \arrow[dl, phantom, "\Swarrow\lambda_{f_{ij}}"] \\
& B_i
\end{tikzcd}
\qquad
\begin{tikzcd}
B_j \arrow[r, "i_{B_j}"] \arrow[dr, "f_{ij}"'] & B_j \arrow[d, "f_{ij}"] \arrow[dl, phantom, "\Swarrow\rho_{f_{ij}}"] \\
& B_i
\end{tikzcd}$$

  where $\rho_{f_{ij}}\colon f_{ij} \circ i_{p_j} \Rightarrow f_{ij}$ and $\lambda_{f_{ij}}\colon i_{p_i} \circ f_{ij} \Rightarrow f_{ij}$ are the components of the right and left identity natural isomorphisms in $\mathbb{B}$.
- (*coherence condition*). For each $i, j, k, l \in [n]$ such that $i \leqslant j \leqslant k \leqslant l$ the following tetrahedron:

$$\begin{tikzcd}
& B_i & \\
B_l \arrow[ur, "f_{il}"] \arrow[rr, "f_{jl}"] \arrow[dr, "f_{kl}"'] & \arrow[ul, phantom, "\Nwarrow\beta_{ijl}"] \arrow[ur, phantom, "\Swarrow\beta_{ijk}"] \arrow[dl, phantom, "\Swarrow\beta_{ikl}"] \arrow[dr, phantom, "\Nwarrow\beta_{jkl}"] & B_j \arrow[ul, "f_{ij}"'] \\
& B_k \arrow[uu, "f_{ik}"] \arrow[ur, "f_{jk}"'] &
\end{tikzcd}$$

  is commutative. This means that we have the identity of 2-cells in the bicategory $\mathbb{B}$:

$$\beta_{ikl}(\beta_{ijk} \circ f_{kl}) = \beta_{ijl}(f_{ij} \circ \beta_{jkl})\alpha_{ijkl}.$$

Remark 4.2. Simplicial sets that are isomorphic to a nerve of a bicategory have been characterized in [**11**] and [**12**]. Simplicial sets that are isomorphic to a nerve of a bicategory form a full subcategory of the category of simplicial sets. This category is equivalent to the category Bicat_1 of bicategories with lax normal functors. Let us recall that a lax functor (F, ϕ) is normal if $F(\mathrm{id}_x) = \mathrm{id}_{Fx}$ and $\phi_x\colon \mathrm{id}_{Fx} \implies F(\mathrm{id}_x)$ is the identity 2-cell, and that oplax means that all the structure maps go in the opposite direction. This equivalence also holds in the topological setting.

Definition 4.3. Let $\mathbb{B}$ be a topological bicategory. The **classifying topos** $\mathcal{B}\mathbb{B}$ of the topological bicategory $\mathbb{B}$ is defined as the topos $\mathrm{Sh}(N\mathbb{B})$.

Definition 4.4. The **classifying space** $B\mathbb{B}$ of a topological bicategory $\mathbb{B}$ is the geometric realization $|N\mathbb{B}|$ of its nerve $N\mathbb{B}$.

With these definitions we have the following corollary of Theorem 2.3.

Corollary 4.5. *For any topological bicategory $\mathbb{B}$, the topos of sheaves $\mathrm{Sh}(B\mathbb{B})$ on the classifying space $B\mathbb{B}$ has the same weak homotopy type as the classifying topos $\mathcal{B}\mathbb{B}$.*

Definition 4.6. For a topological bicategory $\mathbb{B}$ write $\mathrm{Lin}(X, \mathbb{B})$ for the category of linear orders over X equipped with an augmentation $\mathrm{aug}\colon NL \to N\mathbb{B}$. An object E of this category will be called a **Duskin principal $\mathbb{B}$-bundle**. We call two Duskin principal $\mathbb{B}$-bundles E_0 and E_1 on X **concordant**, if there exists a Duskin principal $\mathbb{B}$-bundle on $X \times [0,1]$ such that we have the equivalences $E_0 \simeq i_0^*(E)$ and $E_0 \simeq i_1^*(E)$ under the obvious inclusions $i_0, i_1\colon X \hookrightarrow X \times [0,1]$.

Remark 4.7. We can consider a linear order L as a locally trivial bicategory (with only trivial 2-morphisms). In this case the Duskin nerve of L coincides with the ordinary nerve of L which justifies the same notation NL for both nerves.

Remark 4.8. By the above remark, an augmentation $NL \to N\mathbb{B}$ is the same, by the nerve construction, as a continuous normal lax functor $L \to \mathbb{B}$.

Similarly to Theorem 3.6, we have from Theorem 2.10 the following "classifying" property of the classifying 1-topos $\mathcal{B}\mathbb{B}$.

Theorem 4.9. *For a topological bicategory $\mathbb{B}$ and a topological space X, there is a natural equivalence of categories*

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathcal{B}\mathbb{B}) \simeq \mathrm{Lin}(X, \mathbb{B}).$$

On homotopy classes of topos morphisms we have the natural bijection

$$[\mathrm{Sh}(X), \mathcal{B}\mathbb{B}] \cong \mathrm{Lin}_c(X, \mathbb{B}).$$

Definition 4.10. We say that a topological bicategory $\mathbb{B}$ is **locally contractible** $\mathbb{B}$ if its space of objects B_0, its space of 1-arrows B_1 and its space of 2-arrows B_2 are locally contractible. A topological bicategory $\mathbb{B}$ is a **good topological bicategory**, if its nerve $N\mathbb{B}$ is a good simplicial space.

The "classification" property of the classifying space $B\mathbb{B}$ now follows as a corollary from Corollary 2.11.

Corollary 4.11. *For a locally contractible bicategory $\mathbb{B}$ and a* CW*-complex X, there is a natural bijection*

$$[X, B\mathbb{B}] \cong \mathrm{Lin}_c(X, \mathbb{B}).$$

Remark 4.12. If, in addition, the topological bicategory $\mathbb{B}$ is a good one, then the above is also true if we use the geometric realization of the underlying simplicial set instead of the thickened geometric realization of the nerve. In particular, it does not matter which geometric realization we use if all $N\mathbb{B}_n$ are LEC. The case of a good topological bicategory, as well as the sufficient conditions for a bicategory being a good one, are discussed in [**1**]. Those conditions actually guarantee that all $N\mathbb{B}_n$ are LEC. Thus, our corollary above gives a slight generalization of the result of Baas, Bökstedt and Kro.

5. Principal bundles under a category

Before introducing an alternative notion of a classifying topos of a bicategory in the next section, we will introduce some additional background material. Everything up to and including Remark 5.18 is taken almost verbatim from [**21**], where the interested reader can find the missing proofs (as well as more details). Definition 5.20, Theorem 5.21 and Corollary 5.23 might be new. To make our discussion more complete, we start with the definitions of the classifying topoi in the cases of a small and s-étale category. We also recall the definition of a principal $\mathbb{C}$-bundles in these cases.

Proposition 5.1. *The category of all presheaves on a small category $\mathbb{C}$ is a topos.*

Definition 5.2. The topos $\mathcal{B}\mathbb{C}$ of presheaves on a small category $\mathbb{C}$ is called the **classifying topos** of $\mathbb{C}$.

Remark 5.3. At this point, the reader may wonder how the above Definition 5.2 is related to the definition of the classifying topos of a topological category given in Definition 3.1. We will address this question later in 5.11, after we introduce further relevant material.

Definition 5.4. For a small category $\mathbb{C}$ and a space X, a **$\mathbb{C}$-bundle over** X is a covariant functor $E\colon \mathbb{C} \to \mathrm{Sh}(X)$. Such a $\mathbb{C}$-bundle is called a **principal** (flat, filtering) if for each point $x \in X$ the following conditions — nonemptiness, transitivity and freeness — are satisfied for the stalks $E(c)_x$ for objects $c \in \mathbb{C}$:

(i) There is at least one object c in $\mathbb{C}$ for which the stalk $E(c)_x$ is nonempty.

(ii) For any two points $y \in E(c)_x$ and $z \in E(d)_x$, there are arrows $\alpha\colon b \to c$ and $\beta\colon b \to d$ from some object b of $\mathbb{C}$, and an object $w \in E(b)_x$ such that $\alpha w = y$ and $\beta w = z$.

(iii) For any two parallel arrows $\alpha, \beta\colon c \to d$ and any $y \in E(c)_x$ for which $\alpha y = \beta y$, there is an arrow $\gamma\colon b \to c$ and a point $z \in E(b)_x$ such that $\alpha\gamma = \beta\gamma$ and $\gamma z = y$.

A map between two principal $\mathbb{C}$-bundles is a natural transformation between the corresponding functors. The category of principal $\mathbb{C}$-bundles will be denoted as $\mathrm{Prin}(X, \mathbb{C})$.

Examples 5.5. The following well-known notions are examples of principal $\mathbb{C}$-bundles:

(i) (principal group bundles). Any group G can be seen as a groupoid (and therefore a category) with only one object. In this way, the above definition of a principal $\mathbb{C}$-bundle becomes the usual one, where a *principal left G-bundle* over X is an étale space $p\colon P \to X$ with a fibre-preserving left action $a\colon G \times P \to P$ of G on P for which the induced map $(a, pr_2)\colon G \times P \to P \times P$ is a homeomorphism.

(ii) (principal monoid bundles). Any monoid M can be seen as a category with only one object. If every morphism in a such category is a monomorphism, then the monoid M is said to have left cancellation if $mk = ml$ implies $k = l$ for any $k, l, m \in M$. Segal used such a monoid M in order to introduce a *right principal monoid bundle* in [**22**] as an étale space $p\colon P \to X$ over X with a fibre-preserving right action of M on P, such that each fibre P_x is a principal M-set. A *right principal M-set* S is a set with a right action of M, which is free in the sense that $sm_1 = sm_2$ for any $m_1, m_2 \in M$ and $s \in S$, and is transitive in the sense that for any $s_1, s_2 \in M$ there exist $m_1, m_2 \in M$ and $s \in S$ such that $s_1 = sm_1$ and $s_2 = sm_2$. Although Segal used the right action of a monoid with left cancellation, it is obvious that when $\mathbb{C}$ is a monoid M with right cancellation, the above definition of a left principal $\mathbb{C}$-bundle becomes a left principal monoid bundle.

(iii) (principal poset bundles). Any partially ordered set P may be seen as a category with exactly one morphism $i \to j$ if and only if $i \leqslant j$. A principal P-bundle over a topological space X is a covering $\mathcal{U} = \{U_i\}_{i \in P}$ of X such that when $i \leqslant j$ then $U_i \subseteq U_j$ and which is **locally directed** in the sense that any $U_{ij} = U_i \cap U_j$ is covered by the family $\mathcal{U}_{ij} = \{U_k\colon k \leqslant i \wedge k \leqslant j\}$.

(iv) (principal simplicial sets). Any linear order over a topological space defines a topological category and therefore a simplicial space via its nerve as in Remark 2.6. One can see that a simplicial set $S\colon \Delta^{\mathrm{op}} \to \mathrm{Set}$ is a principal Δ^{op}-bundle if and only if is the nerve of a (uniquely determined) nonempty linear order.

Definition 5.6. A **$\mathbb{C}$-sheaf** is an étale space $p\colon S \to C_0$ equipped with a continuous right $\mathbb{C}$ action

$$\alpha\colon S \times_{C_0} C_1 \to S,$$

which we denote by $\alpha(x, f) = x \cdot f$. This action is defined for all pairs (x, f) for which $p(x) = t(f)$, and it satisfies the following axioms:

$$p(x \cdot f) = s(f), \qquad (x \cdot f) \cdot g = x \cdot (fg), \qquad x \cdot \mathrm{id}_{p(x)} = x.$$

A map between $\mathbb{C}$-sheaves or a $\mathbb{C}$-*equivariant map* is a map of étale spaces over C_0, which is compatible with the $\mathbb{C}$-action.

Proposition 5.7. *The category of $\mathbb{C}$-sheaves is a topos.*

Definition 5.8. The topos $\mathcal{B}\mathbb{C}$ of $\mathbb{C}$-sheaves is called **the classifying topos of the s-étale topological category** $\mathbb{C}$.

Examples 5.9. We now provide some examples of $\mathbb{C}$-sheaves to illustrate their significance:

(i) Any small category $\mathbb{C}$ can be seen as a topological category with the discrete topology. Then a $\mathbb{C}$-sheaf is the same thing as a presheaf on $\mathbb{C}$ which justifies the same notation $\mathcal{B}\mathbb{C}$, as in Definition 5.2, for the classifying topos of a small category $\mathbb{C}$.

(ii) Any topological space X may be seen as a discrete topological category $\mathbb{X}$ (the one for which all morphisms are identities). Then an $\mathbb{X}$-sheaf is just a sheaf on X and the topos $\mathcal{B}\mathbb{X}$ is the topos $\mathrm{Sh}(X)$ of sheaves on X.

(iii) Let $\mathcal{G}$ be an action groupoid coming from the right action of a topological group G on a topological space X. The groupoid $\mathcal{G}$ has X as a space of objects and $X \times G$ as a space of morphisms, where morphisms are of the form $(x, g) \colon x \cdot g \to x$. Then a $\mathcal{G}$-sheaf $p \colon S \to X$ is a sheaf which is G-equivariant. Therefore, $\mathcal{B}\mathcal{G}$ is the category of G-equivariant sheaves.

In the case of an s-étale topological category, i.e., a topological category with the source map $s \colon C_1 \to C_0$ being an étale map, we have the following definition.

Definition 5.10. Let $\mathbb{C}$ be an s-étale topological category. A **$\mathbb{C}$-bundle over a space** X is an étale map (sheaf) $p \colon E \to X$ with a continuous fibrewise left action given by the maps

$$\pi \colon E \to B_0, \qquad \text{and} \qquad a \colon B_1 \times_{B_0} E \to E.$$

Such a $\mathbb{C}$-bundle is called **principal** if the three conditions of nonemptiness, transitivity and freeness hold for each $x \in X$:

(i) The stalk E_x is nonempty.

(ii) For any two points $y \in E_x$ and $z \in E_x$, there are a $w \in E_x$ and arrows $\alpha \colon \pi(w) \to \pi(y)$ and $\beta \colon \pi(w) \to \pi(z)$ such that $\alpha w = y$ and $\beta w = z$.

(iii) For any point $y \in E_x$ and any pair of arrows α, β in $\mathbb{B}$ with $s(\alpha) = \pi(y) = s(\beta)$ and $\alpha y = \beta y$, there is a point $w \in E_x$ and an arrow $\gamma \colon \pi(w) \to \pi(y)$ in $\mathbb{C}$ such that $\gamma w = y$ in E_x and $\alpha\gamma = \beta\gamma$ and $\gamma z = y$ in $\mathbb{B}$.

A map between two principal $\mathbb{C}$-bundles is a sheaf map preserving the $\mathbb{C}$-action. The resulting category of principal $\mathbb{C}$-bundles will again be denoted as $\mathrm{Prin}(X, \mathbb{C})$.

Remark 5.11. A small category can be viewed as an s-étale topological category with the discrete topology. In this case the respective definitions of principal bundles and of classifying topoi are of course equivalent. A topological category is locally connected if the spaces of objects and arrows are locally connected. For a locally connected s-étale topological category, the classifying topos introduced in this section and the one defined as the topos of sheaves on the nerve are weak homotopy equivalent.

In both cases (small and s-étale topological) we have the same notion of concordance of principal $\mathbb{C}$-bundles as in topological case (see 2.9).

For either a small or an s-étale topological category, we have:

Theorem 5.12. *There is a natural equivalence of categories*

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathcal{B}\mathbb{C}) \simeq \mathrm{Prin}(X, \mathbb{C}).$$

On homotopy classes of topos morphisms we have the natural bijection

$$[\mathrm{Sh}(X), \mathcal{B}\mathbb{C}] \cong \mathrm{Prin}_c(X, \mathbb{C}).$$

For a CW*-complex X and any small category or any locally contractible s-étale category $\mathbb{C}$, there is a natural bijection*

$$[X, B\mathbb{C}] \cong \mathrm{Prin}_c(X, \mathbb{C}),$$

where in the s-étale case the fat geometric realization is taken in order to construct the classifying space.

Proposition 5.13. *For either a small category or a locally connected s-étale category, there is a natural weak homotopy equivalence*

$$\mathrm{Sh}(B\mathbb{C}) \to \mathcal{B}\mathbb{C}.$$

Remark 5.14. Definition 2.1 of the topos $\mathrm{Sh}(Y)$ of sheaves on the simplicial space Y generalizes to the case when the opposite simplicial model category Δ^{op} is replaced by an arbitrary small category $\mathbb{K}$. Then, instead of a simplicial space, we have a diagram of spaces indexed by $\mathbb{K}$, i.e., a covariant functor Y from $\mathbb{K}$ into the category Top topological spaces. With an evident modification of Definition 2.6, we obtain the topos of sheaves on the diagram of spaces Y.

Remark 5.15. From a diagram of spaces indexed by a small category $\mathbb{K}$, we can construct a category $Y_{\mathbb{K}}$. The object is a pair (k, y), $k \in \mathbb{K}$, $y \in Y_k$ and the arrow $(k, y) \to (l, z)$ is an arrow in $\mathbb{K}$ $\alpha\colon k \to l$ such that $Y(\alpha)(y) = z$. This is just the Grothendieck construction. The category $Y_{\mathbb{K}}$ can be equipped with an s-étale topology. Further, a diagram of spaces Y is called locally contractible if each Y_k is locally contractible. For a locally contractible Y, the Grothendieck construction gives a locally contractible s-étale topological category $Y_{\mathbb{K}}$.

Proposition 5.16. *Let $\mathrm{Sh}(Y)$ be the category of sheaves on a diagram of spaces Y indexed by a small category $\mathbb{K}$. Then there is a natural equivalence of topoi*

$$\mathrm{Sh}(Y) \simeq \mathcal{B}(Y_{\mathbb{K}}).$$

Hence, for any topological space X, there is a natural equivalence

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathrm{Sh}(Y)) \simeq \mathrm{Prin}(X, Y_{\mathbb{K}}).$$

A principal $Y_{\mathbb{K}}$-bundle can also be characterized as a principal $\mathbb{K}$-bundle equipped with an augmentation. Let us recall that a principal $\mathbb{K}$-bundle over X consists of a system of sheaves E^k for each object k of $\mathbb{K}$ on X and sheaf maps $E(\alpha)\colon E^k \to E^l$ for each arrow $\alpha\colon k \to l$. An augmentation on of E over Y is a system of maps $\mathrm{aug}^k\colon E^k \to Y_k$ such that for any arrow $\alpha\colon k \to l$

$$Y(\alpha)\mathrm{aug}^k = \mathrm{aug}^l E(\alpha).$$

Together with the morphisms of principal bundles that respect augmentations, we have the category

$$\mathrm{AugPrin}(X, \mathbb{K}, Y)$$

of principal $\mathbb{K}$-bundles with an augmentation to Y.

Proposition 5.17. *For X and Y as above, we have a natural equivalence of categories*

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathrm{Sh}(Y)) \simeq \mathrm{Prin}(X, Y_{\mathbb{K}}) \simeq \mathrm{AugPrin}(X, \mathbb{K}, Y).$$

Remark 5.18. The case $\mathbb{K} = \Delta^{\mathrm{op}}$ gives Theorem 2.10 as a corollary. For this, the following equivalence

$$\mathrm{Prin}(X, \Delta^{\mathrm{op}}) \simeq \mathrm{Lin}(X)$$

has to be used. A principal Δ^{op}-bundle E over X is a simplicial sheaf such that each stalk E_x is a principal Δ^{op}-bundle E over the one-point space x, i.e., a principal simplicial set. Finally, a simplicial set is principal only if it is a nerve of a (uniquely determined) nonempty linear order.

Next, let us consider the case $\mathbb{K} = \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$; i.e., in this case a diagram of spaces Y labeled by $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ is just a bisimplicial space. Concerning principal $\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$-bundles over X, we have the following result which follows from [**20**, Chap. VII, exercise 14].

Proposition 5.19. *There are natural equivalences of categories*

$$\mathrm{Prin}(X, \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}) \simeq \mathrm{Prin}(X, \Delta^{\mathrm{op}}) \times \mathrm{Prin}(X, \Delta^{\mathrm{op}}) \simeq \mathrm{Lin}(X) \times \mathrm{Lin}(X).$$

Now, an augmentation is the same thing as a bisimplicial map from the product of two linear orders $NL \times NL'$ to a bisimplicial set Y. Hence, similarly to Definition 2.7, we do have:

Definition 5.20. For any space X and any bisimplicial space Y, write $\mathrm{Lin}^2(X, Y)$ for the product category of linear orders $(L \times L', \mathrm{aug})$ over X equipped with a bisimplicial map (**augmentation**) $\mathrm{aug}\colon NL \times NL' \to Y$ from the product of nerves of L and L' to Y. The morphisms $(L \times L', \mathrm{aug}) \to (L_1 \times L'_1, \mathrm{aug}')$ in $\mathrm{Lin}^2(X, Y)$ are maps of products of the linear orders $L \times L' \to L_1 \times L'_1$ such that the induced map $NL \times NL' \to NL_1 \times NL'_1$ on the products of nerves respects the augmentations.

With the same definition of concordance as in Definition 2.9, we do have similarly to Theorem 2.10:

Theorem 5.21. *Let Y be a bisimplicial space. For any space X there is a natural equivalence of categories*

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathrm{Sh}(Y)) \simeq \mathrm{Lin}^2(X, Y).$$

On homotopy classes of topos morphisms we have the natural bijection

$$[\mathrm{Sh}(X), \mathrm{Sh}(Y)] \cong \mathrm{Lin}^2_c(X, Y).$$

Similarly to Theorem 2.3, we have the following theorem, where the geometric realization $|Y|$ of a a bisimplicial space Y can be taken as the geometric realization of its diagonal. Equivalently, Y can be defined as the "horizontal" geometric realization followed by the the "vertical" one or vice-versa.

Theorem 5.22. *For any bisimplicial space Y the topoi* $\mathrm{Sh}(Y)$ *and* $\mathrm{Sh}(|Y|)$ *have the same weak homotopy type.*

We recall that, in accordance with Remark 5.15, a bisimplicial space Y is locally contractible if all spaces $Y_{n,m}$ are locally contractible. Again, geometric realization of a locally contractible bisimplicial space is locally contractible. Hence, we have similarly to 2.11 the following corollary:

Corollary 5.23. *Let X be a* CW*-complex and Y a locally contractible bisimplicial space. There is a natural bijection between homotopy classes of the maps $[X, |Y|]$ and the concordance classes* $\mathrm{Lin}^2_c(X, Y)$.

6. Classifying spaces and classifying topoi of topological bicategories II

It is beyond the scope of this paper to give a full account of the constructions of Lack and Paoli, Tamsamani and Simpson. Concerning the latter two, the interested reader may find the nice survey of definitions of n-categories by T. Leinster [**19**]. Let Set and Cat_1 denote the categories of (small) sets and (small) categories, respectively, and let Cat denote the 2-category of (small) categories.

6.1. Lack-Paoli nerve as a singular functor

The nerve construction of Lack and Paoli [**17**] is obtained as the singular functor when $\mathcal{V} = \mathrm{Cat}$. In order to define the nerve $N\mathbb{B}$ of a (small) bicategory $\mathbb{B}$, they introduced a (strict) 2-category NHom with bicategories as objects, whose 1-cells are normal homomorphisms (normal lax functors with invertible comparison maps). We will not give the general definition of 2-cells (icons) here. We describe them below explicitly in a special case.

Every category (in particular Cat_1 and the category defined by the ordinal $[n]$) can be seen as a locally discrete bicategory with only identity 2-cells. The normal homomorphism between locally discrete bicategories is just a functor between the corresponding categories, and there are no nontrivial icons between such functors. In this way, we obtain a fully faithful inclusion 2-functor

$$J\colon \Delta \to \mathrm{NHom}$$

and the category Δ can be seen as a full sub-2-category of NHom. The singular 2-functor

$$N_{LP}\colon \mathrm{NHom} \to [\Delta^{\mathrm{op}}, \mathrm{Cat}]$$

of the inclusion J is a Lack and Paoli 2-nerve. The 2-functor N_{LP} is fully faithful.

Definition 6.1. A **normal homomorphism** $(B, f, \beta)\colon [n] \to \mathbb{B}$ from an ordinal $[n]$ to the bicategory $\mathbb{B}$ is a lax normal functor for which each 2-cell β_{ijk} in Definition 4.1 is invertible.

Definition 6.2. An **icon** between normal homomorphisms $F, G\colon [n] \to \mathbb{B}$ of bicategories is a lax natural transformation $\phi\colon F \Rightarrow G$, in which the component $\phi_i\colon B_i \to C_i$ is an identity, for each $i \in [n]$. More explicitly, an icon $\phi\colon (B, f, \beta) \Rightarrow (C, g, \gamma)$ consists of the following data:

(i) For any $i \in [n]$ an identity $B_i = C_i$,

(ii) For each $i, j \in [n]$ such that $i \leqslant j$, a 2-cell $\phi_{ij} \colon f_{ij} \Rightarrow g_{ij}$

$$
\begin{tikzcd}
B_j \arrow[r, bend left, "f_{ij}"] \arrow[r, bend right, "g_{ij}"'] \arrow[r, phantom, "\Downarrow\phi_{ij}"] & B_i
\end{tikzcd}
$$

such that for all $i, j, k \in [n]$ with $i \leqslant j \leqslant k$ we have an equality of pasting diagrams

$$
\begin{tikzcd}
 & B_j \arrow[dr, bend left, "f_{ij}"] \arrow[d, phantom, "\Downarrow\beta_{ijk}"] & \\
B_k \arrow[ur, bend left, "f_{jk}"] \arrow[rr, bend left, "f_{ik}"] \arrow[rr, bend right, "g_{ik}"'] \arrow[rr, phantom, "\Downarrow\phi_{ik}"] & {} & B_i
\end{tikzcd}
=
\begin{tikzcd}
 & B_j \arrow[dr, bend left, "f_{ij}"] \arrow[dr, bend right, "g_{ij}"'] \arrow[dr, phantom, "\Swarrow\phi_{ij}"] & \\
B_k \arrow[ur, bend left, "f_{jk}"] \arrow[ur, bend right, "g_{jk}"'] \arrow[ur, phantom, "\Searrow\phi_{jk}"] \arrow[rr, bend right, "g_{ik}"'] \arrow[rr, phantom, "\Downarrow\gamma_{ijk}"] & & B_i
\end{tikzcd}
$$

which means that the following identity of 2-cells holds in $\mathbb{B}$:

$$\phi_{ik}\beta_{ijk} = \gamma_{ijk}(\phi_{ij} \circ \phi_{jk}).$$

6.2. Characterization of Lack-Paoli 2-nerves of bicategories

In their paper [**17**], Lack and Paoli also described necessary and sufficient conditions for a simplicial object $X \colon \Delta^{\mathrm{op}} \to \mathrm{Cat}$ to be a 2-nerve of a bicategory. In order to provide such characterization, they used *discrete isofibrations* which are functors $P \colon E \to B$ such that for each object e in the category E and each isomorphism $\beta \colon b \to P(e)$ in B there exists a unique isomorphism $\varepsilon \colon e' \to e$ in E with $P(\varepsilon) = \beta$. Further, let $\mathrm{c}_n \colon X_n \to \mathrm{Cosk}_{n-1}(X)_n$ denote the n-component of the simplicial map $c \colon X \to \mathrm{Cosk}_{n-1}(X)$ from a simplicial object X to its $n-1$-coskeleton $\mathrm{Cosk}_{n-1}(X)$, which is the unit of an adjunction between $(n-1)$-truncation tr_n and $(n-1)$-coskeleton Cosk_{n-1}.

Theorem 6.3. *Necessary and sufficient conditions for a* 2*-functor* $X \colon \Delta^{\mathrm{op}} \to \mathrm{Cat}$ *to be a* 2*-nerve of a bicategory are:*

(i) X *is* 3*-coskeletal,*

(ii) X_0 *is discrete,*

(iii) the Segal functors $S_n \colon X_n \to X_1 \times_{X_0} \cdots \times_{X_0} X_1$ *are equivalences of categories,*

(iv) c_2 *and* c_3 *are discrete isofibrations.*

6.3. Lack-Paoli 2-nerve as a bisimplicial set (space)

If we apply the Grothendieck nerve functor at each level of the 2-nerve of Lack and Paoli (6.1), we obtain a functor

$$B_{LP}\colon \mathrm{NHom} \to [\Delta^{\mathrm{op}}, \mathrm{SSet}],$$

where the right-hand side is the category of bisimplicial sets. If we define the 2-nerve in such bisimplicial terms, then the definition also makes sense for a topological bicategory $\mathbb{B}$, in which case the 2-nerve will naturally be a bisimplicial space. Although the above conditions 6.2 can be translated into the bisimplicial language, we will not do it here. From now on we will understand the 2-nerve of Lack and Paoli as a bisimplicial set (bisimplicial space in case of a topological bicategory).

6.4. Tamsamani and Simpson

Let Tam denote the full sub-2-category of $[\Delta^{\mathrm{op}}, \mathrm{Cat}]$ consisting of those X, for which X_0 is discrete and the Segal maps S_n are equivalences. Further, let Simpson denote the smaller full sub-2-category of those X for which the Segal maps S_n are fully faithful and surjective on objects. Also, in these cases we can interpret these "2-nerves" as bisimplicial sets (see [**19, 23, 25**], where the corresponding definitions can be found). We will speak of a Tamsamani 2-nerve (or 2-category) and a Simpson 2-nerve (or 2-category).

Remark 6.4. The Lack-Paoli 2-nerve is also a Simpson 2-nerve and thus also a Tamsamani 2-nerve. To each Tamsamani 2-nerve X there is a bicategory GX (and vice-versa) constructed in [**25**]. For more details on Tamsamani 2-nerves (including a proper notion of equivalence), we refer the reader to this paper.

Here we only mention the following results of Lack and Paoli:

The (Lack-Paoli) 2-nerve 2-functor $N_{LP}\colon \mathrm{NHom} \to \mathrm{Tam}$, seen as landing in the 2-category Tam, has a left 2-adjoint G. Since N_{LP} is fully faithful, the counit $GN_{LP} \to 1$ is invertible. Each component $u\colon X \to N_{LP}G$ of the unit is a pointwise equivalence (i.e., each component u_n is an equivalence) and u_0 and u_1 are identities.

Let $\mathrm{Ps}(\Delta^{\mathrm{op}}, \mathrm{Cat})$ denote the 2-category of 2-functors, pseudonatural transformations and modifications, and let $\mathrm{Tam}_{\mathrm{ps}}$ be its full sub-2-category consisting of Tamsamani 2-categories. Then the 2-nerve 2-functor $N_{LP}\colon \mathrm{NHom} \to \mathrm{Tam}_{\mathrm{ps}}$ is a biequivalence of 2-categories.

Definition 6.5. Let $\mathbb{B}$ be a topological bicategory. The **classifying topos** $\mathcal{B}_{LP}\mathbb{B}$ of the topological bicategory $\mathbb{B}$ is defined as the topos of sheaves $\mathrm{Sh}(B_{LP}\mathbb{B})$ on the bisimplicial space $B_{LP}\mathbb{B}$ (Lack-Paoli bisimplicial nerve).

Definition 6.6. The **classifying space** $B_{LP}\mathbb{B}$ of a topological bicategory $\mathbb{B}$ is the geometric realization $|B_{LP}\mathbb{B}|$ of its bisimplicial nerve $B_{LP}\mathbb{B}$.

With these definitions we have the following corollary of Theorem 5.22.

Corollary 6.7. *For any topological bicategory $\mathbb{B}$, the topos of sheaves $\mathrm{Sh}(B_{LP}\mathbb{B})$ on the classifying space $B_{LP}\mathbb{B}$ has the same weak homotopy type as the classifying topos $\mathcal{B}\mathcal{B}\mathbb{B}$.*

Definition 6.8. For a topological bicategory $\mathbb{B}$, write $\mathrm{Lin}^2(X, \mathbb{B})$ for the product category of linear orders over X equipped with an augmentation $\mathrm{aug}\colon NL \times NL' \to NN\mathbb{B}$. An object E of this category will be called a **Lack-Paoli principal $\mathbb{B}$-bundle**. We call the two Lack-Paoli principal $\mathbb{B}$-bundles E_0 and E_1 on X **concordant**, if there exists a Lack-Paoli principal $\mathbb{B}$-bundle on $X \times [0,1]$ such that we have the equivalences $E_0 \simeq i_0^*(E)$ and $E_0 \simeq i_1^*(E)$ under the obvious inclusions $i_0, i_1\colon X \hookrightarrow X \times [0,1]$.

Similarly to Theorems 3.6 and 4.9 we have from Theorem 5.21 the following "classifying" property of the classifying topos $\mathcal{B}_{LP}\mathbb{B}$.

Theorem 6.9. *For a topological bicategory $\mathbb{B}$ and a topological space X, there is a natural equivalence of categories*

$$\mathrm{Hom}(\mathrm{Sh}(X), \mathcal{B}_{LP}\mathbb{B}) \simeq \mathrm{Lin}^2(X, \mathbb{B}).$$

On homotopy classes of topos morphisms we have the natural bijection

$$[\mathrm{Sh}(X), \mathcal{B}_{LP}\mathbb{B}] \cong \mathrm{Lin}^2_c(X, \mathbb{B}).$$

The "classification" property of the classifying space $B_{LP}\mathbb{B}$ now follows as a corollary from Corollary 5.23.

Corollary 6.10. *For a locally contractible bicategory $\mathbb{B}$ and a* CW*-complex X, there is a natural bijection*

$$[X, B_{LP}\mathbb{B}] \cong \mathrm{Lin}^2_c(X, \mathbb{B}).$$

Remark 6.11 (Tamsamani and Simpson principal $\mathbb{B}$-bundles). In the above Definitions 6.5, 6.6 and 6.8 we could have used, instead of a Lack-Paoli 2-nerve, the Tamsamani or the Simpson 2-nerve (in the case these are bisimplicial spaces). Obviously, for such *Tamsamani* and *Simpson* principal $\mathbb{B}$-bundles, Corollaries 6.7, 6.10 and Theorem 6.9 are still valid.

