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Smooth approximation

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Contents

Foreword

This work consists essentially of Chapter 7 of the book [HJ], with the following differences:

- the last section, dealing with the approximation of norms, is omitted;
- some special results in Section 2 pertaining to class $\mathcal{W}$ are omitted;
- there are some small improvements around Theorem 48.

The omissions were introduced so that the text is more compact, more "self-contained", and also to minimise the input of the coauthor of the book. The work contains results from the author's papers [Jo1], [HJ2], [HJ3], [Jo2], and [Jo3], of course as well as many results of many other mathematicians, usually with reworked and streamlined proofs.

The references to statements whose numbering use the dot convention are references to the book [HJ], e.g. Theorem 1.90 refers to Theorem 90 in Chapter 1 of [HJ].

Notation

We fix some notation for objects and notions that the reader should be familiar with. By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ we denote the sets of natural numbers, integers, rational numbers, reals, and complex numbers respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. By $\mathbb{R}^+$ we denote the set of positive real numbers. By $\mathbb{K}$ we denote the scalar field $\mathbb{R}$ or $\mathbb{C}$. We use the convention that a sum over an empty set is zero and a product over an empty set is equal to 1. Further, $x^0 = 1$ for any $x \in \mathbb{K}$. For $x \in \mathbb{R}$ we denote by $[x]$ the integer part of x, i.e. the unique number $k \in \mathbb{Z}$ satisfying $k \leq x < k + 1$, by $\lceil x \rceil$ we denote the ceiling of x, i.e. the unique number $k \in \mathbb{Z}$ satisfying $k - 1 < x \leq k$.

For a set A we denote its cardinality by $|A|$ or $\operatorname{card} A$. By abusing the notation we write $\{x_\gamma\}_{\gamma\in\Gamma} \subset X$ meaning that $\{x_\gamma\}_{\gamma\in\Gamma}$ is a collection such that $x_\gamma \in X$ for each $\gamma \in \Gamma$.

Let (P, ρ) be a metric space. We denote $B(x, r) = \{y \in P;\ \rho(y, x) \leq r\}$ and $U(x, r) = \{y \in P;\ \rho(y, x) < r\}$ the closed, resp. open ball in P centred at $x \in P$ with radius $r > 0$. In case that it is necessary to distinguish the spaces in which the balls are taken, we will write $B_P(x, r)$, resp. $U_P(x, r)$. By B_X and U_X we denote the closed, resp. open unit ball of a normed linear space X. By S_X we denote the unit sphere of a normed linear space X. An interior of a set A in a topological space is denoted by $\operatorname{Int} A$.

When we speak of a subspace of a Banach space, we always mean a closed subspace. General subspaces will be referred to as "linear subspaces". We define $\operatorname{span} \emptyset = \{0\}$. If X is a normed linear space with a Schauder basis $\{e_n\}$ and $x = \sum_{n=1}^{\infty} x_n e_n \in X$, then $\operatorname{supp} x = \{n \in \mathbb{N};\ x_n \neq 0\}$ is called a support of x; a finitely supported vector is a vector with finite support. A topological dual of a topological vector space is denoted by X^*. Inner product is denoted by $\langle x, y\rangle$. Let X, Y be normed linear spaces. For simplicity we say that X contains Y if X has a subspace isomorphic to Y.

By $C(X;Y)$ we denote the set of continuous mappings between topological spaces X, Y. If Y is a topological vector space, then $C(X;Y)$ is a vector space. For functions, i.e. mappings into the scalars, we use a shortened notation $C(X) = C(X;\mathbb{K})$; from the context it should always be clear whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. For a mapping $f: X \to Y$, where Y is a vector space, we denote $\operatorname{supp}_\circ f = f^{-1}(Y \setminus \{0\})$. If X is a topological space, then we denote $\operatorname{supp} f = \overline{\operatorname{supp}_\circ f}$. An L-Lipschitz mapping is a mapping that is Lipschitz with a constant L. By χ_A we denote the characteristic function of the set A.

The n-dimensional Lebesgue measure will be denoted by λ_n, or just λ if the dimension is clear from the context.

All topological spaces are automatically and without mention assumed to be Hausdorff.

By $\tilde{X}$ we denote the Taylor complexification of a real normed linear space X. By $H(U;Y)$ we denote the vector space of holomorphic mappings from an open set U of a complex normed linear space into a complex Banach space Y. By $C^k(U;Y)$, resp. $C^\omega(U;Y)$ we denote the vector space of C^k-smooth, resp. analytic, mappings from an open set U of a normed linear space into a normed linear space Y. If $Y = \mathbb{K}$, then we write $H(U)$, resp. $C^k(U)$, resp. $C^\omega(U)$ for short.

Let X, Y be normed linear spaces. By $\mathcal{P}(^nX;Y)$ we denote the space of continuous n-homogeneous polynomials from X to Y. By $\mathcal{P}^n(X;Y)$ we denote the space of continuous polynomials of degree at most n from X to Y. By $\mathcal{P}(X;Y)$ we denote the space of continuous polynomials from X to Y. Again, in the scalar case $Y = \mathbb{K}$ we write just $\mathcal{P}(^nX)$, $\mathcal{P}^n(X)$, or $\mathcal{P}(X)$.

Let $U \subset X$ be open, $f: U \to Y$, and $x \in U$. By $\frac{\partial f}{\partial h}(x)$ we denote the directional derivative of f at x in the direction $h \in X$. By $Df(x)$ we denote the Fréchet derivative of f at x, and by $Df(x)[h]$ we denote the evaluation of this derivative in $h \in X$. Similarly we denote by $D^k f(x)$ the kth Fréchet derivative of f at x. By $d^k f(x)$ we denote the k-homogeneous polynomial corresponding to the symmetric k-linear mapping $D^k f(x)$, so $d^k f(x)[h] = D^k f(x)[h, \dots, h]$. For convenience we put $d^0 f = f$.

A modulus is a non-decreasing function $\omega\colon [0, +\infty) \to [0, +\infty]$ continuous at 0 with $\omega(0) = 0$. The set of all moduli will be denoted by $\mathcal{M}$ and the set of all sub-additive moduli will be denoted by $\mathcal{M}_s$. If $f \in C^k(U;Y)$ and $\Omega \subset \mathcal{M}$ is a convex cone, then we say that f is $C^{k,\Omega}$-smooth on U if $d^k f$ is uniformly continuous on U with modulus ω for some $\omega \in \Omega$. We denote by $C^{k,\Omega}(U;Y)$ the vector space of all $C^{k,\Omega}$-smooth mappings from U into Y. Let $\alpha \in [0, 1]$. We say that f is $C^{k,\alpha}$-smooth on U if $f \in C^{k,\Omega}(U;Y)$ for $\Omega = \{Kt^\alpha;\ K \in \mathbb{R}^+\}$, i.e. $d^k f$ is α-Hölder on U. We say that f is $C^{k,+}$-smooth on U if $f \in C^k(U;Y)$ and $d^k f$ is uniformly continuous on U. The vector spaces of all respective mappings are denoted by $C^{k,\alpha}(U;Y)$ and $C^{k,+}(U;Y)$. In particular, $f \in C^{k,1}(U;Y)$ if $d^k f$ is Lipschitz on U. We say that f is locally $C^{k,1}$-smooth on U if for each $x \in U$ there is a neighbourhood of x on which f is $C^{k,1}$-smooth. We denote by $C^{k,1}_{\mathrm{loc}}(U;Y)$ the vector space of all $C^{k,1}_{\mathrm{loc}}$-smooth mappings from U into Y.

Smooth approximation

We are concerned with the general problem of approximating a given mapping from a subset of a Banach space X into a Banach space Y by means of polynomials or C^k-smooth mappings. The classical example is the Weierstraß-type Theorem 9, where arbitrary C^k-smooth mapping from a compact set $K \subset \mathbb{R}^n$ is shown to be approximable by polynomials, uniformly on K together with derivatives of order up to k. The best-known case is when $k = 0$.

In Section 3 we prove one of the highlights in the theory of smoothness, Theorem 20 of Jaroslav Kurzweil. This result claims that if a real separable Banach space X admits a separating polynomial, then every continuous mapping from X into a Banach space can be uniformly approximated by real analytic mappings. By adjusting the proof somewhat it can be shown that the result remains true for Banach spaces that admit a separating analytic function with uniform radii of convergence (e.g. for c_0), provided that the approximated mapping is uniformly continuous.

One of the principal tools for obtaining C^k-smooth approximations of continuous mappings are partitions of unity, which are studied in Section 5. This is a very powerful tool which leads to general positive results in separable spaces, as well as in non-separable WCG or $C(K)$ spaces, admitting a C^k-smooth bump.

The rest is devoted to the study of smooth approximations preserving special properties of the approximated mapping. In order to study C^k-smooth approximations of Lipschitz mappings preserving the Lipschitz condition we introduce the concept of sup-partitions of unity and characterise it by means of componentwise C^k-smooth and bi-Lipschitz embeddings into $c_0(\Gamma)$. We show that every separable Banach space admitting a Lipschitz and C^k-smooth bump admits C^k-smooth sup-partitions of unity. This is applied to establish the existence of C^k-smooth and Lipschitz approximations of a given Lipschitz function in a separable Banach space X admitting a C^k-smooth and Lipschitz bump function. The real analytic case is also included, under the assumption of the existence of a separating polynomial. We also obtain results of this sort for vector valued Lipschitz mappings for certain types of the domain or range spaces.

In Section 8 we prove, again for certain types of the domain or range spaces, the existence of approximations of C^1-smooth mappings by C^k-smooth mappings for the mapping and its first derivative.

The last section is devoted to C^k-smooth (and convex) approximations of convex functions. It is rather easy to see that this problem is essentially equivalent to the statement that every equivalent norm on X can be approximated by C^k-smooth norms. A necessary condition for this result is clearly the existence of at least some equivalent C^k-smooth norm on X (similarly to the existence of a C^k-smooth bump in the general case). We prove that this condition is also sufficient for all separable Banach spaces.

The methods can be essentially divided into two groups: Global methods, in which the approximating mapping is constructed on the whole space at once by a formula – these are represented by integral and infimal convolutions; and local methods, in which we approximate locally and then glue together the approximations using for example partitions of unity. Some of the methods are mixed, for example the real analytic approximations although using the partitions of unity are necessarily retaining the global flavour. A note on the hypotheses on the target space: if the method employed uses a limiting procedure (this includes the Bochner integral, or countable partitions of unity), we need the completeness of the space for the process to converge. This is not necessary if we use a finite procedure such as the locally finite partitions of unity. Likewise, the domain space needs to be complete only in some circumstances, for example when using Schauder bases, or when dealing with real analytic mappings.

All the vector spaces are real unless stated otherwise.

1. Separation

As we shall see, a lot of the approximation methods ultimately boils down to the ability of separation of certain sets by smooth functions. In this section we present some of the separation results that will be used later. Some of them are somewhat technical, as we require rather fine separation properties. Nevertheless, we begin with a lemma that serves as a prominent tool for smoothing up mappings on $\mathbb{R}^n$ and lies behind most of the approximation results. The method was used already by Karl Weierstraß.

Let X be a set, Y a normed linear space, $f\colon X \to Y$, and $S \subset X$. We denote $\|f\|_S = \sup_{x\in S}\|f(x)\|$. In the following lemma we consider $\mathbb{C}^n$ with the Euclidean norm.

LEMMA 1. *Let Y be a Banach space, $C \subset Y$ a closed convex set, and let $f\colon \mathbb{R}^n \to C$ be strongly measurable (with respect to the Lebesgue measure) and bounded. Put $\Psi_\kappa(z) = \exp\left(-\kappa\sum_{j=1}^n z_j^2\right)$ for $z \in \mathbb{C}^n$ and $\kappa \in \mathbb{R}^+$, and define $g_\kappa\colon \mathbb{C}^n \to \tilde{Y}$ by the Bochner integral*

$$g_\kappa(z) = \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(z-y)f(y)\,\mathrm{d}y,$$

where $c_\kappa = \int_{\mathbb{R}^n}\Psi_\kappa(y)\,\mathrm{d}y = \left(\frac{\pi}{\kappa}\right)^{\frac{n}{2}}$. Then $g_\kappa \in H(\mathbb{C}^n;\tilde{Y})$ and $g_\kappa\restriction_{\mathbb{R}^n} \in C^\omega(\mathbb{R}^n;C)$ for every $\kappa \in \mathbb{R}^+$. Further, if f is Bochner integrable, then for every $\delta > 0$

$$\lim_{\kappa\to+\infty}\|g_\kappa\|_{G_\delta} = 0$$

(in case that $G_\delta \neq \emptyset$), where

$$G_\delta = \left\{z \in \mathbb{C}^n;\ \|\operatorname{Im} z\|^2 < \operatorname{dist}(\operatorname{Re} z, \operatorname{supp} f)^2 - \delta^2\right\}.$$

If f is L-Lipschitz, then so is each $g_\kappa\restriction_{\mathbb{R}^n}$, $\kappa \in \mathbb{R}^+$. Finally, if $f \in C^k(\mathbb{R}^n;Y)$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$ with all d^jf, $j = 0,\dots,k$ bounded and uniformly continuous on $\mathbb{R}^n$, then for all $0 \le j \le k$

$$\lim_{\kappa\to+\infty}\left\|d^jg_\kappa - d^jf\right\|_{\mathbb{R}^n} = 0.$$

PROOF. First note that

$$\begin{aligned}|\Psi_\kappa(z)| &= \exp\Bigl(-\kappa\sum_{j=1}^n\operatorname{Re}(z_j^2)\Bigr) = \exp\Bigl(-\kappa\sum_{j=1}^n\bigl((\operatorname{Re} z_j)^2 - (\operatorname{Im} z_j)^2\bigr)\Bigr)\\ &= \exp\Bigl(-\kappa\bigl(\|\operatorname{Re} z\|^2 - \|\operatorname{Im} z\|^2\bigr)\Bigr)\end{aligned}$$

and $D\Psi_\kappa(z)[h] = -2\kappa\Psi_\kappa(z)\sum_{j=1}^n z_j h_j$. Put $F(z,y) = \Psi_\kappa(z-y)f(y)$ and let $K > 0$ be such that $\|f\|_{\mathbb{R}^n} \le K$. Then for $z \in \mathbb{C}^n$, $\|z\| < r$, and $y \in \mathbb{R}^n$ we have

$$\begin{aligned}\|D_1F(z,y)\| &= 2\kappa|\Psi_\kappa(z-y)|\|z-y\|\|f(y)\|\\ &= 2\kappa K\exp\bigl(\kappa\|\mathrm{Im}\, z\|^2\bigr)\exp\bigl(-\kappa\|\mathrm{Re}\, z-y\|^2\bigr)\|z-y\|\\ &\le 2\kappa K\exp(\kappa r^2)\exp\bigl(-\kappa(\max\{\|y\|-r,0\})^2\bigr)(r+\|y\|).\end{aligned}$$

Hence we can apply Theorem 1.90 on bounded subsets of $\mathbb{C}^n$, which gives $g_\kappa \in H(\mathbb{C}^n;\tilde{Y})$ and thus also $g_\kappa\restriction_{\mathbb{R}^n} \in C^\omega(\mathbb{R}^n;Y)$. If $g_\kappa(x) \notin C$ for some $x \in \mathbb{R}^n$, then by the separation theorem there are $\phi \in Y^*$ and $\alpha \in \mathbb{R}$ such that $\phi\bigl(f(y)\bigr) < \alpha < \phi\bigl(g_\kappa(x)\bigr)$ for all $y \in \mathbb{R}^n$. But

$$\phi\bigl(g_\kappa(x)\bigr) = \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(x-y)\phi\bigl(f(y)\bigr)\,\mathrm{d}y < \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\alpha\Psi_\kappa(x-y)\,\mathrm{d}y = \alpha,$$

which is a contradiction. Hence $g_\kappa\restriction_{\mathbb{R}^n} \in C^\omega(\mathbb{R}^n;C)$.

Further, if f is Bochner integrable, then for a fixed $\delta > 0$ and any $z \in G_\delta$ we can estimate

$$\begin{aligned}\|g_\kappa(z)\| &\le \frac{1}{c_\kappa}\int_{\mathbb{R}^n}|\Psi_\kappa(z-y)|\|f(y)\|\,\mathrm{d}y\\ &= \frac{1}{c_\kappa}\int_{\mathrm{supp}\, f}\exp\Bigl(-\kappa\bigl(\|\mathrm{Re}\, z-y\|^2-\|\mathrm{Im}\, z\|^2\bigr)\Bigr)\|f(y)\|\,\mathrm{d}y\\ &\le \frac{1}{c_\kappa}\exp\left(-\kappa\delta^2\right)\int_{\mathrm{supp}\, f}\|f(y)\|\,\mathrm{d}y.\end{aligned}$$

Since $\lim\limits_{\kappa\to+\infty}\frac{1}{c_\kappa}\exp(-\kappa\delta^2) = 0$, it follows that $\lim\limits_{\kappa\to+\infty}\|g_\kappa\|_{G_\delta} = 0$.

For any $x \in \mathbb{R}^n$ we can use the substitution $y \to x-y$ to obtain

$$g_\kappa(x) = \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(y)f(x-y)\,\mathrm{d}y. \tag{1}$$

Thus if f is L-Lipschitz, then for any $u, v \in \mathbb{R}^n$ we have

$$\begin{aligned}\|g_\kappa(u)-g_\kappa(v)\| &= \frac{1}{c_\kappa}\left\|\int_{\mathbb{R}^n}\Psi_\kappa(y)f(u-y)\,\mathrm{d}y - \int_{\mathbb{R}^n}\Psi_\kappa(y)f(v-y)\,\mathrm{d}y\right\|\\ &\le \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(y)\|f(u-y)-f(v-y)\|\,\mathrm{d}y\\ &\le L\|u-v\|\frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(y)\,\mathrm{d}y = L\|u-v\|.\end{aligned}$$

Now suppose that $f \in C^k(\mathbb{R}^n;Y)$ for some $k \in \mathbb{N}_0 \cup \{\infty\}$ with all d^jf, $j = 0,\dots,k$ bounded and uniformly continuous on $\mathbb{R}^n$. Using (1), the boundedness of the differentials, Theorem 1.90, and induction we get

$$d^jg_\kappa(x) = \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(y)d^jf(x-y)\,\mathrm{d}y = \frac{1}{c_\kappa}\int_{\mathbb{R}^n}\Psi_\kappa(x-y)d^jf(y)\,\mathrm{d}y$$

for every $x \in \mathbb{R}^n$ and $1 \le j \le k$. Fix $0 \le j \le k$ and choose an arbitrary $\varepsilon > 0$. Consider $\mathbb{R}^n$ with the Euclidean norm. By the uniform continuity there is $\delta > 0$ such that $\|d^jf(x) - d^jf(y)\| < \frac{\varepsilon}{2}$ whenever $x, y \in \mathbb{R}^n$, $\|x-y\| < \delta$. Moreover there is $M > 0$ such that

$\|d^j f(y)\| \le M$ for all $y \in \mathbb{R}^n$. For κ large enough so that $2Me^{-\frac{1}{2}\kappa\delta^2}2^{\frac{n}{2}} < \frac{\varepsilon}{2}$ we then have

$$\begin{aligned}
\left\|d^j g_\kappa(x) - d^j f(x)\right\| &= \left\|\frac{1}{c_\kappa}\int_{\mathbb{R}^n} \Psi_\kappa(x-y) d^j f(y)\,\mathrm{d}y - \frac{1}{c_\kappa}\left(\int_{\mathbb{R}^n} \Psi_\kappa(x-y)\,\mathrm{d}y\right) d^j f(x)\right\| \\
&= \left\|\frac{1}{c_\kappa}\int_{\mathbb{R}^n} \Psi_\kappa(x-y)\left(d^j f(y) - d^j f(x)\right)\,\mathrm{d}y\right\| \\
&\le \frac{1}{c_\kappa}\int_{\mathbb{R}^n} \Psi_\kappa(x-y)\left\|d^j f(y) - d^j f(x)\right\|\,\mathrm{d}y \\
&\le \frac{1}{c_\kappa}\int_{\|x-y\|<\delta} \Psi_\kappa(x-y)\frac{\varepsilon}{2}\,\mathrm{d}y + \frac{1}{c_\kappa}\int_{\|x-y\|\ge\delta} \Psi_\kappa(x-y) 2M\,\mathrm{d}y \\
&\le \frac{\varepsilon}{2} + \frac{2M}{c_\kappa}\int_{\|x-y\|\ge\delta} \Psi_\kappa(x-y)\,\mathrm{d}y \\
&= \frac{\varepsilon}{2} + \frac{2M}{c_\kappa}\int_{\|x-y\|\ge\delta} \exp\left(-\kappa\|x-y\|^2\right)\,\mathrm{d}y \\
&\le \frac{\varepsilon}{2} + \frac{2M}{c_\kappa}e^{-\frac{1}{2}\kappa\delta^2}\int_{\mathbb{R}^n} \exp\left(-\frac{\kappa}{2}\|x-y\|^2\right)\,\mathrm{d}y \\
&= \frac{\varepsilon}{2} + 2Me^{-\frac{1}{2}\kappa\delta^2}\frac{c_{\kappa/2}}{c_\kappa} = \frac{\varepsilon}{2} + 2Me^{-\frac{1}{2}\kappa\delta^2}2^{\frac{n}{2}} < \varepsilon
\end{aligned}$$

for every $x \in \mathbb{R}^n$.

□

The next fact is used silently many times throughout this work.

FACT 2. *There is a function $\theta \in C^\infty(\mathbb{R};[0,1])$ such that $\theta(t) = 0$ for $t \le 0$, $\theta(t) \in (0,1)$ for $t \in (0,1)$, and $\theta(t) = 1$ for $t \ge 1$.*

PROOF. Define a function θ_0 by $\theta_0(t) = 0$ for $t \le 0$ and $\theta_0(t) = \exp\left(-\frac{1}{t}\right)$ for $t > 0$. It is standard to check that $\theta_0 \in C^\infty(\mathbb{R};[0,1))$. Put $\theta_1(t) = 1 - e^e\theta_0\left(\frac{1}{e} - t\right)$ and finally $\theta = \theta_1 \circ \theta_0$. □

LEMMA 3. *Let $K \subset \mathbb{R}^n$ be a compact set and $U \subset \mathbb{R}^n$ an open neighbourhood of K. Then there is a function $\varphi \in C^\infty(\mathbb{R}^n;[0,1])$ such that $\operatorname{supp}\varphi \subset U$ and $\varphi = 1$ on some neighbourhood of K.*

PROOF. Let θ be the function from Fact 2 and define $\psi \in C^\infty(\mathbb{R}^n)$ by $\psi(x) = \theta(1 - x_1^2 - \cdots - x_n^2)$. Notice that $\operatorname{supp}\psi \subset B(0,1)$ and $\psi(0) = 1$. Let $d = \frac{1}{2}\operatorname{dist}(K, \mathbb{R}^n \setminus U)$. For $w \in K$ define $\psi_w(x) = \psi\left(\frac{x-w}{d}\right)$ and $V_w = \{x \in \mathbb{R}^n;\ \psi_w(x) > \frac{1}{2}\}$. Since K is a compact set there are $w_1, \dots, w_k \in K$ such that $K \subset V_{w_1} \cup \cdots \cup V_{w_k}$. Put

$$\varphi(x) = \theta\left(2\sum_{j=1}^k \psi_{w_j}(x)\right).$$

□

LEMMA 4. *There are functions $\theta_n \in H(\mathbb{C})$, $n \in \mathbb{N}$, with the following properties:*

(T1) *$\theta_n\restriction_{\mathbb{R}}$ maps into $[0,1]$,*
(T2) *$\theta_n\restriction_{\mathbb{R}}$ is 4-Lipschitz,*
(T3) *$|\theta_n(z)| \le 2^{-n}$ for every $z \in \mathbb{C}$, $|z| \le \frac{1}{4}$,*

(T4) $|\theta_n(x) - 1| \le 2^{-n}$ *for every* $x \in \mathbb{R}$, $x \ge 1$,
(T5) $|(\theta_n \restriction_{\mathbb{R}})'(x)| \le 2^{-n}$ *for every* $x \in \mathbb{R}$, $x \le \frac{1}{2}$ *or* $x \ge 1$.

PROOF. Let $f : \mathbb{R} \to [0, 1]$ be defined as $f(t) = 0$ for $t \le \frac{5}{8}$, $f(t) = 4t - \frac{5}{2}$ for $t \in \left(\frac{5}{8}, \frac{7}{8}\right)$, and $f(t) = 1$ for $t \ge \frac{7}{8}$. Obviously f is a 4-Lipschitz function. We put $\theta_n = g_{\kappa_n}$ from Lemma 1, where $\kappa_n \in \mathbb{R}^+$ is chosen so that (T4) holds and

$$\sqrt{2}e^{-\kappa_n/128} \le 2^{-n}, \tag{2}$$

$$\frac{2\kappa_n}{c_{\kappa_n}} \int_{|t| \ge \frac{1}{8}} |t| e^{-\kappa_n t^2} \,\mathrm{d}t = \frac{2e^{-\kappa_n/64}}{c_{\kappa_n}} \le 2^{-n}. \tag{3}$$

The function θ_n clearly has the properties (T1) and (T2). To prove (T3) we use successively the definition of f, the fact that $|\operatorname{Im} z| \le \frac{1}{4}$, $\operatorname{Re} z \le \frac{1}{4}$, and finally (2) to obtain

$$\begin{aligned}
|\theta_n(z)| &\le \frac{1}{c_{\kappa_n}} \int_{\mathbb{R}} f(t) e^{-\kappa_n \operatorname{Re}(z-t)^2} \,\mathrm{d}t = \frac{e^{\kappa_n (\operatorname{Im} z)^2}}{c_{\kappa_n}} \int_{\mathbb{R}} f(t) e^{-\kappa_n (\operatorname{Re} z - t)^2} \,\mathrm{d}t \\
&\le \frac{e^{\frac{1}{16}\kappa_n}}{c_{\kappa_n}} \int_{\frac{5}{8}}^{+\infty} e^{-\kappa_n (\operatorname{Re} z - t)^2} \,\mathrm{d}t \le \frac{e^{\frac{1}{16}\kappa_n}}{c_{\kappa_n}} e^{-\frac{\kappa_n}{2}(\frac{3}{8})^2} \int_{\frac{5}{8}}^{+\infty} e^{-\frac{\kappa_n}{2}(\operatorname{Re} z - t)^2} \,\mathrm{d}t \\
&\le \sqrt{2} e^{-\kappa_n/128} \le 2^{-n}.
\end{aligned}$$

Finally, we show (T5). Suppose that $x \le \frac{1}{2}$ or $x \ge 1$. Differentiating under the integral sign we obtain

$$\theta_n'(x) = \frac{2\kappa_n}{c_{\kappa_n}} \int_{\mathbb{R}} f(t)(t - x) e^{-\kappa_n (t-x)^2} \,\mathrm{d}t.$$

Since $t \mapsto t e^{-\kappa t^2}$ is odd and f is constant on $[x - \frac{1}{8}, x + \frac{1}{8}]$, we get (using also (3))

$$|\theta_n'(x)| = \frac{2\kappa_n}{c_{\kappa_n}} \left| \int_{|x-t| \ge \frac{1}{8}} f(t)(t-x) e^{-\kappa_n (t-x)^2} \,\mathrm{d}t \right| \le \frac{2\kappa_n}{c_{\kappa_n}} \int_{|x-t| \ge \frac{1}{8}} |t - x| e^{-\kappa_n (t-x)^2} \,\mathrm{d}t \le 2^{-n}.$$

□

LEMMA 5. *Let* $\{\varepsilon_n\}_{n=1}^{\infty}$ *and* $\{a_n\}_{n=1}^{\infty}$ *be two sequences of positive real numbers. There are functions* $\zeta_n \in H(\mathbb{C}^n)$, $n \in \mathbb{N}$, *and a sequence* $\{\delta_n\}_{n=1}^{\infty}$ *of positive real numbers with the following properties:*

(Z1) $\zeta_n \restriction_{\mathbb{R}^n}$ *maps into* $[0, 1]$,
(Z2) $\zeta_n \restriction_{\mathbb{R}^n}$ *is* 2*-Lipschitz with respect to the maximum norm,*
(Z3) $|\zeta_n(z)| \le \varepsilon_n$ *for every* $z \in \mathbb{C}^n$ *such that there is* $k \in \{1, \dots, n-1\}$ *for which* $\operatorname{Re} z_k \le \frac{1}{4}$ *and* $\sum_{j=1}^n a_j (\operatorname{Im} z_j)^2 \le \delta_k$.
(Z4) $\zeta_n(x) \ge \frac{1}{2}$ *for every* $x \in \mathbb{R}^n$ *for which* $x_n \le 1$ *and* $x_j \ge 1$, $j = 1, \dots, n-1$,
(Z5) $\zeta_n(x) \le \varepsilon_n$ *for* $x \in \mathbb{R}^n$ *satisfying* $x_n \ge 2$.