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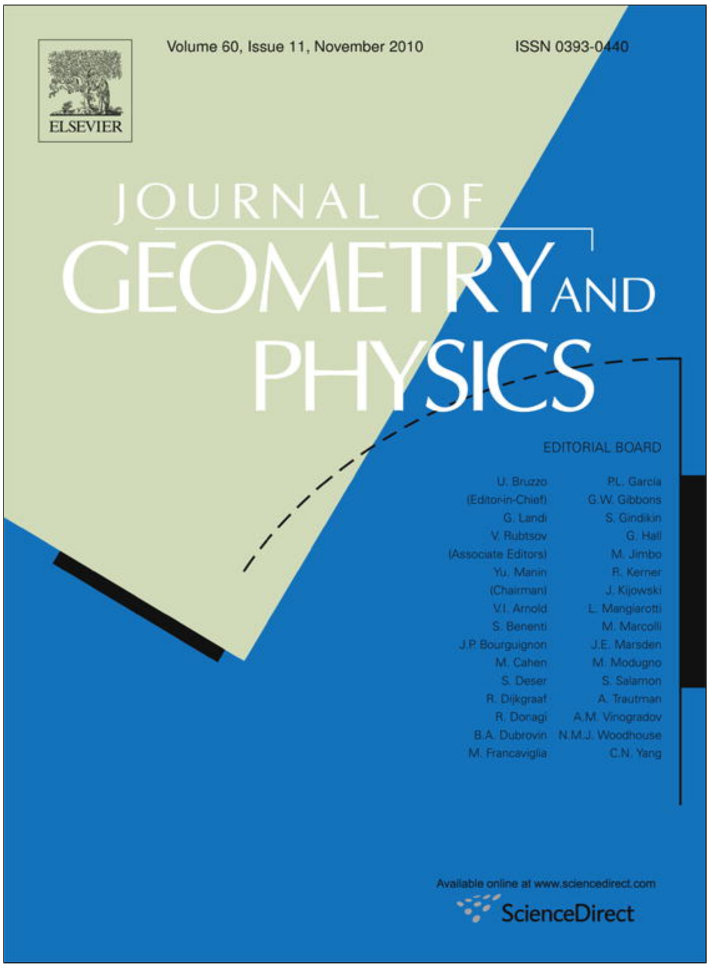
Igor Baković ibakovic@gmail.com

Faculty of Natural Sciences and Mathematics, University of Split, Teslina 12/III, 21000 Split, Croatia

Branislav Jurčo branislav.jurco@googlemail.com

Max Planck Institute for Mathematics, Vivatsgasse 7, 53111 Bonn, Germany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Noncommutative gerbes and deformation quantization

Paolo Aschieri [a,b,*], Igor Baković [c], Branislav Jurčo [d], Peter Schupp [e]

[a] Facoltà di Scienze M.F.N., Dipartimento di Scienze e Tecnologie Avanzate, Viale T. Michel 11, Alessandria, 15121, Italy
[b] INFN, Sezione di Torino, Gruppo collegato di Alessandria, Italy
[c] Department of Mathematics, Faculty of Natural Sciences and Mathematics, University of Split, Teslina 12/III, 21000 Split, Croatia
[d] Max-Planck-Institut für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
[e] Center for Mathematics, Modeling and Computing, Jacobs University Bremen, Campus Ring 1, 28759 Bremen, Germany

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ABSTRACT

We define noncommutative gerbes using the language of star products. Quantized twisted Poisson structures are discussed as an explicit realization in the sense of deformation quantization. Our motivation is the noncommutative description of D-branes in the presence of topologically non-trivial background fields.

1. Introduction

Abelian gerbes, more precisely gerbes with an abelian band [1–5], are the next step up from a line bundle on the geometric ladder in the following sense: A unitary line bundle can be represented by a 1-cocycle in Čech cohomology, i.e., a collection of smooth transition functions $g_{\alpha\beta}$ on the intersections $U_\alpha \cap U_\beta$ of an open cover $\{U_\alpha\}$ of a manifold M satisfying $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Similarly, an abelian gerbe can be represented by a 2-cocycle in Čech cohomology, i.e., by a collection $\lambda = \{\lambda_{\alpha\beta\gamma}\}$ of maps $\lambda_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to U(1)$, valued in the abelian group $U(1)$, satisfying

$$\lambda_{\alpha\beta\gamma} = \lambda_{\beta\alpha\gamma}^{-1} = \lambda_{\alpha\gamma\beta}^{-1} = \lambda_{\gamma\beta\alpha}^{-1} \tag{1}$$

and the 2-cocycle condition

$$\delta\lambda = \lambda_{\beta\gamma\delta}\, \lambda_{\alpha\gamma\delta}^{-1}\, \lambda_{\alpha\beta\delta}\, \lambda_{\alpha\beta\gamma}^{-1} = 1 \tag{2}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$. The collection $\lambda = \{\lambda_{\alpha\beta\gamma}\}$ of maps with the stated properties represents a gerbe in the same sense as a collection of transition functions represents a line bundle. In the special case where λ is a Čech 2-coboundary with $\lambda = \delta h$, i.e., $\lambda_{\alpha\beta\gamma} = h_{\alpha\beta}\, h_{\beta\gamma}\, h_{\gamma\alpha}$, we say that the collection $h = \{h_{\alpha\beta}\}$ of functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ represents a trivialization of a gerbe. Considering the "difference" of two 2-coboundaries $\{h_{\alpha\beta}\}$, $\{h'_{\alpha\beta}\}$ representing two trivializations of a gerbe we step down the geometric ladder again and obtain a 1-cocycle: $g_{\alpha\beta} \equiv h_{\alpha\beta}/h'_{\alpha\beta}$ satisfies the 1-cocycle condition $g_{\alpha\beta}\, g_{\beta\gamma}\, g_{\gamma\alpha} = 1$.

There exists a local trivialization of a 2-cocycle for any particular open set U_0 of the covering: Defining $h_{\beta\gamma} \equiv \lambda_{0\beta\gamma}$ with $\beta, \gamma \neq 0$ we find from the 2-cocycle condition that $\lambda_{\alpha\beta\gamma} = h_{\alpha\beta}\, h_{\beta\gamma}\, h_{\gamma\alpha}$. This observation leads to a definition of an abelian

* Corresponding author at: Facoltà di Scienze M.F.N., Dipartimento di Scienze e Tecnologie Avanzate, Viale T. Michel 11, Alessandria, 15121, Italy.
E-mail address: aschieri@to.infn.it (P. Aschieri).

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gerbe (more precisely "gerbe data") á la Hitchin [3] in terms of line bundles on the double overlaps of the cover. The only difference with respect to line bundles from this point of view is that we step up the geometric ladder, in the sense that now line bundles on $U_\alpha \cap U_\beta$ are used as replacements for transition functions. A gerbe á la Hitchin is then a collection of line bundles $L_{\alpha\beta}$ for each double overlap $U_\alpha \cap U_\beta$, such that:

G1. There is an isomorphism $L_{\alpha\beta} \cong L^{-1}_{\beta\alpha}$.
G2. There is a trivialization $\lambda_{\alpha\beta\gamma}$ of $L_{\alpha\beta} \otimes L_{\beta\gamma} \otimes L_{\gamma\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.
G3. The trivialization $\lambda_{\alpha\beta\gamma}$ satisfies $\delta\lambda = 1$ on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$.

In this paper we will use the term gerbe for an (abelian) gerbe á la Hitchin. In this respect, we should notice that in general the term gerbe is used to name a locally non-empty and locally connected stack in groupoids [1,2,4,5]. We will use the term standard gerbe in order to name such gerbes.

Gerbes are interesting in physics for several reasons: One motivation is the interpretation of D-brane charges in terms of K-theory in the presence of a topologically non-trivial B-field, when the gauge fields living on D-branes become connections on certain noncommutative algebras rather than on a vector bundle [6–14]. Azumaya algebras appear to be a natural choice and give the link to gerbes. Gerbes, rather than line bundles, are the structure that arises in the presence of closed 3-form backgrounds as, e.g., in WZW models and Poisson sigma models with WZW term [11,15,16]. Gerbes help illuminate the geometry of mirror symmetry of 3-dimensional Calabi–Yau manifolds [3] and they provide a language to formulate duality transformations with higher order antisymmetric fields [17]. Our motivation is the noncommutative description of D-branes in the presence of topologically non-trivial background fields.

The paper is organized as follows: In Section 2, we recall the local description of noncommutative line bundles in the framework of deformation quantization. Instead of repeating that construction we shall take the properties that were derived in [18,19] as a formal definition of a noncommutative line bundle. In the same spirit we define noncommutative gerbes in Section 3, using the language of star products and complement this definition, in Section 4, with an explicit realization of noncommutative gerbes as quantizations of twisted Poisson structures as introduced in [21] and further discussed in [22].

Notice that we will use the term noncommutative gerbe to describe a specific non-abelian 2-cocycle. By the correspondence (in the sense of 2-categories, see [5] for details) between degree two non-abelian cohomology classes and equivalence classes of (standard) gerbes understood as locally non-empty and locally connected stack in groupoids there is such a (standard) gerbe corresponding to this specific non-abelian 2-cocycle. Hence our definition of a noncommutative gerbe leads to a non-abelian gerbe in the standard sense of Giraud, Deligne, Breen and Brylinski [1,2,4,5]. We will discuss this shortly in Section 5.

Since the first version of this paper was posted on the arXiv, see hep-th/0206101v1 (where Section 5 was not present), some related work appeared in [23–27].

2. Noncommutative line bundles

Here we collect some facts on noncommutative line bundles [28,18] that we will need in what follows.[1] Let (M, θ) be a general Poisson manifold, and let $\star$ be the Kontsevich's deformation quantization of the Poisson tensor θ. Further let us consider a good covering $\{U^i\}$ of M. For the purposes of this paper a noncommutative line bundle $\mathcal{L}$ is defined by a collection of $\mathbb{C}[[\hbar]]$-valued local transition functions $G^{ij} \in C^\infty(U^i \cap U^j)[[\hbar]]$ (that can be thought valued in the enveloping algebra of $U(1)$, see [29]), and a collection of maps $\mathcal{D}^i : C^\infty(U^i)[[\hbar]] \to C^\infty(U^i)[[\hbar]]$, formal power series in $\hbar$, starting with the identity, and with coefficients being differential operators, such that

$$G^{ij} \star G^{jk} = G^{ik} \tag{3}$$

on $U^i \cap U^j \cap U^k$, $G^{ii} = 1$ on U^i, and

$$\mathrm{Ad}_\star G^{ij} = \mathcal{D}^i \circ (\mathcal{D}^j)^{-1} \tag{4}$$

on $U^i \cap U^j$ or, equivalently, $\mathcal{D}^i(f) \star G^{ij} = G^{ij} \star \mathcal{D}^j(f)$ for all $f \in C^\infty(U^i \cap U^j)[[\hbar]]$. Obviously, with this definition the local maps $\mathcal{D}^i$ can be used to define *globally* a new star product $\star'$ (because the inner automorphisms $\mathrm{Ad}_\star G^{ij}$ do not affect $\star'$)

$$\mathcal{D}^i(f \star' g) = \mathcal{D}^i f \star \mathcal{D}^i g. \tag{5}$$

We say that two line bundles $\mathcal{L}_1 = \{G^{ij}_1, \mathcal{D}^i_1, \star\}$ and $\mathcal{L}_2 = \{G^{ij}_2, \mathcal{D}^i_2, \star\}$ are equivalent if there exists a collection of invertible local functions $H^i \in C^\infty(U^i)[[\hbar]]$ such that

$$G^{ij}_1 = H^i \star G^{ij}_2 \star (H^j)^{-1} \tag{6}$$

and

$$\mathcal{D}^i_1 = \mathrm{Ad}_\star H^i \circ \mathcal{D}^i_2. \tag{7}$$

[1] A noncommutative line bundle is a finite projective module. In the present context it can be understood as a quantization of a line bundle over a compact manifold in the sense of deformation quantization. Here we shall take the properties of quantized line bundles as derived in [18,19] as a formal definition of a noncommutative line bundle.

The tensor product of two line bundles $\mathcal{L}_1 = \{G_1^{ij}, \mathcal{D}_1^i, \star_1\}$ and $\mathcal{L}_2 = \{G_2^{ij}, \mathcal{D}_2^i, \star_2\}$ is well defined if $\star_2 = \star_1'$ (or $\star_1 = \star_2'$). Then the corresponding tensor product is a line bundle $\mathcal{L}_2 \otimes \mathcal{L}_1 = \mathcal{L}_{21} = \{G_{12}^{ij}, \mathcal{D}_{12}^{ij}, \star_1\}$ defined as

$$G_{12}^{ij} = \mathcal{D}_1^i(G_2^{ij}) \star_1 G_1^{ij} = G_1^{ij} \star_1 \mathcal{D}_1^j(G_2^{ij}) \tag{8}$$

and

$$\mathcal{D}_{12}^i = \mathcal{D}_1^i \circ \mathcal{D}_2^i. \tag{9}$$

The order of indices of $\mathcal{L}_{21}$ indicates the bimodule structure of the corresponding space of sections to be defined later, whereas the first index on the G_{12}'s and $\mathcal{D}_{12}$'s indicates the star product (here: $\star_1$) by which the objects multiply.

A section $\Psi = (\Psi^i)$ is a collection of functions $\Psi^i \in C^\infty_{\mathbb{C}}(U^i)[[\hbar]]$ satisfying consistency relations

$$\Psi^i = G^{ij} \star \Psi^i \tag{10}$$

on all intersections $U^i \cap U^j$. With this definition the space of sections $\mathcal{E}$ is a right $\mathfrak{A} = (C^\infty(M)[[\hbar]], \star)$ module. We shall use the notation $\mathcal{E}_{\mathfrak{A}}$ for it. The right action of the function $f \in \mathfrak{A}$ is the regular one

$$\Psi.f = (\Psi^k \star f). \tag{11}$$

Using the maps $\mathcal{D}^i$ it is easy to turn $\mathcal{E}$ also into a left $\mathfrak{A}' = (C^\infty(M)[[\hbar]], \star')$ module ${}_{\mathfrak{A}'}\mathcal{E}$. The left action of $\mathfrak{A}'$ is given by

$$f.\Psi = (\mathcal{D}^i(f) \star \Psi^i). \tag{12}$$

It is easy to check, using (4), that the left action (12) is compatible with (10). From the property (5) of the maps $\mathcal{D}^i$ we find

$$f.(g.\Psi) = (f \star' g).\Psi. \tag{13}$$

Together we have a bimodule structure ${}_{\mathfrak{A}'}\mathcal{E}_{\mathfrak{A}}$ on the space of sections.

There is an obvious way of tensoring sections. The section

$$\Psi_{12}^i = \mathcal{D}_1^i(\Psi_2^i) \star_1 \Psi_1^i \tag{14}$$

is a section of the tensor product line bundle (8) and (9). Tensoring of line bundles naturally corresponds to tensoring of bimodules.

Using the Hochschild complex we can introduce a natural differential calculus on the algebra $\mathfrak{A}$.[2] The p-cochains, elements of $C^p = \mathrm{Hom}_{\mathbb{C}}(\mathfrak{A}^{\otimes p}, \mathfrak{A})$, play the role of p-forms and the derivation $\mathrm{d} : C^p \to C^{p+1}$ is given on $C \in C^p$ as

$$\begin{aligned} \mathrm{d}C(f_1, f_2, \ldots, f_{p+1}) &= f_1 \star C(f_2, \ldots, f_{p+1}) - C(f_1 \star f_2, \ldots, f_{p+1}) + C(f_1, f_2 \star f_3, \ldots, f_{p+1}) - \cdots \\ &\quad + (-1)^p C(f_1, f_2, \ldots, f_p \star f_{p+1}) + (-1)^{p+1} C(f_1, f_2, \ldots, f_p) \star f_{p+1}. \end{aligned} \tag{15}$$

A (contravariant) connection $\nabla : \mathcal{E} \otimes_{\mathfrak{A}} C^p \to \mathcal{E} \otimes_{\mathfrak{A}} C^{p+1}$ can now be defined by a formula similar to (15) using the natural extension of the left and right module structure of $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathfrak{A}} C^p$. Namely, for a $\Phi \in \mathcal{E} \otimes_{\mathfrak{A}} C^p$ we have

$$\begin{aligned} \nabla\Phi(f_1, f_2, \ldots, f_{p+1}) &= f_1.\Phi(f_2, \ldots, f_{p+1}) - \Phi(f_1 \star f_2, \ldots, f_{p+1}) + \Phi(f_1, f_2 \star f_3, \ldots, f_{p+1}) - \cdots \\ &\quad + (-1)^p \Phi(f_1, f_2, \ldots, f_p \star f_{p+1}) + (-1)^{p+1} \Phi(f_1, f_2, \ldots, f_p).f_{p+1}. \end{aligned} \tag{16}$$

We also have the cup product $C_1 \cup C_2$ of two cochains $C_1 \in C^p$ and $C_2 \in C^q$;

$$(C_1 \cup C_2)(f_1, \ldots, f_{p+q}) = C_1(f_1, \ldots, f_p) \star C_2(f_{p+1}, \ldots, f_q). \tag{17}$$

The cup product extends to a map from $(\mathcal{E} \otimes_{\mathfrak{A}} C^p) \otimes_{\mathfrak{A}} C^q$ to $\mathcal{E} \otimes_{\mathfrak{A}} C^{p+q}$. The connection ∇ satisfies the graded Leibniz rule with respect to the cup product and thus defines a bona fide connection on the module $\mathcal{E}_{\mathfrak{A}}$. On the sections, the connection ∇ introduced here is simply the difference between the left and right action of $C^\infty(M)[[\hbar]]$ on $\mathcal{E}$:

$$\nabla\Psi(f) = f.\Psi - \Psi.f = \left(\nabla^i \Psi^i(f)\right) = \left(\mathcal{D}^i(f) \star \Psi^i - \Psi^i \star f\right). \tag{18}$$

As in [19] we define the gauge potential $\mathcal{A} = (\mathcal{A}^i)$, where the $\mathcal{A}^i : C^\infty(U^i)[[\hbar]] \to C^\infty(U^i)[[\hbar]]$ are local 1-cochains, by

$$\mathcal{A}^i \equiv \mathcal{D}^i - \mathrm{id}. \tag{19}$$

Then we have for a section $\Psi = (\Psi^i)$, where the $\Psi^i \in C^\infty_{\mathbb{C}}(U^i)[[\hbar]]$ are local 0-cochains,

$$\nabla^i \Psi^i(f) = \mathrm{d}\Psi^i(f) + \mathcal{A}^i(f) \star \Psi^i, \tag{20}$$

2 Other choices for the differential calculus are of course possible, e.g., the Lie algebra complex.

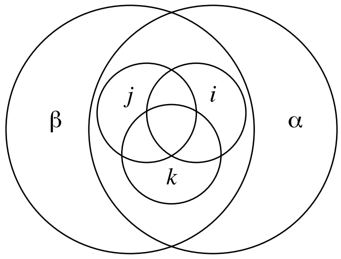

Fig. 1. Double intersection $U_\alpha \cap U_\beta$ equipped with a NC line bundle $G^{ij}_{\alpha\beta} \star_\alpha G^{jk}_{\alpha\beta} = G^{ik}_{\alpha\beta}$.

and more generally $\nabla^i \Phi^i = \mathrm{d}\Phi^i + \mathcal{A}^i \cup \Phi^i$ with $\Phi = (\Phi^i) \in \mathcal{E} \otimes_{\mathfrak{A}} C^p$. In the intersections $U^i \cap U^j$ we have the gauge transformation (cf. (4))

$$\mathcal{A}^i = \mathrm{Ad}_\star G^{ij} \circ \mathcal{A}^j + G^{ij} \star \mathrm{d}(G^{ij})^{-1}. \tag{21}$$

The curvature $K_\nabla \equiv \nabla^2 : \mathcal{E} \otimes_{\mathfrak{A}} C^p \to \mathcal{E} \otimes_{\mathfrak{A}} C^{p+2}$ corresponding to the connection ∇, measures the difference between the two star products $\star'$ and $\star$. On a section Ψ, it is given by

$$(K_\nabla \Psi)(f, g) = \left(\mathcal{D}^i(f \star' g - f \star g) \star \Psi^i\right). \tag{22}$$

The connection for the tensor product line bundle (8) is given on sections as

$$\nabla_{12}\Psi^i_{12} = \mathcal{D}^i_1(\nabla_2 \Psi^i_2) \star_1 \Psi^i_1 + \mathcal{D}^i_1(\Psi_2) \star_1 \nabla_1 \Psi^i_1. \tag{23}$$

Symbolically,

$$\nabla_{12} = \nabla_1 + \mathcal{D}_1(\nabla_2). \tag{24}$$

Let us note that if we assume the base manifold M to be compact, then the space of sections $\mathcal{E}$ as a right $\mathfrak{A}$-module is projective of finite type. Of course, the same holds if $\mathcal{E}$ is considered as a left $\mathfrak{A}'$ module. Also let us note that the two algebras $\mathfrak{A}$ and $\mathfrak{A}'$ are Morita equivalent. Up to a global isomorphism they must be related by an action of the Picard group $\mathrm{Pic}(M) \cong H^2(M, \mathbb{Z})$ as follows. Let $L \in \mathrm{Pic}(M)$ be a (complex) line bundle on M and c its Chern class. Let F be a curvature two form on M whose cohomology class $[F]$ is (the image in $\mathbb{R}$ of) the Chern class c. Consider the formal Poisson structure θ' given by the geometric series

$$\theta' = \theta(1 + \hbar F\theta)^{-1}. \tag{25}$$

In this formula θ and F are understood as maps $\theta : T^*M \to TM, F : TM \to T^*M$ and θ' is the result of the indicated map compositions. Then $\star'$ must (up to a global isomorphism) be the deformation quantization of θ' corresponding to $c \in H^2(M, \mathbb{Z})$. This construction depends only on the integer cohomology class c, indeed if c is the trivial class then $F = \mathrm{d}a$ and the corresponding quantum line bundle is trivial, i.e.,

$$G^{ij} = (H^i)^{-1} \star H^j. \tag{26}$$

In this case the linear map

$$\mathcal{D} = \mathrm{Ad}_\star H^i \circ \mathcal{D}^i \tag{27}$$

defines a global equivalence (a stronger notion than Morita equivalence) of $\star$ and $\star'$.

3. Noncommutative gerbes

Now let us consider any covering $\{U_\alpha\}$ (not necessarily a good one) of a manifold M. Here we switch from upper Latin to lower Greek indices to label the local patches. The reason for the different notation will become clear soon. Consider each local patch equipped with its own star product $\star_\alpha$ the deformation quantization of a local Poisson structure θ_α. We assume that on each double intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the local Poisson structures θ_α and θ_β are related similarly as in the previous section via some integral closed two form $F_{\beta\alpha}$, which is the curvature of a line bundle $L_{\beta\alpha} \in \mathrm{Pic}(U_{\alpha\beta})$

$$\theta_\alpha = \theta_\beta(1 + \hbar F_{\beta\alpha}\theta_\beta)^{-1}. \tag{28}$$

Let us now consider, as in Fig. 1, a good covering $U^i_{\alpha\beta}$ of each double intersection $U_\alpha \cap U_\beta$[3] with a noncommutative line bundle $\mathcal{L}_{\beta\alpha} = \{G^{ij}_{\alpha\beta}, \mathcal{D}^i_{\alpha\beta}, \star_\alpha\}$

$$G^{ij}_{\alpha\beta} \star_\alpha G^{jk}_{\alpha\beta} = G^{ik}_{\alpha\beta}, \qquad G^{ii}_{\alpha\beta} = 1, \tag{29}$$

$$\mathcal{D}^i_{\alpha\beta}(f) \star_\alpha G^{ij}_{\alpha\beta} = G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(f) \tag{30}$$

3 At this point we use so called hypercovers. Hypercovers are defined, e.g., in [20]. A thorough treatment with non-abelian cohomology classes is given in [5]. The hypercovers are not necessary if M is paracompact.

and

$$\mathcal{D}^i_{\alpha\beta}(f \star_\beta g) = \mathcal{D}^i_{\alpha\beta}(f) \star_\alpha \mathcal{D}^i_{\alpha\beta}(g). \tag{31}$$

The opposite order of indices labelling the line bundles and the corresponding transition functions and equivalences simply reflects a choice of convention. As in the previous section the order of indices of $\mathcal{L}_{\alpha\beta}$ indicates the bimodule structure of the corresponding space of sections, whereas the order of Greek indices on G's and D's indicates the star product in which the objects multiply. The product always goes with the first index of the multiplied objects.

A noncommutative gerbe is characterised by the following axioms:

Axiom 1. $\mathcal{L}_{\alpha\beta} = \{G^{ij}_{\beta\alpha}, \mathcal{D}^i_{\beta\alpha}, \star_\beta\}$ and $\mathcal{L}_{\beta\alpha} = \{G^{ij}_{\alpha\beta}, \mathcal{D}^i_{\alpha\beta}, \star_\alpha\}$ are related as follows

$$\{G^{ij}_{\beta\alpha}, \mathcal{D}^i_{\beta\alpha}, \star_\beta\} = \{(\mathcal{D}^j_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta}), (\mathcal{D}^i_{\alpha\beta})^{-1}, \star_\beta\} \tag{32}$$

i.e. $\mathcal{L}_{\alpha\beta} = \mathcal{L}^{-1}_{\beta\alpha}$. (Notice also that $(\mathcal{D}^j_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta}) = (\mathcal{D}^i_{\alpha\beta})^{-1}(G^{ji}_{\alpha\beta})$.)

Axiom 2. On the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ the tensor product $\mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ is equivalent to the line bundle $\mathcal{L}_{\gamma\alpha}$. Explicitly

$$G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(G^{ij}_{\beta\gamma}) = \Lambda^i_{\alpha\beta\gamma} \star_\alpha G^{ij}_{\alpha\gamma} \star_\alpha (\Lambda^j)^{-1}_{\alpha\beta\gamma}, \tag{33}$$

$$\mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma} = \mathrm{Ad}_{\star_\alpha} \Lambda^i_{\alpha\beta\gamma} \circ \mathcal{D}^i_{\alpha\gamma}. \tag{34}$$

Axiom 3. On the quadruple intersection $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$

$$\Lambda^i_{\alpha\beta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\delta} = \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta}, \tag{35}$$

$$\Lambda^i_{\alpha\beta\gamma} = (\Lambda^i_{\alpha\gamma\beta})^{-1} \quad \text{and} \quad \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\alpha}) = \Lambda^i_{\alpha\beta\gamma}. \tag{36}$$

With slight abuse of notation we have used Latin indices $\{i, j, \ldots\}$ to label both the good coverings of the intersection of the local patches U_α and the corresponding transition functions of the consistent restrictions of line bundles $\mathcal{L}_{\alpha\beta}$ to these intersections. A short comment on the consistency of Axiom 3 is in order. Let us define

$$\mathcal{D}^i_{\alpha\beta\gamma} = \mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma} \circ \mathcal{D}^i_{\gamma\alpha}. \tag{37}$$

Then it is easy to see that

$$\mathcal{D}^i_{\alpha\beta\gamma} \circ \mathcal{D}^i_{\alpha\gamma\delta} \circ \mathcal{D}^i_{\alpha\delta\beta} = \mathcal{D}^i_{\alpha\beta} \circ \mathcal{D}^i_{\beta\gamma\delta} \circ \mathcal{D}^i_{\beta\alpha}. \tag{38}$$

In view of (34) this implies that

$$\Lambda^i_{\alpha\beta\gamma\delta} \equiv \mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta} \star_\alpha \Lambda^i_{\alpha\delta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\beta}$$

is central. Using this and the associativity of $\star_\alpha$ together with (33) applied to the triple tensor product $\mathcal{L}_{\delta\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ transition functions

$$G^{ij}_{\alpha\beta\gamma} \equiv G^{ij}_{\alpha\beta} \star_\alpha \mathcal{D}^j_{\alpha\beta}(G^{ij}_{\beta\gamma}) \star_\alpha \mathcal{D}^j_{\alpha\beta}(\mathcal{D}^j_{\beta\gamma}(G^{ij}_{\gamma\delta})) \tag{39}$$

reveals that $\Lambda^i_{\alpha\beta\gamma\delta}$ is independent of i. It is therefore consistent to set $\Lambda^i_{\alpha\beta\gamma\delta}$ equal to 1. A similar consistency check works also for (36). If we replace all noncommutative line bundles $\mathcal{L}_{\alpha\beta}$ in Axioms 1–3 by equivalent ones, we get by definition an equivalent noncommutative gerbe.

There is a natural (contravariant) connection on a noncommutative gerbe. It is defined using the (contravariant) connections $\nabla_{\alpha\beta} = (\nabla^i_{\alpha\beta})$ (cf. (16) and (18)) on quantum line bundles $\mathcal{L}_{\beta\alpha}$. Let us denote by $\nabla_{\alpha\beta\gamma}$ the contravariant connection formed on the triple tensor product $\mathcal{L}_{\alpha\gamma\beta} \equiv \mathcal{L}_{\alpha\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$ with maps $\mathcal{D}^i_{\alpha\beta\gamma}$ and transition functions (39) according to the rule (24). Axiom 2 states that $\Lambda^i_{\alpha\beta\gamma}$ is a trivialization of $\mathcal{L}_{\alpha\gamma\beta}$ and that

$$\nabla^i_{\alpha\beta\gamma} \Lambda^i_{\alpha\beta\gamma} = 0. \tag{40}$$

Using Axiom 2 one can show that the product bundle

$$\mathcal{L}_{\alpha\beta\gamma\delta} = \mathcal{L}_{\alpha\beta\gamma} \otimes \mathcal{L}_{\alpha\gamma\delta} \otimes \mathcal{L}_{\alpha\delta\beta} \otimes \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\delta\gamma} \otimes \mathcal{L}_{\beta\alpha} \tag{41}$$

is trivial: it has transition functions $G^{ij}_{\alpha\beta\gamma\delta} = 1$ and maps $\mathcal{D}^i_{\alpha\beta\gamma\delta} = \mathrm{id}$. The constant unit section is thus well defined on this bundle. On $\mathcal{L}_{\alpha\beta\gamma\delta}$ we also have the section $(\Lambda^i_{\alpha\beta\gamma\delta})$. Axiom 3 implies $(\Lambda^i_{\alpha\beta\gamma\delta})$ to be the unit section. If two of the indices $\alpha, \beta, \gamma, \delta$ are equal, triviality of the bundle $\mathcal{L}_{\alpha\beta\gamma\delta}$ implies (36). Using for example the first relation in (36) one can show that (35) written in the form $\mathcal{D}^i_{\alpha\beta}(\Lambda^i_{\beta\gamma\delta}) \star_\alpha \Lambda^i_{\alpha\beta\delta} \star_\alpha \Lambda^i_{\alpha\delta\gamma} \star_\alpha \Lambda^i_{\alpha\gamma\beta} = 1$ is invariant under cyclic permutations of any three of the four factors appearing on the l.h.s.

If we now assume that all line bundles $\mathcal{L}_{\beta\alpha}$ are trivial (this is for example the case when $\{U_\alpha\}$ is a good covering) then $F_{\alpha\beta} = \mathrm{d}a_{\alpha\beta}$ for each $U_\alpha \cap U_\beta$ and

$$G^{ij}_{\alpha\beta} = (H^i_{\alpha\beta})^{-1} \star_\alpha H^j_{\alpha\beta}$$

$$\mathcal{D}_{\alpha\beta} = \mathrm{Ad}_{\star_\alpha} H^i_{\alpha\beta} \circ \mathcal{D}^i_{\alpha\beta}.$$

It then easily follows that

$$\Lambda_{\alpha\beta\gamma} \equiv H^i_{\alpha\beta} \star_\alpha \mathcal{D}^i_{\alpha\beta}(H^i_{\beta\gamma}) \star_\alpha \mathcal{D}^i_{\alpha\beta}\mathcal{D}^i_{\beta\gamma}(H^i_{\gamma\alpha}) \star_\alpha \Lambda^i_{\alpha\beta\gamma} \tag{42}$$

defines a global function on the triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$. $\Lambda_{\alpha\beta\gamma}$ is just the quotient of the two sections $\left(H^i_{\alpha\beta} \star_\alpha \mathcal{D}^i_{\alpha\beta}(H^i_{\beta\gamma}) \star_\alpha \mathcal{D}^i_{\alpha\beta}\mathcal{D}^i_{\beta\gamma}(H^i_{\gamma\alpha})\right)^{-1}$ and $\Lambda^i_{\alpha\beta\gamma}$ of the triple tensor product $\mathcal{L}_{\alpha\gamma} \otimes \mathcal{L}_{\gamma\beta} \otimes \mathcal{L}_{\beta\alpha}$. On the quadruple overlap $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$ it satisfies conditions analogous to (35) and (36)

$$\Lambda_{\alpha\beta\gamma} \star_\alpha \Lambda_{\alpha\gamma\delta} = \mathcal{D}_{\alpha\beta}(\Lambda_{\beta\gamma\delta}) \star_\alpha \Lambda_{\alpha\beta\delta}, \tag{43}$$

$$\Lambda_{\alpha\beta\gamma} = (\Lambda_{\alpha\gamma\beta})^{-1} \quad \text{and} \quad \mathcal{D}_{\alpha\beta}(\Lambda_{\beta\gamma\alpha}) = \Lambda_{\alpha\beta\gamma}. \tag{44}$$

Also

$$\mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = \mathrm{Ad}_{\star_\alpha} \Lambda_{\alpha\beta\gamma}. \tag{45}$$

So we can take formulas (43)–(45) as a definition of a noncommutative gerbe in the case of a good covering $\{U_\alpha\}$. We say that the gerbe is defined by the local data $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$.

From now on we shall consider only good coverings. A noncommutative gerbe defined by $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ is said to be trivial if there exists a global star product $\star$ on M and a collection of “twisted” transition functions $G_{\alpha\beta}$ defined on each overlap $U_\alpha \cap U_\beta$ and a collection $\mathcal{D}_\alpha$ of local equivalences between the global product $\star$ and the local products $\star_\alpha$

$$\mathcal{D}_\alpha(f) \star \mathcal{D}_\alpha(g) = \mathcal{D}_\alpha(f \star_\alpha g) \tag{46}$$

satisfying the following two conditions:

$$G_{\alpha\beta} \star G_{\beta\gamma} = \mathcal{D}_\alpha(\Lambda_{\alpha\beta\gamma}) \star G_{\alpha\gamma} \tag{47}$$

and

$$\mathrm{Ad}_\star G_{\alpha\beta} \circ \mathcal{D}_\beta = \mathcal{D}_\alpha \circ \mathcal{D}_{\alpha\beta}. \tag{48}$$

Locally, every noncommutative gerbe is trivial as is easily seen from (43), (44) and (45) by fixing the index α. Defining as in (19), $\mathcal{A}_\alpha = \mathcal{D}_\alpha - \mathrm{id}$, $\mathcal{A}_{\alpha\beta} = \mathcal{D}_{\alpha\beta} - \mathrm{id}$ we obtain the “twisted” gauge transformations

$$\mathcal{A}_\alpha = \mathrm{Ad}_\star G_{\alpha\beta} \circ \mathcal{A}_\beta + G_{\alpha\beta} \star \mathrm{d}(G_{\alpha\beta})^{-1} - \mathcal{D}_\alpha \circ \mathcal{A}_{\alpha\beta}. \tag{49}$$

Two noncommutative gerbes defined[4] by their corresponding local data $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ and $\{\star'_\alpha, \mathcal{D}'_{\alpha\beta}, \Lambda'_{\alpha\beta\gamma}\}$ are equivalent if there exist local equivalences $\mathcal{D}_\alpha$ of star products $\star_\alpha$ and $\star'_\alpha$, i.e.,

$$\mathcal{D}_\alpha(f) \star'_\alpha \mathcal{D}_\alpha(g) = \mathcal{D}_\alpha(f \star_\alpha g) \tag{50}$$

and local functions $\Lambda_{\alpha\beta}$ such that

$$\mathrm{Ad}_{\star'_\alpha} \Lambda_{\alpha\beta} \circ \mathcal{D}'_{\alpha\beta} \circ \mathcal{D}_\beta = \mathcal{D}_\alpha \circ \mathcal{D}_{\alpha\beta} \tag{51}$$

and

$$\mathcal{D}_\alpha(\Lambda_{\alpha\beta\gamma}) \star'_\alpha \Lambda_{\alpha\gamma} = \Lambda_{\alpha\beta} \star'_\alpha \mathcal{D}'_{\alpha\beta}(\Lambda_{\beta\gamma}) \star'_\alpha \Lambda'_{\alpha\beta\gamma}. \tag{52}$$

The classical limit of a noncommutative gerbe is the (classical) Hitchin gerbe defined by considering the classical limit (in the deformation quantization sense) of the structures in Axioms 1–3. Correspondingly the classical limit of the local data $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ gives the local data $\{\cdot_\alpha, \mathrm{id}_\alpha, \lambda_{\alpha\beta\gamma}\}$, where $\cdot_\alpha$ is the restriction to U_α of the (globally defined) ordinary point-wise product of functions on the base manifold M, and $\lambda_{\alpha\beta\gamma}$ is the 2-cocycle of the underlying classical Hitchin gerbe. We say that the noncommutative gerbe $\{\star_\alpha, \mathcal{D}_{\alpha\beta}, \Lambda_{\alpha\beta\gamma}\}$ is a trivial deformation quantization of this classical Hitchin gerbe if it is equivalent to it in the sense of (50)–(52). In particular we have a non-trivial deformation quantization of a Hitchin gerbe, whenever the local products $\star_\alpha$ are not a trivial deformation of (i.e. are not equivalent to) the ordinary commutative point-wise product (cf. (46) with $\star$ replaced by $\cdot$). This is the case considered in the next section, concerning quantization of twisted Poisson structures.