PROOF. Let $f_n : \mathbb{R}^n \to [0, 1]$ be a 2-Lipschitz function (with respect to the maximum norm) such that

$$f_n(x) = \begin{cases} 0 & \text{whenever } x_n \ge 2 \text{ or } \exists j \in \{1, \dots, n-1\} \colon x_j \le \frac{1}{2}, \\ 1 & \text{whenever } x_n \le 1 \text{ and } \forall j \in \{1, \dots, n-1\} \colon x_j \ge 1. \end{cases}$$

(See e.g. Lemma 30.) For each $n \in \mathbb{N}$ put $\delta_n = a_n/64$ and

$$\zeta_n(z) = \frac{1}{c_n} \int_{\mathbb{R}^n} f_n(t) \exp\left(-\kappa_n \sum_{j=1}^{n} a_j (z_j - t_j)^2 \right) \mathrm{d}t \quad \text{for } z \in \mathbb{C}^n,$$

where $c_n = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n a_j t_j^2} \, \mathrm{d}t = \sqrt{\left(\frac{\pi}{\kappa_n}\right)^n \prod_{j=1}^n a_j^{-1}}$ and $\kappa_n \in \mathbb{R}^+$ is chosen so that (Z4) and (Z5) hold (analogously as in Lemma 1) and

$$e^{-\kappa_n a_j/64} \leq 2^{-\frac{n}{2}} \varepsilon_n \text{ for } j = 1, \dots, n-1.$$

The function ζ_n belongs to $H(\mathbb{C}^n)$ and has the properties (Z1) and (Z2) (again similarly as in Lemma 1). To prove (Z3) we use successively use the definition of f_n, the fact that $\operatorname{Re} z_k \leq \frac{1}{4}$, and the definition of δ_k to obtain

$$\begin{aligned}
|\zeta_n(z)| &\leq \frac{1}{c_n} \int_{\mathbb{R}^n} f_n(t) e^{-\kappa_n \sum\limits_{j=1}^{n} a_j \operatorname{Re}(z_j - t_j)^2} \mathrm{d}t = \frac{e^{\kappa_n \sum\limits_{j=1}^{n} a_j (\operatorname{Im} z_j)^2}}{c_n} \int_{\mathbb{R}^n} f_n(t) e^{-\kappa_n \sum\limits_{j=1}^{n} a_j (\operatorname{Re} z_j - t_j)^2} \mathrm{d}t \\
&\leq \frac{e^{\kappa_n \delta_k}}{c_n} \int_{\substack{t \in \mathbb{R}^n \\ t_k > \frac{1}{2}}} e^{-\kappa_n \sum\limits_{j=1}^{n} a_j (\operatorname{Re} z_j - t_j)^2} \mathrm{d}t \\
&= \frac{e^{\kappa_n \delta_k}}{c_n} \int_{\substack{t \in \mathbb{R}^n \\ t_k > \frac{1}{2}}} e^{-\frac{\kappa_n}{2} \sum\limits_{j=1}^{n} a_j (\operatorname{Re} z_j - t_j)^2} \cdot e^{-\frac{\kappa_n}{2} \sum\limits_{j=1}^{n} a_j (\operatorname{Re} z_j - t_j)^2} \mathrm{d}t \\
&\leq \frac{e^{\kappa_n \delta_k}}{c_n} e^{-\frac{\kappa_n}{2} a_k \frac{1}{16}} \int_{\mathbb{R}^n} e^{-\frac{\kappa_n}{2} \sum\limits_{j=1}^{n} a_j (\operatorname{Re} z_j - t_j)^2} \mathrm{d}t = 2^{\frac{n}{2}} e^{-\kappa_n a_k/64} \leq \varepsilon_n.
\end{aligned}$$

□

LEMMA 6. *Let K be a compact space such that $C(K)$ admits a C^k-smooth bump function, $k \in \mathbb{N} \cup \{\infty\}$. Then for every $\zeta, \eta \in \mathbb{R}$, $0 < \zeta < \eta$, there is a function $\beta_{\zeta,\eta} \in C^k(C(K); [0,1])$ such that*

$$\beta_{\zeta,\eta}(f) = \begin{cases} 1 & \textit{when } \|f\|_\infty \leq \zeta, \\ 0 & \textit{when } \|f\|_\infty \geq \eta. \end{cases}$$

PROOF. By hypothesis there exists a function $\varphi \in C^k(C(K); [0,1])$ and $\alpha \in \mathbb{R}$, $\alpha > 0$, such that $\varphi(f) = 1$ for $\|f\|_\infty \leq \alpha$, while $\varphi(f) = 0$ for $\|f\|_\infty \geq 1$. Choose $n \in \mathbb{N}$ so that $(\frac{\zeta}{\eta})^n \leq \alpha$ and put

$$\beta_{\zeta,\eta}(f) = \varphi\left(\frac{f^n}{\eta^n}\right).$$

Since the mapping $f \mapsto f^n$ is a continuous n-homogeneous polynomial (and in particular it is C^∞-smooth), the function $\beta_{\zeta,\eta}$ has the required properties.

□

We remark that the Taylor complexification $\tilde{c_0}$ of the real space c_0 is isometric to the complex space c_0.

PROPOSITION 7. *Let $q \geq 1$. There are an open set $W \subset \tilde{c_0}$ and a function $\mu \in H(W)$ with the following properties:*

(M1) *For every $w \in c_0 \setminus \{0\}$ there is $\Delta_w > 0$ such that $U_{\tilde{c_0}}(y, \Delta_w) \subset W$ for every $y \in c_0$ satisfying $|w| \leq |y| \leq q|w|$, where the inequalities are understood in the lattice sense.*

(M2) $\mu(w) \geq 8$ *for* $w \in c_0$, $\|w\| \geq 8$,
(M3) $|\mu(z)| < 2$ *for* $z \in U_{\tilde{c_0}}(y, \Delta_w)$, *where* $y \in c_0$, $\|y\| \leq 1$, *and* $w \in c_0 \setminus \{0\}$, $|w| \leq |y| \leq q|w|$,
(M4) $\mu\restriction_{c_0}$ *is* $\sqrt{2}$*-Lipschitz and maps into* $\mathbb{R}$.

PROOF. Define μ on c_0 as the Minkowski functional of the set $\left\{x \in c_0;\ \sum_{n=1}^{\infty}(x_n)^{2n} \leq 1\right\}$. Then μ is an equivalent norm on c_0 for which $\|x\| \leq \mu(x) \leq \sqrt{2}\|x\|$ (see also Theorem 5.104 and Example 1.137). This gives property (M2) and (M4).

Let $f : \tilde{c_0} \times (\mathbb{C} \setminus \{0\}) \to \mathbb{C}$ be defined as $f(z, u) = \sum_{n=1}^{\infty}(z_n/u)^{2n} - 1$. This function is holomorphic on $\tilde{c_0} \times (\mathbb{C} \setminus \{0\})$ and for every $x \in c_0 \setminus \{0\}$ we have $f(x, \mu(x)) = 0$.

Fix $w \in c_0 \setminus \{0\}$. Put $R = \frac{\|w\|}{2}$, $S = \frac{\|w\|}{4}$, $M = 1 + \sum_{n=1}^{\infty}\left(\frac{1}{2} + \frac{2q}{\|w\|}|w_n|\right)^{2n}$, $a = \frac{1}{\sqrt{2}q\|w\|}$, $r = \min\left\{\frac{1}{2}\frac{aR^2}{aR+M}, 2 - \sqrt{2}\right\}$, and $\Delta_w = s$ as defined in Theorem 1.176. Now choose any $y \in c_0$, $|w| \leq |y| \leq q|w|$. Then $R < \|w\| \leq \|y\| \leq \mu(y)$, thus $B(\mu(y), R) \subset V = \mathbb{C} \setminus \{0\}$. For any $z \in B(y, S)$, $u \in B(\mu(y), R)$ we have $|u| \geq \mu(y) - R \geq \|y\| - R \geq \|w\| - R = \frac{\|w\|}{2}$ and $|z_n| \leq |y_n| + |z_n - y_n| \leq q|w_n| + \|z - y\| \leq q|w_n| + \frac{\|w\|}{4}$, and hence $|f(z, u)| \leq 1 + \sum_{n=1}^{\infty}|\frac{z_n}{u}|^{2n} \leq M$. Finally, $|D_2 f(y, \mu(y))| = \left|-\frac{1}{\mu(y)}\sum_{n=1}^{\infty} 2n\left(\frac{y_n}{\mu(y)}\right)^{2n}\right| \geq \frac{1}{\mu(y)}\sum_{n=1}^{\infty}\left(\frac{y_n}{\mu(y)}\right)^{2n} = \frac{1}{\mu(y)} \geq \frac{1}{\sqrt{2}\|y\|} \geq a$. Thus by Theorem 1.176 the equation $f(z, u) = 0$ uniquely determines a holomorphic function μ_y^w on $U_{\tilde{c_0}}(y, \Delta_w)$ with values in $U(\mu(y), r)$ and this holds for every $y \in c_0$, $|w| \leq |y| \leq q|w|$.

Take any two functions $\mu_1 = \mu_{y_1}^{w_1}$, $\mu_2 = \mu_{y_2}^{w_2}$ defined on open balls U_1 and U_2 respectively. If U_1 and U_2 intersect, then it is easy to check that $U_1 \cap U_2 \cap c_0$ is a non-empty set relatively open in c_0. Since $\mu_1 = \mu$ on $U_1 \cap c_0$ and $\mu_2 = \mu$ on $U_2 \cap c_0$, it follows that both holomorphic functions μ_1 and μ_2 agree on $U_1 \cap U_2 \cap c_0$ and therefore on the whole $U_1 \cap U_2$ (Corollary 1.158). This observation allows us to put $W = \bigcup\{U_{\tilde{c_0}}(y, \Delta_w);\ w \in c_0 \setminus \{0\}, y \in c_0, |w| \leq |y| \leq q|w|\}$ and define μ on W by $\mu(z) = \mu_y^w(z)$ whenever $z \in U(y, \Delta_w)$. This gives property (M1).

To prove (M3) let $w \in c_0 \setminus \{0\}$, $y \in c_0$, $|w| \leq |y| \leq q|w|$, $\|y\| \leq 1$, and $z \in U_{\tilde{c_0}}(y, \Delta_w)$. Then by the choice of r above we have $\mu(z) \in U(\mu(y), 2 - \sqrt{2})$ and therefore $|\mu(z)| < |\mu(y)| + 2 - \sqrt{2} \leq \sqrt{2}\|y\| + 2 - \sqrt{2} \leq 2$.

□

LEMMA 8 ([Ru]). *Let* (P, ρ) *be a metric space and* $\mathcal{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ *an open covering of* P. *Then there are open refinements* $\{V_{n\alpha}\}_{n \in \mathbb{N}, \alpha \in \Lambda}$, $\{W_{n\alpha}\}_{n \in \mathbb{N}, \alpha \in \Lambda}$ *of* $\mathcal{U}$ *that satisfy the following:*

- $V_{n\alpha} \subset W_{n\alpha} \subset U_\alpha$ *for all* $n \in \mathbb{N}$, $\alpha \in \Lambda$,
- $\operatorname{dist}(V_{n\alpha}, P \setminus W_{n\alpha}) \geq 2^{-n}$ *for all* $n \in \mathbb{N}$, $\alpha \in \Lambda$,
- $\operatorname{dist}(W_{n\alpha}, W_{n\beta}) \geq 2^{-n}$ *for any* $n \in \mathbb{N}$ *and* $\alpha, \beta \in \Lambda$, $\alpha \neq \beta$.
- *for each* $x \in P$ *there are an open neighbourhood* U_x *of* x *and a number* $n_x \in \mathbb{N}$ *such that*
  (i) *if* $k > n_x$, *then* $U_x \cap W_{k\alpha} = \emptyset$ *for any* $\alpha \in \Lambda$,
  (ii) *if* $k \leq n_x$, *then* $U_x \cap W_{k\alpha} \neq \emptyset$ *for at most one* $\alpha \in \Lambda$.

PROOF. Choose some well-ordering of the set Λ. Define the sets $V_{n\alpha}$ by induction on $n \in \mathbb{N}$: If $V_{j\beta}$ are already defined for $j < n$ and all $\beta \in \Lambda$, let $V_{n\alpha}$ be the union of all $U(x, 2^{-n})$ such that

(a) α is the smallest with $x \in U_\alpha$,
(b) $x \notin V_{j\beta}$ for all $j < n$, $\beta \in \Lambda$,
(c) $U(x, 5 \cdot 2^{-n}) \subset U_\alpha$.

Further, let $W_{n\alpha} = \bigcup\{U(x, 2^{-n});\ x \in V_{n\alpha}\}$ for all $n \in \mathbb{N}$, $\alpha \in \Lambda$.

Certainly $V_{n\alpha} \subset W_{n\alpha} \subset U_\alpha$ and $\operatorname{dist}(V_{n\alpha}, P \setminus W_{n\alpha}) \geq 2^{-n}$ for all $n \in \mathbb{N}, \alpha \in \Lambda$. To see that $\{V_{n\alpha}\}$ covers P, observe that, for $x \in P$, there is a smallest $\alpha \in \Lambda$ such that $x \in U_\alpha$, and n so large that (c) holds. Then, by (b), $x \in V_{j\beta}$ for some $j \leq n$, $\beta \in \Lambda$.

To prove the third property, suppose that $n \in \mathbb{N}$, $\alpha, \beta \in \Lambda$, $\alpha < \beta$, $p \in W_{n\alpha}$, and $q \in W_{n\beta}$. There is a ball $U(y, 2^{-n})$ in the definition of $V_{n\alpha}$ such that $\rho(p, y) < 2 \cdot 2^{-n}$, and a ball $U(z, 2^{-n})$ in the definition of $V_{n\beta}$ such that $\rho(q, z) < 2 \cdot 2^{-n}$. By (c), $U(y, 5 \cdot 2^{-n}) \subset U_\alpha$ but, by (a), $z \notin U_\alpha$. So $\rho(y, z) \geq 5 \cdot 2^{-n}$ and $\rho(p, q) \geq \rho(y, z) - \rho(p, y) - \rho(q, z) > 2^{-n}$.

Finally assume $x \in P$. Find some pair $n \in \mathbb{N}$, $\beta \in \Lambda$ such that $x \in V_{n\beta}$, and choose $j \in \mathbb{N}$ so that $U(x, 2^{-j+1}) \subset V_{n\beta}$. Put $n_x = n + j - 1$ and $U_x = U(x, 2^{-n-j})$. To show (i), suppose that $k > n_x$ and choose any $\alpha \in \Lambda$ and $z \in W_{k\alpha}$. It follows that there is a ball $U(y, 2^{-k})$ in the definition of $V_{k\alpha}$ such that $\rho(y, z) < 2 \cdot 2^{-k}$. Since $k > n$, by (b), $y \notin V_{n\beta}$. And since $U(x, 2^{-j+1}) \subset V_{n\beta}$ and $k \geq j + 1$,

$$\begin{aligned} \rho(x, z) &\geq \rho(x, y) - \rho(y, z) \geq 2^{-j+1} - \rho(y, z) \\ &> 2^{-j+1} - 2^{-k+1} \geq 2^{-j+1} - 2^{-j} = 2^{-j} > 2^{-n-j}. \end{aligned}$$

From the definition of U_x and n_x it is easy to see that (ii) follows from the third property.

□

2. Approximation by polynomials

In this section we begin by proving the classical Weierstraß-type theorem on the density of polynomials among C^k-smooth functions in the uniform topology (together with its derivatives) on compact subsets of $\mathbb{R}^n$. Of course, the proof relies heavily on the compactness argument. The result can be extended into infinite-dimensional setting if we are interested in uniform topology on compact sets. We then deal with the approximation in uniform topology on bounded sets, which is not always possible. Applying the theory of $\mathcal{W}$-spaces we give a generalisation of the Weierstraß theorem in some special cases. We finish the section by showing that the assumptions used in order to get positive results are close to being optimal.

THEOREM 9. *Let $\Omega \subset \mathbb{R}^n$ be an open set, Y a Banach space, and $f \in C^k(\Omega; Y)$, $k \in \mathbb{N}_0$. For every compact subset $K \subset \Omega$ and every $\varepsilon > 0$ there is a polynomial $p \in \mathcal{P}(\mathbb{R}^n; Y)$ such that $\|d^j f - d^j p\|_K \leq \varepsilon$ for $0 \leq j \leq k$.*

PROOF. By Lemma 3 there is a function $\varphi \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\varphi \subset \Omega$, $\operatorname{supp}\varphi$ is compact, and $\varphi = 1$ on a neighbourhood of K. Since $\varphi f \in C^k(\Omega; Y)$ (Corollary 1.84), φf has a compact support, and $d^j(\varphi f)(x) = d^j f(x)$ for $x \in K$, $0 \leq j \leq k$, replacing f by a function defined as φf on Ω and 0 on $\mathbb{R}^n \setminus \Omega$ we may suppose that $f \in C^k(\mathbb{R}^n; Y)$ and $S = \operatorname{supp} f$ is compact.

By Lemma 1 there is $\kappa \in \mathbb{R}^+$ such that $\left\| d^j g_\kappa - d^j f \right\|_{\mathbb{R}^n} < \frac{\varepsilon}{2}$ for $0 \leq j \leq k$. Put

$$Q_m(y) = \sum_{l=0}^{m} \frac{1}{l!} (-\kappa)^l \left(y_1^2 + \cdots + y_n^2\right)^l.$$

Then $Q_m \in \mathcal{P}(\mathbb{R}^n)$ and $\lim_{m\to\infty} d^j Q_m = d^j \Psi_\kappa$ locally uniformly on $\mathbb{R}^n$ for $0 \leq j \leq k$ (Theorem 1.146). Put $M = \|f\|_{\mathbb{R}^n}$. The set $L = K - S$ is compact. Therefore there is $N \in \mathbb{N}$ such that $\left\| d^j Q_N - d^j \Psi_\kappa \right\|_L \leq \frac{c_\kappa}{M} \frac{1}{\lambda(L)} \frac{\varepsilon}{2}$ for each $0 \leq j \leq k$. Let

$$p(x) = \frac{1}{c_\kappa} \int_{\mathbb{R}^n} Q_N(x - y) f(y) \,\mathrm{d}y.$$

Then $p \in \mathcal{P}(\mathbb{R}^n; Y)$ (e.g. by Theorem 2.49). As

$$d^j g_\kappa(x) = \frac{1}{c_\kappa} \int_{\mathbb{R}^n} d^j \Psi_\kappa(x-y) f(y)\, \mathrm{d}y = \frac{1}{c_\kappa} \int_{\mathbb{R}^n} d^j \Psi_\kappa(y) f(x-y)\, \mathrm{d}y$$

and similarly for $d^j p$, we have for any $x \in K$

$$\begin{aligned} \|d^j g_\kappa(x) - d^j p(x)\| &= \left\| \frac{1}{c_\kappa} \int_{\mathbb{R}^n} \left(d^j \Psi_\kappa(y) - d^j Q_N(y)\right) f(x-y)\, \mathrm{d}y \right\| \\ &= \left\| \frac{1}{c_\kappa} \int_L \left(d^j \Psi_\kappa(y) - d^j Q_N(y)\right) f(x-y)\, \mathrm{d}y \right\| \\ &\le \frac{M}{c_\kappa} \int_L \left\| d^j \Psi_\kappa(y) - d^j Q_N(y) \right\| \, \mathrm{d}y \\ &\le \frac{M}{c_\kappa} \left\| d^j Q_N - d^j \Psi_\kappa \right\|_L \lambda(L) \le \frac{\varepsilon}{2}. \end{aligned}$$

□

DEFINITION 10. Let X, Y be normed linear spaces. By $\mathcal{P}_{\mathrm{f}}({}^nX; Y)$ we denote the linear subspace of $\mathcal{P}({}^nX; Y)$ consisting of all polynomials that can be written in the form $P(x) = \sum_{j=1}^k f_j(x)^n y_j$, where $f_j \in X^*$, $y_j \in Y$. We set $\mathcal{P}_{\mathrm{f}}(X; Y) = \operatorname{span} \bigcup_{n=0}^\infty \mathcal{P}_{\mathrm{f}}({}^nX; Y)$.

If X is finite-dimensional, then $\mathcal{P}_{\mathrm{f}}({}^nX; Y) = \mathcal{P}({}^nX; Y)$. This follows from Proposition 1.23 and the fact that $\operatorname{span}\{\langle y, \cdot\rangle^d;\ y \in \mathbb{R}^m\} = \mathcal{P}({}^d\mathbb{R}^m)$ (Section 2.4).

FACT 11. *Let X, Y be normed linear spaces, $P \in \mathcal{P}_{\mathrm{f}}({}^mX)$, and $Q \in \mathcal{P}_{\mathrm{f}}({}^nX; Y)$. Then $PQ \in \mathcal{P}_{\mathrm{f}}({}^{m+n}X; Y)$. In particular, $\mathcal{P}_{\mathrm{f}}(X)$ is a subalgebra of the algebra $\mathcal{P}(X)$ and $p \circ R \in \mathcal{P}_{\mathrm{f}}(X)$ whenever $R \in \mathcal{P}_{\mathrm{f}}(X)$ and $p \in \mathcal{P}(\mathbb{R})$.*

PROOF. It is clear that it suffices to show that if $f, g \in X^*$, then $x \mapsto f(x)^m g(x)^n \in \mathcal{P}_{\mathrm{f}}({}^{m+n}X)$. The polynomial $q \in \mathcal{P}({}^{m+n}\mathbb{R}^2)$, $q(u,v) = u^m v^n$ can be written as $q(u,v) = \sum_{j=1}^k c_j \langle (a_j, b_j), (u,v)\rangle^{m+n}$ (Section 2.4). Hence $f(x)^m g(x)^n = \sum_{j=1}^k c_j h_j(x)^{m+n}$, where $h_j = a_j f + b_j g \in X^*$.

□

The following is an extension of the Weierstraß theorem into infinite-dimensional spaces. Of course the usefulness of this theorem is limited by the fact that here the compact sets are very small.

THEOREM 12. *Let X and Y be normed linear spaces, $K \subset X$ compact, $f \in C(K; Y)$, and $\varepsilon > 0$. Then there is a polynomial $P \in \mathcal{P}_{\mathrm{f}}(X; Y)$ such that $\|f - P\|_K \le \varepsilon$.*

For the proof we need the following lemma on separation of sets by polynomials.

LEMMA 13. *Let X be a normed linear space, $\Omega \subset X$ a bounded set, $C \subset \Omega$ a closed convex set, $K \subset \Omega$ a weakly compact set satisfying $C \cap K = \emptyset$, and $\delta > 0$. Then there is a polynomial $P \in \mathcal{P}_{\mathrm{f}}(X)$ such that $0 \le P(x) \le 1$ for $x \in \Omega$, $P(x) > 1 - \delta$ for $x \in C$, and $P(x) < \delta$ for $x \in K$.*

PROOF. Without loss of generality we may assume that C is non-empty. By the separation theorem for every $x \in K$ there are $f_x \in X^*$ and $b_x, c_x \in \mathbb{R}$ such that $f_x(y) < b_x < c_x < f_x(x)$ for $y \in C$. Since $\left\{f_x^{-1}\big((c_x, +\infty)\big)\right\}_{x \in K}$ is a weakly open covering of the weak compact K, there are $x_1, \dots, x_n \in K$ such that $K \subset \bigcup_{k=1}^n f_{x_k}^{-1}\big((c_{x_k}, +\infty)\big)$. Denote $M_k = \max_{y \in K} f_{x_k}(y)$ and

$a_k = \inf_{y\in C} f_{x_k}(y)$ for $k = 1,\dots,n$, and notice that $a_k \in \mathbb{R}$ by the boundedness of C. By Theorem 9 there are polynomials $q_k \in \mathcal{P}(\mathbb{R})$ such that $|q_k(t)| < \frac{1}{2n}$ for $t \in [a_k, b_{x_k}]$ and $q_k(t) > 1$ for $t \in [c_{x_k}, M_k]$, $k = 1,\dots,n$. Put

$$Q(x) = \sum_{k=1}^{n} q_k^2\big(f_{x_k}(x)\big).$$

Then $0 \le Q(x) < \frac{1}{4n} \le \frac{1}{4}$ for $x \in C$ and $Q(x) > 1$ for $x \in K$.

Denote $m = \inf_{y\in\Omega} Q(y)$ and $M = \sup_{y\in\Omega} Q(y)$. By the boundedness of Ω both m and M are finite. By Theorem 9 there is a polynomial $p \in \mathcal{P}(\mathbb{R})$ such that $0 \le q(t) \le 1$ for $t \in [m, M]$, $q(t) > 1-\delta$ for $t \in [m, \frac{1}{4}]$, and $q(t) < \delta$ for $t \in [1, M]$. We obtain the desired polynomial P by setting $P = p \circ Q$, noting that $P \in \mathcal{P}_{\mathrm{f}}(X)$ by Fact 11.

□

DEFINITION 14. Let X be a set. A collection $\{\psi_\alpha\}_{\alpha\in\Lambda}$ of functions on X is called a partition of unity if

- $\psi_\alpha \colon X \to [0,1]$ for all $\alpha \in \Lambda$,
- $\sum\limits_{\alpha\in\Lambda} \psi_\alpha(x) = 1$ for each $x \in X$.

Let $\mathcal{U}$ be a covering of X. We say that the partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is subordinated to $\mathcal{U}$ if $\{\operatorname{supp}_{\mathrm{o}} \psi_\alpha\}_{\alpha\in\Lambda}$ refines $\mathcal{U}$.

Note that from the second property it immediately follows that the collection $\{\operatorname{supp}_{\mathrm{o}} \psi_\alpha\}_{\alpha\in\Lambda}$ is point-countable, i.e. for every $x \in X$ the set $\{\alpha \in \Lambda;\ \psi_\alpha(x) \ne 0\}$ is countable. In applications often either the set Λ itself is countable, which allows for "global" constructions (e.g. analytic approximation, Section 3), or the collection $\{\operatorname{supp}_{\mathrm{o}} \psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite, which then preserves local properties, like C^k-smoothness (Section 5).

LEMMA 15. *Let X be a normed linear space, $K \subset X$ compact, $\{U(x_k, r_k)\}_{k=1}^n$ a covering of K, and $\delta > 0$. Then there is a polynomial partition of unity $\{\psi_k\}_{k=1}^n \subset \mathcal{P}_{\mathrm{f}}(X)$ on K such that $\psi_k(x) < \delta$ whenever $x \in K \setminus U(x_k, 2r_k)$, $k = 1,\dots,n$.*

PROOF. By Lemma 13 there are polynomials $\varphi_1,\dots,\varphi_n \in \mathcal{P}_{\mathrm{f}}(X)$ satisfying $0 \le \varphi_k(x) \le 1$ for $x \in K$, $\varphi_k(x) > 1-\delta$ for $x \in B(x_k, r_k)$, and $\varphi_k(x) < \delta$ for $x \in K \setminus U(x_k, 2r_k)$, $k = 1,\dots,n$. We construct inductively polynomials $\psi_1,\dots,\psi_n \in \mathcal{P}_{\mathrm{f}}(X)$ that will form a partition of unity on K. Put $\psi_1 = \varphi_1$ and $\psi_k = \varphi_k \cdot \big(1 - \sum_{j=1}^{k-1} \psi_j\big)$ for $k = 2,\dots,n-1$. Finally set $\psi_n = 1 - \sum_{j=1}^{n-1} \psi_j$. Notice that $\psi_k \in \mathcal{P}_{\mathrm{f}}(X)$ by Fact 11 and

$$\sum_{j=1}^{k} \psi_j = \left(\sum_{j=1}^{k-1} \psi_j\right) \cdot 1 + \left(1 - \sum_{j=1}^{k-1} \psi_j\right)\varphi_k, \quad k = 1,\dots,n-1.$$

Thus we can check by induction that

$$0 \le \varphi_k(x) \le \sum_{j=1}^{k} \psi_j(x) \le 1 \quad \text{for } x \in K, k = 1,\dots,n-1, \tag{4}$$

and consequently

$$0 \le \psi_k(x) \le \varphi_k(x) \le 1 \quad \text{for } x \in K, k = 1,\dots,n-1. \tag{5}$$

It follows that $\psi_1,\dots,\psi_n$ form a partition of unity on K.

Moreover this partition has the property that $\psi_k(x) < \delta$ whenever $x \in K \setminus U(x_k, 2r_k)$, $k = 1, \dots, n$. Indeed, for $k < n$ it follows from (5). If $k = n$ and $x \in K \setminus U(x_n, 2r_n)$, then there is $m \in \{1, \dots, n-1\}$ such that $x \in U(x_m, r_m)$. Thus, by (5) and (4), $\psi_n(x) \le 1 - \sum_{j=1}^{m} \psi_j(x) \le 1 - \varphi_m(x) < \delta$.

□

PROOF OF THEOREM 12. By the compactness of K there is a covering $\{U(x_k, r_k)\}_{k=1}^{n}$ of K such that $\|f(x) - f(x_k)\| < \frac{\varepsilon}{2}$ whenever $x \in U(x_k, 2r_k)$, $k = 1, \dots, n$. Let $M > 0$ be such that $\|f\|_K \le M$ and set $\delta = \varepsilon/(4nM)$. Let $\{\psi_k\}_{k=1}^{n}$ be the partition of unity from Lemma 15. Put

$$P(x) = \sum_{k=1}^{n} \psi_k(x) f(x_k), \quad x \in X.$$

Obviously $P \in \mathcal{P}_{\mathrm{f}}(X; Y)$. To show that P approximates f on K fix any $x \in K$. Let $I = \{1 \le k \le n;\ x \in U(x_k, 2r_k)\}$ and $J = \{1, \dots, n\} \setminus I$. Then

$$\begin{aligned}
\|f(x) - P(x)\| &= \left\| \left(\sum_{k=1}^{n} \psi_k(x) \right) f(x) - \sum_{k=1}^{n} \psi_k(x) f(x_k) \right\| \le \sum_{k=1}^{n} \psi_k(x) \|f(x) - f(x_k)\| \\
&\le \sum_{k \in I} \psi_k(x) \|f(x) - f(x_k)\| + \sum_{k \in J} \psi_k(x) \big(\|f(x)\| + \|f(x_k)\| \big) \\
&\le \frac{\varepsilon}{2} + 2nM\delta = \varepsilon.
\end{aligned}$$

□

Note that from Theorem 9 it follows that for any mapping $f \in C(\Omega; Y)$, where $\Omega \subset \mathbb{R}^n$ is open and Y is a normed linear space, there is a sequence of polynomials $\{P_k\}_{k=1}^{\infty} \subset \mathcal{P}(\mathbb{R}^n; Y)$ such that $P_k \to f$ locally uniformly on Ω. If we are interested only in the pointwise convergence, then we have the following infinite-dimensional result:

THEOREM 16. *Let X, Y be normed linear spaces, X separable, $\Omega \subset X$ open, and $f \in C(\Omega; Y)$. Then there is a sequence of polynomials $\{p_n\}_{n=1}^{\infty} \subset \mathcal{P}_{\mathrm{f}}(X; Y)$ such that $\lim_{n \to \infty} p_n(x) = f(x)$ for every $x \in \Omega$.*

PROOF. Let $\{x_n\}_{n \in \mathbb{N}} \subset X$ be such that $\overline{\operatorname{span}}\{x_n\} = X$. Put

$$K_n = \{x \in \operatorname{span}\{x_1, \dots, x_n\};\ \operatorname{dist}(x, X \setminus \Omega) \ge \frac{1}{n}, \|x\| \le n\}.$$

Then K_n is a compact subset of Ω. By Theorem 12 there are polynomials $p_n \in \mathcal{P}_{\mathrm{f}}(X; Y)$ such that $\|f - p_n\|_{K_n} \le \frac{1}{n}$ for every $n \in \mathbb{N}$. Choose any $x \in \Omega$ and $\varepsilon > 0$. There is $\delta > 0$ such that $U(x, 2\delta) \subset \Omega$ and $\|f(x) - f(y)\| < \frac{\varepsilon}{2}$ whenever $y \in U(x, \delta)$. Further, there is $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$, $\delta > \frac{1}{n_0}$, $\|x\| + \delta \le n_0$, and $\operatorname{span}\{x_1, \dots, x_{n_0}\} \cap U(x, \delta) \ne \emptyset$. Choose $z \in \operatorname{span}\{x_1, \dots, x_{n_0}\} \cap U(x, \delta)$. It follows that $z \in K_n$ for every $n \ge n_0$ and hence

$$\|f(x) - p_n(x)\| \le \|f(x) - f(z)\| + \|f(z) - p_n(z)\| < \frac{\varepsilon}{2} + \frac{1}{n} < \varepsilon \quad \text{for every } n \ge n_0.$$

□

In contrast with that, the norm on $c_0(\Gamma)$, Γ uncountable, is not a pointwise limit of a sequence of polynomials on $S_{c_0(\Gamma)}$. Indeed, given any sequence of polynomials $\{P_n\}$ on $c_0(\Gamma)$, by Corollary 3.62 there is $\gamma \in \Gamma$ such that $P_n(e_\gamma) = P_n(0)$ for each $n \in \mathbb{N}$. This was first observed by Aleksander Pełczyński.

In a general infinite-dimensional normed linear space X there are always continuous functions on X that cannot be uniformly approximated by polynomials on S_X. Indeed, while every polynomial is bounded on S_X, it is easy to construct a continuous function on X that is unbounded on S_X. But typically even much more regular functions cannot be uniformly approximated by polynomials.

THEOREM 17 ([NS]). *Let X be an infinite-dimensional Banach space and let S be one of the spaces $C^k(X)$, $C^\infty(X)$, or $C^{k,\Omega}(X)$, where $k \in \mathbb{N}_0$ and $\Omega \subset \mathcal{M}$ is a convex cone. If X admits a bump function from S, then there is a bump function $f \in S$ such that it cannot be uniformly approximated on B_X by polynomials.*

PROOF. The main ingredient of the proof is the fact that for every degree d there is $n \in \mathbb{N}$ such that the unit ball of $\mathbb{R}^n$ (with an arbitrary norm) contains a $\frac{1}{2}$-separated set of cardinality greater than the dimension of the space $\mathcal{P}^d(\mathbb{R}^n)$. Indeed let $A \subset B_{\mathbb{R}^n}$ be a maximal $\frac{1}{2}$-separated set, i.e. $\|x - y\| \geq \frac{1}{2}$ for every $x, y \in A$, $x \neq y$. By the maximality, $B_{\mathbb{R}^n} \subset \bigcup_{x \in A} U(x, \frac{1}{2})$. Therefore

$$\lambda(B_{\mathbb{R}^n}) \leq \sum_{x \in A} \lambda\big(U(x, \tfrac{1}{2})\big) \leq |A| \lambda\big(B(0, \tfrac{1}{2})\big) = |A| \frac{1}{2^n} \lambda(B_{\mathbb{R}^n}),$$

and hence $|A| \geq 2^n$. On the other hand, $\dim \mathcal{P}^d(\mathbb{R}^n) = \binom{n+d}{d}$ (Section 2.1). Since for every $d \in \mathbb{N}$ there is $n_d \in \mathbb{N}$ such that $2^{n_d} > \binom{n_d+d}{d}$, there is a $\frac{1}{2}$-separated set in $B_{\mathbb{R}^{n_d}}$ of cardinality greater that $\dim \mathcal{P}^d(\mathbb{R}^{n_d})$.

Put $\delta = \frac{3}{4}$. Since X is infinite-dimensional, there is a δ-separated set $\{z_d;\ d \in \mathbb{N}\} \subset B_X(0, \delta)$. Notice that $B\big(z_d, \frac{\delta}{3}\big) \subset B_X$ for every $d \in \mathbb{N}$. Let X_d be some n_d-dimensional subspace of X that contains z_d. By the discussion above there is a $\frac{\delta}{6}$-separated subset A_d of $B_{X_d}\big(z_d, \frac{\delta}{3}\big)$ satisfying $\dim \mathcal{P}^d(X_d) < |A_d| < \infty$. Consider the space $\mathbb{R}^{A_d}$ with the supremum norm. Define $R_d \in \mathcal{L}(\mathcal{P}^d(X_d); \mathbb{R}^{A_d})$ by $R_d(p) = p\restriction_{A_d}$. As $\dim R_d(\mathcal{P}^d(X_d)) \leq \dim \mathcal{P}^d(X_d) < |A_d| = \dim \mathbb{R}^{A_d}$, the space $R_d(\mathcal{P}^d(X_d))$ is a proper subspace of a finite-dimensional space $\mathbb{R}^{A_d}$ and so there is $f_d \in \mathbb{R}^{A_d}$ such that $\operatorname{dist}\big(f_d, R_d(\mathcal{P}^d(X_d))\big) = \|f_d\| = 1$.

Let $\varphi \in S$ be a bump function. By shifting and scaling we may suppose that $\varphi(0) = 1$ and $\operatorname{supp} \varphi \subset B_X\big(0, \frac{\delta}{18}\big)$. Define a function $f\colon X \to \mathbb{R}$ by

$$f(x) = \sum_{d \in \mathbb{N}} \sum_{y \in A_d} f_d(y) \varphi(x - y).$$

By the choice of the set $\{z_d\}$ the set $\bigcup_{d \in \mathbb{N}} A_d$ is a $\frac{\delta}{6}$-separated subset of B_X. Using this and the fact that $|f_d(y)| \leq 1$ for $y \in A_d$ it is easy to check that $f \in S$. Obviously $\operatorname{supp} f$ is bounded.

To see that f cannot be approximated on B_X by polynomials pick any $p \in P(X)$. Let $d \in \mathbb{N}$ be such that $p \in \mathcal{P}^d(X)$. Then $p\restriction_{X_d} \in \mathcal{P}^d(X_d)$ (Fact 1.35). Notice that $f\restriction_{A_d} = f_d$. Thus

$$\sup_{x \in B_X} |f(x) - p(x)| \geq \sup_{x \in A_d} |f(x) - p(x)| = \|f_d - R_d(p\restriction_{X_d})\| \geq \operatorname{dist}\big(f_d, R_d(\mathcal{P}^d(X_d))\big) = 1.$$

$\square$

3. Approximation by real-analytic mappings

We begin with a Whitney-type approximation theorem stating that in a finite-dimensional case any mapping in $C^k(\Omega; Y)$, $k \in \mathbb{N}_0$, can be approximated on the whole Ω, in a fine topology, and together with its derivatives of order up to k by real analytic mappings. Then we present the famous result of Jaroslav Kurzweil which extends this result (for $k = 0$) to infinite-dimensional separable Banach spaces X that admit a separating polynomial. We also show that if we only require uniform approximations for uniformly continuous mappings, then it suffices that X admits a separating real analytic function with uniform radii of convergence.

Let X, Y be normed linear spaces, $\Omega \subset X$ open, and $f \in C^k(\Omega; Y)$ for some $k \in \mathbb{N}_0$. For $S \subset \Omega$ we define

$$\|f\|_{S,k} = \sum_{j=0}^{k} \sup_{x \in S} \|d^j f(x)\|.$$

Clearly $\|\cdot\|_{S,k}$ is a semi-norm on the subspace of $C^k(\Omega; Y)$ consisting of mappings with all derivatives up to k bounded on S.

LEMMA 18. *Let X, Y be normed linear spaces over $\mathbb{K}$, $\Omega \subset X$ open, $k \in \mathbb{N}_0$, $\varphi \in C^k(\Omega)$, $f \in C^k(\Omega; Y)$, and $S \subset \Omega$. Then*

$$\|\varphi f\|_{S,k} \le \binom{k}{\left[\frac{k}{2}\right]} \|\varphi\|_{S,k} \|f\|_{S,k}.$$

PROOF. Fix $x \in \Omega$ and $0 \le j \le k$. By the Leibniz formula (Corollary 1.116)

$$\begin{aligned}\|d^j(\varphi f)(x)\| &\le \sum_{l=0}^{j} \binom{j}{l} \left\|d^{j-l}\varphi(x) \cdot d^l f(x)\right\| \le \sum_{l=0}^{j} \binom{j}{l} \left\|d^{j-l}\varphi(x)\right\| \cdot \left\|d^l f(x)\right\| \\ &\le \binom{j}{\left[\frac{j}{2}\right]} \sum_{l=0}^{j} \left\|d^{j-l}\varphi(x)\right\| \cdot \left\|d^l f(x)\right\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|\varphi f\|_{S,k} &\le \sum_{j=0}^{k} \binom{j}{\left[\frac{j}{2}\right]} \sum_{l=0}^{j} \left\|d^{j-l}\varphi\right\|_S \left\|d^l f\right\|_S \le \binom{k}{\left[\frac{k}{2}\right]} \sum_{j=0}^{k} \sum_{l=0}^{j} \left\|d^{j-l}\varphi\right\|_S \left\|d^l f\right\|_S \\ &\le \binom{k}{\left[\frac{k}{2}\right]} \sum_{j=0}^{k} \sum_{l=0}^{k} \left\|d^j\varphi\right\|_S \left\|d^l f\right\|_S = \binom{k}{\left[\frac{k}{2}\right]} \|\varphi\|_{S,k} \|f\|_{S,k}.\end{aligned}$$

□

In the next theorem we consider $\mathbb{C}^n$ with the Euclidean norm.

THEOREM 19. *Let Y be a Banach space, $\Omega \subset \mathbb{R}^n$ open, $k \in \mathbb{N}_0 \cup \{\infty\}$, $f \in C^k(\Omega; Y)$, and $\varepsilon \in C(\Omega; \mathbb{R}^+)$. Put $G = \left\{z \in \mathbb{C}^n;\ \|\operatorname{Im} z\| < \operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus \Omega)\right\}$. Then there is a mapping $g \in H(G; \tilde{Y})$ such that $g\restriction_\Omega \in \mathbb{C}^\omega(\Omega; Y)$ and $\|d^j f(x) - d^j(g\restriction_\Omega)(x)\| < \varepsilon(x)$ for all $x \in \Omega$, $0 \le j \le \min\{k, 1/\varepsilon(x)\}$.*

Notice in particular, that if in the preceding theorem $\Omega = \mathbb{R}^n$, then the approximating mapping g will be an entire mapping, i.e. a mapping that is holomorphic on the whole $\mathbb{C}^n$.

PROOF. Define $K_{-1} = K_0 = \emptyset$, $K_j = \{x \in \mathbb{R}^n;\ \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega) \geq 2^{-j}\} \cap B(0, j)$, $L_j = K_j \setminus \operatorname{Int} K_{j-1}$, and $U_j = (\operatorname{Int} K_{j+1}) \setminus K_{j-2}$ for $j \in \mathbb{N}$. Note that $K_j \subset K_{j+1}$, L_j is compact, $U_j \subset \Omega$ is an open neighbourhood of L_j, $\Omega = \bigcup_{j=1}^{\infty} L_j$, and $L_j \cap U_l = \emptyset$ for $l > j+1$. By Lemma 3 there are functions $\varphi_j \in C^{\infty}(\mathbb{R}^n; [0,1])$, $j \in \mathbb{N}$, satisfying $\operatorname{supp} \varphi_j \subset U_j$ (hence $\operatorname{supp} \varphi_j$ is compact) and $\varphi_j = 1$ on a neighbourhood of L_j.

Further, we put $\varepsilon_0 = 1$, $\varepsilon_j = \min\{\varepsilon_{j-1}, \min_{x \in L_j} \varepsilon(x)\}$, $k_0 = 0$, $k_j = k$ if $k < \infty$, and finally $k_j = \max\{k_{j-1}, [\max_{x \in L_j} \frac{1}{\varepsilon(x)}]\}$ if $k = \infty$. Notice that the sequence $\{\varepsilon_j\}_{j=1}^{\infty}$ is non-increasing, while the sequence $\{k_j\}_{j=1}^{\infty}$ is non-decreasing. Put $M_j = \nu_{k_j} \|\varphi_j\|_{\mathbb{R}^n, k_j}$, where $\nu_l = \binom{l}{[\frac{l}{2}]}$. For each $j \in \mathbb{N}$ let $\delta_j > 0$ be such that

$$\delta_j (1 + M_{j+1}) < \frac{\varepsilon_j}{2^j}. \tag{6}$$

To slightly shorten our notation we denote $\bar{g} = g\restriction_{M \cap \mathbb{R}^n}$ for $g \colon M \to Y$, where $M \subset \mathbb{C}^n$. For each $j \in \mathbb{N}$ we define inductively mappings $f_j \in C^k(\mathbb{R}^n; Y)$ and $g_j \in H(\mathbb{C}^n; \tilde{Y})$ such that $\overline{g_j}$ maps into Y as follows: We put $f_j = 0$ on $\mathbb{R}^n \setminus \Omega$ and

$$f_j = \varphi_j \cdot \left(f - \sum_{l=1}^{j-1} \overline{g_l} \right) \tag{7}$$

on Ω. Then $f_j \in C^k(\mathbb{R}^n; Y)$ and since $\operatorname{supp} \varphi_j$ is compact, so is $\operatorname{supp} f_j$. By Lemma 1 there is a mapping $g_j \in H(\mathbb{C}^n; \tilde{Y})$ such that $\overline{g_j}$ maps into Y,

$$\|f_j - \overline{g_j}\|_{\mathbb{R}^n, k_j} < \delta_j, \tag{8}$$

and $\|g_j\|_{G_j} < \frac{1}{2^j}$, where $G_j = \left\{z \in \mathbb{C}^n;\ \|\operatorname{Im} z\|^2 < \operatorname{dist}(\operatorname{Re} z, \operatorname{supp} f_j)^2 - \frac{1}{4^j}\right\}$.

Put

$$g = \sum_{j=1}^{\infty} g_j. \tag{9}$$

Fix any $z \in G$ and put $\delta = \frac{1}{2} \min\left\{\operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus \Omega) - \|\operatorname{Im} z\|, 1\right\}$. (We note that the minimum here is to cater for the case when $\operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus \Omega) = +\infty$, i.e. $\Omega = \mathbb{R}^n$.) Further, put $V = \left\{w \in \mathbb{C}^n;\ \|\operatorname{Re} w - \operatorname{Re} z\| + \|\operatorname{Im} w - \operatorname{Im} z\| < \delta\right\}$, which is a neighbourhood of z. Let $j_0 \in \mathbb{N}$ be such that $2^{-j_0} < \frac{\delta}{2}$ and $\|\operatorname{Re} z\| + \|\operatorname{Im} z\| + \frac{3}{2}\delta \leq j_0$. We claim that $V \subset G_j$ for all $j \geq j_0 + 2$. Indeed, pick any $w \in V$. Since j_0 is chosen so that $U_{\mathbb{R}^n}\big(\operatorname{Re} z, \|\operatorname{Im} z\| + \frac{3}{2}\delta\big) \subset K_{j_0}$, we have

$$\begin{aligned} \|\operatorname{Im} w\| &\leq \|\operatorname{Im} z\| + \|\operatorname{Im} w - \operatorname{Im} z\| < \|\operatorname{Im} z\| + \delta - \|\operatorname{Re} w - \operatorname{Re} z\| \\ &\leq \|\operatorname{Im} z\| + \delta + \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j_0}) - \operatorname{dist}(\operatorname{Re} z, \mathbb{R}^n \setminus K_{j_0}) \\ &\leq \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j_0}) - \frac{\delta}{2}. \end{aligned}$$

Hence, using the fact that $(a-b)^2 \leq a^2 - b^2$ whenever $a, b \in \mathbb{R}$, $a - b \geq 0$, and $b \geq 0$,

$$\begin{aligned} \|\operatorname{Im} w\|^2 &\leq \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j_0})^2 - \frac{\delta^2}{4} \leq \operatorname{dist}(\operatorname{Re} w, \mathbb{R}^n \setminus K_{j-2})^2 - \frac{\delta^2}{4} \\ &\leq \operatorname{dist}(\operatorname{Re} w, U_j)^2 - \frac{\delta^2}{4} \leq \operatorname{dist}(\operatorname{Re} w, \operatorname{supp} f_j)^2 - \frac{\delta^2}{4} < \operatorname{dist}(\operatorname{Re} w, \operatorname{supp} f_j)^2 - \frac{1}{4^j} \end{aligned}$$

and the claim follows. This means that $\|g_j\|_V < \frac{1}{2^j}$ for $j \geq j_0 + 2$. Therefore the series (9) converges absolutely locally uniformly on G and so $g \in H(G; \tilde{Y})$. Obviously since each $\bar{g_j}$ maps into Y, so does $\bar{g}$.

To show the approximation property of the mapping $\bar{g}$ fix $x \in \Omega$ and $0 \leq l \leq \min\{k, 1/\varepsilon(x)\}$. There is $p \in \mathbb{N}$ such that $x \in L_p$. Hence $l \leq k_p$ and $\varepsilon_p \leq \varepsilon(x)$. Since $\varphi_p = 1$ on a neighbourhood of L_p, by (7) and (8) we have

$$\left\| f - \sum_{j=1}^{p} \bar{g_j} \right\|_{L_p,k_p} = \|f_p - \bar{g_p}\|_{L_p,k_p} < \delta_p. \tag{10}$$

From Lemma 18, the fact that the sequences $\{k_j\}$ and $\{\nu_j\}$ are non-decreasing, (8), and (10) we obtain

$$\begin{aligned}
\|\overline{g_{p+1}}\|_{L_p,k_p} &\leq \|\overline{g_{p+1}} - f_{p+1}\|_{L_p,k_p} + \|f_{p+1}\|_{L_p,k_p} \\
&\leq \|\overline{g_{p+1}} - f_{p+1}\|_{L_p,k_p} + \nu_{k_p} \|\varphi_{p+1}\|_{L_p,k_p} \left\| f - \sum_{j=1}^{p} \bar{g_j} \right\|_{L_p,k_p} \\
&\leq \|\overline{g_{p+1}} - f_{p+1}\|_{\mathbb{R}^n,k_{p+1}} + \nu_{k_{p+1}} \|\varphi_{p+1}\|_{\mathbb{R}^n,k_{p+1}} \left\| f - \sum_{j=1}^{p} \bar{g_j} \right\|_{L_p,k_p} \\
&< \delta_{p+1} + M_{p+1}\delta_p.
\end{aligned}$$

Finally, for $j > p+1$ we have $U_j \cap L_p = \emptyset$ and since $\operatorname{supp} f_j \subset U_j$, $f_j = 0$ on a neighbourhood of L_p. Hence

$$\|\bar{g_j}\|_{L_p,k_p} = \|\bar{g_j} - f_j\|_{L_p,k_p} \leq \|\bar{g_j} - f_j\|_{\mathbb{R}^n,k_j} < \delta_j.$$

Putting all this together with (6) yields

$$\begin{aligned}
\|d^l f(x) - d^l \bar{g}(x)\| &= \left\| d^l f(x) - \sum_{j=1}^{\infty} d^l \bar{g_j}(x) \right\| \leq \left\| d^l f(x) - \sum_{j=1}^{p} d^l \bar{g_j}(x) \right\| + \sum_{j=p+1}^{\infty} \|d^l \bar{g_j}(x)\| \\
&\leq \left\| f - \sum_{j=1}^{p} \bar{g_j} \right\|_{L_p,k_p} + \sum_{j=p+1}^{\infty} \|\bar{g_j}\|_{L_p,k_p} < \delta_p(1 + M_{p+1}) + \sum_{j=p+1}^{\infty} \delta_j \\
&\leq \sum_{j=p}^{\infty} \delta_j(1 + M_{j+1}) < \sum_{j=p}^{\infty} \frac{\varepsilon_j}{2^j} \leq \sum_{j=p}^{\infty} \frac{\varepsilon_p}{2^j} \leq \varepsilon_p \leq \varepsilon(x).
\end{aligned}$$

We note that the first equality follows from the fact that the series (9) is a locally uniformly convergent series of holomorphic mappings.

□

Now we move on to the infinite-dimensional case. First we state the results and then prove them both together.

THEOREM 20 (Jaroslav Kurzweil, [K1]). *Let X be a separable Banach space that admits a separating polynomial and Y a Banach space. Let $\Omega \subset X$ be open, $f \in C(\Omega; Y)$, and $\varepsilon \in C(\Omega; \mathbb{R}^+)$. Then there is $g \in C^\omega(\Omega; Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

DEFINITION 21. Let X be a Banach space. We say that X has property (K) if there exists a separating real analytic function q on X and $d > 0$ such that for each $x \in X$ the radius of norm convergence of the Taylor series of q at x is at least d.

Given metric spaces P, Q we denote by $C_{\mathrm{u}}(P;Q)$ the space of all uniformly continuous mappings from P to Q.

THEOREM 22 ([CH], [Fry1]). *Let X be a separable Banach space with property (K) and Y a Banach space. Let $\Omega \subset X$ be open, $f \in C_{\mathrm{u}}(\Omega;Y)$, and $\varepsilon > 0$. Then there is $g \in C^{\omega}(\Omega;Y)$ such that $\|f-g\|_{\Omega} \leq \varepsilon$.*

We prove both Theorem 20 and Theorem 22 together, with the help of the next two lemmata.

LEMMA 23. *Let X be a separable Banach space and $\Omega \subset X$ open. Suppose there is $\alpha > 0$ such that for any open covering $\{U(x_n, r_n)\}_{n=1}^{\infty}$ (or for any uniform open covering $\{U(x_n, r)\}_{n=1}^{\infty}$, i.e. $r_n = r$ for all $n \in \mathbb{N}$) of Ω and any sequence $\{w_n\}_{n=1}^{\infty}$ of positive real numbers there exists an open neighbourhood $V \subset \tilde{X}$ of Ω and a sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset H(V)$ with the following properties:*

(i) *The sum $\sum_{n=1}^{\infty} w_n\varphi_n$ converges absolutely locally uniformly on V,*
(ii) *$\varphi_n \restriction_{\Omega}$ maps into $[0,+\infty)$ for every $n \in \mathbb{N}$,*
(iii) *$w_n\varphi_n(x) \leq \frac{1}{4}2^{-n}$ for every $x \in \Omega \setminus U(x_n, \alpha r_n)$, $n \in \mathbb{N}$, and*
(iv) *for every $x \in \Omega$ there is $k \in \mathbb{N}$ such that $x \in U(x_k, \alpha r_k)$ and $\varphi_k(x) \geq w_k$.*

Then for every mapping $f \in C(\Omega;C)$, where C is a closed convex subset of a Banach space Y, (resp. for every $f \in C_{\mathrm{u}}(\Omega;C)$) there is a mapping $g \in C^{\omega}(\Omega;C)$ satisfying $\|f-g\|_{\Omega} \leq 1$.

PROOF. Using the separability of Ω and the continuity (resp. uniform continuity) of f we find a covering (resp. uniform covering) $\{U(x_n, r_n)\}_{n=1}^{\infty}$ of Ω such that

$$\|f(x) - f(x_n)\| < \frac{1}{2} \quad \text{for } x \in U(x_n, \alpha r_n) \cap \Omega. \tag{11}$$

Put $w_n = 1 + \|f(x_n)\|$. Let $\{\varphi_n\}$ be the sequence of functions satisfying (i)–(iv).

The function $\varphi(z) = \sum_{n=1}^{\infty} \varphi_n(z)$ is well-defined for every $z \in V$ by (i) and moreover $\varphi \in H(V)$. Further, by (ii) and (iv), for every $x \in \Omega$

$$\varphi(x) \geq \varphi_k(x) \geq 1. \tag{12}$$

Hence there is an open neighbourhood W of Ω in $\tilde{X}$ such that $W \subset V$ and $\varphi \neq 0$ on W. Define $\psi_n(z) = \varphi_n(z)/\varphi(z)$ for $z \in W$. Then the functions ψ_n are holomorphic on W and have the following properties:

(a) $\{\psi_n \restriction_{\Omega}\}$ is a partition of unity on Ω,
(b) $\psi_n(x)\|f(x_n)\| \leq \frac{1}{4}2^{-n}$ for every $x \in \Omega \setminus U(x_n, \alpha r_n)$, $n \in \mathbb{N}$, and
(c) $\psi_n(x)\|f(x)\| \leq \frac{1}{4}2^{-n}$ for every $x \in \Omega \setminus U(x_n, \alpha r_n)$, $n \in \mathbb{N}$.

Indeed, (a) follows from (ii) and the definition of ψ_n and φ, and (b) follows from (iii) and (12). To prove (c) choose $n \in \mathbb{N}$ and $x \in \Omega \setminus U(x_n, \alpha r_n)$. Then $\varphi(x) \geq \varphi_k(x) \geq w_k = 1 + \|f(x_k)\| > \|f(x)\|$ by (iv) and (11). Thus $\psi_n(x)\|f(x)\| \leq \psi_n(x)\varphi(x) = \varphi_n(x) \leq w_n\varphi_n(x) \leq \frac{1}{4}2^{-n}$ by (iii).

Finally we put $g(z) = \sum_{n=1}^{\infty} \psi_n(z) f(x_n)$ for $z \in W$. As

$$g(z) = (1/\varphi(z)) \sum_{n=1}^{\infty} \varphi_n(z) f(x_n)$$

and the sum converges locally uniformly on W by (i), we obtain $g \in H(W;\tilde{Y})$. Clearly $g\restriction_{\Omega} \in C^{\omega}(\Omega;C)$. Further, choose an arbitrary $x \in \Omega$. Put $I = \{n \in \mathbb{N};\ x \in U(x_n, \alpha r_n)\}$.

Then using (a), (11), (b), and (c) we obtain

$$\begin{aligned}\|f(x)-g(x)\| &= \left\|\sum_{n=1}^{\infty}\psi_n(x)\big(f(x)-f(x_n)\big)\right\| \\ &\le \sum_{n\in I}\psi_n(x)\|f(x)-f(x_n)\| + \sum_{n\in\mathbb{N}\setminus I}\psi_n(x)\big(\|f(x)\|+\|f(x_n)\|\big) < 1.\end{aligned}$$

□

LEMMA 24. *Let X be a Banach space, $\Omega \subset X$ open, and $\{U(x_n, r_n)\}_{n=1}^{\infty}$ an open covering of Ω in X. Suppose that there are a function $q \in H(G)$ and $\alpha > 0$ such that $G = \{z \in \tilde{X};\ \|\operatorname{Im} z\| < \Delta \sup_{n\in\mathbb{N}} \frac{1}{r_n}\}$ for some $\Delta > 0$, $q\restriction_X$ maps into $[0, +\infty)$, $q(x) \ge 2$ for $x \in X \setminus U_X$, $\operatorname{Re} q(z) \le \frac{1}{4}$ for $z \in U_{\tilde{X}}(0, 1/\alpha)$, and suppose there is a sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers such that for each $x \in \Omega$ the function*

$$z \mapsto \sum_{n=1}^{\infty} a_n\big(\operatorname{Im} q\big((x-x_n+z)/(\alpha r_n)\big)\big)^2 \tag{13}$$

is defined on some neighbourhood of 0 in $\tilde{X}$ and is continuous at 0.

Then for every sequence $\{w_n\}_{n=1}^{\infty}$ of positive real numbers there are an open neighbourhood $V \subset \tilde{X}$ of Ω and a sequence of functions $\{\varphi_n\}_{n=1}^{\infty} \subset H(V)$ satisfying the properties (i)–(iv) in Lemma 23.

PROOF. Put $\varepsilon_n = \frac{1}{2w_n^2}\frac{1}{4}2^{-n}$ and let ζ_n be the functions and $\{\delta_n\}$ the sequence from Lemma 5. Denote $\alpha_n = 1/(\alpha r_n)$ and $G_\alpha = \{z \in \tilde{X};\ \|\operatorname{Im} z\| < \alpha\Delta\}$ and put

$$\varphi_n(z) = 2w_n\zeta_n\Big(q\big(\alpha_1(z-x_1)\big), \dots, q\big(\alpha_n(z-x_n)\big)\Big) \quad \text{for } z \in G_\alpha, n \in \mathbb{N}.$$

Then $\varphi_n \in H(G_\alpha)$ and by (Z1), $\varphi_n\restriction_X$ maps into $[0, +\infty)$.

Pick any $x \in \Omega$. Then there is $j \in \mathbb{N}$ such that $x \in U(x_j, r_j)$ and hence $q(\alpha_j(x-x_j)) \le \frac{1}{4} < 1$. Let $k \in \mathbb{N}$ be the smallest index such that $q(\alpha_k(x-x_k)) < 1$. Then $x \in U(x_k, \alpha r_k)$ and property (Z4) implies that $\varphi_k(x) \ge w_k$.

Let $\delta_x > 0$ be such that $\|x - x_j + z\| < r_j$ and $\sum_{n=1}^{\infty} a_n\big(\operatorname{Im} q(\alpha_n(x-x_n+z))\big)^2 \le \delta_j$ whenever $z \in \tilde{X}$, $\|z\| \le \delta_x$. Then $\operatorname{Re} q(\alpha_j(x-x_j+z)) \le \frac{1}{4}$ and hence, by (Z3), $|w_n\varphi_n(x+z)| < 2^{-n}$ for $n > j$. It follows that $\sum_{n=1}^{\infty} w_n\varphi_n$ converges absolutely uniformly on $U_{\tilde{X}}(x, \delta_x)$. We put $V = G_\alpha \cap \bigcup_{x\in\Omega} U_{\tilde{X}}(x, \delta_x)$.

Finally we show that (iii) is satisfied. Fix $n \in \mathbb{N}$. For $x \in \Omega \setminus U(x_n, \alpha r_n)$ we have $q(\alpha_n(x-x_n)) \ge 2$, hence, by (Z5), $w_n\varphi_n(x) \le \frac{1}{4}2^{-n}$.

□

The next lemma shows that in certain circumstances it is possible to pass from uniform approximations to fine approximations.