[4] After passing to a common refinement of respective trivializing coverings, if necessary.

We conclude this section with the following remark concerning the role of local functions $\Lambda_{\alpha\beta\gamma}$ and $\mathcal{D}_{\alpha\beta}$ satisfying relations (43)–(45). These represent a honest non-abelian 2-cocycle, as defined for example in [5]. It follows from the discussion of Section 2, that each $\mathcal{D}_{\alpha\beta}$ defines an equivalence, in the sense of deformation quantization, of star products $\star_\alpha$ and $\star_\beta$ on $U_\alpha \cap U_\beta$. The non-triviality of the non-abelian 2-cocycle (43)–(45) can therefore be seen as an obstruction to gluing the collection of local star products $\{\star_\alpha\}$, i.e., the collection of local rings $C^\infty(U_\alpha)[[\hbar]]$, into a global one. We also mention that in [30] a 2-cocycle similar to that of (43)–(45) represents an obstruction to gluing together certain local rings appearing in quantization of contact manifolds.

4. Quantization of twisted Poisson structures

Let $H \in H^3(M, \mathbb{Z})$ be a closed integral three form on M. Such a form is known to define a gerbe on M. We can find a good covering $\{U_\alpha\}$ and local potentials B_α with $H = \mathrm{d}B_\alpha$ for H. On $U_\alpha \cap U_\beta$ the difference of the two local potentials $B_\alpha - B_\beta$ is closed and hence exact: $B_\alpha - B_\beta = \mathrm{d}a_{\alpha\beta}$. On a triple intersection $U_\alpha \cap U_\beta \cap U_\gamma$ we have

$$a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} = -\mathrm{i}\lambda_{\alpha\beta\gamma}\mathrm{d}\lambda^{-1}_{\alpha\beta\gamma}. \tag{53}$$

The collection of local functions $\{\lambda_{\alpha\beta\gamma}\}$ represents a gerbe.

Let us also assume the existence of a formal antisymmetric bivector field $\theta = \theta^{(0)} + \hbar\theta^{(1)} + \cdots$ on M such that

$$[\theta, \theta] = \hbar\, \theta^* H, \tag{54}$$

where $[\,,\,]$ is the Schouten–Nijenhuis bracket and θ^* denotes the natural map sending n-forms to n-vector fields by "using θ to raise indices". Explicitly, in local coordinates, $\theta^* H^{ijk} = \theta^{im}\theta^{jn}\theta^{ko}H_{mno}$. We call θ a Poisson structure twisted by H [21,11,15]. On each U_α we can introduce a local formal Poisson structure $\theta_\alpha = \theta(1 - \hbar B_\alpha\theta)^{-1}$, $[\theta_\alpha, \theta_\alpha] = 0$. The Poisson structures θ_α and θ_β are related on the intersection $U_\alpha \cap U_\beta$ as in (28)

$$\theta_\alpha = \theta_\beta(1 + \hbar F_{\beta\alpha}\theta_\beta)^{-1}, \tag{55}$$

with an exact $F_{\beta\alpha} = da_{\beta\alpha}$. Now we can use Kontsevich's formality [31] to obtain local star products $\star_\alpha$ and to construct for each intersection $U_\alpha \cap U_\beta$ the corresponding equivalence maps $\mathcal{D}_{\alpha\beta}$. See [19,18] for an explicit formula for the equivalence maps. According to our discussion in the previous section these $\mathcal{D}_{\alpha\beta}$, supplemented by trivial transition functions, define a collection of trivial line bundles $\mathcal{L}_{\beta\alpha}$. On each triple intersection we then have

$$\mathcal{D}_{\alpha\beta} \circ \mathcal{D}_{\beta\gamma} \circ \mathcal{D}_{\gamma\alpha} = \mathrm{Ad}_{\star_\alpha} \Lambda_{\alpha\beta\gamma}. \tag{56}$$

It follows from the discussion after formula (36) that the collection of local functions $\{\Lambda_{\alpha\beta\gamma}\}$ represents a noncommutative gerbe (a deformation quantization of the classical gerbe represented by $\{\lambda_{\alpha\beta\gamma}\}$) if each of the central functions $\Lambda_{\alpha\beta\gamma\delta}$ introduced there can be chosen to be equal to 1. See [22, Section 5] and [32] that this is really the case. As mentioned at the end of the previous section, the non-triviality of the non-abelian 2-cocycle (43)–(45) can be seen as an obstruction to gluing the collection of local star products $\{\star_\alpha\}$, i.e., the collection of local rings $C^\infty(U_\alpha)[[\hbar]]$, into a global one. Hence, in the context of this section, this obstruction comes as a deformation quantization of the classical obstruction to gluing together local formal Poisson structures $\{\,,\,\}_\alpha$ into a global one.

5. Relation to [30] and to gerbes in the sense of Giraud, Deligne, Breen, and Brylinski

In the paper of Kashiwara [30] a 2-cycle, similar to that of (43)–(45), represents an obstruction to gluing together certain local rings appearing in quantization of contact manifolds. In order to make closer contact with [30], and apply its results, we consider for each open U_α the corresponding sheaf of local rings $C^\infty(U_\alpha)[[\hbar]]$. The prestack of left $\star_\alpha$-modules $\mathfrak{M}_\alpha$ on U_α is actually a stack [33]. It follows from [18,34] that $\star_\alpha$ and $\star_\beta$ are Morita equivalent on $U_\alpha \cap U_\beta$. The Morita equivalence is given by the bimodule ${}_\alpha\mathcal{E}_\beta$ of sections of the noncommutative line bundle $\mathcal{L}_{\alpha\beta}$. Therefore, we have a functor (an equivalence of stacks) $\varphi_{\alpha\beta} : \mathfrak{M}_\alpha|U_\alpha \cap U_\beta \to \mathfrak{M}_\beta|U_\alpha \cap U_\beta$ defined by $\mathcal{L}_{\alpha\beta}$. Because of the Axiom 2 of Section 3, $\Lambda_{\alpha\beta\gamma}$ defines on $U_\alpha \cap U_\beta \cap U_\gamma$ an isomorphism of functors $\phi_{\alpha\beta\gamma} : \varphi_{\alpha\beta}\varphi_{\beta\gamma} \to \varphi_{\alpha\gamma}$, which due to Axiom 3 of Section 3 satisfies the associativity condition on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$. Then, according to [30] there exists, up to equivalence, a unique stack $\mathfrak{M}$ such that the stacks $\mathfrak{M}|U_\alpha$ and $\mathfrak{M}_\alpha$ are equivalent.

We now show that our noncommutative gerbe can be seen as a "standard gerbe" in the sense of [1,2,4,5], i.e., a gerbe understood as locally non-empty and locally connected stacks in groupoids. As already mentioned, the local functions $\Lambda_{\alpha\beta\gamma}$ and $\mathcal{D}_{\alpha\beta}$ satisfying relations (43)–(45) represent a honest non-abelian 2-cocycle as defined, e.g., in [5]. Due to the correspondence (in the sense of 2-categories, see [5] for details) between degree two non-abelian cohomology classes and equivalence classes of standard gerbes there exists a standard gerbe corresponding to the non-abelian 2-cocycle (43)–(45). We briefly describe this corresponding standard gerbe.

The collection of data consisting of an open covering $\{U_\alpha\}$ of M, local rings $C^\infty(U_\alpha)[[\hbar]]$, isomorphisms $\mathcal{D}_{\alpha\beta}$ and invertible sections $\Lambda_{\alpha\beta\gamma}$ satisfying 2-cocycle relations (43)–(45) (more precisely the data satisfying relations (34–36)) define up to equivalence an algebroid stack $\mathfrak{C}$ in the terminology of [32] (see also [27]) such that $\mathfrak{C}|_\alpha$ is equivalent to the stack of locally free $\star_\alpha$-modules of rank 1. If we think about this algebroid stack in terms of the corresponding pseudofunctor $U \mapsto \mathfrak{C}(U)$, we can consider in each category $\mathfrak{C}(U)$ its maximal subgroupoid. The associated stack to the corresponding substack is a standard gerbe. Hence noncommutative gerbes, that we introduced in this article as the deformation quantization of abelian gerbes and related to the obstruction of defining a global $\star$-product on M, can be seen as non-abelian gerbes in the standard sense of Giraud, Deligne, Breen, and Brylinski.

We finish with a short remark concerning the relation to the later paper [23], where a more general question of deformation of (descent data for) a special kind of stacks is considered. Results of [23] concerning deformations of gerbes (see Section 2.1. of [23] for a definition of gerbe used there) and the classification of deformations of gerbes (see Section 4 of [23]) apply to the deformation quantization of Hitchin gerbes as well. We then notice that similarly to [30] and also to the present paper, the deformation quantization of a gerbe leads to a "stack of algebras". It would be interesting to compare the approaches of [23] and of the present paper in more detail.

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NONABELIAN BUNDLE 2-GERBES

BRANISLAV JURČO

Max Planck Institute for Mathematics
Vivatsgasse 7, 53111 Bonn, Germany

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We define 2-crossed module bundle 2-gerbes related to general Lie 2-crossed modules and discuss their properties. If $(L \to M \to N)$ is a Lie 2-crossed module and $Y \to X$ is a surjective submersion then an $(L \to M \to N)$-bundle 2-gerbe over X is defined in terms of a so-called $(L \to M \to N)$-bundle gerbe over the fiber product $Y^{[2]} = Y \times_X Y$, which is an $(L \to M)$-bundle gerbe over $Y^{[2]}$ equipped with a trivialization under the change of its structure crossed module from $L \to M$ to $1 \to N$, and which is subjected to further conditions on higher fiber products $Y^{[3]}$, $Y^{[4]}$ and $Y^{[5]}$. String structures can be described and classified using 2-crossed module bundle 2-gerbes.

Keywords: 2-crossed module; nonabelian bundle gerbe and 2-gerbe; nonabelian cohomology; string group; string structure.

Introduction

The modest purpose of this paper is to introduce nonabelian bundle 2-gerbes related to 2-crossed modules [20], simultaneously generalizing abelian bundle 2-gerbes [49, 50, 19] and crossed-module bundle gerbes [1, 30]. The idea is to describe objects in differential geometry, which would, in the terminology of [10], correspond to the Čech cohomology classes in $H^1(X, L \to M \to N)$, i.e. the first Čech cohomology classes on a manifold X with values in a Lie 2-crossed module $L \to M \to N$. What we want is a theory, which in the case of the 2-crossed module $U(1) \to 1 \to 1$ reproduces the theory of abelian bundle 2-gerbes and in the case of a 2-crossed module $1 \to M \to N$ reproduces the theory of crossed module bundle gerbes related to the crossed module $M \to N$ ($(M \to N)$-bundle gerbes). The latter requirement can slightly be generalized as follows. Let us assume a given crossed module $L \xrightarrow{\partial} M$. If we put $A := \ker \partial$ and $Q := \operatorname{coker} \partial$ then we have a 4-term exact sequence of Lie groups $0 \to A \to L \xrightarrow{\partial} M \to Q \to 1$ with abelian A. Let us assume that $A = U(1)$ is in the center of L and that the restriction to $U(1)$ of the action of M on L is trivial. Then we want that an $(U(1) \to L \to M)$-bundle 2-gerbe is the same thing as an $(L \to M)$ bundle gerbe twisted with an abelian bundle 2-gerbe [2].

The paper is organized as follows. In Sec. 2, we briefly recall the relevant notions of a Lie crossed module and Lie 2-crossed module. In Sec. 3, relevant results on crossed module bundles and on crossed module bundle gerbes are collected. Let us mention that crossed module bundles are special kinds of bitorsors [27, 26, 7, 10] and that crossed module bundle gerbes can be seen as a special case of gerbes with constant bands (this follows, e.g. from discussion in Sec. 4.2 of [10] commenting on the abelian bundle gerbes of [39], the cocycle bitorsors of [52, 53], and the bouquets of [24]). In Sec. 4, 2-crossed module bundle gerbes are introduced as crossed module bundles with an additional structure. 2-crossed module bundle gerbes are to 2-crossed module bundle 2-gerbes the same as crossed module bundles are to crossed module bundle gerbes. Finally, in Sec. 5, 2-crossed module bundle 2-gerbes are introduced and their properties discussed, including their local description in terms of 3-cocycles similar to those of [23, 9, 10]. The example of a lifting bundle 2-gerbe is described in some detail. Also, we discuss the relevance of 2-crossed module bundle 2-gerbes to string structures and their classification (see Proposition 4.12 and Remark 4.14). For the relevance of gerbes and abelian 2-gerbes to the string group and string structures see, e.g. [5, 17, 16, 30, 41, 48, 54]. For discussions of abelian 2-gerbes in relation to quantum field theory and string theory see, e.g. [37, 18, 19, 2].

Let us mention that in [9, 10] much more general 2-gerbes were introduced in the language of 2-stacks. These are generalizations of gerbes (defined as locally nonempty and locally connected stacks in groupoids [26, 38, 10, 7]) and seem to be related rather to crossed squares than to 2-crossed modules. We hope to return to a discussion concerning a possible relation of our bundle 2-gerbes and the 2-gerbes of [9, 10] in the future. Also, we hope to discuss the relevant notion of a 2-bouquet elsewhere. Our task here is to describe nonabelian bundle 2-gerbes using a language very close to that of the classical reference books [34, 29]. This will allow us to introduce connection, curvature, curving etc. in the forthcoming paper [32] using the language of differential geometry. For some further related work see, e.g. [45–47, 44, 25].

In this paper, we work in the category of differentiable manifolds. In particular, all groups (with exception of the string group) are assumed to be Lie groups and all maps are assumed to be smooth maps. It would be possible to work with (for instance, paracompact Hausdorff) topological spaces, topological groups and continuous maps too. For this we would have to use a proper replacement of the notion of the surjective submersion $\pi : Y \to X$ in the definitions of crossed module bundle gerbes, 2-crossed module bundle gerbes and 2-crossed module bundle 2-gerbes. For instance, instead of surjective submersions we could consider surjective maps $\pi : Y \to X$ with the property that for each point $y \in Y$ there is a neighborhood O of $\pi(y)$ with a section $\sigma : O \to Y$, such that $s(\pi(y)) = y$. Such map may be called a surjective topological submersion.

1. Crossed Modules, 2-Crossed Modules

Let us recall the notion of a crossed module of Lie groups (see, e.g. [11, 15, 43]).

1.1. Definition. Let L and M be two Lie groups. We say that L is a crossed M-module if there is a Lie group morphism $\partial_1 : L \to M$ and a smooth action of M

on $L(m,l) \mapsto {}^m l$ such that

$$^{\partial_1(l)}l' = ll'l^{-1} \quad \text{(Peiffer condition)}$$

for $l, l' \in L$, and

$$\partial_1({}^m l) = m\partial_1(l)m^{-1}$$

for $l \in L, m \in M$ hold true.

We will use the notation $L \overset{\partial_1}{\to} M$ or $L \to M$ for the crossed module.

Let us also recall that a crossed module is a special case of a pre-crossed module, in which the Peiffer condition does not necessarily hold. There is an obvious notion of a morphism of crossed modules.

1.2. Definition. A morphism between crossed modules $L \overset{\partial_1}{\to} M$ and $L' \overset{\partial_1'}{\to} M'$ is a pair of Lie group morphisms $\lambda : L \to L'$ and $\kappa : M \to M'$ such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\partial_1} & M \\ {\scriptstyle\lambda}\big\downarrow & & \big\downarrow{\scriptstyle\kappa} \\ L' & \xrightarrow{\partial_1'} & M' \end{array}$$

commutes, and for any $l \in L$ and $m \in M$ we have the following identity

$$\lambda({}^m l) = {}^{\kappa(m)}\lambda(l).$$

1.3. Remark. A crossed module of Lie groups defines naturally a strict Lie 2-group C (see, e.g. [6]). The Lie group of objects is $C_0 = \{*\}$, the Lie group of 1-arrows is $C_1 = M$ and the Lie group of 2-arrows is $C_2 = M \ltimes L$. The "vertical" multiplication is given on C_2 by

$$(m, l_1)(\partial_1(l_1)m, l_2) = (m, l_1 l_2)$$

and the "horizontal" multiplication is given by

$$(m_1, l_1)(m_2, l_2) = (m_1 m_2, l_1 \, {}^{m_1} l_2).$$

See, e.g. [12, 14] for more details on the relation between crossed modules and strict Lie 2-groups.

1.4. Definition. The definition of a 2-crossed module of groups is due to Conduché [20]; (see also, e.g. [21, 42, 13, 43, 44]). A Lie 2-crossed module is a complex of Lie groups

$$L \overset{\partial_1}{\to} M \overset{\partial_2}{\to} N \tag{1}$$

together with smooth left actions by automorphisms of N on L and M (and on N by conjugation), and the Peiffer lifting, which is an equivariant map $\{\ ,\ \} : M \times M \to L$, i.e. ${}^n\{m_1, m_2\} = \{{}^n m_1, {}^n m_2\}$ such that:

(i) (1) is a complex of N-modules, i.e. ∂_1 and ∂_2 are N-equivariant and $\partial_1\partial_2(l) = 1$ for $l \in L$,
(ii) $m_1 m_2 m_1^{-1} = \partial_1\{m_1, m_2\}^{\partial_2(m_1)} m_2 =: \langle m_1, m_2\rangle$, for $m_1, m_2 \in M$,
(iii) $[l_1, l_2] := l_1 l_2 l_1^{-1} l_2^{-1} = \{\partial_1 l_1, \partial_1 l_2\}$, for $l_1, l_2 \in L$,
(iv) $\{m_1 m_2, m_3\} = \{m_1, m_2 m_3 m_2^{-1}\}^{\partial_2(m_1)}\{m_2, m_3\}$, for $m_1, m_2, m_3 \in M$,
(v) $\{m_1, m_2 m_3\} = {}^{m_1 m_2 m_1^{-1}}\{m_1, m_3\}\{m_1, m_2\}$, for $m_1, m_2, m_3 \in M$,
(vi) $\{\partial_1(l), m\}\{m, \partial_1(l)\} = l^{\partial_2(m)}(l^{-1})$, for $m \in M, l \in L$,

wherein the notation ${}^n m$ and ${}^n l$ for left actions of the element $n \in N$ on elements $m \in M$ and $l \in L$ has been used. Also, let us note that ${}^m l := l\{\partial_1(l)^{-1}, m\}$ defines a left action of M on L by automorphisms. This is a consequence of the other axioms and is proved in [20, 13], where it is also shown that, equipped with this action, $L \xrightarrow{\partial_1} M$ defines a crossed module.

1.5. Example. Any crossed module $(L \xrightarrow{\delta} M)$ determines a 2-crossed module $A := \ker(\partial) \to L \to M$ with an abelian A.

1.6. Remark. In addition to the crossed module $L \xrightarrow{\partial_1} M$, there is another crossed module that can be associated with the 2-crossed module $L \xrightarrow{\partial_1} M \xrightarrow{\partial_2} N$. By definition, we see that $M \xrightarrow{\partial_2} N$ is a (special) pre-crossed module in which the Peiffer condition is satisfied only up to the Peiffer lifting. Hence, $M/\partial_1(L) \xrightarrow{\partial_2'} N$, with the induced Lie group homomorphism ∂_2' and with the induced action of N on $M/\partial_1(L)$, is a crossed module.

There is an obvious notion of a morphism of 2-crossed modules.

1.7. Definition. A morphism between 2-crossed modules $L \xrightarrow{\partial_1} M \xrightarrow{\partial_2} N$ and $L' \xrightarrow{\partial_1'} M' \xrightarrow{\partial_2'} N'$ is a triple of Lie group morphisms $L \to L'$, $M \to M'$ and $N \to N'$ making up, together with the maps ∂_1, ∂_1', ∂_2 and ∂_2' a commutative diagram

$$\begin{tikzcd} L \arrow[r, "\partial_1"] \arrow[d, "\lambda"'] & M \arrow[r, "\partial_2"] \arrow[d, "\mu"'] & N \arrow[d, "\nu"] \\ L' \arrow[r, "\partial_1'"] & M' \arrow[r, "\partial_2'"] & N' \end{tikzcd} \tag{2}$$

and being compatible with the actions of N on M and L and of N' on M' and L', respectively and with the respective Peiffer liftings.

1.8. Remark. A 2-crossed module of Lie groups defines naturally a Gray (Lie) 3-groupoid with a single object. For the construction and for more details on the relation between 2-crossed modules and Gray 3-groupoids see [33, 13, 42, 25]. There are two "vertical" multiplications and one "horizontal" multiplication on triples (3-cells) $(n, m, l) \in N \times M \times L$. The vertical multiplications are determined by the crossed module $L \to M$. The two vertical multiplications are given by

$$(n, m, l_1)(n, \partial_1(l_1)m, l_2) = (n, m, l_1 l_2)$$

and

$$(n, m_1, l_1)(\partial_2(m_1)n, m_2, l_2) = (n, m_1 m_2, l_1\, {}^{m_1}l_2)$$

and the horizontal multiplication is given by

$$(n_1, m_1, l_1)(n_2, m_2, l_2) = (n_1 n_2, m_1\, {}^{n_1}m_2, l_1\, {}^{m_1}({}^{n_1}l_2)).$$

2. Crossed Module Bundle Gerbes

Let X be a (smooth) manifold. Crossed module bundle gerbes have been introduced, for instance, in [30, 1]. These can be seen as generalizations of abelian bundle gerbes [39, 40]. If $(L \stackrel{\partial_1}{\to} M)$ is a crossed module of Lie groups, X a manifold and $P \to X$ a left principal L-bundle, we can change the structure group of P from L to M, in order to obtain a left principal M-bundle $P' = M \times_{\partial_1} P$ defined as follows. Points $p' \in P'$ correspond to equivalence classes $[m, p] \in M \times_{\partial_1} P$ with the equivalence relation on $M \times P$ given by $(m, p) \sim (m\partial_1(l), l^{-1}p)$. Obviously, the principal left M-action is given by $M \times P' \to P'$, $m' \times [m, p] \mapsto [m'm, p]$.

2.1. Definition. Let $(L \stackrel{\partial_1}{\to} M)$ be a crossed module of Lie groups and X a manifold. Let $P \to X$ be a left principal L-bundle, such that the principal M-bundle $M \times_{\partial_1} P$ is trivial with a trivialization defined by a section (i.e. a left L-equivariant smooth map) $\boldsymbol{m} : P \to M$. We call the pair $(P, \boldsymbol{m})$ an $(L \to M)$-bundle.

2.2. Remark. If we think about the crossed module $L \to M$ as a groupoid with the Lie group of objects M and the Lie group of arrows $M \ltimes L$ then a crossed module bundle is the same thing as a principal groupoid bundle.

2.3. Definition. Two $(L \to M)$-bundles $(P, \boldsymbol{m})$ and $(P', \boldsymbol{m}')$ over X are isomorphic if they are isomorphic as left L-bundles by an isomorphism $\boldsymbol{\ell} : P \to P'$ such that $\boldsymbol{m}'\boldsymbol{\ell} = \boldsymbol{m}$. An $(L \to M)$-bundle is trivial if it is isomorphic to the trivial $(L \to M)$-bundle $(X \times L, \partial_1 \mathrm{pr}_L)$.

2.4. Example. Notice that a general $(L \to M)$-bundle is not necessarily locally trivial, although it is locally trivial as a left principal L-bundle. For instance, for a function $m : X \to M$ such that $\mathrm{Im}(m)$ is not a subset of $\mathrm{Im}(\partial_1)$ the $(L \to M)$-bundle $(X \times L, \partial_1 \mathrm{pr}_L \cdot m \mathrm{pr}_X)$ is locally non-trivial. We will refer to such an $(L \to M)$-bundle as an $(L \to M)$-bundle defined by the M-valued function m. Two such $(L \to M)$-bundles are isomorphic if their respective sections m and m' are related by an L-valued function l on X by $m' = \partial_1(l)m$. Obviously, compositions of isomorphisms corresponds to multiplication of the respective L-valued function defining them.

2.5. Example. A $(1 \to G)$-bundle is the same thing as a G-valued function.

2.6. Example. A pair $(T, \mathbf{l})$, where T is a trivial left principal L-bundle and $\mathbf{l} : T \to L$ its trivializing section, defines an $(L \to M)$-bundle with the section $\boldsymbol{m} = \partial_1 \mathbf{l} : T \to M$. $(T, \mathbf{l})$ is a trivial $(L \to M)$-bundle.

2.7. Example. Let L be a normal subgroup of M. The adjoint action of M restricted to L defines a crossed module structure on $L \to M$ with $\ker \partial_1 = 1$. Let L be also a closed subgroup of M and assume M to be finite-dimensional. We put $G := \frac{L}{M}$, so that we have an exact sequence of Lie groups $1 \to L \to M \xrightarrow{\bar{\pi}} G \to 1$. It follows that $M \to G$ is a left principal L-bundle over G [34] (hence, admitting smooth local sections).[a] Moreover, $(M \to G, \boldsymbol{m})$ with $\boldsymbol{m} = \mathrm{id}_M$ is an $(L \to M)$-bundle.

2.8. Pullback. Obviously, a pullback of an $(L \to M)$-bundle is again an $(L \to M)$-bundle. Pullbacks preserve isomorphisms of crossed module bundles, in particular a pullback of a trivial $(L \to M)$-bundle is again a trivial $(L \to M)$-bundle.

2.9. Change of the structure crossed module. If $(L \to M) \to (L' \to M')$ is a morphism of crossed modules, there is an obvious way to construct, starting from an $(L \to M)$-bundle $(P, \boldsymbol{m})$, an $(L' \to M')$-bundle $(L' \times_\lambda P, \kappa \boldsymbol{m})$ where $\lambda : L \to L'$ and $\kappa : M \to M'$ define the morphism of the two crossed modules. Obviously, the change of the structure crossed module preserves isomorphisms of crossed module bundles.

2.10. 1-cocycles. Consider an $(L \to M)$-bundle $(P, \boldsymbol{m})$ and a trivializing covering $\coprod O_i = X$ of the left principal L-bundle P. Let $\sigma_i : P|O_i \to L$ be the trivializing sections of L and $l_{ij} = \sigma_i^{-1}\sigma_j : O_i \cap O_j \to L$ be the corresponding transition functions. We put $m_i = \partial_1(\sigma_i)^{-1}\boldsymbol{m}$, which obviously gives an L-valued function on O_i. We have $\partial_1(l_{ij}) = m_i m_j^{-1}$. Hence the $(L \to M)$-bundle $(P, \boldsymbol{m})$ can be described by a 1-cocycle given by transition functions (m_i, l_{ij}), $m_i : O_i \to M$, $l_{ij} : O_{ij} = O_i \cap O_j \to L$ satisfying on nonempty O_{ij},

$$\partial_1(l_{ij}) = m_i m_j^{-1},$$

and on nonempty $O_{ijk} = O_i \cap O_j \cap O_k$

$$l_{ij} l_{jk} = l_{ik}.$$

Transition functions (m_i, l_{ij}) and (m'_i, l'_{ij}) corresponding to two isomorphic $(L \to M)$-bundles are related by

$$m'_i = \partial_1(l_i) m_i,$$

[a] More generally, to assure the existence of smooth local sections of $\bar{\pi}$ in a short exact sequence of topological groups $1 \to L \to M \xrightarrow{\bar{\pi}} G \to 1$, we would have to ask the projection $\bar{\pi}$ to be a Hurewicz fibration.

and

$$l'_{ij} = l_i l_{ij} l_i^{-1}.$$

We say that two 1-cocycles (m_i, l_{ij}) and (m'_i, l'_{ij}), related as above, are equivalent. We will denote by $H^0(X, L \to M)$ the set of corresponding equivalence classes. A trivial $(L \to M)$-bundle is described by transition functions $(\partial_1(l_i), l_i l_j^{-1})$.

On the other hand, given transition functions (m_i, l_{ij}) we can reconstruct an $(L \to M)$-bundle. We define a left principal L-bundle P with the total space formed by equivalence classes of triples $[x, l, i]$ with $x \in O_i$, $l \in L$ under the equivalence relation $(x, l, i) \sim (x', l', j)$ iff $x = x'$ and $l' = ll_{ij}$. The principal left L-action is given by $l'[x, l, i] = [x, l'l, i]$. Now we put $\boldsymbol{m}([x, l, i]) = \partial_1(l) m_i(x)$. $(P, \boldsymbol{m})$ is an $(L \to M)$-bundle.

With the two above constructions it is not difficult to prove that the isomorphism classes of $(L \to M)$-bundles are 1-1 with elements of $H^0(X, L \to M)$.

2.11. Lifting crossed module bundle. Let L and M be as above in (2.7). Consider a G-valued function $g : X \to G$. The pullback $g^*(M, \mathrm{id})$ of the $(L \to M)$-bundle $\pi : M \to G$ is an $(L \to M)$-bundle on X (the lifting crossed module bundle). It is the obstruction to a lifting of the G-valued function G to some M-valued function. To go in the opposite direction, we note that we have an obvious morphism of crossed modules $(L \to M) \to (1 \to G)$. Under the change of the structure crossed module of an $(L \to M)$-bundle $(P, \boldsymbol{m})$ to $(1 \to G)$, the section $\boldsymbol{m}$ becomes an L-invariant G-valued function $\pi\boldsymbol{m}$ on P. Hence, it is identified with an G-valued function g on X. Two isomorphic $(L \to M)$-bundles give the same function. The two constructions are inverse to each other up to an isomorphism of $(L \to M)$-bundles.

It is now easy to give a local description of lifting crossed module bundles. Let $\{O_i\}_i$ be an open covering on X. Let P be an $(L \to M)$-bundle described by transition function (l_{ij}, m_i). Since $\pi\partial_1 = 1$, we have $\pi(m_i) = \pi(m_j)$. Hence, the collection of local functions $\{\pi(m_i)\}_i$ defines a G-valued function on X. To go in the opposite direction, let g be a G-valued function on X. Let $\{O_i\}_i$ be a trivializing covering of the pullback principal bundle $g^*(M)$. The function g can now be described by a collection of local functions $g_i : O_i \to G$ such that $g_i = g_j$ on O_{ij}. Hence, we have local functions $m_i : O_i \to M$ the local sections of $g^*(M)$ such that $\pi(m_i) = g_i$, which are related on double intersections O_{ij} by $m_i = \partial_1(l_{ij}) m_j$ with L-valued functions $l_{ij} : O_{ij} \to L$, the transition functions of the principal L-bundle $g^*(M)$, fulfilling the 1-cocycle condition $l_{ij} l_{jk} = l_{ik}$ on O_{ijk}.

Concerning crossed module bundles, we have the following lemma and proposition [1].

Lemma 2.1. *The $(L \to M)$-bundle $(P, \boldsymbol{m})$ is also a right principal L-bundle with the right action of L given by $p.l = {}^{\boldsymbol{m}(p)}(l).p$ for $p \in P, l \in L$. The left and right actions commute. The section $\boldsymbol{m}$ is L-biequivariant.*

Proposition 2.1. *Let $\mathcal{P} = (P, \boldsymbol{m})$ and $\tilde{\mathcal{P}} = (\tilde{P}, \tilde{\boldsymbol{m}})$ are two $(L \to M)$-bundles over X. Let us define an equivalence relation on the Whitney sum $P \oplus \tilde{P} = P \times_X \tilde{P}$ by $(pl, \tilde{p}) \sim (p, l\tilde{p})$, for $(p, \tilde{p}) \in P \oplus \tilde{P}$ and $l \in L$. Then $\mathcal{P}\tilde{\mathcal{P}} := (P\tilde{P} := (P \oplus \tilde{P})/\sim, \boldsymbol{m}\tilde{\boldsymbol{m}})$ with $\boldsymbol{m}\tilde{\boldsymbol{m}}([p, \tilde{p}]) := \boldsymbol{m}(p)\tilde{\boldsymbol{m}}(\tilde{p})$ is an $(L \to M)$-bundle.*

2.12. Remark. Obviously, if $\mathcal{P} \cong \mathcal{Q}$ and $\tilde{\mathcal{P}} \cong \tilde{\mathcal{Q}}$ then also $\mathcal{P}\tilde{\mathcal{P}} \cong \mathcal{Q}\tilde{\mathcal{Q}}$. The set of isomorphism classes of $(L \to M)$-bundles equipped with the above-defined product is a group. The unit is given by the class of the trivial bundle $(X \times L, \partial_1 \mathrm{pr_L})$. The inverse is given by the class of $(L \to M)$-bundle $(P^{-1}, \boldsymbol{m}^{-1})$ with P^{-1} having the same total space as P, the left L-action on P^{-1} being the inverse of the right L-action on P and the trivializing section $\boldsymbol{m}^{-1}$ being the composition of the inverse in M with the trivializing section $\boldsymbol{m}$. Let us note that in the case of an exact sequence $1 \to L \to M \to N \to 1$ as above (2.7) this group structure is compatible with the group structure of $G = \frac{M}{L}$-valued functions with pointwise multiplication.

2.13. Example. If $\mathcal{P} = (P = X \times L, \partial_1 \mathrm{pr}_L \cdot m\mathrm{pr}_X)$ and $\mathcal{P}' = (P' = X \times L, \partial_1 \mathrm{pr}_L \cdot m'\mathrm{pr}_X)$ are $(L \to M)$-bundles defined by two respective M-valued functions m and m' on X (2.4) then the product $\mathcal{P}\mathcal{P}'$ is explicitly described again as an $(L \to M)$-bundle defined by the function mm' by identifying $[(x, l), (x, l')] \in PP'$ with $(x, l\,{}^m l') \in X \times L$.

2.14. Product on 1-cocycles. Transition functions $(\bar{m}_i, \bar{l}_{ij})$ of the product of two $(L \to M)$-bundles described by transition functions (m_i, l_{ij}) and $(\tilde{m}_i, \tilde{l}_{ij})$ are given by

$$\bar{m}_i = m_i \tilde{m}_i$$

and

$$\bar{l}_{ij} = l_{ij}\, {}^{m_i}\tilde{l}_{ij}.$$

Transition functions of the inverse crossed module bundle are $({}^{m_j^{-1}} l_{ij}^{-1} = {}^{m_i^{-1}} l_{ij}^{-1}, m_i^{-1})$.

2.15. 1-cocycles as functors. The crossed module $(L \to M)$ defines naturally a topological category (groupoid) $\mathcal{C}$ with the set of objects $C_0 = L$ and the set of arrows $C_1 = M \times L$. Let us consider the topological category $\mathcal{O}$ (groupoid) defined by the good covering $\{O_i\}$ of X with objects $x_i := (x, i \,|\, x \in O_i)$ and exactly one arrow from x_i to y_j iff $x = y$. Then a 1-cocycle is the same thing as a continuous functor from $\mathcal{O}$ to $\mathcal{C}$. Further, if $2\mathcal{B}$ is a strict topological 2-category, then the category of 2-arrows with the vertical composition is naturally a topological category $\mathcal{B}$. The horizontal composition in $2\mathcal{B}$ defines a continuous functor from the Cartesian product $\mathcal{B} \times \mathcal{B}$ to $\mathcal{B}$. Thus, in case $\mathcal{B} = \mathcal{C}$ it defines naturally a multiplication on functors $\mathcal{O} \to \mathcal{C}$ (i.e. on transition functions), which is the same as the one defined above (2.1).