LEMMA 25 ([K1]). *Let Ω be a topological space and Y a normed linear space. Let $S \subset C(\Omega; Y)$ and $S_1 \subset C(\Omega)$ be such that $h/\eta \in S$ for any positive function $\eta \in S_1$ and any mapping $h \in S$. Suppose that for any $f \in C(\Omega; Y)$ there is $h \in S$ such that $\|f - h\|_\Omega \le 1$ and for any $\varphi \in C(\Omega)$ there is $\eta \in S_1$ such that $|\varphi - \eta|_\Omega \le 1$. Then for any $f \in C(\Omega; Y)$ and any positive function $\varepsilon \in C(\Omega)$ there is $g \in S$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for every $x \in \Omega$.*

PROOF. Define $\varphi \in C(\Omega)$ by $\varphi = 1 + 2/\varepsilon$. According to the assumptions there is $\eta \in S_1$ such that $|\varphi(x) - \eta(x)| \le 1$ for every $x \in \Omega$. Since $\eta f \in C(\Omega; Y)$, there is $h \in S$ such that $\|\eta(x) f(x) - h(x)\| \le 1$ for every $x \in \Omega$. Notice that $\eta(x) \ge \varphi(x) - 1 = 2/\varepsilon(x) > 1/\varepsilon(x) > 0$ for every $x \in \Omega$. Thus $g = h/\eta \in S$ and $\|f(x) - g(x)\| \le 1/\eta(x) < \varepsilon(x)$ for every $x \in \Omega$. $\square$

PROOF OF THEOREM 20. By Fact 4.45 we may assume that there is an m-homogeneous polynomial p on X such that $p(x) \ge 2$ for $x \in X \setminus U_X$. Let $q = \tilde{p}$. Because $q(0) = 0$, by the continuity there is $\alpha > 0$ such that $\operatorname{Re} q(z) \le \frac{1}{4}$ for $z \in U_{\tilde{X}}(0, 1/\alpha)$.

Suppose that $\{U(x_n, r_n)\}_{n=1}^{\infty}$ is an open covering of Ω. Put

$$a_n = \frac{r_n^{2m}}{2^n(1 + \|x_n\|)^{2m}}$$

Then

$$\begin{aligned} a_n\big(\operatorname{Im} q\big((x - x_n + z)/(\alpha r_n)\big)\big)^2 &\le a_n \left|q\big((x - x_n + z)/(\alpha r_n)\big)\right|^2 \le a_n \|q\|^2 \frac{\|x - x_n + z\|^{2m}}{(\alpha r_n)^{2m}} \\ &\le \frac{\|q\|^2}{\alpha^{2m}} \frac{\|x - x_n + z\|^{2m}}{2^n(1 + \|x_n\|)^{2m}} \le \frac{\|q\|^2}{\alpha^{2m}} \big(1 + \|x\| + \|z\|\big)^{2m} \frac{1}{2^n} \end{aligned}$$

and hence for every $x \in X$ the sum in (13) converges absolutely locally uniformly to a continuous function on $\tilde{X}$. Thus the hypotheses of Lemma 24 are satisfied and using it together with Lemma 23 we can conclude that for any Banach space Z and any continuous mapping $f \in C(\Omega; Z)$ there is a mapping $h \in C^{\omega}(\Omega; Z)$ satisfying $\|f - h\|_{\Omega} \le 1$. Finally Lemma 25 applied to $S = C^{\omega}(\Omega; Y)$ and $S_1 = C^{\omega}(\Omega)$ finishes the proof. $\square$

PROOF OF THEOREM 22. By Theorem 1.171 there are $d > 0$ and a function $q \in H(G)$, $G = \{z \in \tilde{X};\ \|\operatorname{Im} z\| < d\}$, such that $q\restriction_X \colon X \to [0, +\infty)$, $q(0) = 0$, $q(x) \ge 2$ for $x \in X \setminus U_X$, and the radius of norm convergence of the Taylor series of q at every point $x \in X$ is at least d. Let $\alpha > 0$ be such that $\operatorname{Re} q(z) \le \frac{1}{4}$ for $z \in U_{\tilde{X}}(0, 1/\alpha)$ and $\frac{1}{2}\alpha d > 1$.

Suppose $\{U(x_n, r)\}_{n=1}^{\infty}$ is a uniform open covering of Ω. Put

$$M_n = \sup\Big\{\big|q\big((x_j - x_n + w)/(\alpha r)\big)\big|\,;\ w \in \tilde{X}, \|w\| \le \tfrac{1}{2}\alpha r d, 1 \le j \le n\Big\}$$

and $a_n = 1/(2^n M_n^2)$. (Note that by the assumption on the radius of the Taylor series $M_n < +\infty$.) Fix $x \in \Omega$. There is $k \in \mathbb{N}$ such that $x \in U(x_k, r)$. For $z \in \tilde{X}$, $\|z\| \le r(\frac{1}{2}\alpha d - 1)$ we have $\|x - x_k + z\| \le \frac{1}{2}\alpha r d$ and hence for $n \ge k$

$$a_n\big(\operatorname{Im} q\big((x - x_n + z)/(\alpha r)\big)\big)^2 \le a_n \left|q\big((x_k - x_n + x - x_k + z)/(\alpha r)\big)\right|^2 \le a_n M_n^2 = \frac{1}{2^n}.$$

Therefore the sum in (13) converges absolutely uniformly on $B_{\tilde{X}}\big(0, r(\frac{1}{2}\alpha d - 1)\big)$ to a continuous function. Using Lemma 24 together with Lemma 23 and a suitable scaling finishes the proof. $\square$

The space c_0 does not admit a separating polynomial (Proposition 5.49 or Corollary 3.59), but it has property (K) (take $P_n(x) = \big(e_n^*(x)\big)^{2n}$ in Example 1.137 and combine it with Corollary 1.165). The property (K) is inherited by subspaces and finite direct sums. In certain circumstances it can also pass to infinite direct sums: Assume that all members of a sequence of Banach spaces $\{X_n\}$ have property (K) witnessed by non-negative functions q_n with radii at least d_n and satisfying $q_n(0) = 0$ and $q_n(x) \ge 1$ whenever $\|x\| \ge 1$. Suppose that there are

$0 < d \leq \frac{1}{2}\inf_{n\in\mathbb{N}} d_n$ and a sequence $\{\alpha_n\} \subset \mathbb{N}$ such that $\sup_{n\in\mathbb{N}} \sup_{z\in B_{\widetilde{X}_n}(0,d)} |\tilde{q_n}(z)|^{\alpha_n} < 1$, where $\tilde{q_n}$ is the analytic extension of q_n to a neighbourhood of X_n in $\widetilde{X}_n$ (Theorem 1.171). Then $\left(\bigoplus_{n=1}^{\infty} X_n\right)_{c_0}$ has property (K) witnessed by $q(x_1, x_2, \dots) = \sum_{n=1}^{\infty} q_n(x_n)^{2n\alpha_n}$ with radii at least d (use Corollary 1.165). Thus for example $\left(c_0 \oplus \bigoplus_{n=1}^{\infty} \ell_{2n}\right)_{c_0}$ has property (K).

By Theorem 5.64 a space with property (K) that does not contain c_0 admits a separating polynomial. By Corollary 5.68 every space with (K) is saturated by spaces from $\{\ell_p;\ p \text{ even}\} \cup \{c_0\}$. Let us mention without proof the next result, which should be compared with Corollary 5.105.

PROPOSITION 26 ([CH]). *Let X be a Banach space with property (K) such that all polynomials on X are weakly sequentially continuous. Then X is isomorphic to a subspace of c_0.*

Whence all Banach spaces with the Dunford-Pettis property and (K) are isomorphic to subspaces of c_0 (Theorem 3.68). In particular, since every $C(K)$ space which is isomorphic to a subspace of c_0 is isomorphic to c_0 [LP], we have the following corollary:

COROLLARY 27. *If the Banach space $C(K)$ has property (K), then it is isomorphic to c_0.*

4. Infimal convolution

The infimal convolution is another global approximation technique, which similarly to the integral convolution preserves certain regularity properties of the approximated function, like for example the Lipschitzness. The undisputable advantage is that it does not need any finite-dimensional structure and it works equally well even on non-separable spaces. The drawback is that this technique is fundamentally scalar and also it usually produces only smoothness of the first order. The notion goes back to Felix Hausdorff around 1919.

DEFINITION 28. Let X be a set, $f\colon X \to \mathbb{R} \cup \{+\infty\}$, and $K\colon X^2 \to \mathbb{R} \cup \{+\infty\}$. We define the infimal convolution of f and K by

$$(f \,\square\, K)(x) = \inf_{y\in X}\bigl(f(y) + K(x,y)\bigr), \quad x \in X.$$

The function K is called a kernel. If $(X, +)$ is a commutative group, then we associate with each $g\colon X \to \mathbb{R} \cup \{+\infty\}$ the kernel $K_g(x,y) = g(x-y)$. We may then define the infimal convolution of f and g as $f \,\square\, g = f \,\square\, K_g$, i.e.

$$(f \,\square\, g)(x) = \inf_{y\in X}\bigl(f(y) + g(x-y)\bigr).$$

Note that in this case $f \,\square\, g = g \,\square\, f$.

FACT 29. *Let X be a set, $f\colon X \to \mathbb{R} \cup \{+\infty\}$, and $K\colon X^2 \to \mathbb{R}$.*

(i) *If $K(x,x) = 0$ for every $x \in X$, then $f \,\square\, K \leq f$.*
(ii) *If f is proper, then $f \,\square\, K < +\infty$ everywhere.*
(iii) *If X is a metric space, f is proper, and the functions $x \mapsto K(x,y)$, $y \in X$, are uniformly continuous with modulus $\omega \in \mathcal{M}$, then either $f \,\square\, K$ is identically $-\infty$, or $f \,\square\, K$ is real-valued and uniformly continuous with modulus ω.*

PROOF. Both (i) and (ii) are obvious.

(iii) Let ρ be the metric on X. Suppose there is $z \in X$ such that $(f \,\square\, K)(z) = -\infty$. Then there is a sequence $\{y_n\}$ in X satisfying $f(y_n) + K(z, y_n) < -n$ for each $n \in N$. Now for any

$x \in X$ we have $(f \,\square\, K)(x) \le f(y_n)+K(x,y_n) = f(y_n)+K(z,y_n)-K(z,y_n)+K(x,y_n) < -n+\omega(\rho(x,z))$, which implies that $(f \,\square\, K)(x) = -\infty$. Therefore $f \,\square\, K$ is either identically $-\infty$, or $f \,\square\, K$ is real-valued and uniformly continuous with modulus ω, as it is an infimum of a family of uniformly continuous functions with modulus ω.

$\square$

The following extension lemma is useful for example when we deal with smooth approximations of Lipschitz functions defined on some subset of a normed linear space X: It suffices to formulate the approximation results for functions defined on the whole of X.

LEMMA 30. *Let (P,ρ) be a metric space, $\emptyset \neq A \subset P$, and $f\colon A \to \mathbb{R}$ a uniformly continuous function with modulus $\omega \in \mathcal{M}_s$. Then there is an extension of f to the whole of P which is uniformly continuous with modulus ω.*

PROOF. Define $\bar{f}\colon P \to \mathbb{R}$ by $\bar{f} = f$ on A and $\bar{f} = +\infty$ on $P \setminus A$. Put $g = \bar{f} \,\square\, (\omega \circ \rho)$. Then

$$g(x) = \inf_{y \in A}\Big(f(y) + \omega\big(\rho(x,y)\big)\Big)$$

For any $x, y \in A$ we have $f(x) - f(y) \le \omega(\rho(x,y))$ and hence $f(x) \le f(y) + \omega(\rho(x,y))$. It follows that $f \le g$ on A. This together with Fact 29(i) implies that $g = f$ on A. Consequently by Fact 29(iii) g is real-valued and uniformly continuous with modulus ω.

$\square$

The next lemma tells us that the results on uniform approximation of Lipschitz functions immediately give also approximation of uniformly continuous functions.

LEMMA 31. *Let (P,ρ) be a metric space, $f\colon P \to \mathbb{R}$ a uniformly continuous function with modulus $\omega \in \mathcal{M}_s$, and $\varepsilon > 0$. Further, let $a \in \mathbb{R}^+$ be such that $\omega(a) \le \varepsilon$. Then there is an $\frac{\varepsilon}{a}$-Lipschitz function $g\colon P \to \mathbb{R}$ such that $|f - g|_P \le \varepsilon$.*

PROOF. We let $g = f \,\square\, \frac{\varepsilon}{a}\rho$. Fix $x \in P$. Clearly $g(x) \le f(x)$ (Fact 29(i)). From the subadditivity of ω it follows that for any $y \in P$

$$f(x) - f(y) \le \omega\big(\rho(x,y)\big) \le \omega\left(\left\lceil \frac{\rho(x,y)}{a} \right\rceil a\right) \le \left\lceil \frac{\rho(x,y)}{a} \right\rceil \omega(a) \le \left(\frac{\rho(x,y)}{a} + 1\right)\varepsilon.$$

Thus $f(y) + \frac{\varepsilon}{a}\rho(x,y) \ge f(x) - \varepsilon$, which implies that $g(x) \ge f(x) - \varepsilon$. Finally, g is $\frac{\varepsilon}{a}$-Lipschitz by Fact 29(iii).

$\square$

THEOREM 32 (Jean-Michel Lasry and Pierre-Louis Lions, [LL]). *Let H be a Hilbert space, $f\colon H \to \mathbb{R}$ an L-Lipschitz function, and $\varepsilon > 0$. Then there is an L-Lipschitz function $g \in C^{1,1}(H)$ satisfying $|f - g|_H \le \varepsilon$.*

For the proof we need a few auxiliary results concerning convex functions.

LEMMA 33. *Let X be a normed linear space, $f \in C(X)$, and suppose there exist functions $\mu, \nu \in C^{1,\alpha}(X)$, $\alpha \in (0,1]$, such that $f + \mu$ is convex and $f - \nu$ is concave. Then $f \in C^{1,\alpha}(X)$.*

PROOF. It clearly suffices to show that $f + \mu \in C^{1,\alpha}(X)$ which we show using Lemma 5.20. Notice that $\mu + \nu = (f + \mu) + (-f + \nu)$ is necessarily convex. From the concavity of $f - \nu$ it follows that $(f-\nu)(x+h)+(f-\nu)(x-h)-2(f-\nu)(x)\le 0$ for any $x,h\in X$ and hence

$$\begin{aligned}(f+\mu)(x+h)&+(f+\mu)(x-h)-2(f+\mu)(x)\\&\le (f+\mu)(x+h)+(f+\mu)(x-h)-2(f+\mu)(x)\\&\quad-\big((f-\nu)(x+h)+(f-\nu)(x-h)-2(f-\nu)(x)\big)\\&=(\mu+\nu)(x+h)+(\mu+\nu)(x-h)-2(\mu+\nu)(x)\le C\|h\|^{1+\alpha},\end{aligned}$$

where the last inequality follows from Lemma 5.20 used on $\mu+\nu$.

□

Let X be a normed linear space. For any $f\colon X\to\mathbb{R}\cup\{+\infty\}$ and $t>0$ we define the Moreau-Yosida regularisation $f_t=f\,\square\,\frac{1}{2t}\|\cdot\|^2$. We note that the constant $\frac{1}{2}$ is useless (and perhaps even annoying) in our proofs but using this particular kernel is customary in convex analysis for many good reasons.

FACT 34. *Let H be a Hilbert space and $f\colon H\to\mathbb{R}\cup\{+\infty\}$ a proper function.*

(i) *The extended real-valued function $-f_t+\frac{1}{2t}\|\cdot\|^2$ is convex for every $t>0$.*
(ii) *Suppose that f is real-valued and $f+\frac{1}{2t}\|\cdot\|^2$ is convex for some $t>0$. Then $f_s+\frac{1}{2(t-s)}\|\cdot\|^2$ is convex for every $0<s<t$.*

PROOF. (i) This follows from the fact that

$$\begin{aligned}-f_t(x)+\frac{1}{2t}\|x\|^2&=-\inf_{y\in H}\left(f(y)+\frac{1}{2t}\|x-y\|^2\right)+\frac{1}{2t}\|x\|^2\\&=\sup_{y\in H}\left(\frac{1}{2t}\big(\|x\|^2-\|x-y\|^2\big)-f(y)\right)\\&=\sup_{y\in H}\left(\frac{1}{t}\langle x,y\rangle-\frac{1}{2t}\|y\|^2-f(y)\right),\end{aligned}$$

which is a supremum of affine functions.

(ii) We have

$$\begin{aligned}f_s(x)+\frac{1}{2(t-s)}\|x\|^2&=\inf_{y\in H}\left(f(y)+\frac{1}{2s}\|x-y\|^2+\frac{1}{2(t-s)}\|x\|^2\right)\\&=\inf_{y\in H}\left(f(y)+\frac{1}{2t}\|y\|^2+\frac{1}{2s}\|x-y\|^2+\frac{1}{2(t-s)}\|x\|^2-\frac{1}{2t}\|y\|^2\right)\\&=\inf_{y\in H}\left(f(y)+\frac{1}{2t}\|y\|^2+\frac{t-s}{2st}\left\|\frac{t}{t-s}x-y\right\|^2\right).\end{aligned}$$

It is easy to verify that for any convex function $\phi\colon X\times Y\to\mathbb{R}$, where X,Y are vector spaces, the function $x\mapsto\inf_{y\in Y}\phi(x,y)$ is also convex ("a convex body casts a convex shadow"), from which the result follows if we set $\phi(x,y)=f(y)+\frac{1}{2t}\|y\|^2+\frac{t-s}{2st}\|\frac{t}{t-s}x-y\|^2$.

□

PROOF OF THEOREM 32. For every C-Lipschitz function $h\colon H \to \mathbb{R}$ we have $\lim_{t\to 0+} |h_t - h|_H = 0$. Indeed,

$$\begin{aligned}
0 \le h(x) - h_t(x) &= h(x) - \inf_{y\in H}\left(h(y) + \frac{1}{2t}\|x-y\|^2\right)\\
&= h(x) + \sup_{y\in H}\left(-h(y) - \frac{1}{2t}\|x-y\|^2\right)\\
&= \sup_{y\in H}\left(h(x) - h(y) - \frac{1}{2t}\|x-y\|^2\right) \le \sup_{y\in H}\left(C\|x-y\| - \frac{1}{2t}\|x-y\|^2\right)\\
&= \sup_{\delta\in[0,+\infty)}\left(C\delta - \frac{1}{2t}\delta^2\right) = \frac{C^2 t}{2}
\end{aligned}$$

for any $x \in H$, where the first inequality follows from Fact 29(i). Moreover, as $h_t = \frac{1}{2t}\|\cdot\|^2 \,\square\, h$, each h_t is C-Lipschitz by Fact 29(iii).

So choose $t > 0$ such that $|f_t - f|_H \le \frac{\varepsilon}{2}$. Then f_t is L-Lipschitz. Next, find $0 < s < t$ such that $\left|(-f_t)_s - (-f_t)\right|_H \le \frac{\varepsilon}{2}$ and put $g = -(-f_t)_s$. Then $|f - g|_H \le \varepsilon$ and the function g is L-Lipschitz.

Further, the function $-f_t + \frac{1}{2t}\|\cdot\|^2$ is convex by Fact 34(i), and hence the function $g - \frac{1}{2(t-s)}\|\cdot\|^2$ is concave by Fact 34(ii). Using Fact 34(i) again, this time on the function $-f_t$, we can conclude that $g + \frac{1}{2s}\|\cdot\|^2$ is convex. Since the function $\|\cdot\|^2$ is a 2-homogeneous polynomial, it belongs to $C^{1,1}(H)$, and so Lemma 33 finishes the proof.

$\square$

Another nice application of the infimal convolution gives the next result.

PROPOSITION 35 ([We]). *Let A be a closed subset of a Hilbert space H. Then there is a function $f \in C^{1,1}(H)$ such that $A = f^{-1}(\{0\})$.*

PROOF. Let $\mathfrak{I}_A\colon H \to \mathbb{R}\cup\{+\infty\}$ be the indicator function of the set A, i.e. $\mathfrak{I}_A(x) = 0$ for $x \in A$ and $\mathfrak{I}_A = +\infty$ for $x \in H \setminus A$. We let $f = -\big(-(\mathfrak{I}_A)_1\big)_{\frac{1}{2}}$. Without loss of generality we may assume that A is non-empty and hence $\mathfrak{I}_A$ is proper. It is easy to see that $(\mathfrak{I}_A)_t(x) = \frac{1}{2t}\operatorname{dist}^2(x, A)$ for every $x \in H$, $t > 0$. Thus $(\mathfrak{I}_A)_1$ is real-valued and using Fact 34 similarly as in the proof of Theorem 32 we can conclude that $f + \|\cdot\|^2$ is convex and $f - \|\cdot\|^2$ is concave.

Next, notice the following observation: Suppose that $h\colon X \to \mathbb{R}\cup\{+\infty\}$ is a proper function on a normed linear space X and $0 < s < t$. For any $x, y, z \in X$ we have

$$\begin{aligned}
\frac{1}{2t}\|z-y\|^2 &= \frac{1}{2t}\left\|\frac{s}{t}\left(\frac{t}{s}(z-x)\right) + \left(1-\frac{s}{t}\right)\left(\frac{t}{t-s}(x-y)\right)\right\|^2\\
&\le \frac{1}{2t}\left(\frac{s}{t}\left\|\frac{t}{s}(z-x)\right\|^2 + \left(1-\frac{s}{t}\right)\left\|\frac{t}{t-s}(x-y)\right\|^2\right)\\
&= \frac{1}{2s}\|z-x\|^2 + \frac{1}{2(t-s)}\|x-y\|^2,
\end{aligned}$$

where we used the convexity of $\|\cdot\|^2$. From this and Fact 29(i) we obtain

$$\begin{aligned} h_t(x) \le -(-h_t)_s(x) &= \sup_{z\in X}\inf_{y\in X}\left(h(y)+\frac{1}{2t}\|z-y\|^2-\frac{1}{2s}\|x-z\|^2\right)\\ &\le \sup_{z\in X}\inf_{y\in X}\left(h(y)+\frac{1}{2(t-s)}\|x-y\|^2\right)=h_{t-s}(x) \quad \text{for every } x\in X. \end{aligned}$$

This gives us $\frac{1}{2}\operatorname{dist}^2(x,A)\le f(x)\le \operatorname{dist}^2(x,A)$ for every $x\in H$. It follows that $f^{-1}(\{0\})=A$ and that $f+\|\cdot\|^2$ is locally bounded and thus convex continuous. Now Lemma 33 implies that $f\in C^{1,1}(H)$.

□

5. Approximation of continuous mappings and partitions of unity

In this section we investigate smooth partitions of unity, the main tool for obtaining C^k-smooth approximations of continuous mappings in Banach spaces. We show that several rather general classes of Banach spaces admit C^k-smooth approximations provided they have a C^k-smooth bump. This applies especially to separable spaces, WCG spaces, or $C(K)$ spaces. We finish by showing that super-reflexive spaces admit partitions of unity consisting of functions with Hölder derivative.

DEFINITION 36. Let $\mathcal{S}$ be a class of functions. We say that a topological space X admits $\mathcal{S}$-partitions of unity if for any open covering $\mathcal{U}$ of X there is a partition of unity on X subordinated to $\mathcal{U}$ such that each member of the partition belongs to $\mathcal{S}$.

DEFINITION 37. A family of subsets of a topological space X is called

- *locally finite* if for each point $x\in X$ there is a neighbourhood of x that meets only finitely many members of this family;
- *discrete* if for each point $x\in X$ there is a neighbourhood of x that meets at most one member of this family;
- σ*-locally finite* if it can be decomposed into countably many locally finite families;
- σ*-discrete* if it can be decomposed into countably many discrete families.

A family of subsets of a metric space P is called

- *uniformly discrete* if there is $d>0$ such that the distance of any two members of this family is at least d;
- σ*-uniformly discrete* if it can be decomposed into countably many uniformly discrete families.

A partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ is called locally finite if $\{\operatorname{supp}_\circ\psi_\alpha\}_{\alpha\in\Lambda}$ is locally finite, it is called σ-discrete if $\{\operatorname{supp}_\circ\psi_\alpha\}_{\alpha\in\Lambda}$ is σ-discrete, and it is called σ-uniformly discrete if $\{\operatorname{supp}_\circ\psi_\alpha\}_{\alpha\in\Lambda}$ is σ-uniformly discrete.

If $\mathcal{S}$ is a class of mappings, then we use the notation $S(X;Y)=\mathcal{S}\cap Y^X$, i.e. $S(X;Y)$ is the set of mappings from X to Y that belong to $\mathcal{S}$. A class of C^k-smooth mappings will be denoted by $\mathcal{C}^k$ and similarly for other smoothness classes from Section 1.5.

DEFINITION 38. Let P be a metric space and $S \subset C(P)$ a ring of functions. We say that S is a *partition ring* if it satisfies the following conditions:

(i) For each $S_0 \subset S$ with $\{\operatorname{supp}_o f;\ f \in S_0\}$ uniformly discrete in P and $\operatorname{supp}_o f$ bounded for each $f \in S_0$ there is a $g \in S$ with $\operatorname{supp}_o g = \bigcup_{f \in S_0} \operatorname{supp}_o f$.

(ii) Let $f \in S$ and $\operatorname{supp}_o f = U_1 \cup U_2$, where U_1 and U_2 are open subsets of P with $\operatorname{dist}(U_1, U_2) > 0$. Then $\chi_{U_1} \cdot f \in S$.

(iii) For each $f \in S$ bounded below and $\varepsilon > 0$ there is a $g \in S$ such that $0 \le g \le 1$, $f^{-1}\big((-\infty, \varepsilon]\big) \subset g^{-1}(\{0\})$ and $f^{-1}\big([2\varepsilon, +\infty)\big) \subset g^{-1}(\{1\})$.

Note that if $S \subset C(P)$ is such that it is for example stable under composition with functions in $C^\infty(\mathbb{R})$ that have zero derivative outside a bounded interval, then S satisfies the condition (iii) in the above definition. Indeed, if $f \in S$ and $\varepsilon > 0$, then we find $\theta \in \mathbb{C}^\infty(\mathbb{R}; [0,1])$ such that $\theta(t) = 0$ for $t \le \varepsilon$ and $\theta(t) = 1$ for $t \ge 2\varepsilon$ (Fact 2), and we set $g = \theta \circ f$. Then $g \in S$ and has the properties required in (iii).

Examples of partition rings: C^k-smooth functions on normed linear spaces, smooth bounded Lipschitz functions, or smooth bounded Lipschitz functions with Hölder derivatives (see the proof of Theorem 48).

DEFINITION 39. Let $\mathcal{S}$ be a class of mappings defined on a topological space X. We say that $\mathcal{S}$ is *determined locally* if whenever f is a mapping defined on X such that for every $x \in X$ there are a neighbourhood U of x and a mapping $g \in \mathcal{S}$ such that $f = g$ on U, then $f \in \mathcal{S}$.

Examples of classes determined locally are $\mathcal{C}^k$ classes or class of continuous Gâteaux differentiable mappings. Note that if a ring of functions on a metric space is determined locally, then conditions (i) and (ii) in the definition of a partition ring are automatically satisfied.

LEMMA 40. *Let P be a metric space and S a partition ring of functions on P. Consider the following statements.*

(i) *For every $A \subset W \subset P$, A closed and W open there is $\varphi \in S$ such that $\varphi = 1$ on A and* $\operatorname{supp}_o \varphi \subset W$.

(ii) *For every $V \subset W \subset P$ bounded open sets satisfying* $\operatorname{dist}(V, P \setminus W) > 0$ *there is $\varphi \in S$ such that* $V \subset \operatorname{supp}_o \varphi \subset W$.

(iii) *For every $V \subset W \subset P$ bounded open sets satisfying* $\operatorname{dist}(V, P \setminus W) > 0$ *there are $\varphi_n \in S$, $n \in \mathbb{N}$, such that* $V \subset \bigcup_{n \in \mathbb{N}} \operatorname{supp}_o \varphi_n \subset W$.

(iv) *The family* $\{\operatorname{supp}_o f;\ f \in S\}$ *contains a σ-uniformly discrete basis for the topology of P.*

(v) *The space P admits locally finite and σ-uniformly discrete S-partitions of unity.*

(vi) *The space P admits locally finite S-partitions of unity.*

(vii) *The family* $\{\operatorname{supp}_o f;\ f \in S\}$ *contains a σ-locally finite basis for the topology of P.*

Then (i)⇒(ii)⇒(iii)⇒(iv)⇔(v)⇒(vi)⇒(vii). If S is moreover determined locally, then all seven statements are equivalent.

We note that the σ-uniformly discrete partitions of unity will prove very useful in Sections 7 and 8, as they allow us to use certain separable techniques in a non-separable setting.

PROOF. (i)⇒(ii)⇒(iii) is obvious.

(iii)⇒(iv) Let $\mathcal{U}_m = \{U_\alpha^m\}_{\alpha \in \Lambda_m}$ be a uniform covering of P by open balls with radius $\frac{1}{m}$. By Lemma 8 there are open refinements $\{V_{n\alpha}^m\}_{n \in \mathbb{N}, \alpha \in \Lambda_m}$, $\{W_{n\alpha}^m\}_{n \in \mathbb{N}, \alpha \in \Lambda_m}$ of $\mathcal{U}_m$ such that $V_{n\alpha}^m \subset W_{n\alpha}^m \subset U_\alpha^m$, $\operatorname{dist}(V_{n\alpha}^m, P \setminus W_{n\alpha}^m) \ge 2^{-n}$ and the family $\{W_{n\alpha}^m\}_{\alpha \in \Lambda_m}$ is uniformly discrete for all $n \in \mathbb{N}$. Thus, by (iii), there are $\varphi_{n\alpha k}^m \in S$ such that $V_{n\alpha}^m \subset \bigcup_{k \in \mathbb{N}} \operatorname{supp}_o \varphi_{n\alpha k}^m \subset W_{n\alpha}^m$. The family $\{\operatorname{supp}_o \varphi^m_{n\alpha k};\ m,n,k \in \mathbb{N}, \alpha \in \Lambda_m\}$ is therefore a σ-uniformly discrete basis for the topology of P.

(iv)⇒(v) Let $\mathcal{U}$ be an open covering of P. We construct a locally finite and σ-uniformly discrete S-partition of unity subordinated to $\mathcal{U}$. Without loss of generality we may assume that $\mathcal{U}$ consists of bounded sets. By (iv) there are $S_j \subset S$, $j \in \mathbb{N}$, such that $\{\operatorname{supp}_o f;\ f \in S_j\}$ are uniformly discrete and $\{\operatorname{supp}_o f;\ f \in S_j, j \in \mathbb{N}\}$ is an open covering of P that refines $\mathcal{U}$. By property (i) of a partition ring there are functions $f_j \in S$ such that $\operatorname{supp}_o f_j = \bigcup_{f \in S_j} \operatorname{supp}_o f$. Replacing f_j by f_j^2 if necessary we may assume that $f_j \geq 0$. By property (iii) of a partition ring there are functions $g_{jk} \in S$ such that $0 \leq g_{jk} \leq 1$, $\operatorname{supp}_o g_{jk} \subset \operatorname{supp}_o f_j$, and $f_j^{-1}\big([\frac{1}{k}, +\infty)\big) \subset g_{jk}^{-1}(\{1\})$. Let $n \mapsto (j_n, k_n)$ be a bijection of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$ and put $\varphi_n = g_{j_n k_n}$.

Now for $n \in \mathbb{N}$ let $\psi_n = \varphi_n \prod_{k=1}^{n-1}(1 - \varphi_k)$. Then $\psi_n \in S$ (as S is a ring) and $\{\psi_n\}_{n\in\mathbb{N}}$ is a locally finite partition of unity on P. Indeed, for any $x \in P$ there is $j \in \mathbb{N}$ such that $x \in \operatorname{supp}_o f_j$ and hence there are a neighbourhood U of x and $k \in \mathbb{N}$ such that $f_j(y) > \frac{1}{k}$ for $y \in U$. It follows that $g_{jk}(y) = 1$ for $y \in U$. Let $m \in \mathbb{N}$ be such that $j = j_m$ and $k = k_m$. Choose any $y \in U$. Then $\varphi_m(y) = g_{jk}(y) = 1$ and hence $\psi_n(y) = 0$ for $n > m$. Since

$$(1 - \varphi_1)(1 - \varphi_2) \cdots (1 - \varphi_m) = 1 - \psi_1 - \cdots - \psi_m,$$

it follows that $\sum_{n=1}^{\infty} \psi_n(y) = \sum_{n=1}^{m} \psi_n(y) = 1$.

Finally, for $n \in \mathbb{N}$ and $f \in S_{j_n}$ let $\psi_{n,f} = \chi_{\operatorname{supp}_o f} \cdot \psi_n$. Using the fact that $\operatorname{supp}_o \psi_n \subset \operatorname{supp}_o \varphi_n \subset \operatorname{supp}_o f_{j_n}$ and the uniform discreteness of $\{\operatorname{supp}_o f;\ f \in S_{j_n}\}$ it follows that $\sum_{f \in S_{j_n}} \psi_{n,f} = \psi_n$ and from property (ii) of a partition ring also that $\psi_{n,f} \in S$. As moreover $\operatorname{supp}_o \psi_{n,f} \subset \operatorname{supp}_o f$, we can conclude that $\{\psi_{n,f}\}_{n\in\mathbb{N}, f \in S_{j_n}}$ is a locally finite, σ-uniformly discrete S-partition of unity on P subordinated to $\mathcal{U}$.

(v)⇒(iv) Let $\mathcal{U}_m$ be a uniform covering of P by open balls with radius $\frac{1}{m}$. By (v) there is an S-partition of unity $\{\psi^m_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda_m}$ subordinated to $\mathcal{U}_m$ such that $\{\operatorname{supp}_o \psi_{n\alpha}\}_{\alpha\in\Lambda_m}$ is uniformly discrete for each $n \in \mathbb{N}$. It follows that the family $\{\operatorname{supp}_o \psi^m_{n\alpha};\ m \in \mathbb{N}, n \in \mathbb{N}, \alpha \in \Lambda_m\}$ is a σ-uniformly discrete basis for the topology of P.

(v)⇒(vi) is obvious.

(vi)⇒(vii) Let $\mathcal{U}_m$ be a uniform covering of P by open balls with radius $\frac{1}{m}$. By (vi) there is a locally finite S-partition of unity $\{\psi^m_\alpha\}_{\alpha\in\Lambda_m}$ subordinated to $\mathcal{U}_m$. It follows that the family $\{\operatorname{supp}_o \psi^m_\alpha;\ m \in \mathbb{N}, \alpha \in \Lambda_m\}$ is a σ-locally finite basis for the topology of P.

Now suppose that S is determined locally.

(vi)⇒(i) Let $\{\psi_\alpha\}_{\alpha\in\Lambda}$ be a locally finite S-partition of unity subordinated to the open covering $\{W, P \setminus A\}$ of P. Let $\Lambda_1 = \{\alpha \in \Lambda;\ \operatorname{supp}_o \psi_\alpha \subset W\}$ and put $\varphi = \sum_{\alpha\in\Lambda_1} \psi_\alpha$. As the sum is locally finite and S is determined locally, $\varphi \in S$. Obviously $\operatorname{supp}_o \varphi \subset W$. Further, $\operatorname{supp}_o \psi_\alpha \subset P \setminus A$ for $\alpha \in \Lambda \setminus \Lambda_1$ and hence $\varphi(x) = \sum_{\alpha\in\Lambda_1} \psi_\alpha(x) = \sum_{\alpha\in\Lambda} \psi_\alpha(x) = 1$ for $x \in A$.

(vii)⇒(iii) By (vii) there is $\{\varphi_{n\alpha};\ n \in \mathbb{N}, \alpha \in \Lambda_n\} \subset S$ such that $V = \bigcup_{n\in\mathbb{N},\alpha\in\Lambda_n} \operatorname{supp}_o \varphi_{n\alpha}$ and $\{\operatorname{supp}_o \varphi_{n\alpha}\}_{\alpha\in\Lambda_n}$ is locally finite for each $n \in \mathbb{N}$. Put $\varphi_n = \sum_{\alpha\in\Lambda_n} \varphi^2_{n\alpha}$. As the sum is locally finite and S is a ring determined locally, $\varphi_n \in S$. Further, $\operatorname{supp}_o \varphi_n = \bigcup_{\alpha\in\Lambda_n} \operatorname{supp}_o \varphi_{n\alpha}$, hence (iii) follows.

□

COROLLARY 41 ([BF]). *Let X be a separable normed linear space and S a partition ring on X such that for every $f \in S$, $a \in \mathbb{R}$, and $b \in X$ the function $g(x) = f(ax + b)$ belongs to S. Then X admits locally finite and σ-uniformly discrete S-partitions of unity if and only there is a bump function in S.*

PROOF. Suppose $\varphi \in S$ is a bump function. Since S is stable under shifts and scaling, we may suppose that $\varphi(0) > 0$ and $\operatorname{supp}_o \varphi \subset U_X$. By the continuity of φ there is $0 < r < 1$ such that $\varphi > 0$ on $U(0, r)$.

We show that (iii) in Lemma 40 is satisfied. So let $V \subset W \subset X$ be bounded open sets satisfying $\operatorname{dist}(V, P \setminus W) = \delta > 0$. By the Lindelöf property of V there is a countable subset $\{x_n\}_{n\in\mathbb{N}}$ of V such that $V \subset \bigcup_{n\in\mathbb{N}} U(x_n, \delta r)$. We put $\varphi_n(x) = \varphi\big((x - x_n)/\delta\big)$. Then $\varphi_n \in S$ and $V \subset \bigcup_{n\in\mathbb{N}} U(x_n, \delta r) \subset \bigcup_{n\in\mathbb{N}} \operatorname{supp}_o \varphi_n \subset \bigcup_{n\in\mathbb{N}} U(x_n, \delta) \subset W$.

The reverse implication is clear for example from Lemma 40(vii).

□

COROLLARY 42. *Let P be a metric space, $Q \subset P$, let R be a partition ring on P and S a partition ring on Q such that each function from R restricted to Q belongs to S. If P admits locally finite and σ-uniformly discrete R-partitions of unity, then Q admits locally finite and σ-uniformly discrete S-partitions of unity.*

PROOF. By the equivalence of (iv) and (v) in Lemma 40 the family $\{\operatorname{supp}_o f;\ f \in R\}$ contains a σ-uniformly discrete basis $\mathcal{B}$ for the topology of P. Consider the family $\mathcal{A} = \{G \cap Q;\ G \in \mathcal{B}\}$. It is clear that $\mathcal{A}$ is a σ-uniformly discrete basis for the topology of Q. Moreover, $\mathcal{A} \subset \{\operatorname{supp}_o f;\ f \in S\}$. Indeed, if $G \in \mathcal{A}$, then there is $f \in R$ such that $G = \operatorname{supp}_o f \cap Q$. Then $\operatorname{supp}_o f\restriction_Q = \{x \in Q;\ f(x) \neq 0\} = \operatorname{supp}_o f \cap Q = G$ and $f\restriction_Q \in S$. Now it suffices to apply Lemma 40 again.

□

DEFINITION 43. Let $\mathcal{Y}$ be a class of normed linear spaces and $\mathcal{S}$ be a class of mappings from a metric space P into spaces from $\mathcal{Y}$. We say that $\mathcal{S}$ is an *approximation class* if

- $\mathcal{S}$ is determined locally,
- $S(P; \mathbb{R})$ is a partition ring,
- $f + g \in \mathcal{S}$ whenever $f, g \in \mathcal{S}$ map into the same space,
- for every $Y \in \mathcal{Y}$, every $y \in Y$, and every $\varphi \in S(P; [0, 1])$ the mapping $x \mapsto \varphi(x)y$ belongs to $\mathcal{S}$.

Notice that the second property implies that the class $\mathcal{Y}$ must contain at least $\mathbb{R}$.

The following theorem goes back to Kazimierz Kuratowski around 1922.

THEOREM 44. *Let P be a metric space and $\mathcal{S}$ an approximation class on P. Then the following statements are equivalent:*

(i) *P admits locally finite $\mathcal{S}$-partitions of unity.*
(ii) *For any convex subset C of a normed linear space of class $\mathcal{Y}$, any $f \in C(P; C)$, and any $\varepsilon \in C(P; \mathbb{R}^+)$ there is $g \in S(P; C)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for every $x \in P$.*
(iii) *For any 1-Lipschitz $f : P \to [0, 1]$ and any $\varepsilon > 0$ there is $g \in S(P; \mathbb{R})$ such that $|f - g|_P \leq \varepsilon$.*

PROOF. (i)⇒(ii) For each $x \in P$ find $r(x) > 0$ such that

$$\varepsilon(y) > \frac{\varepsilon(x)}{2} \quad \text{and} \quad \|f(y) - f(x)\| < \frac{\varepsilon(x)}{2} \quad \text{for each } y \in U(x, r(x)).$$

It follows that

$$\|f(y) - f(x)\| < \varepsilon(y) \quad \text{for each } y \in U(x, r(x)). \tag{14}$$

By (i) there is a locally finite $\mathcal{S}$-partition of unity $\{\psi_\alpha\}_{\alpha\in\Lambda}$ on P subordinated to the covering $\{U(x, r(x));\ x \in P\}$. For each $\alpha \in \Lambda$ let $U_\alpha = U(x_\alpha, r(x_\alpha))$ be such that $\operatorname{supp}_o \psi_\alpha \subset U_\alpha$.

Define

$$g(x) = \sum_{\alpha\in\Lambda} \psi_\alpha(x) f(x_\alpha). \tag{15}$$

By the properties of $\mathcal{S}$ each mapping $\psi_\alpha f(x_\alpha)$ belongs to $\mathcal{S}$ as well as finite sums of these mappings. Since the sum in the definition of g is locally finite and $\mathcal{S}$ is determined locally, $g \in \mathcal{S}$. Moreover, as $g(x)$ is a convex combination of points from C for every $x \in P$, $g \in S(P; C)$. Finally, choose any $x \in P$. Then

$$\begin{aligned}\|f(x) - g(x)\| &= \Big\|\sum_{\alpha\in\Lambda} \psi_\alpha(x)\big(f(x) - f(x_\alpha)\big)\Big\| \le \sum_{\alpha\in\Lambda:\, x\in U_\alpha} \psi_\alpha(x)\|f(x) - f(x_\alpha)\| \\ &< \varepsilon(x) \sum_{\alpha\in\Lambda:\, x\in U_\alpha} \psi_\alpha(x) = \varepsilon(x),\end{aligned}$$

where the last inequality follows from (14).

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i) We show that the condition (ii) in Lemma 40 is satisfied. Let $V \subset W \subset P$ be bounded open sets satisfying $\operatorname{dist}(V, P \setminus W) > 3\delta$ for some $0 < \delta < \frac{1}{3}$. Put $f(x) = \min\{\operatorname{dist}(x, P \setminus W), 1\}$. By (iii) there is $g \in S(P; \mathbb{R})$ such that $|f - g|_P \le \delta$. Then $g \le \delta$ on $P \setminus W$ and $g > 2\delta$ on V. By property (iii) of a partition ring there is $\varphi \in S(P; \mathbb{R})$ such that $\varphi = 0$ on $P \setminus W$ and $\varphi > 0$ on V.

□

Next we show how to construct smooth partitions of unity on various classes of Banach spaces. In the following theorem the mapping Φ introduces a "coordinate system" on X, while the mappings P_F serve as the "projections" associated to this "coordinate system". The requirement is that for every $x \in X$ if we take "large coordinates of x", then the associated "projection" approximates x well.

THEOREM 45 ([H]). *Let X be a normed linear space that admits a C^k-smooth bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Γ be a set and $\Phi\colon X \to c_0(\Gamma)$ a continuous mapping such that for every $\gamma \in \Gamma$ the function $e^*_\gamma \circ \Phi$ is C^k-smooth on the set where it is non-zero. For each finite $F \subset \Gamma$ let $P_F \in C^k(X; X)$ be such that the space* span $P_F(X)$ *admits locally finite $\mathcal{C}^k$-partitions of unity. Assume that for each $x \in X$ and each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - P_F(x)\| < \varepsilon$ if we set $F = \{\gamma \in \Gamma;\ |\Phi(x)(\gamma)| \ge \delta\}$. Then X admits locally finite and σ-uniformly discrete $\mathcal{C}^k$-partitions of unity.*

PROOF. Denote by $\mathcal{F}$ a set of all finite subsets of Γ (including an empty set). For any $q \in \mathbb{R}^+$ let $\zeta_q \in \mathbb{C}^\infty(\mathbb{R}; [0, 1])$ be such that $\zeta_q(t) = 0$ for $|t| \ge q$, $0 < \zeta_q(t) < 1$ for $\frac{q}{2} < |t| < q$, and $\zeta_q(t) = 1$ for $t \in [\frac{q}{2}, \frac{q}{2}]$ (Fact 2). For each $F \in \mathcal{F}$ and $q, r \in \mathbb{R}^+$ we define a function $\varphi_{F,q,r}\colon c_0(\Gamma) \to \mathbb{R}$ by

$$\varphi_{F,q,r}(x) = \prod_{\gamma\in F} \Big(1 - \zeta_{2r}\big(x(\gamma)\big)\Big) \prod_{\gamma\in\Gamma\setminus F} \zeta_q\big(x(\gamma)\big).$$

For $x \in c_0(\Gamma)$ let $H = \{\gamma \in \Gamma;\ |x(\gamma)| \ge \frac{q}{4}\}$. Then $H \in \mathcal{F}$ and $|y(\gamma)| < \frac{q}{2}$ for $y \in U(x, \frac{q}{4})$, $\gamma \in \Gamma \setminus H$. Thus

$$\varphi_{F,q,r}(y) = \prod_{\gamma\in F} \Big(1 - \zeta_{2r}\big(y(\gamma)\big)\Big) \prod_{\gamma\in H\setminus F} \zeta_q\big(y(\gamma)\big) \tag{16}$$

for $y \in U(x, \frac{q}{4})$, which implies that $\varphi_{F,q,r}$ is LFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ and $\varphi_{F,q,r} \in \mathbb{C}^\infty(c_0(\Gamma); [0,1])$. It is easy to check that $\operatorname{supp}_\circ \varphi_{F,q,r} = W_{F,q,r}$, where

$$W_{F,q,r} = \Big\{x \in c_0(\Gamma);\ \min_{\gamma\in F}|x(\gamma)| > r,\ \sup_{\gamma\in\Gamma\setminus F}|x(\gamma)| < q\Big\}.$$

Notice that $\operatorname{dist}(W_{F,q,r}, W_{H,q,r}) \ge r - q$ whenever $F, H \in \mathcal{F}$, $F \ne H$, and $r > q$. Therefore the family $\{W_{F,q,r};\ F \in \mathcal{F}, q, r \in \mathbb{Q}, r > q > 0\}$ is σ-uniformly discrete.

Further, $\varphi_{F,q,r} \circ \Phi \in C^k(X)$. This follows from (16), the fact that $\zeta_s \circ e^*_\gamma \circ \Phi \in C^k(X)$ for each $\gamma \in \Gamma$ and $s \in \mathbb{R}^+$, and the continuity of Φ.

Note that $C^k(Z)$, where Z is a normed linear space, is a partition ring determined locally. By the hypothesis and Lemma 40 for each $F \in \mathcal{F}$ there is a σ-discrete basis $\mathcal{V}_F$ for the topology of $\operatorname{span} P_F(X)$ formed by the sets in $\{\operatorname{supp}_\circ f;\ f \in C^k(\operatorname{span} P_F(X))\}$. Further, as X admits a C^k-smooth bump function, the family $\{\operatorname{supp}_\circ f;\ f \in C^k(X)\}$ contains a neighbourhood basis of 0, say $\{U_m\}_{m\in\mathbb{N}}$.

In X consider the family

$$\Big\{\Phi^{-1}(W_{F,q,r}) \cap P_F^{-1}(V) \cap (Id - P_F)^{-1}(U_m);\ F \in \mathcal{F}, q, r \in \mathbb{Q}, 0 < q < r, V \in \mathcal{V}_F, m \in \mathbb{N}\Big\}.$$

Using the continuity of Φ it is easy to verify that this is a σ-discrete (and in particular σ-locally finite) subfamily of $\{\operatorname{supp}_\circ f;\ f \in C^k(X)\}$ (notice that $\Phi^{-1}(W_{F,q,r}) = \operatorname{supp}_\circ \varphi_{F,q,r} \circ \Phi$). To finish the proof using Lemma 40 we need to show that this family forms a basis for the topology of X.

To this end choose $x \in X$ and $\varepsilon > 0$. Let $m \in \mathbb{N}$ be such that $U_m \subset U(0, \frac{\varepsilon}{6})$ and further let $\delta > 0$ be such that

$$x - P_F(x) \in U_m$$

when we set $F = \{\gamma \in \Gamma;\ |\Phi(x)(\gamma)| \ge \delta\}$. Because $\Phi(x) \in c_0(\Gamma)$, there exist $q, r \in \mathbb{Q}$ with $0 < q < r < \delta$ satisfying $|\Phi(x)(\gamma)| < q$ whenever $\gamma \in \Gamma \setminus F$. Thus $x \in \Phi^{-1}(W_{F,q,r})$. Since $\mathcal{V}_F$ is a basis for the topology of $\operatorname{span} P_F(X)$, there exists $V \in \mathcal{V}_F$ such that

$$P_F(x) \in V \subset U\left(P_F(x), \frac{\varepsilon}{3}\right).$$

It follows that $x \in \Phi^{-1}(W_{F,q,r}) \cap P_F^{-1}(V) \cap (Id - P_F)^{-1}(U_m)$. If y is any other member of this set, then we have $\|P_F(x) - P_F(y)\| < \frac{\varepsilon}{3}$ because $P_F(y) \in V$, while $\|P_F(y) - y\| < \frac{\varepsilon}{3}$ because $y - P_F(y) \in U_m$. Thus $\|x - y\| < \varepsilon$, which is what we wanted to prove.

□

COROLLARY 46 ([H]). *Let X be a normed linear space that admits a C^k-smooth bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let μ be a limit ordinal and let $\{P_\alpha \in C^k(X; X);\ \alpha < \mu\}$ be an equi-continuous family of mappings having the property that for every $x \in X$ the mapping $P_x\colon [0,\mu] \to X$ defined by $P_x(\alpha) = P_\alpha(x)$ for $\alpha < \mu$, $P_x(\mu) = x$, is continuous. If for each $\alpha < \mu$ the space* span $P_\alpha(X)$ *admits locally finite $\mathcal{C}^k$-partitions of unity, then so does X.*

PROOF. Since X admits a C^k-smooth bump function, there exist a function $h \in C^k(X; [0,1])$ and $\eta > 0$ such that $h(x) = 0$ for $\|x\| \le \eta$, while $h(x) = 1$ whenever $\|x\| \ge 1$. We set $\Gamma = [0,\mu) \times \mathbb{N}$ and define $\Phi\colon X \to \ell_\infty(\Gamma)$ by

$$\Phi(x)(\alpha, n) = 2^{-n} h\big(2^n(P_{\alpha+1}(x) - P_\alpha(x))\big).$$

We note that the enlargement of the index set by the factor $\mathbb{N}$ would not be necessary if we knew that h is zero only at the origin. Such function however may not exist, cf. Theorem 5.161.

Given $x \in X$ and $\varepsilon > 0$ we fix $m \in \mathbb{N}$ such that $2^{-m} < \varepsilon$ and note that the quantity $\|P_{\alpha+1}(x) - P_\alpha(x)\|$ can exceed $2^{-m}\frac{\eta}{2}$ only for α in some finite $H \subset [0,\mu)$. Indeed, otherwise there would be an increasing sequence $\{\alpha_n\}$ of ordinals with $\|P_{\alpha_n+1}(x) - P_{\alpha_n}(x)\| > 2^{-m}\frac{\eta}{2}$, which contradicts the continuity of P_x at the ordinal $\sup\alpha_n = \lim\alpha_n$. We thus have $0 \le \Phi(x)(\gamma) < \varepsilon$ except when $\gamma \in K = H \times \{0,1,2,\dots,m-1\}$. This shows that Φ maps into $c_0(\Gamma)$. Furthermore, because of the equi-continuity of $\{P_\alpha\}_{\alpha<\mu}$, there is a neighbourhood U of x such that $\|P_{\alpha+1}(y) - P_\alpha(y)\| < 2^{-m}\eta$ for $y \in U$ and $\alpha \in [0,\mu) \setminus H$. It follows that $|\Phi(x)(\gamma) - \Phi(y)(\gamma)| < \varepsilon$ for $y \in U$, $\gamma \in \Gamma \setminus K$. Since K is finite, there is a neighbourhood V of x such that $\|\Phi(x) - \Phi(y)\|_\infty < \varepsilon$ whenever $y \in V$. This shows that Φ is continuous.

To define the "projections associated to Φ" we set $P_\emptyset = P_0$ and $P_F = P_{\alpha(F)+1}$ where, for a finite non-empty subset F of Γ, $\alpha(F) = \max\{\alpha;\ (\alpha,n) \in F$ for some $n \in \mathbb{N}\}$. Notice that since μ is a limit ordinal, $\alpha(F)+1 < \mu$ and P_F is thus well-defined. We shall show that the hypothesis of Theorem 45 is satisfied. Given $x \in X$ and $\varepsilon > 0$ it may be that $\|x - P_\alpha(x)\| < \varepsilon$ for all $\alpha < \mu$; in this case there is clearly no problem. Otherwise, there is a maximal $\beta < \mu$ with $\|x - P_\beta(x)\| \ge \varepsilon$. (Set $\beta = \min\{\gamma \in [0,\mu);\ \|x - P_\alpha(x)\| < \varepsilon$ for all $\gamma < \alpha < \mu\}$ and use the continuity of P_x.) It follows that $\|P_{\beta+1}(x) - P_\beta(x)\| \ge \|x - P_\beta(x)\| - \|x - P_{\beta+1}(x)\| \ge \varepsilon - \|x - P_{\beta+1}(x)\| > 0$. Now we fix $m \in \mathbb{N}$ such that $2^m\|P_{\beta+1}(x) - P_\beta(x)\| \ge 1$, noting that $\Phi(x)(\beta,m) = 2^{-m}$, and set $\delta = 2^{-m}$. If $F = \{(\alpha,n) \in \Gamma;\ |\Phi(x)(\alpha,n)| \ge \delta\}$, then $(\beta,m) \in F$ and so $\alpha(F) \ge \beta$, whence $\|x - P_F(x)\| < \varepsilon$, as required.

□

We say that a class $\mathcal{X}$ of Banach spaces is a $\overline{\mathcal{P}}$-class if for every non-separable $X \in \mathcal{X}$ there exists a projectional resolution of the identity $\{P_\alpha;\ \omega \le \alpha \le \mu\}$ on X such that $P_\alpha(X) \in \mathcal{X}$ for all $\alpha < \mu$, where μ is the first ordinal with cardinality dens X. (We remark that this definition is slightly different from that of a $\mathcal{P}$-class in [HMVZ].) Examples of $\overline{\mathcal{P}}$-classes are reflexive spaces, WCG spaces, WCD spaces, WLD spaces, 1-Plichko spaces (see [KKL, Theorem 17.6]), or duals of Asplund spaces.

The following theorem was shown for $\overline{\mathcal{P}}$-class spaces by Gilles Godefroy, Stanimir Troyanski, John Whitfield, and Václav Zizler in [GTWZ], for preduals of WCG spaces by David McLaughlin in [McL], and for $C(K)$ spaces by Petr Hájek and Richard Haydon in [HH].

THEOREM 47. *Let X be a Banach space such that*

- *X belongs to a $\overline{\mathcal{P}}$-class, or*
- *$X = C(K)$ for K compact, or*
- *X^* is a WCG space.*

The space X admits a C^k-smooth bump function, $k \in \mathbb{N} \cup \{\infty\}$, if and only if for any open $\Omega \subset X$, any convex subset C of a normed linear space, any $f \in C(\Omega; C)$, and any $\varepsilon \in C(\Omega; \mathbb{R}^+)$ there is $g \in C^k(\Omega; C)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for every $x \in \Omega$.

PROOF. $\Leftarrow$ is obvious – to construct a C^k-smooth bump function we just approximate the norm and then compose the approximation with a suitable function from $C^\infty(\mathbb{R})$.

$\Rightarrow$ Note that the class of C^k-smooth mappings from Ω into normed linear spaces is an approximation class. Further, by Corollary 42 and Lemma 40 the existence of locally finite $\mathcal{C}^k$-partitions of unity on Ω follows from the existence of locally finite $\mathcal{C}^k$-partitions of unity on the whole of X. Thus it remains to prove that X admits locally finite $\mathcal{C}^k$-partitions of unity. The approximation then follows by Theorem 44.

First we consider the case that X belongs to a $\overline{\mathcal{P}}$-class $\mathcal{X}$. We use transfinite induction on the density of X. If X is separable, then by Corollary 41 the space X admits locally finite

$\mathcal{C}^k$-partitions of unity. Now suppose that X is non-separable and each space in $\mathcal{X}$ with density smaller than dens X admits locally finite $\mathcal{C}^k$-partitions of unity. Let μ be the first ordinal with cardinality dens X and $\{P_\alpha;\ \omega \le \alpha \le \mu\}$ be a projectional resolution of the identity on X with $P_\alpha(X) \in \mathcal{X}$ for all $\omega \le \alpha < \mu$. Define $P_\alpha = P_\omega$ for $\alpha \in [0, \omega)$. By the inductive hypothesis all the spaces $P_\alpha(X)$ admit locally finite $\mathcal{C}^k$-partitions of unity, hence the hypotheses of Corollary 46 are satisfied and X admits locally finite $\mathcal{C}^k$-partitions of unity.

Next we consider the case $X = C(K)$, K compact. Since X admits a C^k-smooth bump function, X is an Asplund space (Corollary 5.3), which is equivalent to K being scattered (Theorem 5.125). We will exploit this fact heavily in our construction. In particular, the zero-dimensionality of K provides a rich supply of projections given by restrictions to clopen subsets of K.

First we shall examine the structure of closed subsets of K. For each $t \in K$ there is a unique ordinal $\alpha = \alpha(t)$ such that $t \in K^{(\alpha)} \setminus K^{(\alpha+1)}$. Since t is an isolated point of $K^{(\alpha)}$ and K is zero-dimensional, there is a clopen neighbourhood V of t such that $V \cap K^{(\alpha)} = \{t\}$; we choose such a V and call it V_t. Note that $V_t \cap K^{(\beta)} = \emptyset$ for $\beta > \alpha$. For finite subsets B of K we set $V_B = \bigcup_{t \in B} V_t$. We shall say that a finite subset A of K is admissible if $s \notin V_t$ whenever $s, t \in A$, $s \ne t$.

Suppose that H is a closed subset of K. We claim that there is a unique admissible set A with the property that $A \subset H \subset V_A$. If $H = \emptyset$, then this is obviously satisfied with $A = \emptyset$ and no other. For a non-empty H we construct an admissible A with the required property. Let $\alpha_0 = \max\{\alpha;\ H \cap K^{(\alpha)} \ne \emptyset\}$; thus $H \cap K^{(\alpha_0)}$ is a non-empty finite set, which we shall call A_0. If $H \subset V_{A_0}$, we set $A = A_0$ and stop. Otherwise, we set $H_1 = H \setminus V_{A_0}$, which is a closed set, $\alpha_1 = \max\{\alpha;\ H_1 \cap K^{(\alpha)} \ne \emptyset\}$, and $A_1 = H_1 \cap K^{(\alpha_1)}$, and repeat the procedure. In this way we construct a decreasing (and so necessarily finite) sequence $\alpha_0 > \alpha_1 > \cdots > \alpha_l$ of ordinals, and finite sets $A_j = \big(H \setminus (V_{A_0} \cup \cdots \cup V_{A_{j-1}})\big) \cap K^{(\alpha_j)}$, $j = 1, \ldots, l$, such that $H \subset V_{A_0} \cup \cdots \cup V_{A_l}$. By construction, the set $A = A_0 \cup \cdots \cup A_l$ is admissible and $A \subset H \subset V_A$.

Now suppose that there are two different admissible sets B and D satisfying $B \subset H \subset V_B$ and $D \subset H \subset V_D$. Let $\beta = \max\{\alpha;\ B \cap K^{(\alpha)} \ne D \cap K^{(\alpha)}\}$. Without loss of generality we may assume that there is $u \in (B \setminus D) \cap K^{(\beta)}$. Since $u \in B \subset H \subset V_D$, there is $s \in D$ such that $u \in V_s$. Because $u \in V_s \cap K^{(\beta)} \setminus D$ and $K^{(\alpha(s))} \cap V_s = \{s\}$, it must be that $\alpha(s) > \beta$. By the maximality of β we have $B \cap K^{\alpha(s)} = D \cap K^{\alpha(s)}$, whence $s \in B$, which contradicts the admissibility of B.

We now pass to the construction of partitions of unity. We shall proceed by transfinite induction on the height of K. Let μ be an ordinal satisfying $\mathrm{ht}(K) = \mu + 1$, i.e. $K^{(\mu)}$ is finite and non-empty. If $\mu = 0$, then $C(K)$ is finite-dimensional, and so has the required partitions of unity for example by Corollary 41. For $\mu > 0$ we assume inductively that if L is a compact space with $\mathrm{ht}(L) < \mu + 1$ and such that $C(L)$ has a C^k-smooth bump function, then $C(L)$ admits locally finite $\mathcal{C}^k$-partitions of unity. To show that $C(K)$ also admits locally finite $\mathcal{C}^k$-partitions of unity it will be enough to construct the partitions of unity on the finite-codimensional subspace $Z = \{f \in C(K);\ f(t) = 0 \text{ for all } t \in K^{(\mu)}\}$. (Using Lemma 40 it is not difficult to ascertain that whenever some normed linear spaces Y and Z admit locally finite $\mathcal{C}^k$-partitions of unity, then so does the space $Y \oplus Z$.) To this end we construct the mappings required by Theorem 45.