2.16. Crossed module bundle gerbes. Let Y be a manifold. Consider a surjective submersion $\pi : Y \to X$, which in particular admits local sections. Let $\{O_i\}$ be the corresponding covering of X with local sections $\sigma_i : O_i \to Y$, i.e. $\pi\sigma_i = \mathrm{id}$. We also consider $Y^{[n]} = Y \times_X Y \times_X Y \cdots \times_X Y$, the n-fold fibre product of Y, i.e. $Y^{[n]} := \{(y_1, \ldots, y_n) \in Y^n \mid \pi(y_1) = \pi(y_2) = \cdots \pi(y_n)\}$. Given an $(L \to M)$-bundle $\mathcal{P} = (P, \boldsymbol{m})$ over $Y^{[2]}$ we denote by $\mathcal{P}_{12} = p_{12}^*(\mathcal{P})$ the crossed module bundle on $Y^{[3]}$ obtained as a pullback of $\mathcal{P}$ under $p_{12} : Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{P}_{13}$ and $\mathcal{P}_{23}$. Consider a quadruple $(\mathcal{P}, Y, X, \boldsymbol{\ell})$, where $\mathcal{P} = (P, \boldsymbol{m})$ is a crossed module bundle, $Y \to X$ a surjective submersion and $\boldsymbol{\ell}$ an isomorphism of crossed module bundles $\boldsymbol{\ell} : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$. We now consider bundles $\mathcal{P}_{12}$, $\mathcal{P}_{23}$, $\mathcal{P}_{13}$, $\mathcal{P}_{24}$, $\mathcal{P}_{34}$, $\mathcal{P}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} : Y^{[4]} \to Y^{[2]}$ etc. and also the crossed module isomorphisms $\boldsymbol{\ell}_{123}$, $\boldsymbol{\ell}_{124}$, $\boldsymbol{\ell}_{123}$, $\boldsymbol{\ell}_{234}$ induced by projections $p_{123} : Y^{[4]} \to Y^{[3]}$ etc.

2.17. Definition. The quadruple $(\mathcal{P}, Y, X, \boldsymbol{\ell})$, where $Y \to X$ is a surjective submersion, $\mathcal{P}$ is a crossed module bundle over $Y^{[2]}$, and $\boldsymbol{\ell} : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$ an isomorphism of crossed module bundles over $Y^{[3]}$, is called a crossed module bundle gerbe if $\boldsymbol{\ell}$ satisfies the cocycle condition (associativity) on $Y^{[4]}$

$$\begin{tikzcd} \mathcal{P}_{12}\mathcal{P}_{23}\mathcal{P}_{34} \arrow[r, "\boldsymbol{\ell}_{234}"] \arrow[d, "\boldsymbol{\ell}_{123}"'] & \mathcal{P}_{12}\mathcal{P}_{24} \arrow[d, "\boldsymbol{\ell}_{124}"] \\ \mathcal{P}_{13}\mathcal{P}_{34} \arrow[r, "\boldsymbol{\ell}_{134}"] & \mathcal{P}_{14}\,. \end{tikzcd} \tag{3}$$

2.18. Abelian bundle gerbes. Abelian bundle gerbes as introduced in [39, 40] are $(U(1) \to 1)$-bundle gerbes. More generally, if $A \to 1$ is a crossed module then A is necessarily an abelian group and an abelian bundle gerbe can be identified as an $(A \to 1)$-bundle gerbe.

2.19. Example. A $(1 \to G)$-bundle gerbe is the same thing as a G-valued function g on $Y^{[2]}$ (2.5) satisfying on $Y^{[3]}$ the cocycle relation $g_{12}g_{23} = g_{23}$ and hence, a principal G-bundle on X (more precisely a descent datum of a principal G-bundle).

2.20. Pullback. If $f : X \to X'$ is a map then we can pullback $Y \to X$ to $f^*(Y) \to X'$ with a map $\tilde{f} : f^*(Y) \to Y$ covering f. There are induced maps $\tilde{f}^{[n]} : f^*(Y)^{[n]} \to Y^{[n]}$. Then the pullback $f^*(\mathcal{P}, Y, X, \boldsymbol{\ell}) := (\tilde{f}^{[2]*}\mathcal{P}, f^*(Y), f(X), \tilde{f}^{[3]*}\boldsymbol{\ell})$ is again an $(L \to M)$-bundle gerbe.

2.21. Definition. Two crossed module bundle gerbes $(\mathcal{P}, Y, X, \boldsymbol{\ell})$ and $(\mathcal{P}', Y', X, \boldsymbol{\ell}')$ are stably isomorphic if there exists a crossed module bundle $\mathcal{Q} \to \bar{Y} = Y \times_X Y'$ such that over $\bar{Y}^{[2]}$ the crossed module bundles $q^*\mathcal{P}$ and $\mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ are isomorphic. The corresponding isomorphism $\tilde{\boldsymbol{\ell}} : q^*\mathcal{P} \to \mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1}$ should satisfy on $\bar{Y}^{[3]}$ (with an obvious abuse of notation) the condition

$$\tilde{\boldsymbol{\ell}}_{13}\boldsymbol{\ell} = \boldsymbol{\ell}'\tilde{\boldsymbol{\ell}}_{23}\tilde{\boldsymbol{\ell}}_{12}. \tag{4}$$

Here q and q' are projections onto the first and second factor of $\bar{Y} = Y \times_X Y'$ and $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are the pullbacks of $\mathcal{Q} \to \bar{Y}$ to $\bar{Y}^{[2]}$ under the respective projections from $\bar{Y}^{[2]}$ to $\bar{Y}$ etc.

A crossed module bundle gerbe $(\mathcal{P}, Y, X, \boldsymbol{\ell})$ is called trivial if it is stably isomorphic to the trivial crossed module bundle gerbe $((Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1)$. Pullbacks preserve stable isomorphisms, in particular a pullback of a trivial crossed module bundle gerbe is again a trivial crossed module bundle gerbe. If $Y = X$ then the crossed module bundle gerbe is trivial.

2.22. Definition. Let $(\mathcal{P}, Y, X, \boldsymbol{\ell})$ and $(\mathcal{P}', Y', X, \boldsymbol{\ell}')$ be two crossed module bundle gerbes and $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ two stable isomorphisms between them. We call $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ isomorphic if there is an isomorphism $\underline{\boldsymbol{\ell}} : \mathcal{Q} \to \mathcal{R}$ of crossed module bundles on $\bar{Y} = Y \times_X Y'$ such that (with an obvious abuse of notation) the diagram

$$\begin{array}{ccc} q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{Q}}} & \mathcal{Q}_1 q'^*\mathcal{P}'\mathcal{Q}_2^{-1} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow {\scriptstyle \underline{\boldsymbol{\ell}}_1 \underline{\boldsymbol{\ell}}_2^{-1}} \\ q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{R}}} & \mathcal{R}_1 q'^*\mathcal{P}'\mathcal{R}_2^{-1} \end{array} \tag{5}$$

is commutative.

2.23. Remark. Let $\pi' : Y' \to X$ be another surjective submersion and $f : Y' \to Y$ a map such that $\pi' = \pi f$. Then the crossed module bundle gerbes $\mathcal{G}_f = (f^*\mathcal{P}, Y', X, f^{[3]*}\boldsymbol{\ell})$ and $\mathcal{G} = (\mathcal{P}, Y, X, \boldsymbol{\ell})$ are stably isomorphic. This can be easily seen by noticing first that $\mathcal{G}$ is stably isomorphic to itself and then using the obvious fact that pullbacks of crossed module bundles commute with their products [1]. It follows that locally each crossed module bundle gerbe $\mathcal{G}$ is trivial. For this, take a point $x \in X$ and its neighborhood $O \subset X$ such that there exists a local section $\sigma : O \to Y$ of π. Over O we have the bundle gerbe $\mathcal{G}_O$, the restriction of $\mathcal{G}$ to O. Now we can put $Y' := O$ and $\pi' := \mathrm{id}_O$ and we have $\pi\sigma = \pi'$. It follows that $\mathcal{G}_\sigma$ is stably isomorphic to $\mathcal{G}_O$. However $\mathcal{G}_\sigma$ is trivial, because of $Y' = O$.

2.24. Change of the structure crossed module. If $(L \to M) \to (L' \to M')$ is a morphism of crossed modules, there is an obvious way to construct starting from an $(L \to M)$-bundle gerbe an $(L' \to M')$-bundle gerbe by changing the structure crossed module of the corresponding $(L \to M)$-bundle over $Y^{[2]}$. Obviously, the change of the structure crossed module preserves stable isomorphisms of crossed module bundle gerbes.

2.25. 2-cocycles. Locally, crossed module bundle gerbes can be described in terms of 2-cocycles as follows. First, let us note that the trivializing cover $\{O_i\}$ of the map $\pi : Y \to X$ defines a new surjective submersion $\pi' : Y' = \coprod O_i \to X$. The local sections of $Y \to X$ define a map $f : Y' \to Y$, which is compatible with the maps π and π', i.e. such that $\pi f = \pi'$. We know that the crossed module bundle gerbes $\mathcal{G}_f$ and $\mathcal{G}$ are stably isomorphic. Hence, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one. Then the crossed module bundle gerbe can be described by a 2-cocycle (m_{ij}, l_{ijk}) where the maps $m_{ij} : O_{ij} \to M$ and $l_{ijk} : O_{ijk} \to L$ fulfill the following conditions

$$m_{ij}m_{jk} = \partial_1(l_{ijk})m_{ik} \quad \text{on } O_{ijk}$$

and

$$l_{ijk}l_{ikl} = {}^{m_{ij}}l_{jkl}l_{ijl} \quad \text{on } O_{ijkl}.$$

Two crossed module bundle gerbes are stably isomorphic if their respective 2-cocycles (m_{ij}, l_{ijk}) and (m'_{ij}, l'_{ijk}) are related (equivalent) by

$$m'_{ij} = m_i\partial_1(l_{ij})m_{ij}m_j^{-1} \tag{6}$$

and

$$l'_{ijk} = {}^{m_i}l_{ij}\,{}^{m_im_{ij}}l_{jk}\,{}^{m_i}l_{ijk}\,{}^{m_i}l_{ik}^{-1} \tag{7}$$

with $m_i : O_i \to M$ and $l_{ij} : O_{ij} \to L$.

We will denote by $H^1(X, L \to M)$ the set of the corresponding equivalence classes of 2-cocycles.

A trivial crossed module bundle gerbe is described by transition functions

$$m_{ij} = m_i\partial_1(l_{ij})m_j^{-1}$$

and

$$l_{ijk} = {}^{m_i}l_{ij}\,{}^{m_i}l_{jk}\,{}^{m_i}l_{ik}^{-1}.$$

Two collections of stable isomorphism data (m_i, l_{ij}) and (m'_i, l'_{ij}) are isomorphic if

$$m'_i = \partial_1(l_i)m_i,$$

$$l'_{ij} = l_il_{ij}\,{}^{m_{ij}}l_j^{-1}.$$

Now we briefly describe how an $(L \to M)$-bundle gerbe can be reconstructed from transition functions (m_{ij}, l_{ijk}). Put $Y = \coprod O_i$. On each nonempty O_{ij} consider the $(L \to M)$-bundle $\mathcal{P}_{ij}$ defined by the function $m_{ij} : O_{ij} \to M$ as in (2.4). Hence, on $Y^{[2]}$ we have the $(L \to M)$-bundle given by $\mathcal{P} = \coprod_{ij} \mathcal{P}_{ij}$. Now we recall the explicit descriptions of the multiplication (2.13) and isomorphisms (2.4) of two $(L \to M)$-bundles defined by their respective M-valued functions. Using the 2-cocycle relations, it is now straightforward to show that the collection of functions l_{ijk} defines an isomorphism of $\mathcal{P}_{12}\mathcal{P}_{23}$ and $\mathcal{P}_{13}$ on $Y^{[3]}$ satisfying the associativity condition on $Y^{[4]}$ (compare, e.g. [38, Theorem 3.1]).

Further, two crossed module bundle gerbes corresponding to two equivalent 2-cocycles are stably isomorphic. To show it let us denote, similarly as above, by $\mathcal{P}$ and $\mathcal{P}'$ the two $(L \to M)$-bundles over $Y^{[2]}$ defined by the two respective collections of local functions m_{ij} and m'_{ij}. Note that according to (2.4), the local M-valued functions m_i in (6) define an $(L \to M)$-bundle $\mathcal{Q}$ over $Y = \coprod O_i$ and that the local L-valued functions ${}^{m_i}l_{ij}$ define on $Y^{[2]}$ an isomorphism $\tilde{\boldsymbol{\ell}}$ of the $(L \to M)$ bundles $\mathcal{P}'$ and $\mathcal{Q}_1\mathcal{P}\mathcal{Q}_2^{-1}$. Finally, the relation (7) tells us that the isomorphism $\tilde{\boldsymbol{\ell}}$ fulfills the requested compatibility condition (4) (cf. the last statement in Example 2.4 concerning the composition of isomorphisms).

Hence, the above discussion of 2-cocycles proves the following proposition.

Proposition 2.2. *Stable isomorphism classes of $(L \to M)$-bundle gerbes are in a bijective correspondence with elements of $H^1(X, L \to M)$.*

2.26. Remark. Actually, when considering isomorphisms of stable isomorphisms, we have the respective 2-categories of $(L \to M)$-bundle gerbes and transition functions. The correspondence between $(L \to M)$-bundle gerbes and the transition functions can be formulated in the framework of 2-categories similarly to [9], but we will not do this here. Further, if we consider the topological category $\mathcal{O}$ defined by the good covering $\{O_i\}$ of X. Then a 2-valued cocycle can be seen as a continuous normal pseudo-functor from $\mathcal{O}$ to the bicategory defined by the crossed module $L \to M$.

2.27. Lifting crossed module bundle gerbe. Let $L \to M$ be a crossed module associated with a normal subgroup L of M (cf. Example 2.7). We have a Lie group extension

$$1 \to L \overset{\partial_1}{\to} M \overset{\bar{\pi}}{\to} G \to 1$$

and also the $(L \to M)$-bundle $M \overset{\bar{\pi}}{\to} G$.

The following statement has appeared in the literature before. In [5] (cf. Lemma 2) a version of it is attributed to Larry Breen.

Proposition 2.3. *Let $L \to M \overset{\bar{\pi}}{\to} G$ be an Lie group extension. Let us also assume that the conditions for M being a principal L-bundle over G are satisfied.*[b] *Then the isomorphism classes of G-principal bundles are in bijective correspondence with stable isomorphism classes of $(L \to M)$-bundle gerbes.*

Proof. Let $E \to X$ be a (locally trivial) left principal G-bundle over X. As a principal G-bundle E defines a (division) map $g : E^{[2]} \to G$ which gives for two elements in a fiber of E the group element relating to them. The pullback $\mathcal{P} = g^*(M, \mathrm{id}_M)$ of the $(L \to M)$-bundle $M \to G$ gives an $(L \to M)$-bundle on $E^{[2]}$; $\mathcal{P}$ is by definition the lifting $(L \to M)$-bundle corresponding to the division map g (2.11). It follows

[b] Cf. Example 2.4.

that the crossed module bundles $\mathcal{P}_{12}\mathcal{P}_{23}$ and $\mathcal{P}_{13}$ are isomorphic on $E^{[3]}$. This follows from the above-mentioned fact that, in case of Lie groups L, M and G as above, isomorphism classes of $(L \to M)$-bundles are one-to-one to G-valued functions and that this correspondence respects the respective multiplications (2.11). Such an isomorphism $\boldsymbol{\ell}$ fulfills the requested associativity condition because of $\ker(\partial_1) = 1$. Hence, we have a crossed module bundle gerbe $\mathcal{G}$, which can be seen as an obstruction to a lifting of the principal G bundle E to some principal M-bundle. Also, it is easily seen that lifting two isomorphic G-bundle leads to stably isomorphic $(L \to M)$-bundle gerbes. On the other hand, if we have a crossed module $L \to M$ with a trivial kernel of ∂_1 and hence fitting the exact sequence with $G = \operatorname{coker} \partial_1$ we can change the structure crossed module from $L \to M$ to $1 \to G$ in a crossed module bundle gerbe in order to get a principal G-bundle on X. These two constructions are inverse to each other on sets of stable isomorphism classes of $(L \to M)$-bundle gerbes (with $(L \to M)$ as above) and isomorphism classes of principal G-bundles. $\square$

2.28. Remark. It is also easy to give a local description of lifting crossed module bundle gerbes. Let $\{O_i\}_i$ again be a good covering of X. Let us consider an $(L \to M)$-bundle gerbe described by transition functions (m_{ij}, l_{ijk}). Then $\bar{\pi}(m_{ij})\bar{\pi}(m_{jk}) = \bar{\pi}(m_{ik})$. Hence, we have a principal G-bundle with transition functions $g_{ij} = \bar{\pi}(m_{ij})$. To go in the opposite direction, let g_{ij} be the transition functions of a principal G-bundle. Since the double intersections O_{ij} are contractible we can choose lifts m_{ij} of the transition functions g_{ij}. On O_{ijk} these will be related by $m_{ij}m_{jk} = \partial_1(l_{ijk})m_{ik}$ with L-valued functions l_{ijk} which, because of $\ker \partial_1 = 1$, necessarily satisfy the required compatibility condition on O_{ijkl} (2.16).

2.29. Remark. Note that given three principal G-bundles E, E'' and E''' and isomorphisms $E \xrightarrow{f} E'$, $E' \xrightarrow{f'} E''$ and $E'' \xrightarrow{f''} E'''$ such that $f'f = f''$ the corresponding lifting crossed module bundle gerbes $\mathcal{G}, \mathcal{G}''$ and $\mathcal{G}'''$ will be stably isomorphic, but the respective stable isomorphisms $\boldsymbol{ff'}$ and $\boldsymbol{f''}$ will be only isomorphic in general.

3. 2-Crossed Module Bundle Gerbes

Let $(L \to M \to N)$ be a Lie 2-crossed module and $\mathcal{G}$ be an $(L \to M)$-bundle gerbe over X. From the definition of the 2-crossed module we see immediately that the maps $L \to 1$ and $\partial_2 : M \to N$ define a morphism of crossed modules $\mu : (L \xrightarrow{\partial_1} M) \to (1 \to N)$. Thus, we have the following trivial lemma (2.5):

Lemma 3.1. *$\mu(\mathcal{G})$ is a principal N-bundle on X. If $\mathcal{G}$ and $\mathcal{G}'$ are stably isomorphic, then $\mu(\mathcal{G})$ and $\mu(\mathcal{G}')$ are isomorphic.*

3.1. Definition. Let $\mathcal{G}$ be an $(L \to M)$-bundle gerbe such that the principal bundle $\mu(\mathcal{G})$ over X is trivial with a section $\boldsymbol{n} : \mu(\mathcal{G}) \to N$. We call the pair $(\mathcal{G}, \boldsymbol{n})$ a 2-crossed module bundle gerbe.

3.2. Remark. The following interpretation of the trivializing section $\boldsymbol{n}$ will be useful later. For the $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n}) = (((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ the trivializing section $\boldsymbol{n}$ of the left principal N-bundle $\mu(\mathcal{G})$ is the same thing as an N-valued function $\mathbf{n}$ on Y such that $\partial_2(\boldsymbol{m}) = \mathbf{n}_1\mathbf{n}_2^{-1}$.

3.3. Remark. If we think about the 2-crossed module $L \to M \to N$ as a 2-groupoid with objects in L, 1-arrows in $L \times M$ and 2-arrows in $L \times N \times M$ then $(L \to M \to N)$-bundle gerbes should give an example of the bigroupoid 2-torsors introduced in [3].

3.4. Pullback. If $f : X \to X'$ then we put $f^*(\mathcal{G}, \boldsymbol{n}) = (f^*(\mathcal{G}), f^*\boldsymbol{n})$; this will again be a 2-crossed module bundle gerbe.

3.5. Definition. We call two $(L \to M \to N)$-bundle gerbes $(\mathcal{G}, \boldsymbol{n})$ and $(\mathcal{G}', \boldsymbol{n}')$ over the same manifold X stably isomorphic if there exists a stable isomorphism $\boldsymbol{q} := (\mathcal{Q}, \tilde{\boldsymbol{\ell}}) : \mathcal{G} \to \mathcal{G}'$ of $(L \to M)$-bundle gerbes such that $\boldsymbol{n}'\mu(\boldsymbol{q}) = \boldsymbol{n}$ holds true for the induced isomorphism of trivial bundles $\mu(\boldsymbol{q}) : \mu(\mathcal{G}) \to \mu(\mathcal{G}')$. An $(L \to M \to N)$-bundle gerbe is trivial if it is stably isomorphic to the trivial $(L \to M \to N)$-bundle gerbe $(((Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N)$.

Pullbacks preserve stable isomorphisms, in particular a pullback of a trivial 2-crossed module bundle gerbe is again a trivial 2-crossed module bundle gerbe.

3.6. Example. Note that a general $(L \to M \to N)$-bundle gerbe is not necessarily locally trivial, although it is locally trivial as an $(L \to M)$-bundle gerbe. For a function $n : X \to N$ such that $\mathrm{Im}(n)$ is not a subset of $\mathrm{Im}(\partial_2)$ the $(L \to M \to N)$-bundle gerbe $(((Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N \cdot n\mathrm{pr}_X)$ is locally non-trivial. We will refer to such a 2-crossed module as the 2-crossed module bundle gerbe defined by the N-valued function n on X. Two such 2-crossed module bundle gerbes are stably isomorphic iff their respective functions n and n' are related by an M-valued function m by $n' = \partial_2(m)n$. We will refer to such a stable isomorphism as being defined by the function m. Further, two such stable isomorphisms defined by respective functions m and m' are isomorphic iff they are related by an L-valued function l on X by $m' = \partial_1(l)m$.

3.7. Example. Consider an $(1 \to G \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n})$. As a $(1 \to G)$-bundle gerbe $\mathcal{G}$ gives a principal G-bundle P (more precisely a G-valued function g on $Y^{[2]}$ satisfying the 1-cocycle relation on $Y^{[3]}$). The trivializing section $\boldsymbol{n}$ then gives an N-valued function $\mathbf{n}$ (3.2) on Y such that $\partial_2 g_{12}\mathbf{n}_2 = \mathbf{n}_1$ on $Y^{[2]}$ and hence, a trivialization of the left principal G-bundle P under the map $G \to N$. Hence, a $(1 \to G \to N)$-bundle gerbe is the same thing as a $(G \to N)$-bundle.

Obviously, isomorphic $(G \to N)$-bundles correspond to stably isomorphic $(1 \to G \to N)$-bundle gerbes.

3.8. Remark. Let $\pi' : Y' \to X$ be another surjective submersion and $f : Y' \to Y$ a map such that $\pi' = \pi f$. Then the 2-crossed module bundle gerbes $(f^*\mathcal{G}, \boldsymbol{n})$ and $(\mathcal{G}, \boldsymbol{n})$ are stably isomorphic. This can be shown in a completely analogous way to the case of a crossed module bundle gerbe (2.23).

3.9. Change of the structure 2-crossed module. If $(L \to M \to N) \to (L' \to M' \to N')$ is a morphism of 2-crossed modules, there is an obvious way to construct starting from an $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n})$ and $(L' \to M' \to N')$-bundle gerbe $(\mathcal{G}', \boldsymbol{n}')$ by changing the structure crossed module of $\mathcal{G}$ from $L \to M$ to $L' \to M'$ and putting $\boldsymbol{n}' = \nu \boldsymbol{n}$ where ν is the morphism $\nu : N \to N'$ entering the definition of the morphism of two 2-crossed modules. Obviously, change of the structure 2-crossed module preserves stable isomorphisms of 2-crossed module bundle gerbes.

3.10. Definition. Let $((\mathcal{P}, Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ and $((\mathcal{P}', Y', X, \boldsymbol{\ell}'), \boldsymbol{n}')$ be two 2-crossed module bundle gerbes and $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ two stable isomorphisms between them, see (2.21). We call $(\mathcal{Q}, \tilde{\boldsymbol{\ell}}_{\mathcal{Q}})$ and $(\mathcal{R}, \tilde{\boldsymbol{\ell}}_{\mathcal{R}})$ isomorphic if there is an isomorphism $\underline{\boldsymbol{\ell}} : \mathcal{Q} \to \mathcal{R}$ of crossed module bundles on $\bar{Y} = Y \times{}_X Y'$ such that (with an obvious abuse of notation) the diagram

$$\begin{array}{ccc} q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{Q}}} & \mathcal{Q}_1 q'^* \mathcal{P}' \mathcal{Q}_2^{-1} \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow {\scriptstyle \underline{\boldsymbol{\ell}}_1 \underline{\boldsymbol{\ell}}_2^{-1}} \\ q^*\mathcal{P} & \xrightarrow{\tilde{\boldsymbol{\ell}}_{\mathcal{R}}} & \mathcal{R}_1 q'^* \mathcal{P}' \mathcal{R}_2^{-1} \end{array} \tag{8}$$

is commutative. Obviously, pullbacks preserve isomorphisms of stable isomorphisms.

3.11. 2-cocycles. Let $\pi : Y \to X$ be the surjective submersion, which was implicitly contained in the above-definition of a 2-crossed module bundle gerbe. Since also for 2-crossed module bundle gerbes it holds true that 2-crossed module bundle gerbes $(f^*\mathcal{G}, \boldsymbol{n})$ and $(\mathcal{G}, \boldsymbol{n})$ are stably isomorphic if the respective maps π and π' are related by a compatible map, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one, in which case the $(L \to M \to N)$-bundle gerbe is characterized by transition functions (n_i, m_{ij}, l_{ijk}), $n_i : O_i \to N$, $m_{ij} : O_{ij} \to M$, $l_{ijk} : O_{ijk} \to L$ fulfilling 2-cocycle relations

$$\begin{aligned} n_i &= \partial_2(m_{ij}) n_j, \\ m_{ij} m_{jk} &= \partial_1(l_{ijk}) m_{ik}, \\ l_{ijk} l_{ikl} &= {}^{m_{ij}} l_{jkl} l_{ijl} \end{aligned}$$

on O_{ij}, O_{ijk} and O_{ijkl}, respectively.

In terms of 2-cocycles the stable isomorphism $(l_{ijk}, m_{ij}, n_i) \sim (l'_{ijk}, m'_{ij}, n'_i)$ is expressed by relations

$$n'_i = \partial_2(m_i)n_i, \tag{9}$$

$$m'_{ij} = m_i\partial_1(l_{ij})m_{ij}m_j^{-1}, \tag{10}$$

$${}^{m_i^{-1}}l'_{ijk} = l_{ij}\,{}^{m_{ij}}l_{jk}l_{ijk}l_{ik}^{-1}. \tag{11}$$

Two $(L \to M \to N)$-valued 2-cocycles related as above will be called equivalent. The corresponding set of equivalence classes will be denoted by $H^0(X, L \to M \to N)$.

A trivial 2-crossed module bundle gerbe is described by transition functions

$$n_i = \partial_2(m_i),$$
$$m_{ij} = m_i\partial_1(l_{ij})m_j^{-1},$$

and

$${}^{m_i^{-1}}l_{ijk} = l_{ij}l_{jk}l_{ik}^{-1}.$$

Locally, two collections of stable isomorphism data (m_i, l_{ij}) and (m'_i, l'_{ij}) are isomorphic if

$$m'_i = \partial_1(l_i)m_i,$$
$$l'_{ij} = l_il_{ij}\,{}^{m_{ij}}l_j^{-1}.$$

An $(L \to M \to N)$-bundle gerbe can be reconstructed from transition functions (n_i, m_{ij}, l_{ijk}) in complete analogy with the case of an $(L \to M)$-bundle gerbe. Starting from (10) and (11) we can reconstruct an $(L \to M)$-bundle gerbe $\mathcal{G}$ as in (2.25). Further, the collection of N-valued local functions n_i appearing in (9) defines a trivial principal N-bundle $\mathcal{N}$ on X with transition functions $n_in_j^{-1}$. The relation (9) then guarantees that the principal N-bundle $\mu(\mathcal{G})$ is isomorphic to $\mathcal{N}$. Hence, $\mathcal{G}$ is an $(L \to M \to N)$-bundle.

Further, two 2-crossed module bundle gerbes corresponding to two equivalent 2-cocycles are stably isomorphic. Starting from two equivalent 2-cocycles (9)–(11) we will construct as above the two respective $(L \to M \to N)$-bundle gerbes $\mathcal{G}$ and $\mathcal{G}'$. It follows from (2.25) that $\mathcal{G}$ and $\mathcal{G}'$ will be stably isomorphic as $(L \to M)$-bundle gerbes and because of the relation (9) they will also be stably isomorphic as $(L \to M \to N)$-bundle gerbes.

Hence, the above discussion of 2-cocycles proves the following proposition.

Proposition 3.1. *Stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are in a bijective correspondence with the set $H^0(X, L \to M \to N)$.*

Similarly to the case of crossed module bundles (2.25), we can consider a 2-category of $(L \to M \to N)$-bundle gerbes, with 1-arrows being stable isomorphisms and 2-arrows being isomorphism of stable automorphisms and similarly a 2-category of 2-cocycles, but we will not use these.

3.12. Lifting 2-crossed module bundle gerbe. Consider a Lie 2-crossed module $L \to M \to N$ such that $\ker(\partial_1) = 1$ and $\ker(\partial_2) = \mathrm{Im}(\partial_1)$. Put $G := \frac{M}{L}$ and $Q := \frac{N}{G}$. Assume that the conditions are satisfied for having extensions of Lie groups

$$1 \to L \overset{\partial_1}{\to} M \overset{\partial_2}{\to} N \overset{\pi_2}{\to} Q \to 1, \tag{12}$$

$$1 \to L \overset{\partial_1}{\to} M \overset{\pi_1}{\to} G \to 1 \tag{13}$$

and

$$1 \to G \overset{\partial_2'}{\to} N \overset{\pi_2}{\to} Q \to 1, \tag{14}$$

such that $M \overset{\pi_1}{\to} G$ is an $(L \to M)$-bundle and $N \overset{\pi_1}{\to} Q$ is an $(G \to N)$-bundle (cf. Example 2.7). Also, we have an exact sequence of pre-crossed modules

$$\begin{array}{ccccccccc}
1 & \longrightarrow & L & \xrightarrow{\partial_1} & M & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \\
\big\downarrow & & \big\downarrow & & {\scriptstyle \partial_2}\big\downarrow & & {\scriptstyle \partial_2'}\big\downarrow & & \big\downarrow \\
1 & \longrightarrow & 1 & \longrightarrow & N & \longrightarrow & N & \longrightarrow & 1\,,
\end{array}$$

where G is a normal subgroup of N and also a morphism of 2-crossed modules

$$\begin{array}{ccccc}
L & \xrightarrow{\partial_1} & M & \xrightarrow{\partial_2} & N \\
\big\downarrow & & {\scriptstyle \pi_1}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\
1 & \longrightarrow & G & \xrightarrow{\partial_2'} & N\,.
\end{array}$$

Considering the above extension of Lie groups, we have proved the following proposition.

Proposition 3.2. *Consider an exact sequence* (12) *of Lie groups such that the exact sequences* (13) *and* (14) *define an* $(L \to M)$*-bundle and an* $(G \to N)$*-bundle, respectively.*[c] *Then the stable isomorphism classes of* $(L \to M \to N)$*-bundle gerbes are in bijective correspondence with the isomorphism classes of* $(G \to N)$*-bundles.*

Proof. Let us first note that given a $(G \to N)$-bundle $\mathcal{P} = (P, \tilde{\boldsymbol{n}})$ on X, the left principal G-bundle P can be lifted to an $(L \to M)$-bundle gerbe $\mathcal{G}$ (2.27), which will actually be an $(L \to M \to N)$-bundle gerbe. This is because of the identity $\partial_2'\pi_1 = \partial_2$ the trivialization $\tilde{\boldsymbol{n}}$ of $\mathcal{P}$ under ∂_2' naturally defines a trivialization of the principal N-bundle $\mu(\mathcal{G})$ by putting $\mathbf{n} := \tilde{\boldsymbol{n}}$ (cf. (3.2)). Due to the identification $\mathbf{n} := \tilde{\boldsymbol{n}}$, two isomorphic $(G \to N)$-bundles will lead to two stably isomorphic $(L \to M \to N)$-bundle gerbes. On the other hand, starting with an $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n})$ with the 2-crossed module as above, we can change its structure 2-crossed module to $1 \to G \to N$ in order to obtain a principal $(G \to N)$-bundle

[c] Cf. Example 2.4.

$\mathcal{P}$. The N-valued function $\mathbf{n}$ on Y defined by the trivialization $\boldsymbol{n}$ of $\mu(\mathcal{G})$ will give a trivialization of $\partial_2'(\mathcal{P})$, cf. (3.7). From this it is again easy to see that stably isomorphic $(L \to M \to N)$-bundle gerbes will lead to isomorphic $(G \to N)$-bundles. Lifting an principal $(G \to N)$-bundle to an $(L \to M \to N)$-bundle gerbe followed by the change of structure 2-crossed modules $(L \to M \to N) \to (G \to N)$ will give back the original $(G \to N)$-bundle. $\square$

The local description of the above correspondence is similar to the case of crossed module bundle gerbes (2.28).

Because of (2.7) we also have the following corollary.

Corollary 3.1. *Under the hypothesis of Proposition 3.2, stable isomorphism classes of $(L \to M \to N)$-bundle 2-gerbes are in bijective correspondence with Q-valued functions.*

Concerning the corresponding cocycles we have the following isomorphisms of sets.

Corollary 3.2. *Under the hypothesis of Proposition 3.2, we have*

$$H^0(X, L \to M \to N) \cong H^0(X, G \to N) \cong H^0(X, Q).$$

3.13. Remark. Note that given three $(G \to N)$-bundles $\mathcal{P}, \mathcal{P}'$ and $\mathcal{P}''$ and isomorphisms $\mathcal{P} \xrightarrow{f} \mathcal{P}'$, $\mathcal{P}' \xrightarrow{f'} \mathcal{P}''$ and $\mathcal{P} \xrightarrow{f''} \mathcal{P}''$ such that $f'f = f''$ the corresponding lifting 2-crossed module bundle gerbes will be stably isomorphic, but the respective stable isomorphisms $\boldsymbol{f}\boldsymbol{f}'$ and $\boldsymbol{f}''$ will be only isomorphic in general.

3.14. Remark. Similarly to lifting crossed module bundles (cf. (2.11)), also lifting 2-crossed module bundle gerbes can be interpreted as pullbacks. Starting from exact sequences (13) and (14) we have a lifting $(L \to M)$-bundle gerbe over Q. Due to equality $\partial_2' = \partial_2 \pi_1$ this will actually be an $(L \to M \to N)$-bundle gerbe $\mathfrak{Q}$ over Q. A representative of class of a lifting 2-crossed module bundle gerbe over X corresponding to a Q-valued function $q : X \to Q$ can be obtained as the pullback $q^*(\mathfrak{Q})$.

3.15. Remark. For an $(L \to M \to N)$-bundle gerbe $(\mathcal{G}, \boldsymbol{n}) = (((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ we recall from (3.2) that the trivializing section $\boldsymbol{n}$ of the left principal N-bundle $\mu(\mathcal{G})$ defines an N-valued function $\mathbf{n}$ on Y such that $\partial_2(\boldsymbol{m}) = \mathbf{n}_1\mathbf{n}_2^{-1}$. Let us recall that on the left principal L-bundle P there is a compatible principal right L-action. Using the N-valued function $\mathbf{n}$ we can introduce a further principal right L-action on P, which will again commute with the principal left L-action. We will use the notation $(p, l) \mapsto p \cdot_{\mathbf{n}} l$ for $p \in P$, $l \in L$ for this principal right action of L and put $p \cdot_{\mathbf{n}} l := p^{\mathbf{n}_2(y_1, y_2)} l$, where p lies in the fiber over $(y_1, y_2) \in Y^{[2]}$ and $\mathbf{n}_2$

is the pullback to $Y^{[2]}$ of $\mathbf{n}$ under the projection to the second factor of $Y^{[2]}$. It is easy to check that this formula indeed defines a principal right L-action commuting with the principal left L-action on P.