Put $\Gamma = \mathcal{Q} \times \mathcal{A}$, where $\mathcal{Q}$ is the set of all triples $(\zeta, \eta, \xi) \in \mathbb{Q}^3$ with $0 < \zeta < \eta < \xi$, and $\mathcal{A}$ consists of the admissible subsets A of K for which $A \cap K^{(\mu)} = \emptyset$. Let $n\colon \mathcal{Q} \to \mathbb{N}$ be some one-to-one mapping, $\beta_{\zeta,\eta}$ be as in Lemma 6, and $\theta \in C^\infty(\mathbb{R}; [0,1])$ be such that

$\theta^{-1}(\{0\}) = [-1, 1]$ (Fact 2). We define $\Phi\colon Z \to \ell_\infty(\Gamma)$ by

$$\Phi(f)(\zeta, \eta, \xi, A) = \frac{1}{n(\zeta, \eta, \xi)} \beta_{\zeta,\eta}(\chi_{K\setminus V_A} \cdot f) \prod_{t \in A} \theta\left(\frac{f(t)}{\xi}\right).$$

Notice that as V_A is clopen, $\chi_{K\setminus V_A} \in C(K)$. We shall show that Φ is actually a continuous mapping into $c_0(\Gamma)$. To do so fix $f \in Z$ and $\varepsilon > 0$. The quantity $n(\zeta, \eta, \xi)^{-1}$ is greater than ε only for (ζ, η, ξ) in some finite subset R of $\mathcal{Q}$. Put $\Delta = \min\bigl\{\frac{1}{2}(\xi - \eta);\ (\zeta, \eta, \xi) \in R\bigr\} > 0$. For a given $(\zeta, \eta, \xi) \in R$ we have $\Phi(g)(\zeta, \eta, \xi, A) = 0$ for each $g \in U(f, \Delta)$ unless $A \subset \bigl\{t \in K;\ |f(t)| \geq \frac{1}{2}(\eta + \xi)\bigr\} \subset V_A$, which can happen for at most one set $A \in \mathcal{A}$, as we have shown earlier. It follows easily that Φ is a continuous mapping into $c_0(\Gamma)$. Moreover, $f \mapsto \chi_{K\setminus V_A} \cdot f$ and $f \mapsto f(t)$ are bounded linear operators, whence each $e^*_\gamma \circ \Phi$, $\gamma \in \Gamma$, is C^k-smooth.

Finally we define the associated projections $P_F : Z \to Z$ as follows: if $F \subset \Gamma$ is a finite subset with elements $(\zeta_j, \eta_j, \xi_j, A_j)$, $j = 1, \dots, m$, we set $V(F) = \bigcup_{j=1}^m V_{A_j}$, and define $P_F(f) = \chi_{V(F)} \cdot f$. Because $V(F)$ is a clopen set with $V(F) \cap K^{(\mu)} = \emptyset$, P_F is a well-defined linear projection of norm 1, the image $P_F(Z)$ is isometric to the space $C(V(F))$, and $\mathrm{ht}(V(F)) < \mu + 1$. Hence by the inductive hypothesis $P_F(Z)$ admits locally finite $\mathcal{C}^k$-partitions of unity.

It only remains to check the required relation between Φ and the projections. So let $f \in Z$ and $\varepsilon > 0$ be given. Let $H = \{t \in K;\ |f(t)| \geq \varepsilon\}$ and let A be the admissible set such that $A \subset H \subset V_A$. Then $A \in \mathcal{A}$, since $H \cap K^{(\mu)} = \emptyset$ by the definition of Z. There is $(\zeta, \eta, \xi) \in \mathcal{Q}$ satisfying $\|\chi_{K\setminus V_A} \cdot f\|_\infty \leq \zeta < \eta < \xi < \varepsilon$. It follows that $\Phi(f)(\zeta, \eta, \xi, A) > 0$. We set $\delta = \Phi(f)(\zeta, \eta, \xi, A)$, $F = \{\gamma \in \Gamma;\ |\Phi(x)(\gamma)| \geq \delta\}$, and note that $V(F) \supset V_A$. So

$$\|f - P_F(f)\|_\infty = \|\chi_{K\setminus V(F)} \cdot f\|_\infty \leq \|\chi_{K\setminus V_A} \cdot f\|_\infty < \varepsilon.$$

For the proof of the case of X^* being WCG see [McL].

□

THEOREM 48 ([JTZ], [FWZ]). *Let X be a Banach space and $\alpha \in (0, 1]$. Then every open subset of X admits locally finite and σ-uniformly discrete $\mathcal{C}^{1,\alpha}$-partitions of unity if and only if X admits an equivalent norm with modulus of smoothness of power type $1 + \alpha$. In particular, a super-reflexive Banach space admits locally finite and σ-uniformly discrete $\mathcal{C}^{1,\alpha}$-partitions of unity for some $\alpha \in (0, 1]$.*

For the proof we need two auxiliary statements.

LEMMA 49. *Let Γ be a non-empty set, $p, q \in [1, +\infty)$, and $r \in \mathbb{N}$ odd such that $rq \geq p$. Then the Mazur mapping $\Phi_r\colon \ell_p(\Gamma) \to \ell_q(\Gamma)$ defined by $\Phi_r(x)(\gamma) = x(\gamma)^r$ is a one-to-one r-homogenous polynomial with $\|\Phi_r\| = 1$. Further, Φ_r is τ_p–τ_p continuous and on bounded sets even τ_p–τ_p uniformly continuous, where τ_p is the topology of pointwise convergence. If $q > 1$, then Φ_r is τ_p–w uniformly continuous on bounded sets and in particular w–w sequentially continuous.*

PROOF. Define a mapping $M\colon \ell_p(\Gamma)^r \to \ell_\infty(\Gamma)$ by $M(x_1, \dots, x_r)(\gamma) = x_1(\gamma) \cdots x_r(\gamma)$. Obviously M is an r-linear mapping and $\Phi_r(x) = M(x, \dots, x)$. Hence Φ_r is an r-homogeneous polynomial. Moreover, for any $x \in \ell_p(\Gamma)$,

$$\left(\sum_{\gamma\in\Gamma} |x(\gamma)^r|^q\right)^{\frac{1}{q}} = \|x\|_p^r \left(\sum_{\gamma\in\Gamma} \left|\frac{x(\gamma)}{\|x\|_p}\right|^{rq}\right)^{\frac{1}{q}} \leq \|x\|_p^r \left(\sum_{\gamma\in\Gamma} \left|\frac{x(\gamma)}{\|x\|_p}\right|^{p}\right)^{\frac{1}{q}} = \|x\|_p^r,$$

and hence Φ_r maps into $\ell_q(\Gamma)$ and $\|\Phi_r\| \le 1$. Further, $\Phi_r(e_\gamma) = e_\gamma$, which implies that $\|\Phi_r\| = 1$. As r is odd, Φ_r is obviously one-to-one.

The τ_p–τ_p continuity is obvious. Moreover, if $x, y \in B(0, R)$, then $|\Phi_r(x)(\gamma) - \Phi_r(y)(\gamma)| \le rR^{r-1}|x(\gamma) - y(\gamma)|$ for $\gamma \in \Gamma$, from which the uniform τ_p–τ_p continuity on $B(0, R)$ follows. Further, if $q > 1$, then the weak and pointwise uniformity coincide on bounded subsets of $\ell_q(\Gamma)$ (and of course Φ_r maps bounded sets to bounded sets).

□

PROPOSITION 50. *Let X be a super-reflexive Banach space and Γ a set with* $\operatorname{card}\Gamma = \operatorname{dens} X$.

(i) *There are $p \in (1, +\infty)$ and a one-to-one bounded linear operator $T\colon X \to \ell_p(\Gamma)$.*
(ii) *There is a one-to-one homogeneous continuous polynomial $P\colon X \to \ell_2(\Gamma)$ that is also w–w uniformly continuous on bounded sets.*
(iii) *If X admits an equivalent norm with modulus of smoothness of power type $1+\alpha$, $\alpha \in (0, 1]$, then there is a homeomorphic embedding Ψ of X into $\ell_2(\Gamma)$ such that $\Psi \in C^1(X; \ell_2(\Gamma))$ with $D\Psi$ α-Hölder on bounded sets.*

PROOF. (i) By the Gurariĭ-James theorem ([FHHMZ, Theorem 9.25]) there are $p \in (1, +\infty)$ and $K > 0$ such that for any semi-normalised monotone basic sequence $\{x_n\} \subset X$ (finite or infinite) and $\sum a_n x_n \in X$ we have $\|(a_n)\|_{\ell_p} \le K \|\sum a_n x_n\|$. Using transfinite induction on $\operatorname{dens} Y$ we show that for every subspace Y of X there is a one-to-one bounded linear operator $T\colon Y \to \ell_p(\Lambda)$ for some set Λ with $\operatorname{card}\Lambda = \operatorname{dens} Y$. If Y is separable, then Y is isometric to a subspace of ℓ_∞ and the mapping $T\colon \ell_\infty \to \ell_p$, $T\big((a_n)\big) = \big(\frac{1}{2^n} a_n\big)$ is a one-to-one bounded linear operator.

Now suppose that Y is non-separable and for each subspace of X with density smaller than $\operatorname{dens} Y$ there exists the corresponding one-to-one operator. Let μ be the first ordinal with cardinality $\operatorname{dens} Y$ and let $\{P_\alpha;\ \omega \le \alpha \le \mu\}$ be a projectional resolution of the identity on Y ([FHHMZ, Theorem 13.6]). Denote $S_\alpha = P_{\alpha+1} - P_\alpha$ for $\omega \le \alpha < \mu$. Since $\operatorname{dens} S_\alpha(Y) \le \operatorname{card}\alpha < \operatorname{card}\mu$, by the inductive hypothesis there are sets Λ_α with $\operatorname{card}\Lambda_\alpha < \operatorname{card}\mu$ and one-to-one linear operators $T_\alpha\colon S_\alpha(Y) \to \ell_p(\Lambda_\alpha)$ of norm one.

Fix an arbitrary $x \in Y \setminus \{0\}$. Then $x = \sum_{\omega \le \alpha < \mu} S_\alpha(x) = \sum S_{\alpha_n}(x)$, where $\alpha_1 < \alpha_2 < \cdots$ is the enumeration of the countable set $\{\alpha \in [\omega, \mu);\ S_\alpha(x) \neq 0\}$. Notice that the (finite or infinite) sequence $\{S_{\alpha_n}(x)\}$ is a monotone basic sequence. Indeed, $P_{\alpha_k+1} \circ S_{\alpha_n} = S_{\alpha_n}$ for $n \le k$ and $P_{\alpha_k+1} \circ S_{\alpha_n} = 0$ for $n > k$, and hence $\big\|\sum_{n=1}^k a_n S_{\alpha_n}(x)\big\| = \big\|P_{\alpha_k+1}\big(\sum_{n=1}^m a_n S_{\alpha_n}(x)\big)\big\| \le \big\|\sum_{n=1}^m a_n S_{\alpha_n}(x)\big\|$ for $k < m$ and $a_1, \dots, a_m \in \mathbb{R}$. Thus $\{S_{\alpha_n}(x)/\|S_{\alpha_n}(x)\|\}$ is a normalised monotone basic sequence and so

$$\Big(\sum_{\omega \le \alpha < \mu} \|S_\alpha(x)\|^p\Big)^{\frac{1}{p}} = \Big(\sum \|S_{\alpha_n}(x)\|^p\Big)^{\frac{1}{p}} \le K \bigg\|\sum \|S_{\alpha_n}(x)\| \frac{S_{\alpha_n}(x)}{\|S_{\alpha_n}(x)\|}\bigg\|$$
$$= K \Big\|\sum S_{\alpha_n}(x)\Big\| = K\|x\|.$$

It follows that we can define $T\colon Y \to \big(\bigoplus_{\omega \le \alpha < \mu} \ell_p(\Lambda_\alpha)\big)_p$ by $T(x) = \big(T_\alpha \circ S_\alpha(x)\big)_{\omega \le \alpha < \mu}$. Then T is clearly a bounded linear operator. Finally, if $T(x) = 0$, then $S_\alpha(x) = 0$ for all $\omega \le \alpha < \mu$ and so $x = \sum_{\omega \le \alpha < \mu} S_\alpha(x) = 0$, which means that T is one-to-one.

(ii) By (i) there are $p \in (1, +\infty)$ and a one-to-one bounded linear operator $T\colon X \to \ell_p(\Gamma)$. Let $r \in \mathbb{N}$ be odd and satisfy $2r \ge p$. By Lemma 49 the Mazur mapping $\Phi_r\colon \ell_p(\Gamma) \to \ell_2(\Gamma)$

is a one-to-one homogeneous polynomial that is w–w uniformly continuous on bounded sets and so we may put $P = \Phi_r \circ T$.

(iii) Let $\|\cdot\|$ be an equivalent norm on X which is uniformly rotund and its derivative is α-Hölder on the unit sphere, see e.g. [DGZ, Proposition IV.5.2] and the remark after, combined with [DGZ, Lemma IV.5.1]. It is easy to check that then the function $\|\cdot\|^2 \in C^1(X)$ with its derivative α-Hölder on bounded sets. By (ii) there are a set Λ with $\operatorname{card}\Lambda = \operatorname{dens} X$ and $0 \notin \Lambda$, and a one-to-one continuous polynomial $P : X \to \ell_2(\Lambda)$ that is also w–w continuous on bounded sets. Then DP is Lipschitz on bounded sets and hence also α-Hölder on bounded sets. Put $\Gamma = \Lambda \cup \{0\}$ and define $\Psi : X \to \ell_2(\Gamma)$ by $\Psi(x) = (P(x), \|x\|^2)$. Then obviously Ψ is one-to-one and $\Psi \in C^1(X; \ell_2(\Gamma))$ with $D\Psi$ α-Hölder on bounded sets.

It remains to show that Ψ^{-1} is continuous. Let $x_n, x \in X$ be such that $\Psi(x_n) \to \Psi(x)$. Then $\|x_n\| \to \|x\|$ and $P(x_n) \to P(x)$. Let $B \subset X$ be a closed ball containing $\{x_n\}$ and x. Since B is weakly compact and P is one-to-one, $P\restriction_B$ is a w–w homeomorphism and so $x_n \xrightarrow{w} x$. The weak lower semi-continuity of $\|\cdot\|$ implies $\|x + x\| \leq \liminf\|x_n + x\| \leq \limsup\|x_n + x\| \leq 2\|x\|$, hence $\lim\|x_n + x\| = 2\|x\|$, and by the uniform rotundity we finally get $x_n \to x$.

□

PROOF OF THEOREM 48. $\Rightarrow$ follows from Theorem 5.50 and [DGZ, Lemma IV.5.1].

To prove $\Leftarrow$ first recall the following easy fact used several times in this proof: A bounded Lipschitz mapping is α-Hölder for every $\alpha \in (0, 1]$.

Let $\Omega \subset X$ be open and let S be the set of functions from $C^{1,\alpha}(\Omega)$ that are Lipschitz and bounded. Proposition 1.129 implies that S is a ring. We show that S is a partition ring.

Property (i): Let $\{f_\gamma\}_{\gamma\in\Lambda} \subset S$ be such that $\{\operatorname{supp}_\circ f_\gamma\}_{\gamma\in\Lambda}$ is uniformly discrete. Let $g_\gamma = c_\gamma f_\gamma$ for some suitable constants $c_\gamma \neq 0$ chosen so that $|g_\gamma|_\Omega \leq 1$, g_γ is 1-Lipschitz, and Dg_γ is α-Hölder with constant 1 for all $\gamma \in \Lambda$. Put $g = \sum_{\gamma\in\Lambda} g_\gamma$. It is obvious that $g \in C^1(\Omega)$ and g is bounded. To see that g is Lipschitz and Dg is α-Hölder, pick any $x, y \in \Omega$. Suppose there are $\gamma, \beta \in \Lambda$, $\gamma \neq \beta$, such that $x \in \operatorname{supp} g_\gamma$ and $y \in \operatorname{supp} g_\beta$. Then $|g(x) - g(y)| = |g_\gamma(x) - g_\beta(y)| \leq |g_\gamma(x) - 0| + |0 - g_\beta(y)| = |g_\gamma(x) - g_\gamma(y)| + |g_\beta(x) - g_\beta(y)| \leq 2\|x - y\|$ and similarly $\|Dg(x) - Dg(y)\| = \|Dg_\gamma(x) - Dg_\beta(y)\| \leq \|Dg_\gamma(x) - Dg_\gamma(y)\| + \|Dg_\beta(x) - Dg_\beta(y)\| \leq 2\|x - y\|^\alpha$. The other cases are obvious. So $g \in S$ and clearly $\operatorname{supp}_\circ g = \bigcup_{\gamma\in\Lambda} \operatorname{supp}_\circ f_\gamma$.

Property (ii): Let $f \in S$ and $\operatorname{supp}_\circ f = U_1 \cup U_2$, where U_1 and U_2 are open subsets of Ω with $d = \operatorname{dist}(U_1, U_2) > 0$. Consider the function $g = \chi_{U_1} f$. Then $g = f$ on an open set $\Omega \setminus \overline{U_2}$ and $g = 0$ on some neighbourhood of $\overline{U_2}$, hence $g \in C^1(\Omega)$ and both g and Dg are bounded, say by M. To see that g is Lipschitz and Dg is α-Hölder, pick any $x, y \in \Omega$. Suppose that $x \in \overline{U_1}$ and $y \in \overline{U_2}$. Then $|g(x) - g(y)| = |g(x) - 0| \leq M \leq \frac{M}{d}\|x - y\|$ and similarly $\|Dg(x) - Dg(y)\| = \|Dg(x) - 0\| \leq M \leq \frac{M}{d^\alpha}\|x - y\|^\alpha$. The other cases are obvious and so $g \in S$.

Property (iii) holds by the remark after Definition 38 combined with Proposition 1.128.

To finish the proof we show that (ii) of Lemma 40 is satisfied. Let Ψ be the embedding of X into $\ell_2(\Gamma)$ from Proposition 50. Let $W \subset \Omega$ be an arbitrary bounded open set. Then $\Psi(W)$ is open in $\Psi(X)$ and so there is an open $U \subset \ell_2(\Gamma)$ such that $\Psi(W) = U \cap \Psi(X)$. By Proposition 35 there is $f \in C^{1,1}(\ell_2(\Gamma))$ such that $\operatorname{supp}_\circ f = U$. Put $\varphi = f \circ \Psi$.

The mapping $D\Psi$ is Hölder on bounded sets, therefore bounded on bounded sets. Consequently, Ψ is Lipschitz on bounded sets, hence bounded on bounded sets. Further, Df is Lipschitz and hence bounded on bounded sets. Therefore $D\varphi$ is bounded and α-Hölder on bounded sets by Proposition 1.128. Finally, as Ψ is one-to-one, $\operatorname{supp}_\circ \varphi = W$, which is a bounded set and so $D\varphi$ is globally α-Hölder and bounded. So we have found a function $\varphi \in S$ for which $\operatorname{supp}_{\mathrm{o}} \varphi = W$.

$\square$

We remark that the last part of the proof, namely the fact that for any open $W \subset X$ there is $\varphi \in C^{1,\alpha}(X)$ with $\operatorname{supp}_{\mathrm{o}} \varphi = W$, can be shown directly without embedding X into $\ell_2(\Gamma)$. The proof is similar in spirit to that of Proposition 35 but technically much more involved, see [Ce, Corollary 2].

6. Non-linear embeddings into $c_0(\Gamma)$

We begin by giving a characterisation of the existence of $\mathcal{C}^k$-partitions of unity on a normed linear space X by means of non-linear componentwise C^k-smooth embeddings of X into $c_0(\Gamma)$. This result is not essential in our approach to smooth partitions of unity, but it nicely completes the picture in view of the main result of this section: In our aim towards the approximation of Lipschitz functions by smooth functions preserving the Lipschitz property we introduce an important technique of supremal partitions and characterise it again by means of bi-Lipschitz componentwise C^k-smooth embeddings into $c_0(\Gamma)$. We show that every separable normed linear space with a C^k-smooth Lipschitz bump admits C^k-smooth Lipschitz sup-partitions of unity (and a bi-Lipschitz componentwise C^k-smooth embedding into c_0).

It is useful to explicitly state the following fact.

FACT 51. *For any set Γ the space $c_0(\Gamma)$ admits locally finite and σ-uniformly discrete C^∞-smooth and LFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ partitions of unity.*

PROOF. The family $\{W_{F,q,r};\ F \in \mathcal{F}, q, r \in \mathbb{Q}, r > q > 0\}$ from the proof of Theorem 45 is a σ-uniformly discrete basis for the topology of $c_0(\Gamma)$ such that $W_{F,q,r} = \operatorname{supp}_{\mathrm{o}} \varphi_{F,q,r}$ and each $\varphi_{F,q,r}$ is C^∞-smooth and LFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$, so we can use Lemma 40.

It is also instructive to notice that the uniform refinements from Fact 54 for $r = \frac{1}{n}$, $n \in \mathbb{N}$, form a σ-locally finite basis for the topology of $c_0(\Gamma)$. Thus combined with the following observation it gives another proof: For any $x \in c_0(\Gamma)$ and $r > 0$ there is $\varphi \in C^\infty(c_0(\Gamma))$ which is LFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ and such that $\operatorname{supp}_{\mathrm{o}} \varphi = U(x, r)$. Indeed, it suffices to take $\varphi(y) = \prod_{\gamma\in\Gamma} \theta\big(y(\gamma) - x(\gamma)\big)$, where $\theta \in C^\infty(\mathbb{R}, [0, 1])$ is such that $\theta(t) = 1$ whenever $|t| \le \frac{r}{2}$ and $\theta(t) = 0$ if and only if $|t| \ge r$.

$\square$

PROPOSITION 52 ([T]). *Let X be a normed linear space and $k \in \mathbb{N}_0 \cup \{\infty\}$. The space X admits locally finite $\mathcal{C}^k$-partitions of unity if and only if there are a set Γ and a homeomorphism $\Phi\colon X \to c_0(\Gamma)$ such that $e^*_\gamma \circ \Phi \in C^k(X)$ for every $\gamma \in \Gamma$.*

PROOF. $\Rightarrow$ By Lemma 40 there is a basis $\mathcal{V} \subset \{\operatorname{supp}_{\mathrm{o}} f : f \in C^k(X)\}$ for the topology of X such that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$, where each $\mathcal{V}_n$ is discrete and $\mathcal{V}_n \cap \mathcal{V}_m = \emptyset$ for $m \neq n$. For every $V \in \mathcal{V}$ we choose $\varphi_V \in C^k(X; [0, 1])$ such that $V = \operatorname{supp}_{\mathrm{o}} \varphi_V$. We put $\Gamma = \mathcal{V}$ and define $\Phi\colon X \to \ell_\infty(\Gamma)$ by

$$\Phi(x)(V) = \frac{1}{n}\varphi_V(x),$$

where $n \in \mathbb{N}$ is the uniquely determined number for which $V \in \mathcal{V}_n$.

The mapping Φ is one-to-one. Indeed, if $x, y \in X$, $x \neq y$, then there is $V \in \mathcal{V}$ such that $x \in V$, $y \neq V$. It follows that $\varphi_V(x) > 0$, while $\varphi_V(y) = 0$, and consequently $\Phi(x)(V) \neq \Phi(y)(V)$.

Moreover, Φ is a continuous mapping into $c_0(\Gamma)$. To see this, for a given $x \in X$ and $\varepsilon > 0$ we find $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \varepsilon$. Then $0 \leq \Phi(x)(V) < \varepsilon$ and $|\Phi(x)(V) - \Phi(y)(V)| < \varepsilon$ whenever $y \in X$ and $V \in \mathcal{V}_n$ for $n \geq n_0$. Further, by the discreteness of the families $\mathcal{V}_n$, there is a neighbourhood U of x such that U meets only finitely many sets in $\mathcal{W} = \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{n_0}$, say $V_1, \dots, V_m$, $m \in \mathbb{N}_0$. Then $\Phi(y)(V) = 0$ whenever $y \in U$ and $V \in \mathcal{W} \setminus \{V_1, \dots, V_m\}$. It follows that Φ maps into $c_0(\Gamma)$. Finally, using the continuity of $\varphi_{V_1}, \dots, \varphi_{V_m}$ we can find a neighbourhood $W \subset U$ of x such that $|\Phi(x)(V_n) - \Phi(y)(V_n)| < \varepsilon$ whenever $y \in W$, $n \in \{1, \dots, m\}$. Thus $\|\Phi(x) - \Phi(y)\|_\infty < \varepsilon$ for each $y \in W$. This shows the continuity of Φ.

To show that Φ^{-1} continuous fix $x \in X$ and $\varepsilon > 0$. Since $\mathcal{V}$ is a basis for the topology of X, there is $V \in \mathcal{V}$ such that $x \in V \subset U(x, \varepsilon)$. Let $n \in \mathbb{N}$ be such that $V \in \mathcal{V}_n$ and choose some $0 < \delta < \frac{1}{n}\varphi_V(x)$. Suppose $y \in X$ is such that $\|\Phi(x) - \Phi(y)\|_\infty < \delta$. Then $\frac{1}{n}\varphi_V(x) - \frac{1}{n}\varphi_V(y) < \delta$ and hence $\varphi_V(y) > 0$. It follows that $y \in V \subset U(x, \varepsilon)$.

$\Leftarrow$ Denote $S = \{f \in C^\infty(c_0(\Gamma));\ f \text{ is LFC-}\{e_\gamma^*\}_{\gamma \in \Gamma}\}$. By Fact 51 and Lemma 40 there is a σ-locally finite basis $\mathcal{V}$ for the topology of $c_0(\Gamma)$ consisting of the sets $\operatorname{supp}_\circ f$ with $f \in S$. Using the homeomorphism Φ we pull this basis back onto X. Moreover, if $f \in S$, then $f \circ \Phi \in C^k(X)$ (Lemma 5.81). Lemma 40 now finishes the proof.

□

In particular, when $k = 0$, the previous result together with Lemma 8 implies that any normed linear space is homeomorphic to a subset of $c_0(\Gamma)$ for some set Γ. This is no longer true for uniform homeomorphisms: the space $C([0, \omega_1])$ is not uniformly homeomorphic to a subset of any $c_0(\Gamma)$. This is a result of Jan Pelant, [PHK]. However, for any separable normed linear space X there is a bi-Lipschitz homeomorphism $\Phi\colon X \to c_0$. This result of Israel Aharoni, [Ah], can be recovered from Corollary 57 when $k = 0$.

DEFINITION 53. Let X be a set. A collection $\{\psi_\alpha\}_{\alpha \in \Lambda}$ of functions on X is called a supremal partition (sup-partition) if

- $\psi_\alpha\colon X \to [0, 1]$ for all $\alpha \in \Lambda$,
- there is a $Q > 0$ such that $\sup_{\alpha \in \Lambda} \psi_\alpha(x) \geq Q$ for each $x \in X$,
- for each $x \in X$ and for each $\varepsilon > 0$ the set $\{\alpha \in \Lambda;\ \psi_\alpha(x) > \varepsilon\}$ is finite (or in other words $(\psi_\alpha(x))_{\alpha \in \Lambda} \in c_0(\Lambda)$).

If in the second property $Q = 1$, then $\{\psi_\alpha\}_{\alpha \in \Lambda}$ is called a sup-partition of unity.

Let $\mathcal{U}$ be a covering of X. We say that the sup-partition $\{\psi_\alpha\}_{\alpha \in \Lambda}$ is subordinated to $\mathcal{U}$ if $\{\operatorname{supp}_\circ \psi_\alpha\}_{\alpha \in \Lambda}$ refines $\mathcal{U}$. We say that $\{\psi_\alpha\}_{\alpha \in \Lambda}$ is locally finite if $\{\operatorname{supp}_\circ \psi_\alpha\}_{\alpha \in \Lambda}$ is locally finite.

Notice that in fact in the above definition for each $x \in X$ there is $\alpha \in \Lambda$ such that $\psi_\alpha(x) \geq Q$.

For a metric space P we denote by $\mathcal{U}(r) = \{U(x, r);\ x \in P\}$ the full uniform covering of P.

FACT 54. *Let Γ be an infinite set, $r > 0$, and $0 < \delta < \frac{r}{2}$. There is an open locally finite uniform refinement $\mathcal{V} = \{V_\gamma\}_{\gamma \in \Gamma}$ of the uniform covering $\mathcal{U}(r)$ of $c_0(\Gamma)$ such that $\mathcal{U}(\frac{r}{2} - \delta)$ refines $\mathcal{V}$. Moreover, $\mathcal{V}$ is formed by the translates of the open ball $U(0, r - \delta)$. Further, there is a C^∞-smooth LFC-$\{e_\gamma^*\}_{\gamma \in \Gamma}$ and $(\frac{2}{r} + \delta)$-Lipschitz locally finite sup-partition of unity $\{\psi_\gamma\}_{\gamma \in \Gamma}$ on $c_0(\Gamma)$ subordinated to $\mathcal{U}(r)$.*

PROOF. Notice that by the homogeneity it suffices to prove all the statements only for $r = 1$.

Let $\{a_\gamma\}_{\gamma\in\Gamma}$ be the set of all vectors in $c_0(\Gamma)$ with coordinates in $\mathbb{Z}$. (Notice that the cardinality of such set is $|\Gamma|$ and so we may index its points by Γ.) We claim that $\mathcal{V} = \{U(a_\gamma, 1-\delta)\}_{\gamma\in\Gamma}$ is the desired refinement.

Clearly, $\mathcal{V}$ is an open refinement of $\mathcal{U}(1)$. To see that it is locally finite, pick any $x \in c_0(\Gamma)$ and find a finite $F \subset \Gamma$ such that $|x(\gamma)| < \frac{\delta}{2}$ whenever $\gamma \in \Gamma \setminus F$. Suppose that $\alpha \in \Gamma$ is such that $y \in U(a_\alpha, 1-\delta)$ for some $y \in U(x, \frac{\delta}{2})$. If $\gamma \in \Gamma \setminus F$, then $|a_\alpha(\gamma)| \le |a_\alpha(\gamma) - y(\gamma)| + |y(\gamma) - x(\gamma)| + |x(\gamma)| < 1 - \delta + \frac{\delta}{2} + \frac{\delta}{2} = 1$ and so $a_\alpha(\gamma) = 0$. From $|x(\gamma) - a_\alpha(\gamma)| < 1 - \frac{\delta}{2}$ and $a_\alpha(\gamma) \in \mathbb{Z}$ it follows that there are at most two possibilities for $a_\alpha(\gamma)$ for each $\gamma \in F$. From this we can conclude that $\left|\{\alpha;\ U(a_\alpha, 1-\delta) \cap U(x, \frac{\delta}{2}) \neq \emptyset\}\right| \le 2^{|F|}$.