Let now $(\mathcal{G}, \boldsymbol{n}) = (((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ and $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = (((\tilde{P}, \tilde{\boldsymbol{m}}), Y, X, \tilde{\boldsymbol{\ell}}), \tilde{\boldsymbol{n}})$ be two 2-crossed module bundle gerbes. Let us again consider on $Y^{[2]}$ the Whitney sum $P \oplus \tilde{P}$ and introduce an equivalence relation on $P \oplus \tilde{P}$ by

$$(p \cdot_{\mathbf{n}} l, \tilde{p}) \sim {}_{\mathbf{n}}(p, l\tilde{p}),$$

and define $\bar{P} = P \cdot_{\mathbf{n}} \tilde{P} = \frac{P \oplus \tilde{P}}{\tilde{n}}$. We will denote an element of $P \cdot_{\mathbf{n}} \tilde{P}$ defined by the equivalence class of $(p, \tilde{p}) \in P \oplus \tilde{P}$ as $[\,p, \tilde{p}\,]_{\mathbf{n}}$ in order to distinguish it from equivalence class $[\,p, \tilde{p}\,] \in P\tilde{P}$ defined previously in (2.1). Also, put

$$\bar{\boldsymbol{m}} = \boldsymbol{m}^{\,\mathbf{n}_2}\tilde{\boldsymbol{m}}.$$

It is easy to see that $\bar{\mathcal{P}} := (\bar{P}, \bar{\boldsymbol{m}})$ is an $(L \to M)$-bundle on $Y^{[2]}$. Let us note that also $\partial_2(\bar{\boldsymbol{m}}) = \bar{\mathbf{n}}_1 \bar{\mathbf{n}}_2$ on $Y^{[2]}$ with

$$\bar{\boldsymbol{n}} = \boldsymbol{n}\tilde{\boldsymbol{n}}.$$

Now on $Y^{[3]}$ we do have the pullbacks $\mathcal{P}_{12}$, $\tilde{\mathcal{P}}_{12}$, $\bar{\mathcal{P}}_{12}$, etc. An element of $\bar{P}_{12}\bar{P}_{23}$ is then given by $((y_1, y_2, y_3), [[\,p, \tilde{p}\,]_{\mathbf{n}}, [\,p', \tilde{p}'\,]_{\mathbf{n}}])$ with $(y_1, y_2, y_3) \in Y^{[3]}$, $p \in P$ and $\tilde{p} \in \tilde{P}$ in the respective fibers of P and $\tilde{P}$ over $(y_1, y_2) \in Y^{[2]}$, and $p' \in P$ and $\tilde{p}' \in \tilde{P}$ are in the respective fibers of P and $\tilde{P}$ over $(y_2, y_3) \in Y^{[2]}$. Finally, we define $\bar{\boldsymbol{\ell}} : \bar{P}_{12}\bar{P}_{23} \to \bar{P}_{13}$ as

$$\bar{\boldsymbol{\ell}}((y_1, y_2, y_3), [[\,p, \tilde{p}\,]_{\mathbf{n}}, [\,p', \tilde{p}'\,]_{\mathbf{n}}]) := ((y_1, y_2, y_3), [\boldsymbol{\ell}([\,p, p'\,], \tilde{\boldsymbol{\ell}}[\tilde{p}, \tilde{p}'\,]\,]_{\mathbf{n}}).$$

Now it is a rather lengthy but straightforward check to establish the following proposition.

Proposition 3.3. $(\bar{\mathcal{G}}, \bar{\boldsymbol{n}}) := (((\bar{P}, \bar{\mathbf{m}}), Y, X, \bar{\boldsymbol{\ell}}), \bar{\boldsymbol{n}})$ *defines an* $(L \to M \to N)$-*bundle gerbe, the product of* $(L \to M \to N)$-*bundle gerbes* $(\mathcal{G}, \boldsymbol{n}) = (((P, \boldsymbol{m}), Y, X, \boldsymbol{\ell}), \boldsymbol{n})$ *and* $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = (((\tilde{P}, \tilde{\boldsymbol{m}}), Y, X, \tilde{\boldsymbol{\ell}}), \tilde{\boldsymbol{n}})$.

3.16. Example. If $(\mathcal{G}, \boldsymbol{n}) = (((P = Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N \cdot n\mathrm{pr}_X)$ and $(\tilde{\mathcal{G}}, \tilde{\boldsymbol{n}}) = (((\tilde{P}Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X), \mathrm{pr}_N \cdot \tilde{n}\mathrm{pr}_X)$ are two $(L \to M \to N)$-bundle gerbes defined by two respective N-valued functions n and $\tilde{n}$ on X (3.6) then their product is explicitly described again as an $(L \to M \to N)$-bundle gerbe defined by the function $n\tilde{n}$ by identifying $[(y_1, y_2, l), (y_1, y_2, \tilde{l})] \in PP'$ with $(y_1, y_2, l^{\,n(x)}\tilde{l}) \in (Y^{[2]} \times L)$. Here $(y_1, y_2) \in Y^{[2]}$ live in the fiber over $x \in X$.

3.17. Remark. The above product defines a group structure on stable isomorphism classes of $(L \to M \to N)$-bundle gerbes. The unit is given by the class of the trivial $(L \to M \to N)$-bundle gerbe $(((Y^{[2]} \times L, \partial_1 \mathrm{pr}_L), Y, X, 1), \mathrm{pr}_N)$. We will give an explicit (local) formula for the inverse later. Let us note that the relation between the stable isomorphism classes of lifting $(L \to M \to N)$-bundle gerbes described above (3.12) and Q-valued functions (and stable isomorphism classes of $(G \to N)$-bundle gerbes) is compatible with the respective multiplications.

3.18. Product on 2-cocycles. The product formulas for the corresponding transition functions (2-cocycles) of the product $\bar{\mathcal{G}} = \mathcal{G}\tilde{\mathcal{G}}$ of two 2-crossed module bundles are given by

$$\bar{n}_i = n_i \tilde{n}_i,$$

$$\bar{m}_{ij} = m_{ij}\, {}^{n_j}\tilde{m}_{ij},$$

$$\bar{l}_{ijk} = l_{ijk}\, {}^{m_{ik}}\{m_{jk}{}^{-1}, {}^{n_j}\tilde{m}_{ij}\}\, {}^{n_i}\tilde{l}_{ijk}.$$

The inverse $(n_i, m_{ij}, l_{ijk})^{-1}$ is given by

$$(n_i^{-1}, {}^{n_j^{-1}}m_{ij}^{-1}, {}^{n_k^{-1}}\{m_{jk}^{-1}, m_{ij}^{-1}\}^{-1}\, {}^{n_i^{-1}}l_{ijk}^{-1}).$$

3.19. Remark. Let us forget, for the moment, about the "horizontal" composition in the Gray 3-groupoid corresponding to the 2-crossed module $L \to M \to N$. The two "vertical" compositions define a strict 2-groupoid (a strict topological 2-category), which we will denote as $2\mathcal{C}$. Let us again consider the topological $\mathcal{O}$ category defined by the good covering $\{O_i\}$. A 2-cocycle is the same thing as a continuous, normal pseudo-functor from $\mathcal{O}$ to $2\mathcal{C}$. Now we can use the fact that the horizontal composition in a topological Gray 3-category defines a continuous cubical functor $\mathfrak{F} : 2\mathcal{C} \times 2\mathcal{C} \to 2\mathcal{C}$ from the Cartesian product $2\mathcal{C} \times 2\mathcal{C}$ to $2\mathcal{C}$ [22]. We may use the following property of cubical functors, which follows almost immediately from definition. If $\mathcal{F}$ and $\mathcal{G}$ are two continuous normal pseudo-functors from $\mathcal{O}$ to $2\mathcal{C}$ then $\mathfrak{F}(\mathcal{F}, \mathcal{G})$ is a pseudo-functor from $\mathcal{O}$ to $2\mathcal{C}$. Hence, we obtain a product on 2-cocycles, which is the same as the one given above (3.3).

4. 2-Crossed Module Bundle 2-Gerbes

Consider again a surjective submersion $\pi : Y \to X$. Let, as before, $p_{ij} : Y^{[3]} \to Y^{[2]}$ denote the projection to the ith and jth component, and similarly for projections of higher fibered powers $Y^{[n]}$ of Y. Let $L \overset{\partial_1}{\to} M \overset{\partial_2}{\to} N$ be a 2-crossed module.

4.1. Definition. A 2-crossed module bundle 2-gerbe is defined by a quintuple $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$, where $\mathfrak{G} = (\mathcal{G}, \boldsymbol{n})$ is a 2-crossed module bundle gerbe over $Y^{[2]}$,

$$\boldsymbol{m} : \mathfrak{G}_{12}\mathfrak{G}_{23} \to \mathfrak{G}_{13}$$

is a stable isomorphism on $Y^{[3]}$ of the product $\mathfrak{G}_{12}\mathfrak{G}_{23}$ of the pullback 2-crossed module bundle gerbes $\mathfrak{G}_{12} = p_{12}^*\mathfrak{G}$ and $\mathfrak{G}_{23} = p_{23}^*\mathfrak{G}$ and the pullback 2-crossed module bundle gerbe $\mathfrak{G}_{13} = p_{13}^*\mathfrak{G}$, and

$$\boldsymbol{\ell} : \boldsymbol{m}_{124}\boldsymbol{m}_{234} \to \boldsymbol{m}_{134}\boldsymbol{m}_{123}$$

is an isomorphism of the composition of pullbacks of stable isomorphisms $p_{124}^*\boldsymbol{m}$ and $p_{234}^*\boldsymbol{m}$ and the composition of pullbacks of stable isomorphisms $p_{123}^*\boldsymbol{m}$ and

$p_{134}^*\boldsymbol{m}$ on $Y^{[4]}$. On $Y^{[5]}$, the isomorphism $\boldsymbol{\ell}$ should satisfy the obvious coherence relation

$$\boldsymbol{\ell}_{1345}\boldsymbol{\ell}_{1235} = \boldsymbol{\ell}_{1234}\boldsymbol{\ell}_{1245}\boldsymbol{\ell}_{2345}.$$

4.2. Abelian bundle 2-gerbes. Abelian bundle 2-gerbes as introduced in [49, 50, 19] are $(U(1) \to 1 \to 1)$-bundle 2-gerbes. If $A \to 1 \to 1$ is a 2-crossed module then A is necessarily an abelian group and an abelian bundle 2-gerbe can be identified as an $(A \to 1 \to 1)$-bundle 2-gerbe.

4.3. Example. Consider an $(1 \to G \to N)$-bundle 2-gerbe $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$. The $(1 \to G \to N)$-bundle gerbe on $Y^{[2]}$ gives a $(G \to N)$-bundle $\mathcal{P}$ on $Y^{[2]}$. The stable isomorphism $\boldsymbol{m} : \mathfrak{G}_{12}\mathfrak{G}_{23} \to \mathfrak{G}_{13}$ gives on $Y^{[3]}$ an isomorphism $\boldsymbol{g} : \mathcal{P}_{12}\mathcal{P}_{23} \to \mathcal{P}_{13}$ satisfying on $Y^{[4]}$ the associativity condition $\boldsymbol{g}_{124}\boldsymbol{g}_{234} = \boldsymbol{g}_{134}\boldsymbol{g}_{123}$ since the first Lie group of the 2-crossed module $(1 \to G \to N)$ is trivial. Hence, a $(1 \to G \to N)$-bundle 2-gerbe is the same thing as a $(G \to N)$-bundle gerbe. Obviously, stably isomorphic $(1 \to G \to N)$-bundle 2-gerbes correspond to stably isomorphic $(G \to N)$-bundle gerbes.

4.4. Pullback. If $f : X \to X'$ is a map then we can pullback $Y \to X$ to $f^*(Y) \to X'$ with a map $\tilde{f} : f^*(Y) \to Y$ covering f. There are induced maps $\tilde{f}^{[n]} : f^*(Y)^{[n]} \to Y^{[n]}$. The pullback $f^*(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell}) := (\tilde{f}^{[2]*}\mathfrak{G}, f^*(Y), f(X), \tilde{f}^{[3]*}\boldsymbol{m}, \tilde{f}^{[4]*}\boldsymbol{\ell})$ is again an $(L \to M \to N)$-bundle 2-gerbe.

4.5. Definition. Two 2-crossed module bundle 2-gerbes $((\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ and $(\mathfrak{G}', Y', X, \boldsymbol{m}', \boldsymbol{\ell}'))$ are stably isomorphic if there exists a 2-crossed module bundle gerbe $\mathfrak{Q} \to \bar{Y} = Y \times {}_X Y'$ such that over $\bar{Y}^{[2]}$ the 2-crossed module bundle gerbes $q^*\mathfrak{G}$ and $\mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1}$ are stably isomorphic. Let $\tilde{\boldsymbol{m}}$ be the stable isomorphism $\tilde{\boldsymbol{m}} : q^*\mathfrak{G} \to \mathfrak{Q}_1 q'^*\mathfrak{G}'\mathfrak{Q}_2^{-1}$. Then we ask on $Y^{[3]}$ (with an obvious abuse of notation) for the existence of an isomorphism $\tilde{\boldsymbol{\ell}}$ of stable isomorphisms

$$\tilde{\boldsymbol{\ell}} : \boldsymbol{m}'\tilde{\boldsymbol{m}}_{23}\tilde{\boldsymbol{m}}_{12} \to \tilde{\boldsymbol{m}}_{13}\boldsymbol{m},$$

fulfilling on $Y^{[4]}$

$$\boldsymbol{\ell}_{1234}\tilde{\boldsymbol{\ell}}_{124}\tilde{\boldsymbol{\ell}}_{234} = \tilde{\boldsymbol{\ell}}_{134}\tilde{\boldsymbol{\ell}}_{123}\boldsymbol{\ell}'_{1234}.$$

Here q and q' are projections onto first and second factor of $\bar{Y} = Y \times {}_X Y'$ and $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ are the pullbacks of $\mathfrak{Q}$ to $\bar{Y}^{[2]}$ under respective projections p_1, p_2 from $\bar{Y}^{[2]}$ to $\bar{Y}$, etc.

A 2-crossed module bundle 2-gerbe $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ is called trivial if it is stably isomorphic to the trivial 2-crossed module bundle 2-gerbe $(\mathcal{T}, Y, X, 1, 1)$, where $\mathcal{T}$ is the trivial 2-crossed module bundle gerbe $(((Z^{[2]} \times L, \partial_1 \mathrm{pr}_L), Z, Y^{[2]}, 1), \mathrm{pr}_N)$. Pullbacks preserve stable isomorphisms, a pullback of a trivial 2-crossed module bundle 2-gerbe is again a trivial 2-crossed module bundle 2-gerbe.

If $Y = X$ then the 2-crossed module bundle 2-gerbe is trivial.

4.6. Definition. Let $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ and $(\mathfrak{G}', Y', X, \boldsymbol{m}', \boldsymbol{\ell}')$ be two 2-crossed module bundle 2-gerbes and $(\mathfrak{Q}, \tilde{\boldsymbol{m}}_{\mathfrak{Q}}, \tilde{\boldsymbol{\ell}}_{\mathfrak{Q}})$ and $(\mathfrak{R}, \tilde{\boldsymbol{m}}_{\mathfrak{R}}, \tilde{\boldsymbol{\ell}}_{\mathfrak{R}})$ two stable isomorphisms between them. We call these two stable isomorphisms stably isomorphic if there is a stable isomorphism $\underline{\boldsymbol{m}} : \mathfrak{Q} \to \mathfrak{R}$ of 2-crossed module bundles on $\bar{Y} = Y \times {}_X Y'$ such that (with an obvious abuse of notation) the diagram

$$\begin{array}{ccc} q^*\mathfrak{G} & \xrightarrow{\tilde{\boldsymbol{m}}_{\mathfrak{Q}}} & \mathfrak{Q}_1 q'^* \mathfrak{G}' \mathfrak{Q}_2^{-1} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow {\scriptstyle \underline{\boldsymbol{m}}_1 \underline{\boldsymbol{m}}_2^{-1}} \\ q^*\mathfrak{G} & \xrightarrow{\tilde{\boldsymbol{m}}_{\mathfrak{R}}} & \mathfrak{R}_1 q'^* \mathfrak{G}' \mathfrak{R}_2^{-1}\,, \end{array}$$

commutes up to an isomorphism of stable isomorphisms

$$\underline{\boldsymbol{\ell}} : \tilde{\boldsymbol{m}}_{\mathfrak{Q}} \underline{\boldsymbol{m}}_1 \underline{\boldsymbol{m}}_2^{-1} \to \tilde{\boldsymbol{m}}_{\mathfrak{R}}$$

on $\bar{Y}^{[2]}$, fulfilling on $\bar{Y}^{[3]}$

$$\tilde{\boldsymbol{\ell}}_{\mathfrak{Q}} \underline{\boldsymbol{\ell}}_{13} = \underline{\boldsymbol{\ell}}_{12} \underline{\boldsymbol{\ell}}_{23} \tilde{\boldsymbol{\ell}}_{\mathfrak{R}}.$$

4.7. Remark. Let $\pi' : Y' \to X$ be another surjective submersion and $f : Y' \to Y$ a map such that $\pi' = \pi f$. Then the 2-crossed module bundle 2-gerbes $(f^*\mathfrak{G}, Y', X, f^{[3]*}\boldsymbol{m}, f^{[4]*}\boldsymbol{\ell})$ and $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ are stably isomorphic. It follows that locally each 2-crossed module bundle 2-gerbe is trivial. The arguments to show the above two statements are completely analogous to the case of a crossed module bundle gerbe (3.8).

4.8. Change of the structure 2-crossed module. If $(L \to M \to N) \to (L' \to M' \to N')$ is a morphism of crossed modules, there is an obvious way to construct, starting from an $(L \to M \to N)$-bundle 2-gerbe $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$, an $(L' \to M' \to N')$-bundle 2-gerbe $(\mathfrak{G}', Y, X, \boldsymbol{m}', \boldsymbol{\ell}')$ by changing the structure 2-crossed module of $\mathfrak{G}$ from $(L \to M \to N)$ to $(L' \to M' \to N')$.

4.9. 3-cocycles. Let $\pi : Y \to X$ be the surjective submersion, which was implicitly contained in the above definition of a 2-crossed module bundle 2-gerbe. Let us recall (4.7) that also for 2-crossed module bundle 2-gerbes it holds true that 2-crossed module bundle 2-gerbes $(f^*\mathfrak{G}, Y', X, f^{[3]*}\boldsymbol{m}, f^{[4]*}\boldsymbol{\ell})$ and $(\mathfrak{G}, Y, X, \boldsymbol{m}, \boldsymbol{\ell})$ are stably isomorphic if the respective maps π and π' are related by a compatible map f. Hence, we can again assume $Y = \coprod O_i$. For simplicity, we assume that the covering $\{O_i\}$ is a good one, in which case the $(L \to M \to N)$-bundle gerbe can be described by transition functions $(n_{ij}, m_{ijk}, l_{ijkl}) n_{ij} : O_{ij} \to N$, $m_{ijk} : O_{ijk} \to M$ and $l_{ijkl} : O_{ijkl} \to L$ satisfying

$$\begin{aligned} n_{ij} n_{jk} &= \partial_2(m_{ijk}) n_{ik}, \\ m_{ijk} m_{ikl} &= \partial_1(l_{ijkl})^{\,n_{ij}} m_{jkl} m_{ijl}, \\ l_{ijkl} \;{}^{n_{ij} m_{jkl}}(l_{ijlm})^{n_{ij}} l_{jklm} &= {}^{m_{ijk}} l_{iklm} \{m_{ijk}, {}^{n_{ik}} m_{klm}\}^{n_{ij} n_{jk} m_{klm}} (l_{ijkm}). \end{aligned} \tag{15}$$

We shall not give explicit formulas relating transition functions (3-cocycles) of two stably isomorphic 2-crossed module bundle 2-gerbes. We introduce the notation $H^1(X, L \to M \to N)$ for the equivalence classes of 3-cocycles. We just give the formulas for transition functions $(n_{ij}, m_{ijk}, l_{ijkl})$ of a trivial 2-crossed module bundle 2-gerbe:

$$\begin{aligned} n_{ik} &= n_i^{-1}\partial_2(m_{ij})n_j, \\ {}^{n_i}m_{ijl} &= \partial_1(l_{ijk}^{-1})\, m_{ij}m_{jk}m_{ik}^{-1}, \\ {}^{n_i}l_{ijkl} &= {}^{n_i m_{ijk}}(l_{ikl}^{-1})l_{ijk}^{-1}\, {}^{m_{ij}}l_{jkl}\{m_{ij},\, {}^{n_j}m_{jkl}\}^{n_i n_{ij} m_{jkl}}(l_{ijl}). \end{aligned} \tag{16}$$

We introduce the notation $H^1(X, L \to M \to N)$ for the corresponding equivalence classes of 3-cocycles.

Now we briefly describe how an $(L \to M \to N)$-bundle 2-gerbe can be reconstructed from transition functions $(n_{ij}, m_{ijk}, l_{ijkl})$. This is analogous to the case of an $(L \to M)$-bundle gerbe (2.25). Put $Y = \coprod O_i$. On each nonempty O_{ij} consider the $(L \to M \to N)$-bundle gerbe $\mathfrak{G}_{ij}$ defined by the function $n_{ij} : O_{ij} \to N$ as in (3.6). Hence, on $Y^{[2]}$ we have the $(L \to M \to N)$-bundle gerbe given by $\mathfrak{G} = \coprod_{ij} \mathfrak{G}_{ij}$. Now, we recall the explicit descriptions of the multiplication (3.16) and stable isomorphisms (3.6) of two $(L \to M \to N)$-bundle gerbes defined by their respective N-valued functions. Also, recall the description of isomorphisms of stable isomorphism in case of such $(L \to M \to N)$-bundle gerbes. Using the 3-cocycle relations, it is now straightforward to show that the collection of functions m_{ijk} defines a stable isomorphism of $\mathfrak{G}_{12}\mathfrak{G}_{23}$ and $\mathfrak{G}_{13}$ on $Y^{[3]}$ satisfying on $Y^{[4]}$ the associativity condition up to an isomorphism defined by the collection of functions l_{ijkl}, which fulfills the coherence relation on $Y^{[5]}$. It is now clear that, in a complete analogy to the case of a crossed module bundle gerbe (2.25), starting from two equivalent 3-cocycles, we obtain stably isomorphic 2-crossed module bundle 2-gerbes. This is however a tedious check and we shall omit it.

Hence, we can summarize the discussion in the following proposition.

Proposition 4.1. *Stable isomorphism classes of $(L \to M \to N)$-bundle 2-gerbes are in a bijective correspondence with the set $H^1(X, L \to M \to N)$.*

It might be interesting to examine possible 3-categorical aspects of the above constructions.

4.10. Lifting 2-crossed module bundle 2-gerbe. As before (cf. (3.12)), consider a Lie 2-crossed module $L \to M \to N$ such that $\ker(\partial_1) = 1$ and $\ker(\partial_2) = \mathrm{Im}(\partial_1)$. Put $G := \frac{M}{L}$ and $Q := \frac{N}{G}$. Assume that the conditions are satisfied for having extensions of Lie groups

$$1 \to L \overset{\partial_1}{\to} M \overset{\partial_2}{\to} N \overset{\pi_2}{\to} Q \to 1, \tag{17}$$

$$1 \to L \overset{\partial_1}{\to} M \overset{\pi_1}{\to} G \to 1 \tag{18}$$

and

$$1 \to G \overset{\partial_2'}{\to} N \overset{\pi_2}{\to} Q \to 1, \tag{19}$$

such that $M \overset{\pi_1}{\to} G$ is an $(L \to M)$-bundle and $N \overset{\pi_1}{\to} Q$ is an $(G \to N)$-bundle (cf. Example 2.7). Also, we have an exact sequence of pre-crossed modules

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & L \arrow[r, "\partial_1"] \arrow[d] & M \arrow[r, "\pi_1"] \arrow[d, "\partial_2"'] & G \arrow[r] \arrow[d, "\partial_2'"'] & 1 \arrow[d] \\
1 \arrow[r] & 1 \arrow[r] & N \arrow[r] & N \arrow[r] & 1\,,
\end{tikzcd}$$

where G is a normal subgroup of N and also a morphism of 2-crossed modules

$$\begin{tikzcd}
L \arrow[r, "\partial_1"] \arrow[d] & M \arrow[r, "\partial_2"] \arrow[d, "\pi_1"'] & N \arrow[d, "\mathrm{id}"'] \\
1 \arrow[r] & G \arrow[r, "\partial_2'"] & N\,.
\end{tikzcd}$$

Proposition 4.2. *Consider an exact sequence (17) of Lie groups such that the exact sequences (18) and (19) define an $(L \to M)$-bundle and an $(G \to N)$-bundle, respectively.*[d] *Then the stable isomorphism classes of $(L \to M \to N)$-bundle 2-gerbes are in bijective correspondence with the isomorphism classes of Q-bundles and hence also with $(G \to N)$-bundle gerbes.*

Proof. Recall that in accordance with (3.14) from the 3-term exact sequence $1 \to L \overset{\partial_1}{\to} M \overset{\pi_1}{\to} G \to 1$ and the right principal $(G \to N)$-bundle $N \to Q$ (given by the 3-term exact sequence $1 \to G \overset{\partial_2'}{\to} N \overset{\pi_2}{\to} Q \to 1$) we can construct a lifting $(L \to M)$-bundle gerbe on Q. This lifting bundle gerbe will actually be an $(L \to M \to N)$-bundle gerbe $\mathfrak{G}$ (3.12). If P is now a left principal Q-bundle over X then we can use the corresponding division map $d : P^{[2]} \to G$ to pull-back the 2-crossed module gerbe $\mathfrak{G}$ from G to $P^{[2]}$. It follows that the 2-crossed module bundle gerbes $\mathfrak{G}_{12}\mathfrak{G}_{23}$ and $\mathfrak{G}_{13}$ are stably isomorphic on $P^{[3]}$. This follows from the above-mentioned fact that, in case of Lie groups L, M, N, and Q as above, stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are one-to-one to Q-valued functions (3.1) and that this correspondence respects the respective multiplications. Such a stable isomorphism in general fulfills on $P^{[4]}$ the associativity condition only up to an isomorphism, which however, due to $\ker(\partial_1) = 1$, will fulfill the requested coherence condition on $P^{[5]}$. Hence, we have obtained 2-crossed module bundle 2-gerbe, the so-called lifting 2-crossed module bundle 2-gerbe. Starting from an isomorphic principal Q-bundle P' we obtain a stably isomorphic $(L \to M \to N)$-bundle 2-gerbes. This follows from the fact that on $\bar{P}^{[2]}$, where $\bar{P} := P \times_X P'$ the pullbacks of respective division functions d and d'

[d] Cf. Example 2.4.

are related by $d(\bar{p}_1, \bar{p}_2) = \bar{d}(\bar{p}_1)d'(\bar{p}_1, \bar{p}_2)\bar{d}(\bar{p}_2)^{-1}$ with some Q-valued function $\bar{d}$ on $\bar{P}$. To finish the argumentation, we refer again to the 1-1 relation between stable isomorphism classes of $(L \to M \to N)$-bundle gerbes to Q-valued functions (3.1) and the fact that this respects the respective multiplication.

Going in the opposite direction, let us consider an $(L \to M \to N)$-bundle 2-gerbe $(\mathfrak{G}, X, Y, \boldsymbol{m}, \boldsymbol{\ell})$ with the 2-crossed module $(L \to M \to N)$ as above. Changing the structure 2-crossed module to $1 \to G \to N$, we obtain a $(G \to N)$-bundle gerbe $(\mathcal{G}, n)$ on X. After changing its structure crossed module to $1 \to Q$ we obtain a left principal Q-bundle on X. Since all steps in the construction preserve the respective stable isomorphisms and isomorphisms, starting from stably isomorphic 2-crossed module bundle 2-gerbes we will obtain isomorphic Q-bundles.

It is a rather tedious task to check that starting from a principal Q-bundle, constructing the lifting 2-crossed module bundle 2-gerbe and going back will result in the same principal Q-bundle. $\square$

Corollary 4.1. *Under the hypothesis of the above proposition,*

$$H^1(X, L \to M \to N) \cong H^1(X, G \to N) \cong H^1(X, Q).$$

4.11. Remark. We can also reinterpret the above-described lifting 2-crossed module bundle 2-gerbe as follows. We start again with a principal Q-bundle P as above. Let us consider the corresponding lifting $(G \to N)$-bundle gerbe $\mathfrak{P}$. This in particular means that on $P^{[2]}$ we have a $(G \to N)$-bundle $\mathcal{P}$ which can be lifted to an $(L \to M \to N)$-bundle gerbe $\mathfrak{G}$ on $P^{[2]}$ (3.12). It follows that the 2-crossed module bundle gerbes $\mathfrak{G}_{12}\mathfrak{G}_{23}$ and $\mathfrak{G}_{13}$ are stably isomorphic with a stable isomorphism $\boldsymbol{m}$. This follows, again, from the above-mentioned fact that in case of Lie groups L, M, N and Q as above stable isomorphism classes of $(L \to M \to N)$-bundle gerbes are one-to-one with Q-valued functions and that this correspondence respects the respective multiplications. Again, such a stable isomorphism $\boldsymbol{m}$ fulfills the associativity condition on $P^{[4]}$ only up to an isomorphism fulfilling the coherence relation on $P^{[5]}$ because of $\ker \partial_1 = 1$. This way we obtain an 2-crossed module bundle 2-gerbe stably isomorphic to the lifting 2-crossed module bundle 2-gerbe (4.10).

4.12. Twisting crossed module bundle gerbes with abelian bundle 2-gerbes. Twisted crossed module bundle gerbes as discussed here were introduced in [2]. A more general concept of twisting has been introduced recently in [47].

Let us consider a 2-crossed module $A \to L \xrightarrow{\delta} M$ associated to the crossed module $L \to M$ (1.5). Recall that in this case A is necessarily abelian. Putting $Q := \operatorname{coker} \delta$ we have an exact sequence

$$0 \to A \xrightarrow{\partial} L \xrightarrow{\delta} M \to Q \to 1.$$

Assume, similarly to (4.10), that extensions of Lie groups

$$1 \to A \xrightarrow{\partial} L \xrightarrow{\pi_1} G \to 1 \tag{20}$$

and

$$1 \to G \xrightarrow{\delta'} M \xrightarrow{\pi_2} Q \to 1, \tag{21}$$

define an $(A \to L)$-bundle and $(G \to M)$-bundle, respectively (cf. (2.7)). However, now we have an exact sequence of crossed modules

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \xrightarrow{\partial} & L & \xrightarrow{\pi_1} & G & \longrightarrow & 1 \\ \big\downarrow & & \big\downarrow & & \delta\big\downarrow & & \delta'\big\downarrow & & \big\downarrow \\ 1 & \longrightarrow & 1 & \longrightarrow & M & \longrightarrow & M & \longrightarrow & 1\,. \end{array}$$

As before, we have a morphism of 2-crossed modules

$$\begin{array}{ccccc} A & \xrightarrow{\partial} & L & \xrightarrow{\delta} & M \\ \big\downarrow & & \pi_1\big\downarrow & & \mathrm{id}\big\downarrow \\ 1 & \longrightarrow & G & \xrightarrow{\delta'} & M\ . \end{array}$$

Hence, starting from a principal Q-bundle P we can construct a lifting $(A \to L \xrightarrow{\delta} M)$-bundle 2-gerbe (recall that according to (4.2) there is a 1-1 correspondence between stable isomorphism classes of $(A \to L \xrightarrow{\delta} M)$-bundle 2-gerbes and isomorphism classes of principal Q-bundles). Let us further assume that what we have here is a central extension of L by A, and that M acts trivially on A. Let us assume that the lifting bundle 2-gerbe G) is described locally, with respect to a good covering, by a 3-cocycle $(m_{ij}, l_{ijk}, a_{ijkl})$

$$\begin{aligned} m_{ij}m_{jk} &= \delta(l_{ijk})m_{ik}, \\ l_{ijk}l_{ikl} &= \partial(a_{ijkl})\,{}^{m_{ij}}l_{jkl}l_{ijl}, \\ a_{ijkl}a_{ijlm}a_{jklm} &= a_{iklm}a_{ijkm}. \end{aligned} \tag{22}$$

The collection of A-valued functions a_{ijkl} on the quadruple intersections represents a Čech class in $H^3(X, A)$ (which in the case $A = U(1)$ would correspond to a class in $H^4(X, \mathbb{Z})$). We may think of it as representing an abelian bundle 2-gerbe A. If we assume that A is trivial, we have

$$a_{ijkl} = \tilde{a}_{ijk}\tilde{a}_{ikl}\tilde{a}^{-1}_{jkl}\tilde{a}^{-1}_{ijl}.$$

Also, we see that we have a 2-cocycle $(l_{ijk}\partial(a_{ijk})^{-1}, m_{ij})$ representing a possibly non-trivial $(L \to M)$-bundle gerbe $\mathcal{G}$. Obviously, the $(A \to 1 \to 1)$-bundle 2-gerbes represented by non-trivial classes in $H^3(X, A)$ represent obstructions to lift a $(G \to M)$-bundle gerbe (and hence also a Q-bundle) to an $(L \to M)$-bundle gerbe. Further, if $\tilde{a}_{ijk}$ and $\tilde{a}'_{ijk}$ represent two trivializations of a_{ijkl} then $\tilde{a}_{ijk}(\tilde{a}'_{ijk})^{-1}$ represents a Čech class in $H^2(X, A)$. We may think of it as representing an abelian bundle gerbe, i.e. the $(A \to 1)$-bundle gerbe, $\mathcal{A}$. We can summarize the above discussion in the following proposition.

Proposition 4.3. *Let $A \to L \xrightarrow{\delta} M$ be a 2-crossed module originating from the crossed module $L \xrightarrow{\delta} M$*[e] *such that the extensions of Lie groups (20) and (21) define an $(A \to L)$-bundle and $(G \to M)$-bundle, respectively.*[f] *Let us also assume that (20) is a central extension of L by A and that M acts trivially on A:*

(i) *A principal Q-bundle on X can be lifted to an $(L \to M)$-bundle gerbe if and only if the corresponding obstruction $(A \to 1 \to 1)$-bundle 2-gerbe* A *is trivial.*
(ii) *If nonempty, the set of stable isomorphism classes of those $(L \to M)$-bundle gerbes, which are liftings of Q-principal bundles from the same isomorphism class, is freely and transitively acted on by the group of stable isomorphism classes of $(A \to 1)$-bundle gerbes.*

Corollary 4.2. *Under the assumptions of Proposition 4.3, there is an exact sequence*

$$H^1(X, A \to 1 \to 1) \to H^1(X, L \to M) \to H^1(X, Q).$$

The above proposition and corollary remain true also in cases when the principal Q-bundles and their isomorphism classes are replaced by $(G \to M)$-bundle gerbes and their stable isomorphism classes (cf. 2.27).

4.13. Remark. Of course, the above lifting always exists when the 4-term exact sequence $1 \to A \to L \to M \to Q \to 1$ corresponds to a trivial class in $H^3(Q, A)$ [36, 11], the third Q-cohomology with values in A. The above lifting also trivially exists when X does not admit non-trivial $(A \to 1 \to 1)$-bundle 2-gerbe, i.e. when $[X, B^2A]$ is trivial.