Finally, we show that $\mathcal{U}(\frac{1}{2} - \delta)$ refines $\mathcal{V}$. Choose any $x \in c_0(\Gamma)$ and find $\beta \in \Gamma$ such that $\|x - a_\beta\| \le \frac{1}{2}$. This is always possible, since there is a finite $F \subset \Gamma$ such that $|x(\gamma)| < \frac{1}{2}$ whenever $\gamma \notin F$, and so $a_\beta(\gamma) = 0$ for such γ. Suppose $z \in U(x, \frac{1}{2} - \delta)$. Then $\|a_\beta - z\|_\infty \le \|a_\beta - x\|_\infty + \|x - z\|_\infty < \frac{1}{2} + \frac{1}{2} - \delta = 1 - \delta$, which implies that $U(x, \frac{1}{2} - \delta) \subset U(a_\beta, 1-\delta)$.

To construct the sup-partition of unity subordinated to $\mathcal{U}(1)$ find $\varepsilon > 0$ and $0 < \eta < \frac{1}{2}$ such that $0 < 1/\left(1 - \eta - \frac{1+\varepsilon}{2}\right) < 2 + \frac{\delta}{4}$ and $(1+\varepsilon)\left(2 + \frac{\delta}{2}\right) \le 2 + \delta$. Let $\mathcal{W} = \{U(a_\gamma, 1-\eta)\}_{\gamma\in\Gamma}$ be the locally finite refinement of $\mathcal{U}(1)$ from the first part of the proof such that $\mathcal{U}(\frac{1}{2} - \eta)$ refines $\mathcal{W}$. Further, let $\|\cdot\|$ be an equivalent C^∞-smooth LFC-$\{e_\gamma^*\}_{\gamma\in\Gamma}$ norm $\|\cdot\|$ on $c_0(\Gamma)$ such that $\|x\|_\infty \le \|x\| \le (1+\varepsilon)\|x\|_\infty$ for all $x \in c_0(\Gamma)$. (To construct such a norm, take for example the Minkowski functional of the set $\{x \in c_0(\Gamma);\ \sum_{\gamma\in\Gamma} \varphi(x(\gamma)) \le 1\}$, where $\varphi \in C^\infty(\mathbb{R})$, φ is convex and even, $\varphi(1) = 1$, and $\varphi(t) = 0$ for $t \in [-\frac{1}{1+\varepsilon}, \frac{1}{1+\varepsilon}]$.) For each $\gamma \in \Gamma$ we put $\psi_\gamma(x) = q\left(\|x - a_\gamma\|\right)$, where $q \in C^\infty(\mathbb{R}; [0,1])$, q is $\left(2 + \frac{\delta}{2}\right)$-Lipschitz, $q(t) = 0$ for $t \ge 1 - \eta$, and $q(t) = 1$ for $t \le \frac{1+\varepsilon}{2}$. The collection $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a locally finite sup-partition of unity. Indeed, clearly $\operatorname{supp}_o \psi_\gamma \subset U(a_\gamma, 1-\eta)$ for each $\gamma \in \Gamma$. It also follows that the set $\{\gamma \in \Gamma;\ \psi_\gamma(x) > 0\}$ is finite for each $x \in X$. Further, fix any $x \in X$. There is an $\alpha \in \Gamma$ such that $U(x, \frac{1}{2} - \eta) \subset U(a_\alpha, 1-\eta)$, which gives $\|x - a_\alpha\|_\infty \le \frac{1}{2}$. Hence $\|x - a_\alpha\| \le (1+\varepsilon)\|x - a_\alpha\|_\infty \le \frac{1+\varepsilon}{2}$, which in turn implies that $\psi_\alpha(x) = 1$.

As the function q is $\left(2 + \frac{\delta}{2}\right)$-Lipschitz and the function $\|\cdot\|$ is $(1+\varepsilon)$-Lipschitz (with respect to the norm $\|\cdot\|_\infty$), the functions ψ_γ are $(2+\delta)$-Lipschitz according to the choice of ε. The rest of the properties of the functions ψ_γ is obvious.

$\square$

THEOREM 55. *Let X be a normed linear space, Γ an infinite set, and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the following statements are equivalent:*

(i) *There is $M \in \mathbb{R}$ such that there is a C^k-smooth and M-Lipschitz sup-partition $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ on X subordinated to $\mathcal{U}(1)$.*
(ii) *There is $M \in \mathbb{R}$ such that there is a C^k-smooth and M-Lipschitz locally finite sup-partition of unity $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ on X subordinated to $\mathcal{U}(1)$.*
(iii) *X is uniformly homeomorphic to a subset of $c_0(\Gamma)$ and for each $\varepsilon > 0$ there is $K > 0$ such that for each 1-Lipschitz function $f\colon X \to [0,1]$ there is a K-Lipschitz function $g \in C^k(X)$ such that $|g - f|_X \le \varepsilon$.*
(iv) *There is a bi-Lipschitz homeomorphism $\Phi\colon X \to c_0(\Gamma)$ such that the component functions $e_\gamma^* \circ \Phi \in C^k(X)$ for every $\gamma \in \Gamma$.*

PROOF. (ii)$\Rightarrow$(i) is obvious.

(i)⇒(iv) Let Q be the quantity from the definition of the sup-partition. Then there is $\beta \in \Gamma$ such that $\varphi_\beta(0) \geq Q$. By scaling and composing φ_β with a suitable function we construct a C-Lipschitz function $h \in C^k(X;[0,1])$ such that $h = 0$ on $B(0,r)$ and $h = 1$ outside $U(0,1)$ for some constants $C, r \in \mathbb{R}$, $r > 0$. (We may for example choose r such that $Q - 2Mr > 0$ and take $h(x) = q(\varphi_\beta(2x))$, where $q \in C^\infty(\mathbb{R})$, q is Lipschitz, $q([0,1]) = [0,1]$, $q(0) = 1$, and $q(s) = 0$ for $s \geq Q - 2Mr$.)

Choose $t > 1$ and for each $n \in \mathbb{Z}$ and $\gamma \in \Gamma$ define functions $\varphi_\gamma^n \in C^k(X)$ by

$$\varphi_\gamma^n(x) = t^n \varphi_\gamma\left(\frac{x}{t^n}\right) h\left(\frac{x}{t^n}\right).$$

The properties of the functions φ_γ and h guarantee that each φ_γ^n is $(M + C)$-Lipschitz. Let $d\colon \mathbb{Z} \times \Gamma \to \Gamma$ be some one-to-one mapping and define $\Phi\colon X \to \mathbb{R}^\Gamma$ by $\Phi(x)(\alpha) = \varphi_\gamma^n(x)$ if $\alpha = d(n,\gamma)$ for some $n \in \mathbb{Z}$, $\gamma \in \Gamma$; $\Phi(x)(\alpha) = 0$ otherwise.

We show that Φ actually maps into $c_0(\Gamma)$. Choose an arbitrary $x \in X$ and $\varepsilon > 0$. There is $n_0 \in \mathbb{Z}$ such that $t^n < \varepsilon$ for all $n < n_0$ and $n_1 \in \mathbb{Z}$ such that $\|x\| \leq rt^n$ for all $n > n_1$. It follows that $|\varphi_\gamma^n(x)| < \varepsilon$ for all $n < n_0$ and $\gamma \in \Gamma$, and, by the properties of h, $\varphi_\gamma^n(x) = 0$ for all $n > n_1$ and $\gamma \in \Gamma$. As for each $n_0 \leq n \leq n_1$, $\varphi_\gamma(x/t^n) > \varepsilon/t^n$ only for finitely many $\gamma \in \Gamma$, we can conclude that $\Phi\colon X \to c_0(\Gamma)$. Since each φ_γ^n is $(M + C)$-Lipschitz, the mapping Φ is $(M + C)$-Lipschitz as well.

To prove that Φ is one-to-one and Φ^{-1} is Lipschitz too, choose any two points $x, y \in X$, $x \neq y$, and find $m \in \mathbb{Z}$ such that $2t^m \leq \|x - y\| < 2t^{m+1}$. Without loss of generality we may assume that $\|x\| \geq t^m$. Then $h(x/t^m) = 1$ and so there is $\gamma \in \Gamma$ such that $\varphi_\gamma^m(x) \geq Qt^m$. Now suppose that $z \in X$ is such that $\varphi_\gamma^m(z) > 0$. As $\operatorname{supp}_\circ \varphi_\gamma \subset U(w,1)$ for some $w \in X$, $\|\frac{x}{t^m} - \frac{z}{t^m}\| < 2$ and consequently $\|x - z\| < 2t^m$. But this means that $\varphi_\gamma^m(y) = 0$ and therefore

$$\|\Phi(x) - \Phi(y)\|_\infty \geq |\varphi_\gamma^m(x) - \varphi_\gamma^m(y)| = \varphi_\gamma^m(x) \geq Qt^m > \frac{Q}{2t}\|x - y\|.$$

(iv)⇒(ii) Let $A, B > 0$ be such that $A\|x - y\| \leq \|\Phi(x) - \Phi(y)\|_\infty \leq B\|x - y\|$ for all $x, y \in X$. By Fact 54 there are $C > 0$ and a C^∞-smooth LFC-$\{e_\gamma^*\}_{\gamma\in\Gamma}$ and C-Lipschitz locally finite sup-partition of unity $\{\psi_\gamma\}_{\gamma\in\Gamma}$ on $c_0(\Gamma)$ subordinated to $\mathcal{U}(\frac{A}{2})$. Putting $\varphi_\gamma = \psi_\gamma \circ \Phi$, $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ is a BC-Lipschitz locally finite sup-partition of unity subordinated to $\mathcal{U}(1)$. Moreover, each φ_γ is C^k-smooth by Lemma 5.81.

(ii)⇒(iii) We already know that (iv) holds and from this the first part of (iii) follows immediately. To prove the second part of (iii), let $\varepsilon > 0$. The basic idea of the proof is that Lipschitz functions are stable under the operation of pointwise supremum. To preserve the smoothness, we will use a "smoothened supremum", or an equivalent smooth norm on $c_0(\Gamma)$. Let $\|\cdot\|$ be an equivalent C^∞-smooth LFC-$\{e_\gamma^*\}_{\gamma\in\Gamma}$ norm on $c_0(\Gamma)$ and let $C > 0$ be such that $\|x\|_\infty \leq \|x\| \leq C\|x\|_\infty$ for all $x \in c_0(\Gamma)$. We will show that $K = 4C^3M/\varepsilon$ satisfies our claim.

By adding the constant 1 we may and do assume that f maps into $[1,2]$. Put $\delta = \frac{\varepsilon}{C}$ and $\psi_\gamma(x) = \varphi_\gamma(\frac{x}{\delta})$ for all $x \in X$, $\gamma \in \Gamma$. It follows that $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is a C^k-smooth and M/δ-Lipschitz sup-partition of unity subordinated to $\mathcal{U}(\delta)$. Recall that $(\psi_\gamma(x))_{\gamma\in\Gamma} \in c_0(\Gamma)$ for each $x \in X$. For each $\gamma \in \Gamma$ there is a point $x_\gamma \in X$ such that $\operatorname{supp}_\circ \psi_\gamma \subset U(x_\gamma, \delta)$. The boundedness of the function f guarantees that also $\big(f(x_\gamma)\psi_\gamma(x)\big)_{\gamma\in\Gamma} \in c_0(\Gamma)$ for each $x \in X$.

Therefore we can define the function $g\colon X \to \mathbb{R}$ by

$$g(x) = \frac{\big\|\big(f(x_\gamma)\psi_\gamma(x)\big)_{\gamma\in\Gamma}\big\|}{\big\|\big(\psi_\gamma(x)\big)_{\gamma\in\Gamma}\big\|}.$$

As

$$\|(\psi_\gamma(x))\| \geq \|(\psi_\gamma(x))\|_\infty = \sup_{\gamma\in\Gamma}\psi_\gamma(x) = 1 \quad \text{for each } x \in X, \tag{17}$$

the function g is well-defined on all of X.

The mapping $x \mapsto \big(\psi_\gamma(x)\big)$ and, by the boundedness of f, also the mapping $x \mapsto \big(f(x_\gamma)\psi_\gamma(x)\big)$ are Lipschitz mappings from X into $c_0(\Gamma) \setminus U(0,1)$. (Notice that for each $x \in X$ there is $\gamma \in \Gamma$ such that $\psi_\gamma(x) = 1$ and $f(x_\gamma)\psi_\gamma(x) \geq 1$.) Since $\|\cdot\|$ is C^∞-smooth and depends locally on finitely many coordinates away from the origin, and since $\psi_\gamma \in C^k(X)$ and $f(x_\gamma)\psi_\gamma \in C^k(X)$ for each $\gamma \in \Gamma$, using Lemma 5.81 we infer that $g \in C^k(X)$.

Using the facts that f maps into $[1,2]$, the functions ψ_γ are M/δ-Lipschitz and map into $[0,1]$, and $\|\cdot\|$ is C-Lipschitz as a function on $(c_0(\Gamma), \|\cdot\|_\infty)$, we obtain that the function $x \mapsto \big\|\big(f(x_\gamma)\psi_\gamma(x)\big)\big\|$ is $2CM/\delta$-Lipschitz and bounded by $2C$. Similarly, the function $x \mapsto \big\|\big(\psi_\gamma(x)\big)\big\|$ is CM/δ-Lipschitz and bounded below by 1. It follows that the function g is K-Lipschitz.

Finally, to show that g approximates f, choose an arbitrary $x \in X$. Applying successively the inequality (17) and the facts that $\operatorname{supp}_\circ \psi_\gamma \subset U(x_\gamma, \delta)$ and f is 1-Lipschitz, we obtain

$$\begin{aligned}
|g(x) - f(x)| &= \left|\frac{\big\|\big(f(x_\gamma)\psi_\gamma(x)\big)\big\|}{\big\|\big(\psi_\gamma(x)\big)\big\|} - f(x)\frac{\big\|\big(\psi_\gamma(x)\big)\big\|}{\big\|\big(\psi_\gamma(x)\big)\big\|}\right| \leq \frac{\big\|\big((f(x_\gamma) - f(x))\psi_\gamma(x)\big)\big\|}{\big\|\big(\psi_\gamma(x)\big)\big\|} \\
&\leq C\, \big\|\big((f(x_\gamma) - f(x))\psi_\gamma(x)\big)\big\|_\infty = C \sup_{\gamma\in\Gamma}\big\{|f(x_\gamma) - f(x)|\psi_\gamma(x)\big\} \\
&= C \sup_{\substack{\gamma\in\Gamma \\ x\in U(x_\gamma,\delta)}} \big\{|f(x_\gamma) - f(x)|\psi_\gamma(x)\big\} \leq C \sup_{\substack{\gamma\in\Gamma \\ x\in U(x_\gamma,\delta)}} \big\{\|x_\gamma - x\|\big\} \leq C\delta = \varepsilon.
\end{aligned}$$

(iii)$\Rightarrow$(ii) Let Φ be the uniform homeomorphism and let $\eta > 0$ be such that $\|\Phi^{-1}(x) - \Phi^{-1}(y)\| < 1$ whenever $x, y \in \Phi(X)$ are such that $\|x - y\| < 2\eta$. Take an open locally finite uniform refinement of the uniform covering $\mathcal{U}(\eta)$ of $c_0(\Gamma)$ from Fact 54 and pull it back onto X using Φ. We obtain an open locally finite uniform refinement $\mathcal{V} = \{V_\gamma\}_{\gamma\in\Gamma}$ of the covering $\mathcal{U}(1)$ of X. Let $0 < \delta \leq 1$ be such that $\mathcal{U}(\delta)$ refines $\mathcal{V}$. For each $\gamma \in \Gamma$ we define the function $f_\gamma\colon X \to [0,1]$ by $f_\gamma(x) = \min\{\operatorname{dist}(x, X \setminus V_\gamma), \delta\}$.

Choose some $0 < \theta < \frac{\delta}{2}$. For each $\gamma \in \Gamma$ the function f_γ is 1-Lipschitz and so, by (iii), there is a K-Lipschitz function $g_\gamma \in C^k(X)$ such that $|g_\gamma - f_\gamma|_X \leq \theta$. Let $q \in C^k(\mathbb{R}; [0,1])$ be a C-Lipschitz function for some $C \in \mathbb{R}$, such that $q(t) = 0$ for $t \leq \theta$ and $q(t) = 1$ for $t \geq \delta - \theta$. Finally, we let $\varphi_\gamma(x) = q(g_\gamma(x))$ for each $\gamma \in \Gamma$. Clearly, each function φ_γ belongs to $C^k(X; [0,1])$ and is M-Lipschitz, where $M = CK$. Further, for any $x \in X$ there is $\alpha \in \Gamma$ such that $U(x, \delta) \subset V_\alpha$, hence $f_\alpha(x) = \delta$ and consequently $\varphi_\alpha(x) = 1$. As $\operatorname{supp}_\circ \varphi_\gamma \subset V_\gamma$ for all $\gamma \in \Gamma$ and $\mathcal{V}$ is locally finite, $\{\varphi_\gamma\}_{\gamma\in\Gamma}$ is a locally finite sup-partition of unity subordinated to $\mathcal{U}(1)$.

□

We note that the proof could be made considerably shorter by proving (iv)$\Rightarrow$(iii) directly using Theorem 71 instead of (ii)$\Rightarrow$(iii) and (iv)$\Rightarrow$(ii). However, the reasons for our strategy of the proof were two: First, we do not need the full generality (and associated machinery) of

Theorem 71 (or Theorem 66) and second, the proof of (ii)$\Rightarrow$(iii) shows an interesting technique for constructing smooth Lipschitz approximations (due to Robb Fry, [Fry2]), and in fact shows the reason for the definition of the notion of sup-partition of unity.

THEOREM 56 (Robb Fry, [Fry2]). *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N}_0 \cup \{\infty\}$. Then there is $M \in \mathbb{R}$ such that there is a C^k-smooth M-Lipschitz sup-partition of unity $\{\psi_j\}_{j=1}^{\infty}$ on X subordinated to $\mathcal{U}(1)$.*

PROOF. Using the C^k-smooth Lipschitz bump function on X as a start, by shifting, scaling, and composing with a suitable real function we construct two functions $f, g \in C^k(X; [0,1])$ along with real numbers $C > 0$ and $0 < \delta < r < 1$ such that $f(x) = 0$ for all $x \in X \setminus U(0,1)$, $f(x) = 1$ for all $x \in B(0,r)$, $g(x) = 1$ for all $x \in X \setminus U(0,r)$, $g(x) = 0$ for all $x \in B(0,\delta)$, and both functions are C-Lipschitz (see also the proof of Theorem 55).

Let $\{x_j\}_{j=1}^{\infty} \subset X$ be such that $\{U(x_j,\delta)\}_{j=1}^{\infty}$ is a covering of X. We put $f_j(x) = f(x - x_j)$ and $g_j(x) = g(x - x_j)$ for each $x \in X$, $j \in \mathbb{N}$. Choose $0 < \eta < 1$ and for each $j \in \mathbb{N}$ let $\varphi_j \in C^k(\mathbb{R}^j)$ be a 1-Lipschitz function (with respect to the maximum norm) such that

$$\min\{w_1, \dots, w_j\} \leq \varphi_j(w) \leq \min\{w_1, \dots, w_j\} + \eta \quad \text{for each } w \in [0,1]^j$$

(use Lemma 1). We note that the functions φ_j will serve as a "smoothened minimum". Finally, to confine the sup-partition into the interval $[0,1]$, let $h \in C^k(\mathbb{R}; [0,1])$ be a D-Lipschitz function such that $h(t) = 0$ for $t \leq \eta$ and $h(t) = 1$ for $t \geq 1$.

For each $j \in \mathbb{N}$ we define

$$\psi_j(x) = h\Big(\varphi_j\big(g_1(x), \dots, g_{j-1}(x), f_j(x)\big)\Big) \quad \text{for each } x \in X.$$

Clearly, $\psi_j \in C^k(X; [0,1])$ and ψ_j is M-Lipschitz for each $j \in \mathbb{N}$, where $M = CD$. Moreover, $\{\psi_j\}_{j=1}^{\infty}$ is a sup-partition of unity. Indeed, choose an arbitrary $x \in X$. Let $k \in \mathbb{N}$ be the smallest index for which $x \in U(x_k, \delta)$. Then $g_k(x) = 0$, which implies that for $j > k$, $\varphi_j\big(g_1(x), \dots, g_{j-1}(x), f_j(x)\big) \leq \eta$ and so $\psi_j(x) = 0$. Therefore the set $\{j \in \mathbb{N};\ \psi_j(x) > 0\}$ is finite. Further, let $n \in \mathbb{N}$ be the smallest index for which $x \in U(x_n, r)$. It follows that $g_j(x) = 1$ for each $j < n$ and $f_n(x) = 1$, hence $\psi_n(x) = 1$.

Finally, if $\|x - x_j\| \geq 1$, then $f_j(x) = 0$ and hence $\psi_j(x) = 0$, which shows that $\{\psi_j\}_{j=1}^{\infty}$ is subordinated to $\mathcal{U}(1)$.

$\square$

COROLLARY 57. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N}_0 \cup \{\infty\}$. Then there is a bi-Lipschitz homeomorphism $\Phi\colon X \to c_0$ such that the component functions $e_j^* \circ \Phi \in C^k(X)$ for every $j \in \mathbb{N}$.*

7. Approximation of Lipschitz mappings

In this section we turn our attention to the problem of approximating Lipschitz mappings by smooth Lipschitz mappings, preferably keeping the control over the Lipschitz constant. Such approximations have applications for example in the theory of Banach manifolds. The finite-dimensional case is easy – the integral convolution respects the Lipschitz property and in fact it preserves the Lipschitz constant.

In the infinite-dimensional case, the infimal convolution preserves the Lipschitz constant too, but unfortunately it gives only the first order smoothness and works only for functions (i.e. mappings into $\mathbb{R}$). Using the local techniques (partitions of unity) alone to obtain the global property (Lipschitzness) presents some insurmountable obstacles. First, it is essentially impossible to gain any global control over the Lipschitz constant of the individual functions in the partition and regardless, there is no control over the cardinality of the (locally finite) sum in (15). Therefore we have to develop several alternative approaches to this problem.

The first one is using the integral convolution even in the infinite-dimensional setting. We show two cases when this is possible. The first one is for separable spaces, where we can use "convolution in a dense set of directions" and then exploit the Lipschitz property of the approximated mapping. The gain is however not particularly strong, as we obtain merely Gâteaux (or uniformly Gâteaux) smooth approximation. More interesting is the use of the integral convolution in the space $c_0(\Gamma)$, which is possible thanks to the very strong LFC structure in this space. The latter result has interesting corollaries when either the source or the target space have certain special properties.

The above techniques are somewhere in-between local and global – they use approximation on finite-dimensional subspaces, which are then somehow "glued together". Another example of this approach is a technique of Nicole Moulis that uses an unconditional basis for gluing together the finite-dimensional approximations. There is also a "local" procedure: the supremal partitions, developed in the previous section, which essentially replace the sum in (15) by supremum, which preserves the Lipschitz property.

All the above results give approximations in the uniform topology. Using the σ-discrete partitions of unity we show how to proceed from uniform approximations to the approximation in fine topology. Finally, we prove an analogue of Theorem 22 (the real analytic approximation) for Lipschitz functions.

We start with a notion of a uniform Gâteaux differentiability. If f is Gâteaux differentiable and for a fixed x in the domain we require the uniformity of the limit defining $\frac{\partial f}{\partial h}(x)$ in $h \in B_X$, we obtain the notion of Fréchet differentiability. If, on the other hand, for each fixed $h \in B_X$ we require the uniformity in x, then we obtain uniform Gâteaux differentiability. Uniformity of this type will prove important later, for example in the applications of Theorem 76.

DEFINITION 58. Let X, Y be normed linear spaces, $U \subset X$ open, and $f\colon U \to Y$ a Gâteaux differentiable mapping. We say that f is uniformly Gâteaux differentiable (UG for short) if for each fixed $h \in S_X$ the limit defining $\frac{\partial f}{\partial h}(x)$ is uniform for $x \in U$.

LEMMA 59. *Let X, Y be normed linear spaces, $U \subset X$ open, and let $f\colon U \to Y$ be a Gâteaux differentiable mapping. If for each $h \in S_X$ the mapping $x \mapsto \delta f(x)[h]$ is uniformly continuous on U, then f is uniformly Gâteaux differentiable on U provided that U is convex; otherwise f is uniformly Gâteaux differentiable on any open $V \subset U$ satisfying* $\operatorname{dist}(V, X \setminus U) > 0$. *Conversely, if f is uniformly Gâteaux differentiable and uniformly continuous on U, then for each $h \in X$ the mapping $x \mapsto \delta f(x)[h]$ is uniformly continuous on any $A \subset U$ satisfying* $\operatorname{dist}(A, X \setminus U) > 0$.

PROOF. Choose $h \in S_X$ and $\varepsilon > 0$, and find $\theta > 0$ such that $\|\delta f(x + th)[h] - \delta f(x)[h]\| < \varepsilon$ for all $x \in U$ and $t \in (-\theta, \theta)$ satisfying $x + th \in U$. If U is convex we set $V = U$ and $\eta = \theta$, otherwise we let $\eta = \min\{\theta, \operatorname{dist}(V, X \setminus U)\}$. Fix $x \in V$ and define a mapping $g\colon I \to Y$ by $g(t) = f(x + th) - t\delta f(x)[h]$, where $I = \{t \in (-\eta, \eta);\ x + th \in U\}$. Notice that I is an open interval containing 0 and $g'(t) = \delta f(x + th)[h] - \delta f(x)[h]$ for $t \in I$. By the assumption, $\|g'(t)\| \le \varepsilon$ for $t \in I$, hence g is ε-Lipschitz on I, and so $\bigl\|\frac{1}{t}\bigl(f(x + th) - f(x)\bigr) - \delta f(x)[h]\bigr\| = \bigl\|\frac{1}{t}\bigl(g(t) - g(0)\bigr)\bigr\| \le \varepsilon$ for all $t \in I$.

To prove the converse statement, choose $h \in X$, $h \neq 0$, a subset $A \subset U$ for which $\operatorname{dist}(A, X \setminus U) > 0$, and $\varepsilon > 0$. Find $0 < \eta < \operatorname{dist}(A, X \setminus U)/\|h\|$ such that $\big\|\frac{1}{\eta}\big(f(x+\eta h) - f(x)\big) - \delta f(x)[h]\big\| < \frac{\varepsilon}{4}$ for any $x \in A$. Let $\theta > 0$ be such that $\|f(x) - f(y)\| < \eta\frac{\varepsilon}{4}$ whenever $x, y \in A$ are such that $\|x - y\| < \theta$. Then, for such x, y, we have

$$\big\|\delta f(x)[h] - \delta f(y)[h]\big\| < \frac{\varepsilon}{2} + \frac{1}{\eta}\big\|f(x+\eta h) - f(x) - f(y+\eta h) + f(y)\big\| < \varepsilon.$$

□

We remark that if $f\colon U \to Y$, $U \subset X$ open, is such that for each $h \in S_X$ the mapping $x \mapsto \frac{\partial f}{\partial h}(x)$ is uniformly continuous on U, then $f\restriction_{E \cap U}$ is $C^{1,+}$-smooth for each finite-dimensional affine subspace $E \subset X$ (Theorem 1.96). In particular, if X is a Banach space and f is Baire measurable, then in view of Theorem 1.101 we do not need to assume that f is Gâteaux differentiable in Lemma 59.

LEMMA 60. *Let X, Y be normed linear spaces, H a dense subset of X, $U \subset X$ open, and let $f\colon U \to Y$ be a Gâteaux differentiable Lipschitz mapping such that for each $h \in H$ the mapping $x \mapsto \delta f(x)[h]$ is uniformly continuous on U. Then the mapping $x \mapsto \delta f(x)[h]$ is uniformly continuous on U for every $h \in X$.*

PROOF. Let L be a Lipschitz constant of f. Pick an arbitrary $h \in X$ and let $\varepsilon > 0$. Find $h_0 \in H$ such that $\|h - h_0\| < \frac{\varepsilon}{4L}$. By the uniform continuity of $x \mapsto \delta f(x)[h_0]$ there is $\eta > 0$ such that $\|\delta f(x)[h_0] - \delta f(y)[h_0]\| < \frac{\varepsilon}{2}$ whenever $x, y \in U$, $\|x - y\| < \eta$. Then

$$\begin{aligned}\big\|\delta f(x)[h] - \delta f(y)[h]\big\| &\le \big\|\delta f(x)[h_0] - \delta f(y)[h_0]\big\| + \big\|\delta f(x)[h - h_0]\big\| + \big\|\delta f(y)[h - h_0]\big\| \\ &< \frac{\varepsilon}{2} + 2L\|h - h_0\| < \varepsilon\end{aligned}$$

whenever $x, y \in U$, $\|x - y\| < \eta$.

□

The following approximation lemma introduces the technique of "convolution in a dense set of directions".

LEMMA 61. *Let X be a separable normed linear space, Y a Banach space, $U \subset X$ open, $f\colon U \to Y$ an L-Lipschitz mapping, $\varepsilon > 0$, and let $V \subset U$ be open such that* $\operatorname{dist}(V, X \setminus U) > \frac{\varepsilon}{2L}$. *Let $\{h_j\}$ be a dense subset of S_X and let $\varphi_j \in C^\infty(\mathbb{R})$, $j \in \mathbb{N}$, be such that $\varphi_j \ge 0$, $\int_{\mathbb{R}} \varphi_j = 1$, and* $\operatorname{supp}\varphi_j \subset \big[-\frac{\varepsilon}{2L2^j}, \frac{\varepsilon}{2L2^j}\big]$. *Extend f to the whole of X by setting $f(x) = 0$ for $x \in X \setminus U$ and define $g_n\colon V \to Y$, $n \in \mathbb{N}$, by*

$$g_n(x) = \int_{\mathbb{R}^n} f\Big(x - \sum_{j=1}^{n} t_j h_j\Big) \prod_{j=1}^{n} \varphi_j(t_j)\,\mathrm{d}\lambda_n(t), \tag{18}$$

where λ_n denotes the n-dimensional Lebesgue measure. Then $g_n \to g$ uniformly on V and the mapping $g\colon V \to Y$ has the following properties: It is L-Lipschitz, Gâteaux differentiable, satisfies $\|f - g\|_V < \varepsilon$, and for each $h \in X$ the mapping $x \mapsto \delta g(x)[h]$ is uniformly continuous on V. Moreover, if $Y = \mathbb{R}$ and U, V, f are convex, then so is g.