4.14. A remark on string structures. Let Q be a simply-connected compact simple Lie group. Associated to Q there is a crossed module $L \to M$ of infinite-dimensional Fréchet Lie groups with $L := \widehat{\Omega Q}$ and $M := P_0Q$, where $\widehat{\Omega Q}$ is the centrally extended group of based smooth loops in Q and P_0Q is the group of smooth paths in Q that start at the identity [4]. Hence in the notation of (4.12) we have $A = U(1)$, and $G = \Omega Q$. Let us note (see [51, 4, 28]) that, in the situation as above (4.12), the classifying space $BU(1) = K(\mathbb{Z}, 2)$ can be equipped with a proper group structure and a topological group $String(Q)$ can be defined fitting an exact sequence of groups $1 \to K(\mathbb{Z}, 2) \to String(Q) \to Q \to 1$. Also, it is known [30, 5] that the categories of $(L \to M)$-bundle gerbes and principal $String(Q)$-bundles are equivalent. A string structure is, by definition, a lift of a principal Q-bundle to a principal $String(Q)$-bundle and hence equivalently a lift of a $(G \to M)$-bundle

[e] Cf. Example 1.5.
[f] Cf. Example 2.4.

gerbe to an $(L \to M)$-bundle gerbe. Thus, the above discussion applies to the existence of string structures and their classification as well.

4.15. Remark. A crossed square $(L \to A) \to (B \to N)$ [35] of Lie groups gives a 2-crossed module, namely $L \to A \rtimes B \to N$ (see, e.g. [42]). A definition of a crossed square bundle 2-gerbe could possibly be read from [8, 10, 9]. It would be interesting to compare these bundle 2-gerbes with $L \to A \rtimes B \to N$-bundle 2-gerbes defined in this paper.

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CROSSED MODULE BUNDLE GERBES; CLASSIFICATION, STRING GROUP AND DIFFERENTIAL GEOMETRY

BRANISLAV JURČO

Max Planck Institute for Mathematics
Vivatsgasse 7, 53111 Bonn, Germany

and

Mathematical Institute, Charles University
Sokolovská 83, 186 73 Prague, Czech Republic

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We discuss nonabelian bundle gerbes and their differential geometry using simplicial methods. Associated to a (Lie) crossed module $(H \to D)$ there is a simplicial group $N\mathcal{C}_{(H\to D)}$, the nerve of the groupoid $\mathcal{C}_{(H\to D)}$ defined by the crossed module, and its geometric realization, the topological group $|N\mathcal{C}_{(H\to D)}|$. We introduce crossed module bundle gerbes so that their (stable) equivalence classes are in a bijection with equivalence classes of principal $|N\mathcal{C}_{(H\to D)}|$-bundles. We discuss the string group and string structures from this point of view. Also, we give a simplicial interpretation to the bundle gerbe connection and bundle gerbe B-field.

Keywords: Nonabelian gerbes; bundle gerbes; B-field.

0. Introduction

Nonabelian gerbes arose in the realms of nonabelian cohomology [1, 2] and higher category theory [3]. Their differential geometry was described thoroughly by Breen and Messing [4] from the algebraic geometry point of view (see also [5], for a combinatorial description). In [6], nonabelian bundle gerbes, generalizing the nice concept of an abelian bundle gerbe [7], were introduced. The nonabelian bundle gerbes have to be shown (along with their connections and curvings) very natural objects from the point of view of classical fiber bundle theory. There is hope that in this form gerbes can be of some use in physics (see, e.g. examples of higher Yang–Mills theories [8] and anomaly cancelation of M5-branes [9]). Closely related to crossed modules bundle gerbes are two bundles introduced in [10] and discussed together with their connections and curvings in [11]. For an independent approach, see [17].

The purpose of this paper is to introduce bundle gerbes associated with crossed modules and discuss their classification. We refer to these bundle gerbes as crossed module bundle gerbes. In the terminology of [6], the crossed module bundle gerbes can be understood as bundle gerbes equipped with modules. We relate the crossed module bundle gerbes to simplicial principal bundles and interpret their differential geometry in simplicial terms.

The first section is devoted to simplicial principal bundles. We describe them as twisted Cartesian products following [18]. Also, we recall the construction of the universal simplicial bundle.

Connections on simplicial bundles are introduced in Sec. 2. This is done in a straightforward way, which we believe, is the relevant one for our purposes. Next, we shortly discuss the corresponding curvature.

Our task in Sec. 3 is to define a simplicial B-field.

In Sec. 4, we describe some simplicial constructions related to a crossed module $(H \to D)$. We can view a crossed module as a one-groupoid $\mathcal{C}_{(H\to D)}$ or as a strict two-group $\tilde{\mathcal{C}}_{(H\to D)}$. We can form the corresponding nerves $N\mathcal{C}_{(H\to D)}$ and $N\tilde{\mathcal{C}}_{(H\to D)}$, respectively. The geometric realization $|N\mathcal{C}_{(H\to D)}|$ is the classifying space of H-principal bundles with a chosen trivialization under the change in the structure group from H to D. If H and D are Lie groups, $\mathcal{C}_{(H\to D)}$ is a simplicial Lie group and its geometric realization $|N\mathcal{C}_{(H\to D)}|$ is a topological group. String group of [12, 13] is an example. We remark on how the construction of [12] relates to the one of Stolz and Teichner [14].

Crossed module bundle gerbes, also referred to as $(H \to D)$-bundle gerbes, are introduced in Sec. 5. The geometric realization $|N\tilde{\mathcal{C}}_{(H\to D)}|$ of the nerve of the two-group $\tilde{\mathcal{C}}_{(H\to D)}$ corresponding to the crossed module $(H \to D)$ is identified as the classifying space of $(H \to D)$-bundle gerbes. Stable equivalence classes of $(H \to D)$-crossed module bundle gerbes are one-to-one with equivalence classes of principal bundles with structure group $|N\mathcal{C}_{(H\to D)}|$. In particular, string structures [14] can be described equivalently in terms of crossed module bundle gerbes. Locally, crossed module bundle gerbes can be described using simplicial maps from the nerve of an open covering $\{O_\alpha\}_\alpha$ of the manifold X to the nerve of the two-group $|\mathcal{C}_{(H\to D)}|$, which can also be identified as the classifying space $\overline{W}N\mathcal{C}_{(H\to D)}$ of principal $N\mathcal{C}_{(H\to D)}$-bundles. Here we have to mention closely related work of Stevenson [19].

In the last section, we describe how the simplicial connection and B-field (introduced in Secs. 2 and 3) relate to the bundle gerbe connection and the bundle gerbe B-field. Here as well as in Secs. 2 and 3 we work in the category of manifolds. However, as Baez pointed out it might be more appropriate to work in the category of "smooth spaces" studied in the Appendix of [11].

Finally, we should again mention work of Breen [2], where the group $|N\mathcal{C}_{(H\to D)}|$ and the classifying space $\overline{W}N\mathcal{C}_{(H\to D)}$ are discussed. I thank Stevenson for pointing out this to me. Also, I am very much indebted to Henriques for help with Secs. 4 and 5.

This paper is a (slightly) revised and updated version of the preprint arXiv.math/05110078. Some results of this paper we generalized to the case of two-crossed module bundle two-gerbes in [20]. I am thankful to the referee of [20] for encouraging me, in his report concerning [20], to try to publish the present paper.

1. Simplicial Principal Bundles

We start by recalling some relevant properties of simplicial principal bundles following mainly [18]. Let $\pi : P \to X$ be a simplicial (left) principal G-bundle, with P and X simplicial sets and G a simplicial group. We will denote by ∂_i and s_i the corresponding face and degeneracy maps. In the rest of the paper we always assume, without spelling it out explicitly, $P \to X$ to possess a pseudo-cross-section $\sigma : X \to P$ such that $\pi\sigma = id_X$, $\partial_i\sigma = \sigma\partial_i$ if $i > 0$ and $s_i\sigma = \sigma s_i$ if $i \geq 0$. Associated with a pseudo-cross-section σ we have the twisting function $\tau : X_n \to G_{n-1}$ defined by

$$\partial_0\sigma(x) = \tau(x) \cdot \sigma(\partial_0 x).$$

We will use the following description of G-bundles which we alternatively can use as a definition.

1.1. *Twistings*

To make this section self-contained we have to describe the twisting first. For a function $\tau : X_n \to G_{n-1}$ to be a twisting the following conditions should be fulfilled:

$$\begin{aligned} \partial_0\tau(x) &= \tau(\partial_1 x)(\tau(\partial_0 x))^{-1}, \\ \partial_i\tau(x) &= \tau(\partial_{i+1}x) \quad \text{for } i \geq 0, \\ s_i\tau(x) &= \tau(s_{i+1}x) \quad \text{for } i \geq 0, \\ \tau(s_0 x) &= e_n \quad \text{for } x \in X_n. \end{aligned}$$

1.2. *Principal bundles as twisted Cartesian products*

A principal G-bundle $p : P \to X$ with a pseudo-cross-section can be identified with the simplicial set $P(\tau) = G \times_\tau X$, with simplices

$$P(\tau)_n = G_n \times X_n$$

and with the following face and degeneracy maps

$$\begin{aligned} \partial_i(g, x) &= (\partial_i g, \partial_i x) \quad \text{for } i \geq 0, \\ \partial_0(g, x) &= (\partial_0 g \cdot \tau(x), \partial_0 x), \\ s_i(g, x) &= (s_i g, s_i x) \quad \text{for } i \geq 0. \end{aligned}$$

There is a canonical choice for the pseudo-cross-section $\sigma(x) = (e_n, x)$, $x \in X_n$ and e_n the identity in G_n.

Equivalence of two G-bundles $P(\tau)$ and $P(\tau')$ over the same X is described in terms of twisting as follows.

1.3. *Equivalence of principal bundles*

We call two twistings τ' and τ equivalent if there exists a map $\psi : X \to G$ such that

$$\begin{aligned} \partial_0\psi(x) \cdot \tau'(x) &= \tau(x) \cdot \psi(\partial_0 x), \\ \partial_i\psi(x) &= \psi(\partial_i x) \quad \text{if } i > 0, \\ s_i\psi(x) &= \psi(s_i x) \quad \text{if } i \geq 0. \end{aligned}$$

It will be convenient to introduce the equivariant map $\bar\sigma : P \to G$, $\bar\sigma(gp) = g \cdot \bar\sigma(p)$, by the equation $p = \bar\sigma(p)\sigma(x)$. In the rest we will always assume the canonical choice of the pseudo-cross-section has been made, in which case $\bar\sigma(g_n, g_{n-1}, \ldots, g_0) = g_n$. We have

$$\partial_0\bar\sigma(p) = \bar\sigma(\partial_0 p)\tau(x)^{-1}.$$

As with ordinary bundles, simplicial bundles can be pulled back and their structure groups can be changed using simplicial group homomorphisms. Pseudo-cross-sections and twistings transform under these operations in an obvious way.

1.4. *Universal G-bundle*

There is a canonical choice of the classifying space of G-bundles denoted as $\overline{W}G$ and constructed as follows. $\overline{W}G_0$ has one element $*$ and $\overline{W}G_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0$ for $n > 0$. Face and degeneracy maps are

$$s_0(*) = (e_0), \quad \partial_i(g_0) = * \quad \text{for } i = 0 \text{ or } 1$$

and

$$\begin{aligned} \partial_0(g_n, \ldots, g_0) &= (g_{n-1}, \ldots, g_0), \\ \partial_{i+1}(g_n, \ldots, g_0) &= (\partial_i g_n, \ldots, \partial_1 g_{n-i+1},\ \partial_0 g_{n-i} \cdot g_{n-i-1},\ g_{n-i-2}, \ldots, g_0), \\ s_0(g_{n-1}, \ldots, g_0) &= (e_n, g_{n-1}, \ldots, g_0), \\ s_{i+1}(g_{n-1}, \ldots, g_0) &= (s_i, g_n, \ldots, s_0 g_{n-i}, e_{n-i},\ g_{n-i-1}, \ldots, g_0), \end{aligned}$$

if $n > 0$. With the choice of a twisting given by

$$\tau(g_{n-1}, \ldots, g_0) = g_{n-1}$$

we have the universal G-principal bundle

$$WG = G \times_\tau \overline{W}G.$$

As with ordinary bundles, we have that WG is contractible and is universal in the following sense.

Theorem 1.1. *Let us assign to any simplicial map*

$$f : X \to \overline{W}G$$

the induced bundle $f^*(WG) \to X$. This defines a one-to-one correspondence between homotopy classes of maps $[X, \overline{W}G]$ and the equivalence classes of principal G-bundles over the base X.

1.5. *Remark*

The proof of the above theorem is based on the following observation:

The principal G-bundle $G \times_\tau X$ corresponding to the twisting τ is obtained as a pullback under the map $f : X \to \overline{W}G$ given by

$$x \mapsto (\tau(x), \tau(\partial_0 x), \ldots, \tau(\partial_0^i x), \ldots, \tau(\partial_0^n x)).$$

It follows (cf. also [21]).

Proposition 1.1. *Homotopy classes of twistings are one-to-one to equivalence classes of twistings.*

2. Simplicial Connection, Curvature

In this section we introduce the notion of a connection on a simplicial bundle. Of course, now we have to assume that G is a simplicial Lie group and P and X are simplicial manifolds. Also, all maps and actions are assumed to be smooth. For a simplicial manifold Y, we will use the notation $\Omega^k(Y) \otimes \mathrm{Lie}(G)$ for the collection, for all n, of all $\mathrm{Lie}(G_n)$-valued k-forms on Y_n. Here, $\mathrm{Lie}(G)$ is the corresponding simplicial Lie algebra $\mathrm{Lie}(G)_n := \mathrm{Lie}(G_n)$ with the induced face and degeneracy maps. For purposes of this paper the following definition of a simplicial connection seems to be adequate.

2.1. *Definition*

Let $\mathcal{A} \in \Omega^1(P) \otimes \mathrm{Lie}(G)$ be a collection of one-forms $\mathcal{A}_n \in \Omega^1(P_n) \otimes \mathrm{Lie}(G_n)$. We call $\mathcal{A}$ a connection on the simplicial principal G-bundle $P \to X$ if it fulfills the following conditions:

(i)
$$\partial_i^* \mathcal{A} = \partial_i \mathcal{A} \quad \text{and} \quad s_i^* \mathcal{A} = s_i \mathcal{A}$$

where ∂_i^* on the left is the pullback of the face map acting on the one-form part of $\mathcal{A}$ and $\partial_i \mathcal{A}$ on the right is the simplicial Lie algebra face map acting on the simplicial Lie algebra part of $\mathcal{A}$ and similarly for degeneracies

(ii) $\mathcal{A}$ is equivariant with respect to the left G-action on P

$$g^* \mathcal{A} = g \mathcal{A} g^{-1}$$

and

(iii) its pullback to the fiber under $\bar{\sigma} : P \to G$ is the Cartan–Maurer form gdg^{-1}, i.e. the collection of elements $g_n dg_n^{-1} \in \Omega^1(G_n) \otimes \mathrm{Lie}(G_n)$.

2.2. *Local connection forms*

Let us consider a collection of one-forms $A \in \Omega^1(X) \otimes \mathrm{Lie}(G)$ with the property

$$\partial_0 A = \tau \partial_0^* A \tau^{-1} + \tau d\tau^{-1}, \quad \partial_i^* A = \partial_i A \quad \text{for } i > 0$$

and

$$s_i^* A = s_i A \quad \text{for } i \geq 0.$$

We call such an A a local connection.

The following proposition is obvious.

Proposition 2.1. *Any connection $\mathcal{A}$ is of the form*

$$\mathcal{A} = \bar{\sigma} A \bar{\sigma}^{-1} + \bar{\sigma} d\bar{\sigma}^{-1}$$

with

$$A = \sigma^* \mathcal{A}.$$

Pullbacks and change of the structure group work as usual.

2.3. *Curvature*

Curvature is defined exactly in the same way as in the case of ordinary bundles. It is a collection of two-forms $\mathcal{F} \in \Omega^2(P) \otimes \mathrm{Lie}(G)$ defined as $\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ and it has the following properties:

(i) $\partial_i^* \mathcal{F} = \partial_i \mathcal{F}$ and $s_i^* \mathcal{F} = s_i \mathcal{F}$

(ii) $\mathcal{F}$ is equivariant with respect to the left G-action on P

$$g^* \mathcal{F} = g \mathcal{F} g^{-1}$$

and

(iii) $\mathcal{F}$ is of the form $\mathcal{F} = \bar{\sigma} F \bar{\sigma}^{-1}$ with $F \in \Omega^2(X) \otimes \mathrm{Lie}(G)$, i.e. it is horizontal. Obviously, $F = dA + A \wedge A$. Notice that

$$\partial_0 F = \tau \partial_0^* F \tau^{-1} \quad \text{and} \quad \partial_i F = \partial_i^* F \quad \text{for } i > 0$$

and

$$s_i F = s_i^* F \quad \text{for } i \geq 0.$$

3. $\bar{B}$-Field

Let

$$\bar{G}_0 = 1,$$

$$\bar{G}_n = \ker \partial_1 \cdots \partial_n \subset G_n.$$

Let us note that $\partial_0 \bar{G}_{n+1}$ is a normal subgroup in G_n. Also, $\partial_i \bar{G}_{n+1} \subset \bar{G}_n$ for $i > 0$. From now on we will assume that there exists an action of G_n on $\bar{G}_{n+1}$; $g_n \times \bar{g}_{n+1} \mapsto {}^{g_n}\bar{g}_{n+1}$ such that

$$\partial_0({}^{g_n}\bar{g}_{n+1}) = g_n \partial_0(\bar{g}_{n+1}) g_n^{-1}$$

and

$${}^{\partial_0 \bar{g}_{n+1}}\bar{g}'_{n+1} = \bar{g}_{n+1} \bar{g}'_{n+1} \bar{g}_{n+1}^{-1}.$$

These conditions will be automatically satisfied in the next sections when we consider simplicial groups originating from crossed modules.

3.1. *Definition*

$\bar{B}$-field is a collection of two-forms $\bar{B}_{n+1} \in \Omega^2(X_n) \otimes (\bar{G}_{n+1})$ such that

$$^{\tau}\partial_0^* \bar{B} = \partial_1 \bar{B}$$

and

$$\partial_i^* \bar{B} = \partial_{i+1} \bar{B} \quad \text{for } i > 0$$

and

$$s_i^* \bar{B} = s_{i+1} \bar{B} \quad \text{for } i \geq 0.$$

Finally we introduce collection of two-forms $\nu \in \Omega^2(X) \otimes \mathrm{Lie}(G)$ as

$$\nu_n = F_n + \partial_0 \bar{B}_{n+1}.$$

Obviously, ν has the same properties with respect to face and degeneration maps as F.

3.2. *Remark*

Of course, there is no reason to stop with connection A and B-field here. One can introduce C-field etc. *ad infinitum*. We will however not do so here as we are really interested only in simplicial groups which are algebraic models of homotopy two-type (crossed modules).

4. Crossed Modules

4.1. *Definition*

Let H and D be two Lie groups. We say that H is a crossed D-module if there is a Lie group homomorphism $\alpha : H \to D$ and an action of D on H denoted by $(d, h) \mapsto {}^d h$ such that

$$^{\alpha(h)}h' = hh'h^{-1} \quad \text{for } h, h' \in H$$

and

$$\alpha({}^d h) = d\alpha(h)d^{-1} \quad \text{for } h \in H, \quad d \in D$$

holds true.

We will use the following notation $(H \to D)$ for a crossed module. If the groups are infinite-dimensional we will assume that these are Frechét Lie groups.

There are two canonical categorical construction associated with any crossed module.

4.2. *Crossed module as a one-groupoid*

Let us denote $\mathcal{C}_{(H\to D)}$ the (topological) groupoid with objects being group elements $d \in D$ and morphisms (one-arrows) group elements (h, d) of the semidirect product $H \rtimes D$.

As with any groupoid (more generally category), we can form the simplicial space, the nerve $N\mathcal{C}_{(H\to D)}$ of $\mathcal{C}_{(H\to D)}$ and its (fat) geometric realization $|N\mathcal{C}_{(H\to D)}|$. The nerve is naturally a simplicial Lie group and its geometric realization becomes naturally a topological group [12]. We will use the following pictorial representation for the simplicial group $N\mathcal{C}_{(H\to D)}$:

$$\xrightarrow{d_0}$$

for the zeroth component,

$$\begin{tikzcd} \cdot \arrow[r, bend left=60, "d_1"{name=U}] \arrow[r, bend right=60, "d_0"'{name=D}] & \cdot \arrow[Rightarrow, from=U, to=D, "h_{01}"] \end{tikzcd}$$

for the first component,

$$\begin{tikzcd} \cdot \arrow[r, bend left=70, "d_2"{name=U}] \arrow[r, "d_1"{name=M}] \arrow[r, bend right=70, "d_0"'{name=D}] & \cdot \arrow[Rightarrow, from=U, to=M, "h_{12}"] \arrow[Rightarrow, from=M, to=D, "h_{01}"] \end{tikzcd}$$

for the second component etc. with the obvious face and degeneracy maps.

The (opposite) group multiplication is given by horizontal composition

$$\xrightarrow{d_1} \cdot \xrightarrow{d_0} = \xrightarrow{d_0 d_1}$$

$$\begin{tikzcd} \cdot \arrow[r, bend left=60, "d_1'"{name=U1}] \arrow[r, bend right=60, "d_0'"'{name=D1}] & \cdot \arrow[r, bend left=60, "d_1"{name=U2}] \arrow[r, bend right=60, "d_0"'{name=D2}] & \cdot \arrow[Rightarrow, from=U1, to=D1, "h_{01}'"] \arrow[Rightarrow, from=U2, to=D2, "h_{01}"] \end{tikzcd} = \begin{tikzcd} \cdot \arrow[rr, bend left=50, "d_1 d_1'"{name=U}] \arrow[rr, bend right=50, "d_0 d_0'"'{name=D}] & & \cdot \arrow[Rightarrow, from=U, to=D, "h_{01}\,{}^{d_1}h_{01}'"] \end{tikzcd}$$

etc.

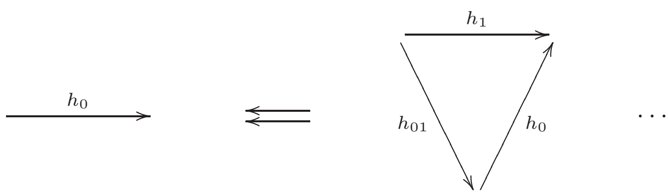
Simplicial homotopy groups of $N\mathcal{C}_{(H\to D)}$ are trivial except $\pi_0(N\mathcal{C}_{(H\to D)}) = \operatorname{coker}\alpha$ and $\pi_1(N\mathcal{C}_{(H\to D)}) = \ker\alpha$.

4.3. *Definition*

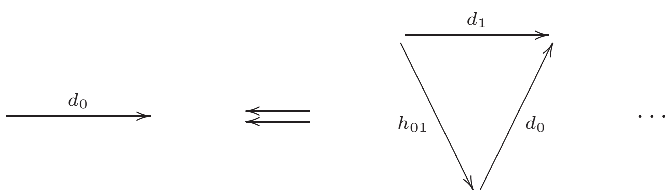
Let $(H \to D)$ be a crossed module of Lie groups and X a manifold. Let $P \to X$ be a left principal H-bundle, such that the principal D-bundle $D \times_\partial P$ is trivial with a trivialization defined by a section (i.e. a left H-equivariant smooth map) $\boldsymbol{d}: P \to D$. We call the pair $(P, \boldsymbol{d})$ an $(H \to D)$-bundle or a crossed module bundle.

Proposition 4.1. $|N\mathcal{C}_{(H\to D)}|$ *is the classifying space of* $(H \to D)$*-bundles.*

Proof. In other words $|N\mathcal{C}_{(H\to D)}|$ is the homotopy fiber of $BH \to BD$. This is the pullback under $B\alpha: BH \to BD$ of the bundle of based paths $P_0BD \to BH$. As a principal $\Omega BD \sim D$-bundle it can be identified with the homotopy quotient $D/\!\!/H = EH \times_\alpha D$ of D by H. Let us recall that EH is the (fat) geometric realization of the following simplicial space (here and in all following pictures, we shall omit the arrows for degeneracy maps)



From here we get $EH \times_\alpha D$ as the geometric realization of the simplicial space



and we see that this is really identical to the simplicial group $N\mathcal{C}_{(H\to D)}$. $\square$

4.4. *Remark*

Bundles defined in 4.3 are automatically left and right H-principal bundles with the two principal H-actions commuting. Moreover the multiplication in Subsec. 4.2 gives naturally a multiplication of such bundles. This follows from [6, Proposition 4].

If P and P' are two crossed module bundles and f and f' the corresponding classifying maps, then the pointwise product map $f \cdot f'$ is a classifying map for a bundle equivalent to the product bundle $P \cdot P'$.

4.5. *String group*

Together with a crossed module $(H \to D)$ we can consider also crossed modules $(H \to \operatorname{Im}\alpha)$ and $(1 \to \operatorname{coker}\alpha)$. This gives an exact sequence of (topological) groups

$$1 \to |N\mathcal{C}_{(H\to \operatorname{Im}\alpha)}| \to |N\mathcal{C}_{(H\to D)}| \to |N\mathcal{C}_{(1\to \operatorname{coker}\alpha)}| = \operatorname{coker}\alpha \to 1.$$

String group *String* is a nice example of the above construction. Let G be a simply connected compact simple Lie group. The crossed module in question is given by $H = \widehat{\Omega G}$, the centrally extended group of based loops, and $D = P_0 G$, the group of based paths [12, 13] or some modification of these [14].

Of course, we can consider crossed modules $(\ker\alpha \to e)$ and $(\operatorname{Im}\alpha) \to D$ as well, in which case we obtain the exact sequence

$$1 \to |N\mathcal{C}_{(\ker\alpha\to e)}| = B\ker\alpha \to |N\mathcal{C}_{(H\to D)}| \to |N\mathcal{C}_{(\operatorname{Im}\alpha\to D)}| \to 1.$$

Let us notice, that the two above exact sequences are (homotopy) equivalent to

$$1 \to B\ker\alpha \to |N\mathcal{C}_{(H\to D)}| \to \operatorname{coker}\alpha \to 1.$$

Hence, we can view the homotopy quotient $D/\!\!/H = EH \times_\alpha D$ as a principal $B\ker\alpha$-bundle over the base space $\operatorname{coker}\alpha$. Since EH is the universal bundle for any subgroup of H it is also the universal bundle for the normal subgroup $\ker\alpha \subset H$. The action of H on EH descents to an action of $H/\ker\alpha = \alpha(H)$ on $B\ker\alpha$ and we see that we have $|N\mathcal{C}_{(H\to D)}| \sim B\ker\alpha \times_{\alpha(H)} D$.

There is another nice description of the above (topological) group structure on $EH \times_\alpha D$.[a] EH itself can be thought of as $|N\mathcal{C}_{(H\to H)}|$. Hence it is a topological group. The action of D on H naturally extends to EH and we can form the semidirect product $EH \rtimes D$. This group structure factors to $EH \rtimes_\alpha D$. Now, if we equip $B\ker\alpha$ with the factor group structure then the D-action factors to $B\ker\alpha$ as it preserves $\ker\alpha$.

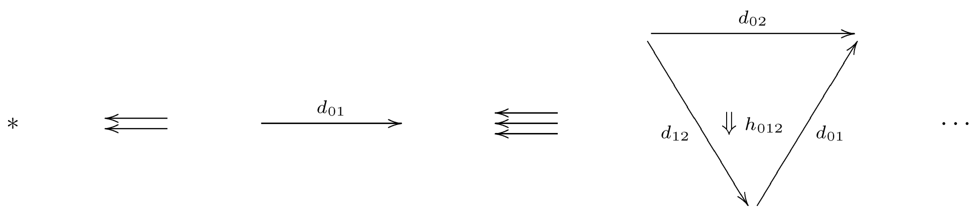
This description of *String* is very close to the one of Stolz and Teichner [14]. Very briefly, in their construction of *String*, H is $\tilde{L}_I G$, the central extension of $L_I G$ (group of all piecewise smooth loops $\gamma : S^1 \to G$ with the support in the upper semicircle $I \in S^1$). Here G is a compact, simply connected Lie group. Their D is the group of based paths $P_e^I G = \{\gamma : I \to G \,|\, \gamma(1) = e\}$. With these choices, they can take $PU(A_\rho)$ as a model for $B\ker\alpha$, where A_ρ is certain von Neumann algebra (type III_1 factor) associated with the vacuum representation of the loop group LG at some fixed level $l \in H^4(BG)$. See [14] for details.

4.6. *Crossed module as a (strict) two-group*

Similarly we denote $\tilde{\mathcal{C}}_{(H\to D)}$ the (topological, strict) two-group, i.e. the (topological) two-category with just one object, one-arrows group elements $d \in D$ and two-arrows group elements (h, d) of $H \rtimes D$. Again, we can form the corresponding nerve

[a] I thank Stevenson for noticing this to me.

$N\tilde{\mathcal{C}}_{(H\to D)}$ [24]. This simplicial manifold can be pictorially represented as:



Simplicial homotopy groups of $N\tilde{\mathcal{C}}_{(H\to D)}$ are trivial except $\pi_1(N\tilde{\mathcal{C}}_{(H\to D)}) = \operatorname{coker}\alpha$ and $\pi_2(N\tilde{\mathcal{C}}_{(H\to D)}) = \ker\alpha$.

5. Crossed Module Bundle Gerbes

We will introduce a slight generalization of the definition of a nonabelian bundle gerbe from [6]. Consider a surjective submersion $f : Y \to X$ (i.e. a map onto with differential onto). It follows that we can always find an open covering $\{O_\alpha\}$ of X with local sections $\sigma_\alpha : O_\alpha \to Y$, i.e. $f \circ \sigma_\alpha = id$. We also consider $Y^{[n]} = Y \times_X Y \times_X Y \cdots \times_X Y$ the n-fold fiber product of Y, i.e. $Y^{[n]} \equiv \{(y_1, \ldots, y_n) \in Y^n \mid f(y_1) = f(y_2) = \cdots f(y_n)\}$.

Given a $(H \to D)$-bundle $\mathcal{E}$ over $Y^{[2]}$ we denote by $\mathcal{E}_{12} = p_{12}^*(\mathcal{E})$ the $(H \to D)$-bundle on $Y^{[3]}$ obtained as pullback of $p_{12} : Y^{[3]} \to Y^{[2]}$ (p_{12} is the identity on its first two arguments); similarly for $\mathcal{E}_{13}$ and $\mathcal{E}_{23}$.

Consider the quadruple $(\mathcal{E}, Y, X, \boldsymbol{h})$ where $\mathcal{E}$ is a crossed module bundle, $Y \to X$ a submersion and $\boldsymbol{h}$ an isomorphism of crossed module bundles $\boldsymbol{h} : \mathcal{E}_{12}\mathcal{E}_{23} \to \mathcal{E}_{13}$ (let us recall that two crossed module bundles can be multiplied to obtain again a crossed module bundle). We now consider $Y^{[4]}$ and the crossed module bundles $\mathcal{E}_{12}$, $\mathcal{E}_{23}$, $\mathcal{E}_{13}$, $\mathcal{E}_{24}$, $\mathcal{E}_{34}$, $\mathcal{E}_{14}$ on $Y^{[4]}$ relative to the projections $p_{12} : Y^{[4]} \to Y^{[2]}$ etc. and also the crossed module isomorphisms $\boldsymbol{h}_{123}$, $\boldsymbol{h}_{124}$, $\boldsymbol{h}_{223}$, $\boldsymbol{h}_{234}$ induced by projections $p_{123} : Y^{[4]} \to Y^{[3]}$ etc.

5.1. *Definition*

The quadruple $(\mathcal{E}, Y, X, \boldsymbol{h})$ is called a crossed module bundle gerbe (or an $(H \to D)$-bundle gerbe) if $\boldsymbol{h}$ satisfies the cocycle condition (associativity) on $Y^{[4]}$

$$\begin{array}{ccc}
\mathcal{E}_{12}\mathcal{E}_{23}\mathcal{E}_{34} & \xrightarrow{\boldsymbol{h}_{234}} & \mathcal{E}_{12}\mathcal{E}_{24} \\
\boldsymbol{h}_{123}\big\downarrow & & \big\downarrow\boldsymbol{h}_{124} \\
\mathcal{E}_{13}\mathcal{E}_{34} & \xrightarrow{\boldsymbol{h}_{134}} & \mathcal{E}_{14}.
\end{array}$$

5.2. *Definition*

Two crossed module bundle gerbes $(\mathcal{E}, Y, X, \boldsymbol{h})$ and $(\mathcal{E}', Y', X, \boldsymbol{h}')$ are stably isomorphic if there exist a crossed module bundle $\mathcal{N} \to Z = Y \times_X Y'$ such that over $Z^{[2]}$

the crossed module bundles $q^*\mathcal{E}\mathcal{N}_2$ and $\mathcal{N}_1 q'^*\mathcal{E}'$ are isomorphic. The corresponding isomorphism $\boldsymbol{\ell} : q^*\mathcal{E}\mathcal{N}_2 \to \mathcal{N}_1 q'^*\mathcal{E}'$ should satisfy on $Y^{[3]}$ the condition

$$\boldsymbol{\ell}_{13} \circ \boldsymbol{h} = \boldsymbol{h}' \circ \boldsymbol{\ell}_{12} \circ \boldsymbol{\ell}_{23}.$$

Here q and q' are projections onto first and second factor of $Z = Y \times_X Y'$. $\mathcal{N}_1$ and $\mathcal{N}_2$ are the pullbacks of $\mathcal{N} \to Z$ to $Z^{[2]}$ under respective projections from $Z^{[2]}$ to Z etc.

5.3. *Remark*

Locally, crossed module bundle gerbes can be described in terms of two-cocycles as follows. We can consider the trivializing cover $\{O_\alpha\}$ of the submersion $Y \to X$ be a good one. Then a crossed module bundle gerbe can be described by an $(H \to D)$-valued two-cocycle $\{d_{\alpha\beta}, h_{\alpha\beta\gamma}\}$ where the maps $d_{\alpha\beta} : O_\alpha \cap O_\beta \to D$ and $h_{\alpha\beta\gamma} : O_\alpha \cap O_\beta \cap O_\gamma \to H$ fulfill the following cocycle condition

$$d_{\alpha\beta} d_{\beta\gamma} = \alpha(h_{\alpha\beta\gamma}) d_{\alpha\gamma} \quad \text{on } O_\alpha \cap O_\beta \cap O_\gamma$$

and

$$h_{\alpha\beta\gamma} h_{\alpha\gamma\delta} = {}^{d_{\alpha\beta}} h_{\beta\gamma\delta} h_{\alpha\beta\delta} \quad \text{on } O_\alpha \cap O_\beta \cap O_\gamma \cap O_\delta.$$

Two crossed module bundle gerbes are stably equivalent if their respective two-cocycles $\{d_{\alpha\beta}, h_{\alpha\beta\gamma}\}$ and $\{d'_{\alpha\beta}, h'_{\alpha\beta\gamma}\}$ are related as

$$d'_{\alpha\beta} = d_\alpha \alpha(h_{\alpha\beta}) d_{\alpha\beta} d_\beta^{-1}$$

and

$$h'_{\alpha\beta\gamma} = {}^{d_\alpha} h_{\alpha\beta} \, {}^{d_\alpha d_{\alpha\beta}} h_{\beta\gamma} \, {}^{d_\alpha} h_{\alpha\beta\gamma} \, {}^{d_\alpha} h_{\alpha\beta}^{-1}$$

with $d_\alpha : O_\alpha \to D$ and $h_{\alpha\beta} : O_\alpha \cap O_\beta \to H$. We call two two-cocycles equivalent if they are related as above.

Pullback of a bundle gerbe is obtained pulling back the corresponding cocycle.

We have the following proposition (cf. [20]).

Proposition 5.1. *Stable equivalence classes of $(H \to D)$-bundle gerbes are one-to-one with equivalence classes of $(H \to D)$-valued two-cocycles.*

5.4. *Universal $N\mathcal{C}_{(H\to D)}$ bundle*

In Subsec. 4.2 we have described the simplicial Lie group $N\mathcal{C}_{(H\to D)}$. Now we can construct the corresponding universal bundle. As a result we get simplicial manifolds $\overline{W}N\mathcal{C}_{H\to D}$ and $WN\mathcal{C}_{H\to D}$ which are pictorially represented as

$$\begin{tikzcd}[column sep=small]
 & & & & & \cdot \arrow[rr, "d_{02}"] \arrow[dr, "d_{12}"'] & & \cdot & \\
* & \arrow[l, shift left] \arrow[l, shift right] & \cdot \arrow[r, "d_{01}"] & \cdot & \arrow[l] \arrow[l, shift left] \arrow[l, shift right] & & \cdot \arrow[ur, "d_{01}"'] \arrow[u, phantom, "\Downarrow h_{012}"] & & \cdots
\end{tikzcd}$$

$$
\begin{tikzcd}
\cdot \arrow[r, "d_0"] & \cdot & & \arrow[ll, shift left] \arrow[ll, shift right] & \cdot \arrow[rr, "d_1"] \arrow[dr, "d_{01}"'] & \arrow[d, phantom, "\Downarrow h_{01}"] & \cdot & \cdots \\
& & & & & \cdot \arrow[ur, "d_0"'] & &
\end{tikzcd}
$$

Comparing to Subsec. 4.6 gives (cf. also [22]).

Proposition 5.2. $\overline{W}N\mathcal{C}_{(H\to D)} = N\tilde{\mathcal{C}}_{(H\to D)}$.

The proof of the following theorem in the original version arXiv: math/0510078 contained some gaps as pointed out in [24], where the proof for a more general case of a crossed module of topological groups has been given. Hence we omit the formal proof and make some remarks on the original proof at the end of this section in Subsec. 5.9. For related statements concerning classification of principal bundles for topological bicategories, see [15] and [16].

Theorem 5.1. *Equivalence classes of principal* $|N\mathcal{C}_{(H\to D)}|$*-bundles are one-to-one with stable equivalence classes of* $(H \to D)$*-bundle gerbes. The geometric realization* $|WN\mathcal{C}_{(H\to D)}| = E|N\mathcal{C}_{(H\to D)}| \to |\overline{W}N\mathcal{C}_{(H\to D)}| = B|N\mathcal{C}_{(H\to D)}|$ *gives the universal* $|N\mathcal{C}_{(H\to D)}|$*-bundle as well as the universal crossed module bundle gerbe.*

5.5. *Remark*

Let us recall that, by definition, under a nonabelian H-bundle gerbe we understand an $(H \to \mathrm{Aut}(H))$-crossed module bundle gerbe [6]. So the universal H-bundle gerbe is the same as the universal $|N\mathcal{C}_{(H\to\mathrm{Aut}(H))}|$-bundle.

5.6. *String structures*

Now we can apply the classifying space functor B to the exact sequence of 4.5. Hence, have the following exact sequence ($\ker\alpha$ is abelian)

$$
\begin{aligned}
1 &\to B\ker\alpha \to |N\mathcal{C}_{(H\to D)}| \to \operatorname{coker}\alpha \to B^2\ker\alpha \to B|N\mathcal{C}_{(H\to D)}| \\
&\to B\operatorname{coker}\alpha \to B^3\ker\alpha.
\end{aligned}
$$

It follows that a lift of a principal $\operatorname{coker}\alpha$-bundle to a principal $|N\mathcal{C}_{(H\to D)}|$-bundle is the same thing as a lift of an $(\alpha(H) \to D)$-bundle gerbe to an $(H \to D)$-bundle gerbe.

In the case of *String* we do have

$$
1 \to K(\mathbb{Z},2) \to String \to Spin \to K(\mathbb{Z},3) \to BString \to BSpin \to K(\mathbb{Z},4).
$$

String structure is a lift of the structure group of a principal *Spin*-bundle to the string group *String* [14]. So the string structure is also lift of an $(\Omega Spin \to P_0Spin)$-bundle gerbe to an $(\widehat{\Omega Spin} \to P_0Spin)$-bundle gerbe.

5.7. *Remark*

A crossed module bundle gerbe is canonically equipped with a module (see Sec. 6 of [6] for the definition of a bundle gerbe module). The trivial D-principal bundle $D \times Y \to Y$ fulfills all the axioms of a module. This is shown in [6] in the case $D = \mathrm{Aut}(H)$ and applies word by word to the more general situation as well.

5.8. *Remark*

Let us consider the (topological) one-category (actually one-groupoid) $\mathcal{C}_{\{O_\alpha\}}$, related to an open covering $\{O_\alpha\}$, described as follows. Objects are pairs (x, O_α) with $x \in O_\alpha$ and there is unique morphism $(x, O_\alpha) \to (y, O_\beta)$ iff $x = y \in O_\alpha \cap O_\beta$. Let $N\mathcal{C}_{\{O_\alpha\}}$ denote the nerve of this category. Consider a map of simplicial sets $N\mathcal{C}_{\{O_\alpha\}} \to \overline{W}N\mathcal{C}_{(H\to D)}$. Then the maps between one-, two- and three-simplices give us the gerbe transition functions (5.3). We also see that the simplicial τ_0 is identified with $d_{\alpha\beta}$, τ_1 identifies with $d_{\alpha\gamma}d_{\beta\gamma}^{-1} \xrightarrow{h_{\alpha\beta\gamma}}$. A similar identification can easily be done also for the equivalence data of (1.3) and the local stable equivalence data of (5.3). Hence, we can conclude that locally the stable equivalence classes of crossed module gerbes are described by homotopy classes of simplicial maps $N\mathcal{C}_{\{O_\alpha\}} \to \overline{W}N\mathcal{C}_{(H\to D)} = N\tilde{\mathcal{C}}_{(H\to D)}$, i.e. they are the same things as equivalence classes of principal $N\mathcal{C}_{(H\to D)}$-bundles over $N\mathcal{C}_{\{O_\alpha\}}$.

5.9. *Remark*

The incomplete proof of the Theorem 5.1 in the original version arXiv: math/0510078 was based on a generalization of [6, Sec. 5], where the lifting bundle gerbes (crossed module bundle gerbes with $\ker\alpha = 0$) were discussed in detail. Here we first repeat the incomplete argument.

Let $f : X \to B|N\mathcal{C}_{(H\to D)}|$ be the classifying map for an $|N\mathcal{C}_{(H\to D)}|$-principal bundle P. Associated with P there is a map $P^{[2]} \to |N\mathcal{C}_{(H\to D)}|$ which sends $(p, p') \in P$ in the same fiber into unique group element $g \in |N\mathcal{C}_{(H\to D)}|$ which relates p and p'. As $|N\mathcal{C}_{(H\to D)}|$ is the classifying space for crossed module bundles, we obtain that way a crossed module bundle $\mathcal{E} \to P^{[2]}$. It follows from Remark 4.4, $\mathcal{E}_{12}\mathcal{E}_{23}$ is isomorphic to $\mathcal{E}_{13}$, with an isomorphism fulfilling the cocycle condition of Definition 5.1. So we obtain a bundle gerbe with $Y = P$. If we start with an equivalent bundle P' we obtain a stably equivalent gerbe.

Conversely, if we start with a crossed module bundle gerbe, the classifying map of the crossed module bundle $\mathcal{E} \to Y^{[2]}$ is a map from $f : Y^{[2]} \to |N\mathcal{C}_{(H\to D)}|$ fulfilling the on $Y^{[3]}$ the cocycle condition $f(y_1, y_2)f(y_2, y_3) = f(y_1, y_3)$ and hence, giving the descent data of an $|N\mathcal{C}_{(H\to D)}|$-principal bundle. Starting from a stably equivalent gerbe we get an equivalent bundle. Thus we have transition functions for an $|N\mathcal{C}_{(H\to D)}|$-bundle.

It has been correctly noticed in [16] that it is not obvious that the classifying map f can be chosen to satisfy the cocycle condition. Also, as noticed there, it has not been shown that the above two constructions are inverse to each other. Since [16] already gives a detailed proof of the Theorem 5.1, we will not try to fill these gaps. We just notice that one possible way to construct an $|N\mathcal{C}_{(H\to D)}|$-bundle starting from an $(H \to D)$-bundle gerbe could be to apply the geometric realization to the corresponding principal $NC_{(H\to D)}$-bundle over $NC_{\{O_\alpha\}}$ (5.8) and use the fact that for a good covering $\{O_\alpha\}$ of X, X is weakly homotopy equivalent to $NC_{\{O_\alpha\}}$.

6. Connection and B-Field on a Bundle Gerbe

In the previous section we have established a correspondence between $|N\mathcal{C}_{(H\to D)}|$-principal bundles and $(H \to D)$-crossed module bundle gerbes. Now we would like to extend this relationship to connections, and also discuss the B-field from this point of view. Let us recall that $|N\mathcal{C}_{(H\to D)}|$ is only a topological group so in general there is no differential geometric connection on a principal $|N\mathcal{C}_{(H\to D)}|$-bundle over a manifold X. But we can use the simplicial connection as described in Sec. 2 on any simplicial $N\mathcal{C}_{(H\to D)}$-bundle $P \to X$.

The notion of a bundle gerbe connection (and that of a bundle gerbe B-field as well) are quite subtle and we are not going to repeat them here in their global formulations (see [6, 4] for that). Instead, we will give their local description using cocycles. This description sees to be well suited for our purposes. We will relate the bundle gerbe connection and B-field to the simplicial connection and simplicial $\bar{B}$-field as they were introduced in Secs. 2 and 3 in the case of a simplicial $N\mathcal{C}_{(H\to D)}$-bundle over $N\mathcal{C}_{\{O_\alpha\}}$ described by a classifying map $N\mathcal{C}_{\{O_\alpha\}} \to \overline{W}N\mathcal{C}_{(H\to D)} = N\tilde{\mathcal{C}}_{(H\to D)}$ (see Remark 5.8).

Let us now recall the local cocycle description of a connection on an crossed module bundle gerbe. Again let $\{O_\alpha\}$ be an open covering of a manifold X.

6.1. *Bundle gerbe connection*

A collection $\{A_\alpha, a_{\alpha\beta}\}$, with $A_\alpha \in \Omega^1(O_\alpha) \otimes \mathrm{Lie}(D)$ and $a_{\alpha\beta} \in \Omega^1(O_\alpha \cap O_\beta) \otimes \mathrm{Lie}(H)$ is called a connection on crossed module bundle gerbe (characterized by a nonabelian cocycle $\{d_{\alpha\beta}, h_{\alpha\beta\gamma}\}$) if it fulfills the following conditions

$$A_\alpha = d_{\alpha\beta} A_\beta d_{\alpha\beta}^{-1} + d_{\alpha\beta} d d_{\alpha\beta}^{-1} + \alpha(a_{\alpha\beta}) \quad \text{on } O_\alpha \cap O_\beta$$

and

$$a_{\alpha\beta} + {}^{d_{\alpha\beta}}a_{\beta\gamma} = h_{\alpha\beta\gamma} a_{\alpha\gamma} h_{\alpha\beta\gamma}^{-1} + h_{\alpha\beta\gamma} d h_{\alpha\beta\gamma}^{-1} + T_{A_\alpha}(h_{\alpha\beta\gamma}^{-1}) \quad \text{on } O_\alpha \cap O_\beta \cap O_\gamma.$$

Here for A a $\mathrm{Lie}(D)$-valued one-form and $h \in H$ the $\mathrm{Lie}(H)$-valued one-form $T_A(h)$ is defined as follows. For $X \in \mathrm{Lie}(D)$ we put $T_X(h) = [h^{\exp(tX)}(h^{-1})]$, where the square bracket [ ] means the tangent vector to the curve at the group identity 1_H. For $\mathrm{Lie}(D)$-valued one-form $A = A^\rho X^\rho$, with $\{X^\rho\}$ a basis of $\mathrm{Lie}(D)$, we put $T_A \equiv A^\rho T_{X^\rho}$.

The curvature F is given by a collection of local two-forms $F_\alpha \in \Omega^2(O_\alpha) \otimes \mathrm{Lie}(D)$ defined as $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$; the corresponding cocycle conditions follow from the definition. We will not repeat the explicit formulas here, interested reader can find them in e.g. [6, 4]. Now we can compare the above definition with the definition of a simplicial connection on a $N\mathcal{C}_{(H \to D)}$-principal bundle $P \to N\mathcal{C}_{\{O_\alpha\}}$. Realizing that τ_0 corresponds $d_{\alpha\gamma}$, τ_1 corresponds to $d_{\alpha\gamma} d_{\beta\gamma}^{-1} \xrightarrow{h_{\alpha\beta\gamma}} d_{\alpha\beta}$, A_0 corresponds to A_α, a_{01} of $A_1 = (\partial_0 A_1 \xrightarrow{a_{01}} \partial_1 A_1)$ corresponds to $-a_{\alpha\beta}$ etc. we easily obtain.

Proposition 6.1. *A connection on a crossed module bundle gerbe defines a simplicial connection on the corresponding $N\mathcal{C}_{(H \to D)}$-principal bundle over $N\mathcal{C}_{\{O_\alpha\}}$ and vice versa.*

Similar discussion applies to B-field as well.

6.2. *Bundle gerbe B-field*

B-field on a crossed module bundle gerbe equipped with a connection is a collection $\{B_\alpha, \delta_{\alpha\beta}\}$ of local two-forms $B_\alpha \in \Omega^2(O_\alpha) \otimes \mathrm{Lie}(H)$ and $\delta_{\alpha\beta} \in \Omega^2(O_{\alpha\beta}) \otimes \mathrm{Lie}(H)$ such that

$$B_\alpha = {}^{d_{\alpha\beta}} B_\beta + \delta_{\alpha\beta} \quad \text{on } O_\alpha \cap O_\beta$$

and

$$\delta_{\alpha\beta} + {}^{d_{\alpha\beta}} \delta_{\beta\gamma} = h_{\alpha\beta\gamma} \delta_{\alpha\gamma} h_{\alpha\beta\gamma}^{-1} + B_\alpha - h_{\alpha\beta\gamma} B_\alpha h_{\alpha\beta\gamma}^{-1} \quad \text{on } O_\alpha \cap O_\beta \cap O_\gamma.$$

Given a simplicial $\bar{B}$ in the present case then the bundle gerbe B-field is identified as the morphism B in the $\bar{B}_1 = (\partial_0 \bar{B}_1 \xrightarrow{-B} 0)$ part of the simplicial $\bar{B}$ and the simplicial $(\partial_2 \bar{B}_2 - \partial_1 \bar{B}_2)$ is identified with the bundle gerbe δ, we obtain the following proposition.

Proposition 6.2. *A simplicial $\bar{B}$-field on a $N\mathcal{C}_{(H \to D)}$ principal bundle over $N\mathcal{C}_{\{O_\alpha\}}$ gives a B-field on the corresponding bundle gerbe and vice versa.*

The bundle gerbe ν-field is defined as $\nu = F + \alpha(B)$. This definition guarantees that it is the same as the simplicial one in the present case.

6.3. *Remark*

It is generally true only in the case of abelian H that connection A and the B-field can be chosen such that $\nu_\alpha = d_{\alpha\beta} \nu_\beta d_{\alpha\beta}^{-1}$. We are are not sure what kind of condition should replace this in the case of nonabelian H.

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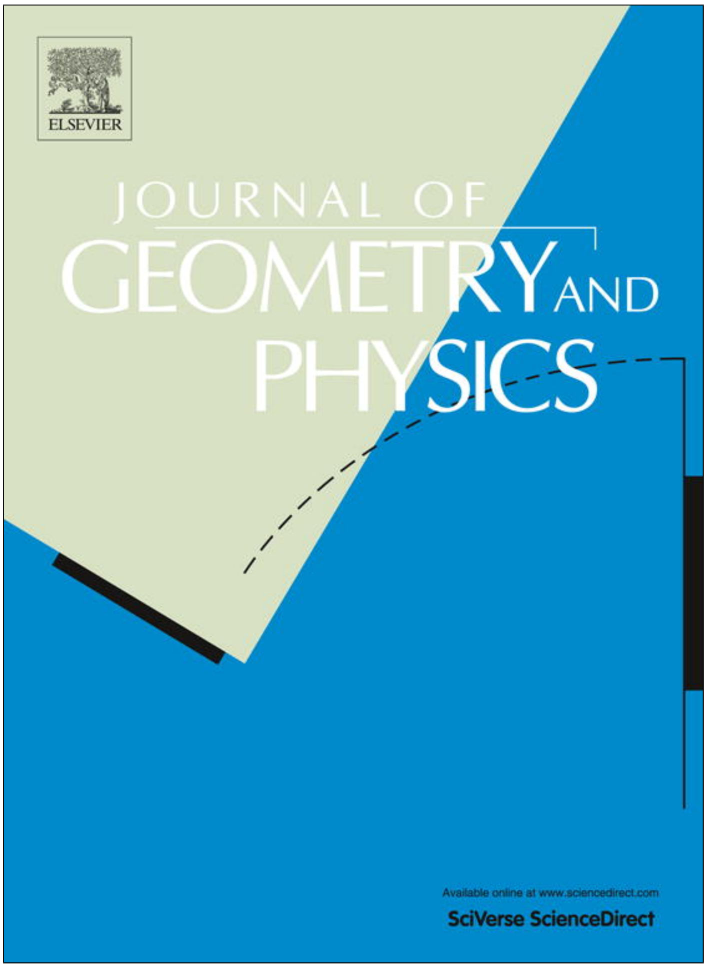
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From simplicial Lie algebras and hypercrossed complexes to differential graded Lie algebras via 1-jets

Branislav Jurčo

Mathematical Institute, Charles university, Sokolovská 83, Prague 186 75, Czech Republic

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ABSTRACT

Let $\mathfrak{g}$ be a simplicial Lie algebra with Moore complex $N\mathfrak{g}$ of length k. Let G be the simplicial Lie group integrating $\mathfrak{g}$, such that each G_n is simply connected. We use the 1-jet of the classifying space $\overline{W}G$ to construct, starting from $\mathfrak{g}$, a Lie k-algebra L. The so constructed Lie k-algebra L is actually a differential graded Lie algebra. The differential and the brackets are explicitly described in terms (of a part) of the corresponding k-hypercrossed complex structure of $N\mathfrak{g}$. The result can be seen as a geometric interpretation of Quillen's (purely algebraic) construction of the adjunction between simplicial Lie algebras and dg-Lie algebras.

1. Introduction

In this paper we describe a geometric construction leading to Quillen's relation between simplicial Lie algebras and differential graded Lie algebras (DGLAs) [1]. We do that following the ideas of Ševera [2], which lead to a construction of L_∞-algebras (or, more generally, L_∞-algebroids) as 1-jets (differentiation) of simplicial manifolds. Here, we apply Ševera's construction to the case when the simplicial manifold in question is the classifying space $\overline{W}G$ of a simplicial Lie group G, the simplicial Lie group G having Moore complex of length k. Main results are Proposition 5.2 and Proposition 5.3. In Proposition 5.2 we describe explicitly the dg manifold representing the 1-jet functor $F_1^{\overline{W}G}$ and in Proposition 5.3 we describe explicitly the corresponding L_∞-algebra as a k-term differential graded Lie algebra L with the differential and brackets given in terms the hypercrossed complex structure of $N\mathfrak{g}$. The result is the same as the one described by the N-functor in the Quillen's adjunction between simplicial Lie algebras and dg-Lie algebras (see Proposition 4.4 of [1]. The construction can equivalently be viewed as an assignment of a k-term DGLA to a k-hypercrossed complex $\underline{\mathfrak{g}}$. The paper is organized as follows.

In Section 2, we recall the relevant material concerning simplicial Lie groups. In particular, we describe the Moore complex of a simplicial Lie group and illustrate its Lie hypercrossed complex structure in the low dimensional case of Lie crossed modules an Lie 2-crossed modules.

In Section 3, we recall the relevant facts regarding simplicial principal bundles.

In Section 4, we summarize Ševera's construction and give the relevant examples following [2]. In particular, we describe in detail the construction of a Lie algebra $\mathfrak{g}$ as a 1-jet of the classifying space BG of the corresponding Lie group G. Also, we describe in detail the construction of a Lie 2-algebra corresponding to a crossed module of Lie algebras $\mathfrak{h} \to \mathfrak{d}$ as a 1-jet of the functor associating to a surjective submersion $M \to N$ of (super)manifolds the set of $(H \to D)$-descent data over $M \to N$.

The examples mentioned above are the starting point of this paper. For this, we note that also the second example can be reinterpreted as the 1-jet of a simplicial manifold. The relevant simplicial manifold is the Duskin nerve of the strict Lie 2-group defined by the Lie crossed module $H \to D$, which is isomorphic to $\overline{W}G$, the classifying space of the simplicial Lie

E-mail addresses: jurco@karlin.mff.cuni.cz, branislav.jurco@googlemail.com.

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group associated to the Lie crossed module $H \to D$. Therefore, it is natural to generalize the above examples by applying Ševera's construction to the case of any simplicial Lie group G and describe explicitly the corresponding 1-jet of $\overline{W}G$. This is done in Section 5. The resulting dg manifold is described in Proposition 5.2 and the corresponding DGLA in Proposition 5.3. This DGLA is the same as the one described by Quillen in Section 4 of [1].

In this paper we do not discuss, up to occasional remarks,[1] applications to the higher gauge theory. These will be given in a forthcoming paper.

All commutators are implicitly assumed to be graded. Although we do not mention it explicitly, all constructions extend more or less straightforwardly to the case when all in involved Lie groups and Lie algebras are super. Hopefully, this is a wormless paper [3].

2. Simplicial groups and higher crossed modules

Here we briefly sketch the relation between simplicial groups and hypercrossed complexes of groups. The basic idea comes from [4] and is further developed and formalized in [5]. We follow [4,6,7].

Although the above mentioned references[2] work with simplicial sets, the constructions and statements relevant for our purposes can be straightforwardly formulated in the context of simplicial manifolds. Let G be a simplicial Lie group. We denote the corresponding face and degeneracy mappings ∂_i and s_i, respectively.

Definition 2.1. The *Moore complex* NG of G is the Lie group chain complex (NG, δ) with

$$NG_n := \bigcap_{i=1}^{n} \ker \partial_i$$

and the differentials $\delta_n : NG_n \to NG_{n-1}$ induced from the respective 0th face maps ∂_0 by restriction. It is a normal complex, i.e. $\delta_n NG_n$ is a normal subgroup of NG_{n-1}.[3] Of course, $NG_0 = G_0$. The Moore complex NG is said to be of length k if NG_n is trivial for $n > k$.[4]

The Moore complex NG carries a structure of a Lie *hypercrossed complex* structure, form which it can be reconstructed [4,5]. To describe the idea behind this, we will need following lemma.

Lemma 2.2. *Let G be a simplicial Lie group. Then G_n can be decomposed as a semidirect product of Lie groups*

$$G_n \cong \ker \partial_n \rtimes s_{n-1} G_{n-1}.$$

Explicitly, for $g \in G_n$, the isomorphism is given by

$$g \mapsto (g s_{n-1} \partial_n g^{-1}, s_{n-1} \partial_n g).$$

The following proposition [4] is the result of a repetitive application on the above lemma.

Proposition 2.3. *For a simplicial Lie group G,*

$$G_n \cong (\ldots (NG_n \rtimes s_0 NG_{n-1}) \rtimes \cdots \rtimes s_{n-1} \ldots s_0 NG_0).$$

The bracketing an ordering of the terms should be clear from the first few terms of the sequence:

$$\begin{aligned}
G_1 &\cong NG_1 \rtimes s_0 NG_0 \\
G_2 &\cong (NG_2 \rtimes s_0 NG_1) \rtimes (s_1 NG_1 \rtimes s_1 s_0 NG_0) \\
G_3 &\cong ((NG_3 \rtimes s_0 NG_2) \rtimes (s_1 NG_2 \rtimes s_1 s_0 NG_1)) \rtimes ((s_2 NG_2 \rtimes s_2 s_0 NG_1) \rtimes (s_2 s_1 NG_1 \rtimes s_2 s_1 s_0 NG_0)). \qquad (2.1)
\end{aligned}$$

We are not going to spell out the rather complicated definition of a hypercrossed complex [5]. Instead, we give some examples.

Example 2.4. A 1-hypercrossed complex of Lie groups is the same thing as a Lie crossed module.

Definition 2.5. Let H and D be two Lie groups. We say that H is a crossed D-module if there is a Lie group morphism $\delta_1 : H \to D$ and a smooth action of D on $H (d, h) \mapsto {}^d h$ such that

$${}^{\delta_1(h)}h' = hh'h^{-1} \quad \text{(Peiffer condition)}$$

1 Cf. Remarks 5.6 and 5.7.

2 And also our basic reference regarding simplicial objects [8] as well as other useful Refs. [9,10].

3 It is a normal subgroup of G_{n-1} too.

4 The objects of the full subcategory of simplicial groups with Moore complex of length k are also called *k-hypergroupoids* of groups [11].

for $h, h' \in H$, and

$$\delta_1({}^d h) = d\delta_1(h)d^{-1}$$

for $h \in H, d \in D$ hold true.

We will use the notation $H \xrightarrow{\delta_1} D$ or $H \to D$ for a crossed module.

Definition 2.6. A morphism between Lie crossed modules $H \xrightarrow{\delta_1} D$ and $H' \xrightarrow{\delta_1'} D'$ is a pair of Lie group morphisms $\lambda : H \to H'$ and $\kappa : D \to D'$ such that the diagram

$$\begin{tikzcd} H \arrow[r, "\delta_1"] \arrow[d, "\lambda"'] & D \arrow[d, "\kappa"] \\ H' \arrow[r, "\delta_1'"] & D' \end{tikzcd}$$

commutes, and for any $h \in H$ and $d \in D$ we have the following identity

$$\lambda({}^d h) = {}^{\kappa(d)}\lambda(h).$$

Starting from a Lie crossed module $H \to D$ we can build up the corresponding simplicial Lie group. Explicitly, cf. Proposition 2.3,

$$G_0 = D, \qquad G_1 = (H \rtimes D), \qquad G_2 = (H \rtimes (H \rtimes D)), \qquad \text{etc.}$$

The construction can be interpreted as the internal nerve of the associated internal category in the category of Lie groups (a strict Lie 2-group).

Example 2.7. A Lie 2-hypercrossed complex is the same thing as a Lie 2-crossed module [4].

Definition 2.8. A Lie 2-crossed module is a complex of Lie groups

$$H \xrightarrow{\delta_2} D \xrightarrow{\delta_1} K \tag{2.2}$$

together with smooth left actions by automorphisms of K on H and D (and on K by conjugation), and the Peiffer pairing, which is an smooth equivariant map $\{\ ,\ \} : D \times D \to H$, i.e., ${}^k\{d_1, d_2\} = \{{}^k d_1, {}^k d_2\}$ such that:

(i) (2.2) is a complex of K-modules, i.e., δ_2 and δ_1 are K-equivariant and $\delta_2\delta_1(h) = 1$ for $h \in H$,
(ii) $d_1 d_2 d_1^{-1} = \delta_2\{d_1, d_2\}^{\delta_1(d_1)} d_2$, for $d_1, d_2 \in D$,
(iii) $h_1 h_2 h_1^{-1} h_2^{-1} = \{\delta_2 h_1, \delta_2 h_2\}$, for $h_1, h_2 \in H$,
(iv) $\{d_1 d_2, d_3\} = \{d_1, d_2 d_3 d_2^{-1}\}^{\delta_1(d_1)}\{d_2, d_3\}$, for $d_1, d_2, d_3 \in D$,
(v) $\{d_1, d_2 d_3\} = {}^{d_1 d_2 d_1^{-1}}\{d_1, d_3\}\{d_1, d_2\}$, for $d_1, d_2, d_3 \in D$,
(vi) $\{\delta_2(h), d\}\{d, \delta_2(h)\} = h^{\delta_1(d)}(h^{-1})$, for $d \in D, h \in H$,

wherein the notation ${}^k d$ and ${}^k h$ for left actions of the element $k \in K$ on elements $d \in D$ and $h \in H$ has been used.

There is an obvious notion of a morphism of Lie 2-crossed modules.

Definition 2.9. A morphism between Lie 2-crossed modules $H \xrightarrow{\delta_2} D \xrightarrow{\delta_1} K$ and $H' \xrightarrow{\delta_2'} D' \xrightarrow{\delta_1'} K'$ is a triple of smooth group morphisms $H \to H'$, $D \to D'$ and $K \to K'$ making up, together with the maps $\delta_2, \delta_2', \delta_1$ and δ_1', a commutative diagram

$$\begin{tikzcd} H \arrow[r, "\delta_2"] \arrow[d, "\lambda"'] & D \arrow[r, "\delta_1"] \arrow[d, "\mu"'] & K \arrow[d, "\nu"] \\ H' \arrow[r, "\delta_2'"] & D' \arrow[r, "\delta_1'"] & K' \end{tikzcd} \tag{2.3}$$

and being compatible with the actions of K on D and H and of K' on D' and H', respectively and with the respective Peiffer pairings.

The corresponding simplicial Lie group is given explicitly by, cf. Proposition 2.3,

$$G_0 = K, \qquad G_1 = (D \rtimes K), \qquad G_2 = ((H \rtimes D) \rtimes (D \rtimes K)),$$
$$G_3 = (H \rtimes (H \rtimes D)) \rtimes ((H \rtimes D) \rtimes (D \rtimes K)), \qquad \text{etc.}$$

This can be interpreted as an internal Duskin nerve [12].

Example 2.10. A Lie 3-hypercrossed complex is the same thing as a Lie 3-crossed module of [13].

We refer the interested reader to [5] for a thorough discussion of hypercrossed complexes of groups and their relation to simplicial groups.

At each level n, there is an lexicographically ordered set $S(n)$ of 2^n sets, which defines the compositions of the degeneracy maps appearing in the decomposition of G_n. Explicitly for $S(n)$ we have:

$$\{\emptyset < \{0\} < \{1\} < \{1,0\} < \{2\} < \{2,0\} < \{2,1\} < \{2,1,0\} < \cdots < \{n-1,\ldots,1\} < \{n-1,\ldots,0\}\}.$$

The important role in the theory of hypercrossed complexes is played by the actions $G_0 \times NG_n \to NG_n$ defined by

$$g_0 \times g_n \mapsto {}^{g_0}g_n : (s_{n-1}\ldots s_0 g_0) g_n (s_{n-1}\ldots s_0 g_0)^{-1}$$

and the so called *Peiffer pairings*. In order to define these, we will use the multi-indices like α and β from $\bigcup_n S(n)$ to write s_α for products of degeneracy maps

$$s_0, s_1, s_1s_0, s_2, s_2s_0, s_2s_1, s_2s_1s_0, \ldots.$$

In particular, for $g \in NG_{n-\sharp\alpha}$ we have $s_\alpha g \in G_n$. For each n consider the set $P(n)$ of pairs (α,β) such that $\emptyset < \alpha < \beta$ and $\alpha \cap \beta = \emptyset$, where $\alpha \cap \beta$ is the set of indices belonging to both α and β.

Definition 2.11. The Peiffer pairing (or lifting) $F_{\alpha,\beta}(g,h) \in NG_n$ for $g \in NG_{n-\sharp\alpha}$, $h \in NG_{n-\sharp\beta}$ and $(\alpha,\beta) \in P(n)$ is defined by

$$F_{\alpha,\beta}(g,h) = p_n(s_\alpha(g)s_\beta(h)s_\alpha(g)^{-1}s_\beta(h)^{-1}),$$

where p_n is the projection to NG_n. For the projector p_n, we have $p_n = p_n^1 \ldots p_n^n$ with $p_n^i(g) = gs_{i-1}\partial_i g^{-1}$.

For us, the relevant Peiffer pairings at each level n will be those defined for pairs $(\alpha,\beta) \in P(n)$ such that $\alpha \cup \beta = \{0,\ldots,n\}$. We shall denote the set of such pairs $\bar{P}(n)$.

Remark 2.12. For a simplicial Lie algebra $\mathfrak{g}$, we have the corresponding Moore complex $N\mathfrak{g}$ of Lie algebras, which carries a structure of a hypercrossed complex of Lie algebras, cf. [14]. All the definitions and statements of this section have, of course, their infinitesimal counterparts. Since these are obvious, we shall not formulate them explicitly.

As shown by Quillen [1] there is an adjunction between simplicial Lie algebras and dg-Lie algebras. The part of the adjunction going from simplicial Lie algebras to dg-Lie algebras acts on the underlying simplicial vector spaces as the Moore complex functor N.

3. Simplicial principal bundles

Let G be a simplicial Lie group and X a simplicial manifold. In this paper we use the name *principal G-bundle* for a *twisted Cartesian product*. Therefore, we start with defining twisting functions. Again, we will denote by ∂_i and s_i the corresponding face and degeneracy maps. We follow [8].[5]

Definition 3.1. For a smooth function $\tau : X_n \to G_{n-1}$ to be a twisting, the following conditions should be fulfilled:

$$\begin{aligned}
&\partial_0\tau(x)\tau(\partial_0 x) = \tau(\partial_1 x),\\
&\partial_i\tau(x) = \tau(\partial_{i+1}x) \quad \text{for } i > 0,\\
&s_i\tau(x) = \tau(s_{i+1}x) \quad \text{for } i \geq 0,\\
&\tau(s_0 x) = e_n \quad \text{for } x \in X_n.
\end{aligned}$$

Definition 3.2. Let τ be a twisting function. A twisted Cartesian product $P(\tau) = G \times_\tau X$ (alternatively a principal G-bundle, or simply G-bundle, $P \to X$) is the simplicial manifold with simplices

$$P(\tau)_n = G_n \times X_n$$

and with the following face and degeneracy maps

$$\begin{aligned}
&\partial_i(g,x) = (\partial_i g, \partial_i x) \quad \text{for } i > 0,\\
&\partial_0(g,x) = (\partial_0 g \cdot \tau(x), \partial_0 x),\\
&s_i(g,x) = (s_i g, s_i x) \quad \text{for } i \geq 0.
\end{aligned}$$

[5] Again, passing from sets to manifolds is straightforward.

The principal (left) G-action

$$G_n \times P(\tau)_n \to P(\tau)_n, \qquad g'_n \times (g_n, x_n) \mapsto (g'_n g_n, x_n)$$

and the projection

$$\pi_n : P_n \to X_n, \qquad (g_n, x_n) \mapsto x_n$$

are smooth simplicial maps.

Equivalence of two G-bundles $P(\tau)$ and $P(\tau')$ over the same X is described in terms of twisting as follows.

Definition 3.3. We call two twistings τ' and τ equivalent if there exists a smooth map $\psi : X \to G$ such that

$$\begin{aligned} &\partial_0\psi(x) \cdot \tau'(x) = \tau(x) \cdot \psi(\partial_0 x), \\ &\partial_i\psi(x) = \psi(\partial_i x) \quad \text{if } i > 0, \\ &s_i\psi(x) = \psi(s_i x) \quad \text{if } i \geq 0. \end{aligned}$$

In particular a twisting or the corresponding G-bundle $P(\tau)$ is trivial iff

$$\tau(x) = \partial_0\psi(x)^{-1} \cdot \psi(\partial_0 x),$$

with ψ as above.

As with ordinary bundles, simplicial principal bundles can be pulled back and their structure groups can be changed using simplicial Lie group morphisms. Twistings transform under these operations in an obvious way.

There is a canonical construction of the classifying space $\overline{W}G$ and of the universal G-bundle WG.

Definition 3.4. The classifying space $\overline{W}G$ is defined as follows. $\overline{W}G_0$ has one element $*$ and $\overline{W}G_n = G_{n-1} \times G_{n-2} \times \cdots \times G_0$ for $n > 0$. Face and degeneracy maps are

$$s_0(*) = e_0, \qquad \partial_i(g_0) = * \quad \text{for } i = 0 \text{ or } 1$$

and

$$\begin{aligned} &\partial_0(g_n, \ldots, g_0) = (g_{n-1}, \ldots, g_0), \\ &\partial_{i+1}(g_n, \ldots, g_0) = (\partial_i g_n, \ldots, \partial_1 g_{n-i+1}, \partial_0 g_{n-i}.g_{n-i-1}, g_{n-i-2}, \ldots, g_0), \\ &s_0(g_{n-1}, \ldots, g_0) = (e_n, g_{n-1}, \ldots, g_0), \\ &s_{i+1}(g_{n-1}, \ldots, g_0) = (s_i g_{n-1}, \ldots, s_0 g_{n-i}, e_{n-i}, g_{n-i-1}, \ldots, g_0), \end{aligned}$$

for $n > 0$. With the choice of a twisting given by

$$\tau(g_{n-1}, \ldots, g_0) = g_{n-1}$$

we have the universal G-principal bundle

$$WG = G \times_\tau \overline{W}G.$$

We have a relation between twistings and simplicial maps $X \to \overline{W}G$ given by the following proposition.

Proposition 3.5. *The map $f_\tau : X \to \overline{W}G$ given by*

$$x \mapsto * \quad \textit{for } x \in X_0$$

and

$$x \mapsto (\tau(x), \tau(\partial_0 x), \ldots, \tau(\partial_0^i x), \ldots, \tau(\partial_0^{n-1} x)) \quad \textit{for } x \in X_n,\ n > 0$$

is a smooth simplicial map.

Vice versa, a smooth simplicial map $f : X \to \overline{W}_G$, given by

$$x \mapsto * \quad \textit{for } x \in X_0$$

and

$$x \mapsto (g_{n-1}^{(n)}(x), \ldots, g_0^{(n)}(x)) \quad \textit{for } x \in X_n,\ n > 0$$

defines a twisting by

$$\tau_f(x) = g_{n-1}^{(n)}(x) \quad \textit{for } x \in X_n,\ n > 0.$$

We have $\tau_{f_\tau} = \tau$ and $f_{\tau_f} = f$.

The role of the universal bundle is the following.

Theorem 3.6. *The principal G-bundle $G \times_\tau X$ corresponding to the twisting τ is obtained from the universal bundle WG as a pullback under the simplicial map f_τ.*

4. L_∞-algebroids as 1-jets of simplicial sets

This section is completely based on [2], to which we also refer for the proofs. We keep, maybe with an occasional exception, the notation and terminology used there. Let SSM denote the category with objects being surjective submersions between supermanifolds and morphisms commutative squares. Any surjective submersion $M \to N$ gives a simplicial supermanifold X, the nerve of the groupoid $M \times_N M \rightrightarrows N$. Further, let SSM_1 denote the full subcategory of SSM with objects $\mathbb{R}^{0|1} \times N \xrightarrow{\mathrm{pr}_2} N$, where N is running through all supermanifolds. Let $\mathrm{SM}_{[1]}$ be the category of supermanifolds with a right action of the supersemigroup $\mathbf{Hom}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$. Put in other words, $\mathrm{SM}_{[1]}$ is the category of differential non-negatively graded supermanifolds. We have the following lemma.

Lemma 4.1. *The category $\widehat{\mathrm{SSM}}_1$ of presheaves on SSM_1 and the category $\widehat{\mathrm{SM}}_{[1]}$ of presheaves on $\mathrm{SM}_{[1]}$ are equivalent.*

Remark 4.2. The above lemma follows from the useful observation

$$\mathrm{Hom}(\mathbb{R}^{0|1} \times N \to N, \mathbb{R}^{0|1} \times X \to X) \simeq \mathrm{Hom}(N, X) \times \mathbf{Hom}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})(N),$$

which just says that the object $\mathbb{R}^{0|1} \times X \in \mathrm{SSM}_1$ corresponds to the object $X \times \mathbf{Hom}(\mathbb{R}^{0|1}, \mathbb{R}^{0|1})$ in $\mathrm{SM}_{[1]}$.

Definition 4.3. Let F be a presheaf on SSM. Its restriction to SSM_1 is an object in $\widehat{\mathrm{SSM}}_1$. The corresponding object F_1 in $\widehat{\mathrm{SM}}_{[1]}$ is called the 1-*jet* of F.

Remark 4.4. The representable 1-jets are of particular interest, since they are represented by differential non-negatively graded supermanifolds.[6] Hence, they can provide us with interesting examples of those. If the $\mathbb{Z}_2$ is given by the parity of the $\mathbb{Z}$-degree, which will be always the case in our examples, then we have a differential non-negatively graded manifold. Let us recall, that a finite-dimensional, positively graded differential manifold is the same thing as an L_∞-algebra. If it is only a non-negatively graded one then it could be, for good reasons explained in [2], referred to as an L_∞-algebroid, cf. also [16] for a formal definition.

Particular examples of presheaves on SSM come from simplicial supermanifolds. If K is a simplicial supermanifold and X the nerve of the groupoid defined by the surjective submersion $M \to N$, the corresponding sheaf $F^K \in \widehat{\mathrm{SSM}}$ is defined by

$$F^K(M \to N) = \mathrm{Hom}(X, K),$$

i.e. it associates with the surjective submersion $M \to N$ the set of all simplicial maps $X \to K$.

In [2], also the following sufficient condition for the 1-jet F_1^K of F^K to be representable is given.

Theorem 4.5. *Let K be a simplicial supermanifold fulfilling the Kan conditions, which is moreover m-truncated for some $m \in \mathbb{N}$. Then the 1-jet F_1^K is representable.*[7]

Another construction described in [2] is the so called 1-approximation of a presheaf $F \in \widehat{\mathrm{SSM}}$. The restriction of F to SSM_1 admits a right adjoint, the induction.

Definition 4.6. The presheaf $\mathrm{app}_1 F \in \widehat{\mathrm{SSM}}$ is defined by successively applying the restriction and induction functors to $F \in \widehat{\mathrm{SSM}}$.

Proposition 4.7. *If the jet 1-jet F_1 is represented by the differential non-negatively graded supermanifold X_F then the sheaf $\mathrm{app}_1 F \in \widehat{\mathrm{SSM}}$ is given by*

$$\begin{aligned}\mathrm{app}_1 F(M \to N) &= \{\textit{morphisms of dg manifolds } T[1](M \to N) \to X_F\}\\ &= \{\textit{morphisms of dg algebras } C^\infty(X_F) \to \Omega(M \to N)\},\end{aligned}$$

where $T[1](M \to N)$ is the shifted fibrewise tangent bundle of M and $\Omega(M \to N) = C^\infty(T[1](M \to N))$ are the fibrewise differential forms on M.

If X_F is positively graded then it can be identified with an L_∞-algebra L_F and we have

$$\mathrm{app}_1 F(M \to N) = \{\textit{Maurer–Cartan elements of } L_F \otimes \Omega(M \to N)\}.$$

6 Differential graded supermanifolds, i.e. Q-supermanifolds provide a natural framework for the Batalin–Vilkovisky formalism [15].

7 We refer to the proof as well for explanation of the Kan conditions and m-truncatedness to the Appendix of [2]. These notions were, for simplicial manifolds, first introduced in [17]. If G is a simplicial Lie group then G, $\overline{W}G$ and WG fulfill the Kan conditions[18].

Example 4.8. Consider the presheaf in $\widehat{SSM}$ represented by $Y \to X$. Its 1-jet is 1-representable by $T[1](Y \to X)$ (equipped with the canonical differential), the shifted fibrewise tangent bundle. This is just a fibrewise version of the following well known fact $\mathrm{Hom}(R^{0|1} \times N \to N, Y \to *) = \mathbf{Hom}(R^{0|1}, Y)(N) \cong \mathrm{Hom}(N, T[1]Y)$, i.e. that "maps from $\mathbb{R}^{0|1}$ to M are the same things as 1-forms on M".

Example 4.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $M \to N$ a surjective submersion. A G-descent data on $M \to N$, i.e a descent of a trivial G-bundle on M to a G-bundle on N, is a map $g : M \times_N M \to G$ satisfying $g(x, x) = e$ and $g(x, y)g(y, z) = g(x, z)$ for $(x, y, z) \in M \times_N M \times_N M$. The G-descent is the same thing as a groupoid morphism from $M \times_N M \to G$. Let us consider the presheaf

$$F(M \to N) = \{G - \text{descent data on } M \to N\},$$

which is in the above notation $F^{\mathcal{N}G}$, with $\mathcal{N}G$ the nerve of G.

By definition, we have for the 1-jet $F_1^{\mathcal{N}G}$

$$F(\mathbb{R}^{0|1} \times N \to N) = \{G - \text{descent data on } \mathbb{R}^{0|1} \times N \to N\}.$$

Such a G-descent data is a map $g : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to G^N$ satisfying the above descent (1-cocycle) condition and is equivalent to a map $\bar{g} : \mathbb{R}^{0|1} \to G^N$, such that $\bar{g}(0) = e$. The relation between maps g and $\bar{g}$ is[8]

$$g(\theta_0, \theta_1) = \bar{g}(\theta_0)^{-1}\bar{g}(\theta_1),$$
$$\bar{g}(\theta) = g(0, \theta).$$

One way to see what the dg manifold representing the 1-jet looks like is the following. Instead of imposing the condition $\bar{g}(0) = e$, we can consider arbitrary functions $\bar{g}(\theta)$ modulo left multiplications with the constant ones. So what we have is the shifted tangent bundle $T[1]G$ equipped with the canonical differential induced from the de Rham differential on G modulo the left G-action. This observation immediately leads to the dg algebra of functions on $\mathfrak{g}[1] \cong \mathbf{Hom}(\mathbb{R}^{0|1}, G)/G$ — the wedge algebra of left invariant forms on G with the de Rham differential, which is just the Chevalley–Eilenberg complex of $\mathfrak{g}$.

Equivalently, with an obvious abuse of notation, which we will commit also in the rest of the paper, we note that we can write

$$\bar{g}(\theta) = e - a\theta,$$

with $a \in \mathfrak{g}^N[1]$. Hence, the 1-jet F_1 is represented by the shifted Lie algebra $\mathfrak{g}[1]$. The differential is computed from

$$g(\theta_0, \theta_1) = 1 + a(\theta_0 - \theta_1) + \frac{1}{2}[a, a]\theta_0\theta_1$$

by computing

$$-(da)\theta_1 = (\delta_\epsilon \bar{g}(\theta_1) - e) = \frac{d}{d\epsilon}(g(\theta_0 + \epsilon, \theta_1 + \epsilon) - e)|_{\epsilon=\theta_0=0} = -\frac{1}{2}[a, a]\theta_1.$$

Hence, the differential is

$$da = \frac{1}{2}[a, a].$$

Finally, the functor $\mathrm{app}_1 F$ associates to a surjective submersion $M \to N$ the set of flat fibrewise connections.[9]

Example 4.10. Let $H \xrightarrow{\delta_1} D$ be a crossed module of Lie groups with the induced crossed module of Lie algebras

$$\mathfrak{h} \xrightarrow{\delta_1} \mathfrak{d}$$

and $M \to N$ a surjective submersion. An $H \to D$-descent data on $M \to N$, is an $(H \to D)$-valued 1-cocycle on the groupoid $Y = M \times_N M$. Such 1-cocycles describe bundle gerbes, similarly as transition functions describe principal bundles [19]. More explicitly, we have a pair of maps (h, d), $d : Y_1 \to D$ and $h : Y_2 \to H$, such that

$$d(y_1)d(y_2) = \delta_1(h(y_1, y_2))d(y_1 \circ y_2), \quad \text{for } (y_1, y_2) \in Y_2,$$
$$h(y_1, y_2)h(y_1 \circ y_2, y_3) = {}^{d(y_1)}h(y_2, y_3)h(y_1, y_2 \circ y_3) \quad \text{for } (y_1, y_2, y_3) \in Y_3,$$

and

$$d(e_x) = e \quad \text{and} \quad h(e_{s(y)}, y) = h(y, e_{t(y)}) = e.$$

8 Notice that $\bar{g}$ is just a trivialization of the 1-cocycle g over $\mathbb{R}^{0|1}$.

9 Vice versa, if G and all the fibers are 1-connected flat fibrewise connections give us G-descents.

Let us consider the presheaf

$$F(M \to N) = \{(H \to D) - \text{descent data on } M \to N\},$$

which is in the above notation $F^{\mathcal{N}(H\to D)}$, with $\mathcal{N}(H \to D)$ the Duskin nerve of $H \to D$.

By definition, we have for the 1-jet $F_1^{\mathcal{N}(H\to D)}$

$$F(\mathbb{R}^{0|1} \times N \to N) = \{(H \to D) - \text{descent data on } \mathbb{R}^{0|1} \times N \to N\}.$$

Such an $(H \to D)$-descent data is a pair of maps (h, d), $d : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to D^N$ and $h : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to H^N$ satisfying the 1-cocycle condition, and is equivalent to a pair of maps $(\bar{h}, \bar{d})$, $\bar{d} : \mathbb{R}^{0|1} \to D^N$ and $\bar{h} : \mathbb{R}^{0|1} \times \mathbb{R}^{0|1} \to H^N$ such that

$$\bar{d}(0) = e, \qquad \bar{h}(\theta, \theta) = \bar{h}(0, \theta) = e.$$

The relation between pairs of maps (d, h) and $(\bar{d}, \bar{h})$ is[10]

$$d(\theta_0, \theta_1) = \bar{d}(\theta_0)^{-1}\delta_1(\bar{h}(\theta_0, \theta_1))\bar{d}(\theta_1),$$
$$^{\bar{d}(\theta_0)}h(\theta_0, \theta_1, \theta_2) = \bar{h}(\theta_0, \theta_1)\bar{h}(\theta_1, \theta_2)\bar{h}(\theta_0, \theta_2)^{-1}$$

and

$$\bar{d}(\theta) = d(0, \theta),$$
$$\bar{h}(\theta_0, \theta_1) = h(0, \theta_0, \theta_1).$$

Obviously, we can write

$$\bar{d}(\theta) = e - a\theta,$$

with $a \in \mathfrak{d}[1]$ and

$$\bar{h}(\theta_0, \theta_1) = e + b\theta_0\theta_1,$$

with $b \in \mathfrak{h}[2]$. Hence, the 1-jet F_1 is represented by the graded vector space $\mathfrak{d}[1] \oplus \mathfrak{h}[2]$.

The differential is computed in a complete analogy with Example 4.9 using expressions

$$d(\theta_0, \theta_1) = e + a(\theta_0 - \theta_1) + \left(\frac{1}{2}[a, a] + \delta_1 b\right)\theta_0\theta_1,$$
$$h(\theta_0, \theta_1, \theta_2) = e + b(\theta_0\theta_1 + \theta_1\theta_2 - \theta_0\theta_2) - {}^{a}b\,\theta_0\theta_1\theta_2.$$

The resulting differential is:

$$da = \frac{1}{2}[a, a] + \delta_1 b,$$
$$db = {}^{a}b.$$

Since we have a positively graded dg manifold, we can describe it as an L_∞-algebra. It is actually a DGLA with generators only in lowest two degrees, i.e a strict Lie 2-algebra. The nonzero components are $L_0 = \mathfrak{d}$ and $L_{-1} = \mathfrak{h}$. The differential is $\delta_1 : \mathfrak{h} \to \mathfrak{d}$. The bracket on $\mathfrak{d}$ is given by its own Lie bracket, and the bracket between $\mathfrak{d}$ and $\mathfrak{h}$ is given by the action of $\mathfrak{d}$ on $\mathfrak{h}$. Let us note that Lie 2-algebras are one to one to crossed modules of Lie groups, cf. [20].

Finally, the functor $\mathrm{app}_1 F$ associates to a surjective submersion $M \to N$ the set of $(\mathfrak{h} \to \mathfrak{d})$-valued flat fibrewise connections.

5. L_∞-algebra of $\overline{W}G$

In this section we generalize Examples 4.9 and 4.10 to the case of a G-descent, where G is a simplicial Lie group with Moore complex of length k. The associated simplicial Lie algebra will be denoted by $\mathfrak{g}$. Examples 4.9 and 4.10 correspond to $k = 1$ and $k = 2$ respectively. Let $M \to N$ be a surjective submersion. We define a G-descent data on $M \to N$ as a G-valued twisting on the nerve of the groupoid $N \times_M N$. We recall, cf. Definition 3.1, that for $\tau : X_n \to G_{n-1}$ to be a twisting, the following conditions should be fulfilled:

$$\partial_0\tau(x)\tau(\partial_0 x) = \tau(\partial_1 x),$$
$$\partial_i\tau(x) = \tau(\partial_{i+1}x) \quad \text{for } i > 0,$$
$$s_i\tau(x) = \tau(s_{i+1}x) \quad \text{for } i \geq 0,$$
$$\tau(s_0 x) = e_n \quad \text{for } x \in X_n.$$

10 Again, the pair $(\bar{h}, \bar{d})$ is just a trivialization of the 1-cocycle (h, d).

Let us consider the presheaf

$$F(M \to N) = \{G - \text{descent data on } M \to N\},$$

which is in the notation of the previous section $F^{\overline{W}G}$, i.e. the sheaf associating with the surjective submersion $N \to M$ the set of all simplicial maps from the nerve of the groupoid $N \times_M N$ to the classifying space $\overline{W}G$.

By definition, we have for the 1-jet $F_1^{\overline{W}G}$

$$F(\mathbb{R}^{0|1} \times N \to N) = \{G - \text{descent data on } \mathbb{R}^{0|1} \times N \to N\}.$$

Such a G-descent data is described by a twisting[11] $\tau : (\mathbb{R}^{0|1})^n \to G^N_{n-1}$ and is equivalent to a function $\psi : (\mathbb{R}^{0|1})^n \to G^N_n$ such that

$$\partial_i \psi(\theta_0, \dots \theta_n) = \psi(\theta_0, \dots, \hat{\theta}_i, \dots \theta_n) \quad \text{if } i > 0,$$
$$s_i \psi(\theta_0, \dots \theta_n) = \psi(\theta_0, \dots, \theta_i, \theta_i \dots \theta_n) \quad \text{if } i \geq 0.$$

We have the following relation between τ and ψ[12]

$$\tau(\theta_0, \dots \theta_n) = \partial_0 \psi(\theta_0, \dots, \theta_n)^{-1} \psi(\theta_1, \dots, \theta_n),$$
$$\psi(\theta_0, \dots, \theta_n) = \tau(0, \theta_0, \dots, \theta_n).$$

From the definition of ψ it follows that

$$\psi(0, \theta_1, \dots, \theta_n) = \tau(0, 0, \theta_1, \dots, \theta_n) = \tau(s_0(0, \theta_1, \dots, \theta_n)) = e_n.$$

Therefore, we write

$$\psi(\theta_0, \dots, \theta_n) = 1 - a(\theta_1, \dots \theta_n)\theta_0,$$

with $a(\theta_1, \dots \theta_n) \in \oplus_{i=0}^n \binom{n}{i} \mathfrak{g}_n[i+1]$. The function a fulfills the following identities

$$\partial_i a(\theta_1, \dots \theta_n) = a(\theta_1, \dots, \hat{\theta}_i, \dots \theta_n) \quad \text{if } i > 0, \tag{5.1}$$
$$\partial_0 a(0, \theta_1, \dots \theta_n) = a(\theta_1, \dots \theta_n) \tag{5.2}$$
$$s_i a(\theta_1, \dots \theta_n) = a(\theta_1, \dots, \theta_i, \theta_i, \dots, \theta_n) \quad \text{if } i > 0, \tag{5.3}$$
$$s_0 a(\theta_1, \dots, \theta_n) = a(0, \theta_1, \dots, \theta_n). \tag{5.4}$$

In the above list, the only possibly not completely obvious one is the ∂_0 Eq. (5.2). However, this one follows from the s_0 Eq. (5.4) by an application of ∂_0. From (5.1) we immediately see that

$$a^n \in N\mathfrak{g}_n[n+1],$$

for the top component a^n of $a(\theta_1, \dots \theta_n) = a^n \theta_1 \dots \theta_n + \cdots$.

To proceed further, it will be more convenient to change the Grassmann coordinates by $\bar{\theta}_0 = \theta_1$ and $\bar{\theta}_i = \theta_{i+1} - \theta_i$ for $i > 1$. In terms of $\bar{\theta}$s, we get the following lemma for the decomposition of $a(\bar{\theta}_0 \dots \bar{\theta}_{n-1}) \in \oplus_{i=0}^n \binom{n}{i} \mathfrak{g}_n[i+1]$ in terms of the shifted Moore complex $N\mathfrak{g}_k[k+1] \oplus \dots \oplus N\mathfrak{g}_0[1]$.

Lemma 5.1. *For $n \leq k$*

$$a(\bar{\theta}_0, \dots, \bar{\theta}_{n-1}) = \sum_{\alpha \in S(n)} s_\alpha a^{n - \sharp\alpha} \bar{\theta}^{S(n) \setminus \alpha},$$

where $\bar{\theta}^\beta := \bar{\theta}_{i_1} \dots \bar{\theta}_{i_l}$ for $\beta = \{i_n, \dots, i_1\} \in S(n)$.

Proof. Straightforward computation using the fact that with the new Grassmann variables $\bar{\theta}$ we have nice simplicial relations $s_i a(\bar{\theta}_0, \dots \bar{\theta}_{n-1}) = a(\bar{\theta}_0, \dots, \bar{\theta}_{i-1}, 0, \bar{\theta}_i, \dots, \bar{\theta}_{n-1})$ and $s_{i-1}\partial_i a(\bar{\theta}_0, \dots, \bar{\theta}_{n-1}) = a(\bar{\theta}_0, \dots, \bar{\theta}_{n-1})|_{\bar{\theta}_{i-1}=0}$. □

We see that, for $n \leq k$, the only independent component of $a \in \oplus_{i=0}^n \binom{n}{i} \mathfrak{g}_n[i+1]$ is the top one $a^n \in \mathfrak{g}_n[n+1]$. Hence, the 1-jet F_1 in this case is represented by $N\mathfrak{g}_k[k+1] \oplus \cdots \oplus N\mathfrak{g}_0[1]$ as a graded manifold.

The differential can be obtained in analogy with Examples 4.9 and 4.10. We write

$$\begin{aligned}
\tau(\theta_0, \dots, \theta_n) &= e + \partial_0 a(\theta_1, \dots, \theta_n)\theta_0 - a(\theta_2, \dots, \theta_n)\theta_1 + \frac{1}{2}[\partial_0 a(\theta_1, \dots \theta_n), a(\theta_2, \dots, \theta_n)]\theta_0\theta_1 \\
&= e + \partial_0 a(\theta_1, \dots, \theta_n)\theta_0 - a(\theta_2, \dots, \theta_n)\theta_1 + \frac{1}{2}[\partial_0 a(0, \theta_2 \dots \theta_n), a(\theta_2, \dots, \theta_n)]\theta_0\theta_1 \\
&= e + \partial_0 a(\theta_1, \dots, \theta_n)\theta_0 - a(\theta_2, \dots, \theta_n)\theta_1 + \frac{1}{2}[a(\theta_2, \dots \theta_n), a(\theta_2, \dots, \theta_n)]\theta_0\theta_1.
\end{aligned}$$

[11] From now on we will omit the annoying N and assume it everywhere implicitly.

[12] As before, ψ is a trivialization of τ, cf. Definition 3.3.

Now, using the above expression for τ,

$$\begin{aligned} -(da(\theta_2,\dots,\theta_n))\theta_1 &= \delta_\epsilon(\psi(\theta_1,\dots,\theta_n)-e) = \frac{d}{d\epsilon}(\tau(\theta_0+\epsilon,\dots,\theta_n+\epsilon)-e)|_{\epsilon=\theta_0=0} \\ &= \frac{d}{d\theta_1}\partial_0 a(\theta_1,\dots,\theta_n)\theta_1 + \sum_{i=2}^{n}\frac{d}{d\theta_i}a(\theta_2,\dots,\theta_n)\theta_1 - \frac{1}{2}[a(\theta_2,\dots,\theta_n),a(\theta_2,\dots,\theta_n)]\theta_1. \end{aligned}$$

Hence, the differential is

$$da(\theta_1,\dots,\theta_n) = -\frac{d}{d\theta_0}\partial_0 a(\theta_0,\dots,\theta_n) - \sum_{i=1}^{n}\frac{d}{d\theta_i}a(\theta_1,\dots,\theta_n) + \frac{1}{2}[a(\theta_1,\dots,\theta_n),a(\theta_1,\dots,\theta_n)]. \tag{5.5}$$

Now, we proceed in extracting the action da^n of differential d on the top component a^n. For this note: the first term gives $-\partial_0 a^{n+1}$ and the second does not contribute to da^n at all. What remains is to determine the top component of the commutator $[a(\theta_1,\dots,\theta_n),a(\theta_1,\dots,\theta_n)]$. This leads to the following proposition.

Proposition 5.2. *Let G be a simplicial group with the simplicial Lie algebra $\mathfrak{g}$. Assume that its Moore complex NG is of length k. Then the 1-jet F_1 of the simplicial manifold $\overline{W}G$ is representable by the dg manifold $\oplus_{n=0}^{k} N\mathfrak{g}_n[n+1]$. The differential da^n on $a^n \in N\mathfrak{g}_n[n+1]$ is described in terms of the face map ∂_0, commutator of $\mathfrak{g}_0$, action of $N\mathfrak{g}_0$ on $N\mathfrak{g}_n$ and Peiffer pairings $f_{\alpha,\beta}$ with $(\alpha,\beta)\in\bar{P}(n)$ as follows:*

For $n=0$

$$da^0 = -\partial_0 a^1 + \frac{1}{2}[a^0,a^0],$$

for $n>0$

$$da^n = -\partial_0 a^{n+1} + {}^{a^0}a^n + \sum_{(\alpha,\beta)\in\bar{P}(n)} \pm f_{\alpha,\beta}(a^{n-\sharp\alpha},a^{n-\sharp\beta}),$$

where the sign is given by the product of parity of $(n-\sharp\alpha)(n-\sharp\beta+1)$ and the parity of the shuffle defined by the pair $(S(n)\setminus\alpha, S(n)\setminus\beta)$.

Proof. The 0th component is clear. What is left is to justify the form of the second and third term in the above expression for da^n, $n>0$. However, this is easily done using the above Lemma 5.1. We just have to be careful about the degrees and signs. We have

$$\frac{1}{2}[a(\bar\theta_0,\dots,\bar\theta_{n-1}),a(\bar\theta_0,\dots,\bar\theta_{n-1})]^n = [s_{n_1}\dots s_0a^0,a^n] + \sum_{(\alpha,\beta)\in\bar{P}(n)} \pm[s_\alpha a^{n-\sharp\alpha}, s_\beta a^{n-\sharp\beta}],$$

with the sign given as the product of parities of $(n-\sharp\alpha)(n-\sharp\beta+1)$ and of the shuffle defined by the pair $(S(n)\setminus\alpha, S(n)\setminus\beta)$. Of course $\sharp\alpha+\sharp\beta=n$.

The first term is just ${}^{a^0}a^n$, i.e. describing the action of $N\mathfrak{g}_0$ on $N\mathfrak{g}_0$ shifted by 1 in degree. Further, note that, by construction, the face ∂_i and degeneracy maps s_i commute with the differential d. In particular, it follows that da^n must be in $\in\bigcap_{i>0}\ker\partial_i[n+2]$. Moreover, since $\partial_i\partial_0=\partial_0\partial_{i+1}$, we also have $\partial_0 a^{n+1}\in\bigcap_{i>0}\ker\partial_i[n+2]$. Therefore, we conclude that the sum over pairs $(\alpha,\beta)\in\bar{P}(n)$ in the above equation is also in $\bigcap_{i>0}\ker\partial_i[n+2]$ and as such can be written, trivially inserting the projection $p_n:\mathfrak{g}_n\to N\mathfrak{g}_n$, as $\sum_{(\alpha,\beta)\in\bar{P}(n)}\pm p_n[s_\alpha a^{n-\sharp\alpha}, s_\beta a^{n-\sharp\beta}]$. □

It is now straightforward to describe the L_∞-algebra corresponding to the above dg manifold explicitly. What we have is a k-term DGLA $L=\oplus_{n=0}^{k}L_{-n}$ with components in degrees $0,-1,\dots-k$, given by $L_{-n}=N\mathfrak{g}_n$. The differentials $d_n: N\mathfrak{g}_n\to N\mathfrak{g}_{n+1}$ are given by the restrictions $d_n=\partial_0|_{N\mathfrak{g}_n}$ of the zeroth face maps, i.e by the differentials δ_n of the Moore complex $N\mathfrak{g}$, i.e, for $x_n\in N\mathfrak{g}_n$

$$d_nx_n=\delta_nx_n. \tag{5.6}$$

The only nonzero brackets are the binary brackets. The nonzero binary brackets are determined by the following prescription:

The bracket $N\mathfrak{g}_0\times N\mathfrak{g}_0\to N\mathfrak{g}_0$ is just the Lie bracket on $N\mathfrak{g}_0$, i.e for $x_0\in N\mathfrak{g}_0$ and $y_0\in N\mathfrak{g}_0$

$$[x_0,y_0]. \tag{5.7}$$

The brackets $N\mathfrak{g}_0\times N\mathfrak{g}_n\to N\mathfrak{g}_n$: $(x,y)\mapsto[x_0,x_n]=-[x_n,x_0]$ are given by the action of $N\mathfrak{g}_0$ on $N\mathfrak{g}_n$

$$[x_0,x_n]=-[x_n,x_0]={}^{x_0}x_n. \tag{5.8}$$

The bracket $N\mathfrak{g}_{n_1} \times N\mathfrak{g}_{n_2} \to N\mathfrak{g}_n$ with $n = n_1 + n_2$, for n_1 and n_2 nonzero, is described as follows: for $x_{n_1} \in N\mathfrak{g}_{n_1}$ and $x_{n_2} \in N\mathfrak{g}_{n_2}$

$$[x_{n_1}, x_{n_2}] = \sum_{(\alpha,\beta)\in\bar{P}(n_1,n_2)} \pm f_{\alpha,\beta}(x_{n_1}, x_{n_2}) + (-1)^{(n_1+1)(n_2+1)} \sum_{(\alpha,\beta)\in\bar{P}(n_2,n_1)} \pm f_{\alpha,\beta}(x_{n_2}, x_{n_1}) \tag{5.9}$$

The $\pm$ sign is given by the product of parity of $n_1(n_2+1)$ and the parity of the shuffle defined by the pair $(\alpha, \beta) \in \bar{P}(n_1, n_2)$. Here $\bar{P}(n_1, n_2) \subset \bar{P}(n)$ denotes the subset of $P(n)$ consisting of those pairs $(\alpha, \beta) \in \bar{P}(n)$, for which $n-\sharp\alpha = n_1$, $n-\sharp\beta = n_2$.

Let us now consider an arbitrary simplicial Lie algebra $\mathfrak{g}$ with Moore complex of length k. Associated to $\mathfrak{g}$ we have the (unique) simplicial group G integrating it, such that all its components are simply connected. Therefore, starting with $\mathfrak{g}$, we can consider the functor $F_1^{\overline{W}G}$. Correspondingly, we have the following.

Proposition 5.3. *Let $\mathfrak{g}$ be a simplicial Lie group with Moore complex $N\mathfrak{g}$ of length k. Then $N\mathfrak{g}$ or becomes a DGLA. The differential and the binary brackets are explicitly given by formulas (5.6)–(5.9). This DGLA structure on $N\mathfrak{g}$ is the same one as described by Quillen's construction in Proposition 4.4 of [1].*

We finish with some remarks:

Remark 5.4. Obviously, one can reformulate the above theorem in terms of a k-hypercrossed complex of Lie algebras $\underline{\mathfrak{g}}$. Such a k-hypercrossed complex $\underline{\mathfrak{g}}$ has a structure of a k-term DGLA described by (5.6)–(5.9).

Remark 5.5. As noted above, crossed modules and Lie 2-algebras are one to one. From the above theorem, we see that for $n > 2$ only a part of the full hypercrossed complex structure enters the description of the DGLA L. For instance, already for $n = 3$, only the symmetric part of the Peiffer pairing appears. Nevertheless, the simplicial (Kan) manifold $\overline{W}G$ can be interpreted as an integration of the DGLA $N\mathfrak{g}$.

On the other hand, any L_∞-algebra L, in particular any DGLA, can be integrated to a (Kan) simplicial manifold $\int L$ [17]. So one might try to compare the integration $\int N\mathfrak{g}$ with the Kan simplicial manifold $\overline{W}G$, or the corresponding 1-jets (differentiations). Here we restrict ourselves only to two related (obvious) remarks.

First, there is the following observation: Let M be a simplicial manifold. Assume that its corresponding 1-jet functor F_1^M is representable by an L_∞ algebra L, with Chevalley-Eilenberg complex $C(L)$. Also, let $\Omega(\Theta_N)$ be the DGA of (normal) forms on the simplicial supermanifold Θ_N, the nerve associated to the surjective submersion $\mathbb{R}^{0|1} \times N \to N$.[13] Then, for the 1-jet corresponding to $\int L$, we have $\mathrm{Hom}(\Theta_N, \int L) = \{$morphisms of dg algebras $C(L) \to \Omega(\Theta_N)\}$. This has to be compared to 1-jet corresponding to M, i.e. to the set $\mathrm{Hom}(\Theta_N, M) = \{$morphisms of dg algebras $C(L) \to \Omega(\mathbb{R}^{0|1} \times N \to N)\}$.

Second, in [17], simplicial homotopy groups $\pi_n^{\mathrm{spl}} \int L$ of $\int L$ have been shown to be finite dimensional diffeological groups. Lie algebra of a diffeological group is defined as the Lie algebra of its universal cover, which is a Lie group. It is a result of [17] that the Lie algebra of $\pi_n^{\mathrm{spl}} \int L$ is canonically isomorphic to $H_{n-1}(L)$. Specified to the case of our interest: the Lie algebra of $\pi_n^{\mathrm{spl}} \int N\mathfrak{g}$ is canonically isomorphic to $H_{n-1}(N\mathfrak{g})$. If we now consider $\overline{W}G$ just as a simplicial set then for its simplicial homotopy groups we have $\pi_n\overline{W}G = \pi_{n-1}G = H_{n-1}(NG)$. Given the Lie structure of G, $H_{n-1}(NG)$ and hence $\pi_n\overline{W}G$ can be considered as diffeological groups. In this sense we can talk about the Lie algebra of $\pi_n\overline{W}G$, which is again $H_{n-1}(N\mathfrak{g})$.

Remark 5.6. The functor $\mathrm{app}_1 F^{\overline{W}G}$ associates to a surjective submersion $M \to N$ the set of L-valued flat fibrewise connections. To obtain also non-flat connections one may use the Weil algebra of L, similarly as in [22]. This, as well as applications to higher gauge theory, will be described elsewhere.

Remark 5.7. Regarding applications to higher gauge theory, in the forthcoming work we plan to extend the results presented here to the case of simplicial groupoids. In particular, we hope to describe a proper generalization of the Atiyah groupoid to the simplicial case and then obtain the Atiyah L_∞-algebroid as a 1-jet of a properly defined simplicial classifying space. This would lead us directly to the notion of an L-valued connection on a simplicial principal G-bundle (or on the corresponding hypercrossed complex bundle gerbe). That this should be possible can be seen from the description of connections and curvings of crossed module bundle gerbes in [23], where a categorification of the Atiyah algebroid is presented.

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13 See, e.g., [21] for the definition of forms on simplicial sets.

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