PROOF. Denote $K_m = \prod_{j=1}^{m}\big[-\frac{\varepsilon}{2L2^j}, \frac{\varepsilon}{2L2^j}\big] \subset \mathbb{R}^m$. Since $x - \sum_{j=1}^{m} t_j h_j \in U$ for $x \in V$ and $(t_1, \dots, t_m) \in K_m$, using the Fubini theorem and the fact that $\int_{\mathbb{R}} \varphi_j = 1$ we obtain for $m > n$

and any $x \in V$

$$\begin{aligned}\|g_m(x) - g_n(x)\| &= \left\| \int_{\mathbb{R}^m} \left(f\left(x - \sum_{j=1}^{m} t_j h_j\right) - f\left(x - \sum_{j=1}^{n} t_j h_j\right)\right) \prod_{j=1}^{m} \varphi_j(t_j)\, \mathrm{d}\lambda_m \right\| \\ &\le L \int_{K_m} \left\| \sum_{j=n+1}^{m} t_j h_j \right\| \prod_{j=1}^{m} \varphi_j(t_j)\, \mathrm{d}\lambda_m \le L \int_{K_m} \left(\sum_{j=n+1}^{m} |t_j| \right) \prod_{j=1}^{m} \varphi_j(t_j)\, \mathrm{d}\lambda_m \\ &\le L \left(\sum_{j=n+1}^{m} \frac{\varepsilon}{2L2^j} \right) \int_{K_m} \prod_{j=1}^{m} \varphi_j(t_j)\, \mathrm{d}\lambda_m < \frac{\varepsilon}{2 \cdot 2^n}.\end{aligned}$$

It follows that there is $g\colon V \to Y$ such that $g_n \to g$ uniformly on V.

The mappings g_n are L-Lipschitz on V. Indeed, for any $x, y \in V$ we have

$$\begin{aligned}\|g_n(x) - g_n(y)\| &\le \int_{K_n} \left\| f\left(x - \sum_{j=1}^{n} t_j h_j\right) - f\left(y - \sum_{j=1}^{n} t_j h_j\right) \right\| \prod_{j=1}^{n} \varphi_j(t_j)\, \mathrm{d}\lambda_n \\ &\le L\|x - y\| \int_{K_n} \prod_{j=1}^{n} \varphi_j(t_j)\, \mathrm{d}\lambda_n = L\|x - y\|.\end{aligned}$$

Therefore g is also L-Lipschitz. Similarly we can check that the functions g_n and g are convex under the additional convexity assumptions.

Moreover, $\|f - g\|_V < \varepsilon$. Indeed, pick $n \in \mathbb{N}$ such that $\|g_n - g\|_V < \frac{\varepsilon}{2}$. Then

$$\begin{aligned}\|f(x) - g(x)\| &\le \|f(x) - g_n(x)\| + \|g_n(x) - g(x)\| \\ &< \int_{\mathbb{R}^n} \left\| f(x) - f\left(x - \sum_{j=1}^{n} t_j h_j\right) \right\| \prod_{j=1}^{n} \varphi_j(t_j)\, \mathrm{d}\lambda_n + \frac{\varepsilon}{2} \\ &\le L \int_{K_n} \left(\sum_{j=1}^{n} |t_j| \right) \prod_{j=1}^{n} \varphi_j(t_j)\, \mathrm{d}\lambda_n + \frac{\varepsilon}{2} < \varepsilon\end{aligned}$$

for any $x \in V$.

Next we show that g is Gâteaux differentiable on V. Fix $n \in \mathbb{N}$, $x \in V$ and define $T\colon \mathbb{R}^n \to X$ by $T(t) = x + \sum_{j=1}^{n} t_j h_j$. Let $\Delta > 0$ be such that $T\big((-\Delta, \Delta)^n\big) \subset V$. Using substitution $t \to s - t$ we obtain

$$g_n \circ T(s) = \int_{R^n} f \circ T(s - t) \prod_{j=1}^{n} \varphi_j(t_j)\, \mathrm{d}\lambda_n(t) = \int_{K} f \circ T(t) \prod_{j=1}^{n} \varphi_j(s_j - t_j)\, \mathrm{d}\lambda_n(t)$$

for any $s \in (-\Delta, \Delta)^n$, where $K = \prod_{j=1}^{n} \big[-\frac{\varepsilon}{2L2^j} - \Delta, \frac{\varepsilon}{2L2^j} + \Delta\big]$. It follows from Corollary 1.91 that the mapping $g_n \circ T$ is C^1-smooth on $(-\Delta, \Delta)^n$. Since by the definition $\frac{\partial g_n}{\partial h}(x) = D(g_n \circ T)(0)[(s_1, \dots, s_n)]$ for all $h = s_1 h_1 + \dots + s_n h_n$, it follows that $h \mapsto \frac{\partial g_n}{\partial h}(x)$ is linear on

$\operatorname{span}\{h_1, \dots, h_n\}$ and

$$\begin{aligned}\frac{\partial g_n}{\partial h_i}(x) &= \int_K f \circ T(t)\varphi_i'(-t_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(-t_j)\,\mathrm{d}\lambda_n(t)\\ &= \int_{\mathbb{R}^n} f\Big(x - \sum_{j=1}^{n} t_j h_j\Big)\varphi_i'(t_i) \prod_{\substack{j=1\\ j\neq i}}^{n} \varphi_j(t_j)\,\mathrm{d}\lambda_n(t)\end{aligned}$$

for each $i \in \{1, \dots, n\}$.

Further, $\{\frac{\partial g_n}{\partial h_i}\}_{n=i}^{\infty}$ converges uniformly for $x \in V$. Indeed, using the Fubini theorem and the fact that $\int_{\mathbb{R}} \varphi_j = 1$ we have for $m > n \geq i$ and any $x \in V$

$$\begin{aligned}\Big\|\frac{\partial g_m}{\partial h_i}(x) - \frac{\partial g_n}{\partial h_i}(x)\Big\| &= \Big\|\int_{\mathbb{R}^m}\Big(f\Big(x - \sum_{j=1}^{m} t_j h_j\Big) - f\Big(x - \sum_{j=1}^{n} t_j h_j\Big)\Big)\varphi_i'(t_i) \prod_{\substack{j=1\\ j\neq i}}^{m} \varphi_j(t_j)\,\mathrm{d}\lambda_m\Big\|\\ &\leq L \int_{K_m}\Big(\sum_{j=n+1}^{m} |t_j|\Big)|\varphi_i'(t_i)| \prod_{\substack{j=1\\ j\neq i}}^{m} \varphi_j(t_j)\,\mathrm{d}\lambda_m \leq \frac{\varepsilon}{2\cdot 2^n}\int_{\mathbb{R}} |\varphi_i'(t)|\,\mathrm{d}t.\end{aligned}$$

By Theorem 1.85 (used on the restrictions to $x + \operatorname{span}\{h_i\}$) we obtain that $\frac{\partial g}{\partial h_i}(x)$ exists for all $x \in V$, $i \in \mathbb{N}$. From the above it also follows that $\frac{\partial g}{\partial (h_i + h_j)}(x) = \frac{\partial g}{\partial h_i}(x) + \frac{\partial g}{\partial h_j}(x)$ for all $x \in V$, $i, j \in \mathbb{N}$.

To see that for given $x \in V$ the derivative $\frac{\partial g}{\partial h}(x)$ exists for all $h \in S_X$ choose $\eta > 0$ and let $i \in \mathbb{N}$ be such that $\|h - h_i\| < \frac{\eta}{3L}$. Then for any $\tau \in \mathbb{R} \setminus \{0\}$ small enough so that $x + \tau h \in V$, $x + \tau h_i \in V$ we have

$$\Big\|\frac{1}{\tau}\big(g(x + \tau h) - g(x)\big) - \frac{1}{\tau}\big(g(x + \tau h_i) - g(x)\big)\Big\| \leq \frac{L}{|\tau|}\|\tau(h - h_i)\| < \frac{\eta}{3}.$$

Thus there is $\theta > 0$ such that

$$\begin{aligned}&\Big\|\frac{1}{\tau_1}\big(g(x + \tau_1 h) - g(x)\big) - \frac{1}{\tau_2}\big(g(x + \tau_2 h) - g(x)\big)\Big\|\\ &\qquad < \frac{2}{3}\eta + \Big\|\frac{1}{\tau_1}\big(g(x + \tau_1 h_i) - g(x)\big) - \frac{1}{\tau_2}\big(g(x + \tau_2 h_i) - g(x)\big)\Big\| < \eta\end{aligned}$$

for $0 < |\tau_1| < \theta$, $0 < |\tau_2| < \theta$.

Next we show that the mapping $h \mapsto \frac{\partial g}{\partial h}(x)$ is L-Lipschitz. For arbitrary $u, v \in X$ and $\eta > 0$ we have

$$\begin{aligned}\Big\|\frac{\partial g}{\partial u}(x) - \frac{\partial g}{\partial v}(x)\Big\| &\leq \Big\|\frac{\partial g}{\partial u}(x) - \frac{1}{\tau}\big(g(x + \tau u) - g(x)\big)\Big\| + \Big\|\frac{1}{\tau}\big(g(x + \tau u) - g(x + \tau v)\big)\Big\|\\ &\quad + \Big\|\frac{\partial g}{\partial v}(x) - \frac{1}{\tau}\big(g(x + \tau v) - g(x)\big)\Big\|\\ &\leq \eta + L\|u - v\|\end{aligned}$$

for τ small enough. Thus $\left\|\frac{\partial g}{\partial u}(x)-\frac{\partial g}{\partial v}(x)\right\| \leq L\|u-v\|$. It follows that $h \mapsto \frac{\partial g}{\partial h}(x) \in \mathcal{L}(X;Y)$, since it is a Lipschitz mapping that is linear on a dense subset of X. Therefore g is Gâteaux differentiable on V.

It remains to prove that $x \mapsto \delta g(x)[h]$ is uniformly continuous on V for any $h \in X$. To this end, first note that the mapping $x \mapsto \frac{\partial g_n}{\partial h_i}(x)$ is L_i-Lipschitz for any $n \geq i$, where $L_i = L\int_{\mathbb{R}}|\varphi_i'(t)|\,\mathrm{d}t$:

$$\left\|\frac{\partial g_n}{\partial h_i}(x)-\frac{\partial g_n}{\partial h_i}(y)\right\| \leq \left\|\int_{\mathbb{R}^n}\left(f\left(x-\sum_{j=1}^n t_j h_j\right)-f\left(y-\sum_{j=1}^n t_j h_j\right)\right)\varphi_i'(t_i)\prod_{\substack{j=1\\ j\neq i}}^n \varphi_j(t_j)\,\mathrm{d}\lambda_n\right\|$$

$$\leq L\|x-y\|\int_{\mathbb{R}}|\varphi_i'(t)|\,\mathrm{d}t = L_i\|x-y\|.$$

Thus the mapping $x \mapsto \delta g(x)[h_i]$ is L_i-Lipschitz for each $i \in \mathbb{N}$. It follows from Lemma 60 that $x \mapsto \delta g(x)[h]$ is uniformly continuous on V for any $h \in X$.

□

COROLLARY 62. *Let X be a separable normed linear space, Y a Banach space, $U \subset X$ open, $k \in \mathbb{N}_0$, $f \in C^k(U;Y)$ such that $d^j f$ is L_j-Lipschitz for $j = 0,\dots,k$, $\varepsilon > 0$, and let $V \subset U$ be open such that* $\operatorname{dist}(V, X\setminus U) > 0$. *Then there is $g \in C^k(V;Y)$ such that $d^j g$ is L_j-Lipschitz for $j = 0,\dots,k$, $d^k g$ is uniformly Gâteaux differentiable (in particular, g is G^{k+1}-smooth), and $\|d^j g - d^j f\|_V < \varepsilon$ for all $j \in \{0,\dots,k\}$.*

PROOF. Let $W \subset U$ be open such that $\operatorname{dist}(W, X\setminus U) > 0$ and $\operatorname{dist}(V, X\setminus W) > 0$. We define mappings $g_n\colon W \to Y$ by formula (18). By Corollary 1.91 we have $g_n \in C^k(W;Y)$ and

$$d^j g_n(x) = \int_{\mathbb{R}^n} d^j f\left(x-\sum_{l=1}^n t_l h_l\right)\prod_{l=1}^n \varphi_l(t_l)\,\mathrm{d}\lambda_n(t) \tag{19}$$

for $x \in W$, $j = 0,\dots,k$. Since each $d^j f$ is L_j-Lipschitz, by Lemma 61 used on (19) we obtain that there are L_j-Lipschitz mappings $q_j\colon W \to \mathcal{P}({}^j X;Y)$ such that $d^j g_n \to q_j$ uniformly on W and $\|d^j f - q_j\|_W < \varepsilon$. Moreover, q_k is Gâteaux differentiable on W and $x \mapsto \delta q_k[h]$ is uniformly continuous on W for each $h \in X$. Therefore q_k is uniformly Gâteaux differentiable on V by Lemma 59. Theorem 1.85 implies that $g_n \to g \in C^k(W;Y)$ uniformly on W and $d^j g = q_j$, $j = 0,\dots,k$.

□

The following version of Lemma 61 is for mappings that are only locally Lipschitz.

LEMMA 63. *Let X be a separable Banach space, Y a Banach space, $U \subset X$ open, $f\colon U \to Y$ a locally Lipschitz mapping, and let $V \subset U$ be open such that $\delta =$* $\operatorname{dist}(V, X\setminus U) > 0$. *Let $\{h_j\}$ be a dense subset of S_X and let $\varphi_j \in C^\infty(\mathbb{R})$, $j \in \mathbb{N}$, be such that $\varphi_j \geq 0$, $\int_{\mathbb{R}}\varphi_j = 1$, and* $\operatorname{supp}\varphi_j \subset \left[-\frac{\delta}{2^j}, \frac{\delta}{2^j}\right]$. *Extend f to the whole of X by setting $f(x) = 0$ for $x \in X\setminus U$ and define $g_n\colon V \to Y$, $n \in \mathbb{N}$, by formula* (18). *Then $g_n \to g$ locally uniformly on V and the mapping $g\colon V \to Y$ is locally Lipschitz and Gâteaux differentiable. Moreover, if $Y = \mathbb{R}$ and U, V, f are convex, then so is g.*

PROOF. Let $K = \left\{\sum_{j=1}^\infty t_j h_j;\ |t_j| \leq \frac{\delta}{2^j}\right\}$. Then K is a compact subset of X and so it is easy to show that for each $x \in V$ there is a neighbourhood $V_x \subset V$ of x such that f is Lipschitz on $V_x - K$. Note that for $y \in V_x$ each $g_n(y)$ is defined using values of f on $V_x - K$ only. So we may repeat the proof of Lemma 61 with the following differences:

- $g_n \to g$ only locally uniformly on V.
- g_n are only locally Lipschitz and so is g.
- $\{\frac{\partial g_n}{\partial h_i}\}_{n=i}^{\infty}$ converges only locally uniformly on V.
- In the proof of the Gâteaux differentiability of g we use the fact that g is locally Lipschitz.

□

COROLLARY 64. *Let X be a separable Banach space that admits a $C^{k,1}_{\text{loc}}$-smooth bump for some $k \in \mathbb{N}_0$. Then X admits a $C^{k,1}_{\text{loc}}$-smooth bump with Gâteaux differentiable kth derivative (in particular it admits G^{k+1}-smooth bump).*

PROOF. Let $f \in C^{k,1}_{\text{loc}}$ be a non-negative bump function with $\operatorname{supp} f \subset B(0,1)$. Let $\varphi_j \in C^\infty(\mathbb{R})$, $j \in \mathbb{N}$, be such that $\varphi_j \geq 0$, $\int_{\mathbb{R}} \varphi_j = 1$, and $\operatorname{supp} \varphi_j \subset \left[-\frac{1}{2^j}, \frac{1}{2^j}\right]$. Define mappings $g_n \colon X \to Y$ by formula (18). By Corollary 1.91 we have $g_n \in C^k(X)$ and (19) holds for $x \in X$, $j = 0, \dots, k$. Since each $d^j f$ is locally Lipschitz, by Lemma 63 used on (19) we obtain that there are locally Lipschitz mappings $q_j : X \to \mathcal{P}(^jX)$ such that $d^j g_n \to q_j$ locally uniformly on X. Moreover, q_k is Gâteaux differentiable on X. Theorem 1.85 implies that $g_n \to g \in C^k(X)$ locally uniformly on X and $d^j g = q_j$, $j = 0, \dots, k$. Finally, since by the definition each g_n is zero outside $B(0,2)$, g is a bump.

□

To proceed to integral convolutions in $c_0(\Gamma)$ we need an auxiliary notion. Let X be a topological vector space, $\Omega \subset X$ an open subset, E an arbitrary set, $M \subset X^*$, and $g \colon \Omega \to E$. Let U be a neighbourhood of zero in X. We say that g depends U-uniformly locally on finitely many coordinates from M (U-ULFC-M for short) if for each $x \in \Omega$ there is a finite subset $F \subset M$ such that g depends only on F on $(x+U) \cap \Omega$ (cf. Definition 5.78).

For any subset $F \subset \Gamma$ we denote the associated projection in $c_0(\Gamma)$ by P_F, i.e. $P_F(x) = \sum_{\gamma \in F} e^*_\gamma(x) e_\gamma$ for $x \in c_0(\Gamma)$. By $c_{00}(\Gamma)$ we denote the linear subspace of $c_0(\Gamma)$ consisting of finitely supported vectors.

LEMMA 65. *Let Γ be an arbitrary set, Y a Banach space, and let $f : c_0(\Gamma) \to Y$ be a mapping that is $U(0,r)$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ for some $r > 0$. Further, let $\Omega \subset c_0(\Gamma)$ be open, let f be uniformly continuous on Ω with modulus ω, and suppose that $f = 0$ on $c_0(\Gamma) \setminus \Omega$. Then for every $V \subset \Omega$ with $\operatorname{dist}(V, c_0(\Gamma) \setminus \Omega) > 0$ and for every $\varepsilon > 0$ there is a $U(0, \frac{r}{2})$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ mapping $g \in C^\infty(c_0(\Gamma); Y)$ such that $\|f - g\|_V \leq \varepsilon$, g is uniformly continuous on V with modulus ω, and the mapping $x \mapsto Dg(x)[h]$ is uniformly continuous on V for any $h \in c_{00}(\Gamma)$. If f is even, then so is g. If moreover $Y = \mathbb{R}$ and f is convex, then so is g.*

PROOF. Let $\eta = \operatorname{dist}(V, c_0(\Gamma) \setminus \Omega)$ and find $0 < \delta < \min\{\eta, \frac{r}{2}\}$ such that $\omega(\delta) < \varepsilon$. Choose an even C^∞-smooth non-negative function φ on $\mathbb{R}$ such that $\operatorname{supp} \varphi \subset [-\delta, \delta]$ and $\int_{\mathbb{R}} \varphi = 1$. We denote $C = \int_{\mathbb{R}} |\varphi'(t)| \, \mathrm{d}\lambda$. Let $\mathcal{F} \subset 2^\Gamma$ be a partially ordered set of non-empty finite subsets of Γ ordered by inclusion. For any $F \in \mathcal{F}$ we define the mapping $g_F : c_0(\Gamma) \to Y$ by

$$g_F(x) = \int_{\mathbb{R}^{|F|}} f\left(x - \sum_{\gamma \in F} t_\gamma e_\gamma\right) \prod_{\gamma \in F} \varphi(t_\gamma) \, \mathrm{d}\lambda_{|F|}(t).$$

Notice that the integral is well-defined, since $f = 0$ on the closed set $c_0(\Gamma) \setminus \Omega$ and f is uniformly continuous on Ω and so it is bounded on totally bounded sets.

The net $\{g_F\}_{\mathcal{F}}$ converges on $c_0(\Gamma)$ to a mapping $g \colon c_0(\Gamma) \to Y$. In fact, we claim that for any $x \in c_0(\Gamma)$ there is an $F \in \mathcal{F}$ such that $g_F(y) = g_H(y)$ for any $F \subset H \in \mathcal{F}$ and any $y \in U(x, \frac{r}{2})$. Indeed, for a fixed $x \in c_0(\Gamma)$ let $F \in \mathcal{F}$ be such that f depends only on

$\{e^*_\gamma\}_{\gamma\in F}$ on $U(x,r)$ and $\|x - P_F(x)\| < \frac{r}{2}$. Choose any $y \in U(x,\frac{r}{2})$ and $H \in \mathcal{F}$, $H \supset F$. Suppose that $t_\gamma \in [-\frac{r}{2},\frac{r}{2}]$ for all $\gamma \in H$. Then $\|x - (y - \sum_{\gamma\in H} t_\gamma e_\gamma)\| < r$ and consequently $f\,(y - \sum_{\gamma\in H} t_\gamma e_\gamma) = f\,(y - \sum_{\gamma\in F} t_\gamma e_\gamma)$. Thus, by the Fubini theorem,

$$\begin{aligned} g_H(y) &= \int_{[-\delta,\delta]^{|H|}} f\Big(y - \sum_{\gamma\in H} t_\gamma e_\gamma\Big) \prod_{\gamma\in H} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|H|}(t) \\ &= \int_{[-\delta,\delta]^{|F|}} f\Big(y - \sum_{\gamma\in F} t_\gamma e_\gamma\Big) \prod_{\gamma\in F} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|F|}(t) \prod_{\gamma\in H\setminus F} \int_{[-\delta,\delta]} \varphi(t_\gamma)\,\mathrm{d}\lambda = g_F(y). \end{aligned}$$

Moreover, $\|x - P_F(y)\| \le \|x - P_F(x)\| + \|P_F\|\|x - y\| < r$ and so we can easily see that $g_F(y) = g_F(P_F(y))$. The mapping $g_F \restriction_{P_F(c_0(\Gamma))}$ is in fact a finite-dimensional convolution with a smooth kernel on $R^{|F|}$, and so g_F is a C^∞-smooth mapping on $U(x,\frac{r}{2})$ (Corollary 1.91; recall that a uniformly continuous mapping is bounded on totally bounded sets). The mapping g is therefore $U(0,\frac{r}{2})$-ULFC-$\{e^*_\gamma\}_{\gamma\in\Gamma}$ and $g \in C^\infty(c_0(\Gamma); Y)$, as for any $x \in c_0(\Gamma)$, $g = g_F \circ P_F$ on $U(x,\frac{r}{2})$ for some $F \in \mathcal{F}$.

To show that $\|f - g\|_V \le \varepsilon$ choose any $x \in V$. Let $F \in \mathcal{F}$ be such that $g(x) = g_F(x)$. Notice that $\|x - (x - \sum_{\gamma\in F} t_\gamma e_\gamma)\| = \|\sum_{\gamma\in F} t_\gamma e_\gamma\| \le \delta < \eta$ whenever $t_\gamma \in [-\delta,\delta]$ for all $\gamma \in F$. Hence $x - \sum_{\gamma\in F} t_\gamma e_\gamma \in \Omega$ and

$$\begin{aligned} \|f(x) - g(x)\| &= \|f(x) - g_F(x)\| \\ &= \left\| \int_{\mathbb{R}^{|F|}} f(x) \prod_{\gamma\in F} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|F|}(t) - \int_{\mathbb{R}^{|F|}} f\Big(x - \sum_{\gamma\in F} t_\gamma e_\gamma\Big) \prod_{\gamma\in F} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|F|}(t) \right\| \\ &\le \int_{[-\delta,\delta]^{|F|}} \left\| f(x) - f\Big(x - \sum_{\gamma\in F} t_\gamma e_\gamma\Big) \right\| \prod_{\gamma\in F} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|F|}(t) \\ &\le \int_{[-\delta,\delta]^{|F|}} \omega(\delta) \prod_{\gamma\in F} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|F|}(t) = \omega(\delta) < \varepsilon. \end{aligned}$$

To see that g is uniformly continuous on V with modulus ω, choose $x, y \in V$ and find $F, H \in \mathcal{F}$ such that $g(x) = g_F(x)$ and $g(y) = g_H(y)$. Then for $K = F \cup H$ we have $g(x) = g_K(x)$ and $g(y) = g_K(y)$. As $x - \sum_{\gamma\in K} t_\gamma e_\gamma \in \Omega$ and $y - \sum_{\gamma\in K} t_\gamma e_\gamma \in \Omega$ whenever $t_\gamma \in (-\eta,\eta)$ for all $\gamma \in K$,

$$\begin{aligned} \|g(x) - g(y)\| &= \|g_K(x) - g_K(y)\| \\ &\le \int_{[-\delta,\delta]^{|K|}} \left\| f\Big(x - \sum_{\gamma\in K} t_\gamma e_\gamma\Big) - f\Big(y - \sum_{\gamma\in K} t_\gamma e_\gamma\Big) \right\| \prod_{\gamma\in K} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|K|}(t) \\ &\le \omega\big(\|x - y\|\big). \end{aligned}$$

Similarly we can check that g is even if f is even and g is convex under the additional assumptions that $Y = \mathbb{R}$ and f is convex.

We finish the proof by showing that the directional derivatives of g in the directions of $c_{00}(\Gamma)$ are uniformly continuous on V. So first, choose any $\alpha \in \Gamma$. For $x, y \in V$ find $F, H \in \mathcal{F}$ such that $g(x) = g_F(x)$ on $U(x,\frac{r}{2})$ and $g(y) = g_H(y)$ on $U(y,\frac{r}{2})$. Put $K = F \cup H \cup \{\alpha\}$. Using

Corollary 1.91 and substitution we obtain

$$Dg_K(x)[e_\alpha] = \int_{\mathbb{R}^{|K|}} f\Bigl(x - \sum_{\gamma\in K} t_\gamma e_\gamma\Bigr)\varphi'(t_\alpha) \prod_{\gamma\in K\setminus\{\alpha\}} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|K|}(t).$$

Hence, similarly as above,

$$\begin{aligned}
&\bigl\|Dg(x)[e_\alpha] - Dg(y)[e_\alpha]\bigr\| \\
&\quad\le \int_{\mathbb{R}^{|K|}} \Bigl\| f\Bigl(x - \sum_{\gamma\in K} t_\gamma e_\gamma\Bigr) - f\Bigl(y - \sum_{\gamma\in K} t_\gamma e_\gamma\Bigr)\Bigr\| |\varphi'(t_\alpha)| \prod_{\gamma\in K\setminus\{\alpha\}} \varphi(t_\gamma)\,\mathrm{d}\lambda_{|K|}(t) \\
&\quad\le \omega\bigl(\|x-y\|\bigr)\int_{\mathbb{R}} |\varphi'(t)|\,\mathrm{d}\lambda = C\omega\bigl(\|x-y\|\bigr).
\end{aligned}$$

Finally, choose any $h \in c_{00}(\Gamma)$ and $x, y \in V$. It follows from the above estimate that

$$\begin{aligned}
\bigl\|Dg(x)[h] - Dg(y)[h]\bigr\| &\le \sum_{\gamma\in\operatorname{supp} h} \bigl\|Dg(x)[e_\gamma^*(h)e_\gamma] - Dg(y)[e_\gamma^*(h)e_\gamma]\bigr\| \\
&\le C\omega\bigl(\|x-y\|\bigr)\sum_{\gamma\in\operatorname{supp} h} |e_\gamma^*(h)| = C\|h\|_{\ell_1}\omega\bigl(\|x-y\|\bigr).
\end{aligned}$$

□

THEOREM 66. *Let Γ be an arbitrary set, Y a Banach space, $U \subset c_0(\Gamma)$ open, and let $f\colon U \to Y$ be a uniformly continuous mapping with modulus ω. Then for every $V \subset U$ with $\operatorname{dist}(V, c_0(\Gamma) \setminus U) > 0$ and every $\varepsilon > 0$ there is a mapping $g \in C^\infty(c_0(\Gamma); Y)$ which uniformly locally depends on finitely many coordinates $\{e_\gamma^*\}_{\gamma\in\Gamma}$, such that $\|f - g\|_V \le \varepsilon$, and g is uniformly continuous on V with modulus ω. If f is moreover L-Lipschitz, then g is L-Lipschitz on V and uniformly Gâteaux differentiable on* $\operatorname{Int} V$.

PROOF. Let $r = \operatorname{dist}(V, c_0(\Gamma) \setminus U)$ and find $0 < \eta \le \frac{r}{2}$ such that $\omega(\eta) < \frac{\varepsilon}{2}$. Define $\varphi\colon \mathbb{R} \to \mathbb{R}$ by $\varphi(t) = \max\{0, t-\eta\} + \min\{0, t+\eta\}$. Then φ is 1-Lipschitz and $|\varphi(t) - t| \le \eta$ for all $t \in \mathbb{R}$. Further, define a mapping $\Phi\colon c_0(\Gamma) \to c_0(\Gamma)$ by $\Phi(x) = \sum_{\gamma\in\Gamma} \varphi(e_\gamma^*(x))e_\gamma$. (Notice that in fact Φ maps into $c_{00}(\Gamma)$.) Then Φ is 1-Lipschitz and $\|\Phi(x) - x\| \le \eta$ for all $x \in c_0(\Gamma)$. Moreover, we claim that Φ is $U(0, \frac{\eta}{2})$-ULFC-$\{e_\gamma^*\}_{\gamma\in\Gamma}$.

Indeed, fix $x \in c_0(\Gamma)$ and find a finite $F \subset \Gamma$ such that $\|x - P_F(x)\| < \frac{\eta}{2}$. Then for any $y \in U(x, \frac{\eta}{2})$ we have $\|y - P_F(y)\| < \eta$. This means that if $y, z \in U(x, \frac{\eta}{2})$ are such that $e_\gamma^*(y) = e_\gamma^*(z)$ for all $\gamma \in F$, then $\varphi(e_\gamma^*(y)) = 0 = \varphi(e_\gamma^*(z))$ for all $\gamma \in \Gamma \setminus F$ and of course $\varphi(e_\gamma^*(y)) = \varphi(e_\gamma^*(z))$ for all $\gamma \in F$. Hence $\Phi(y) = \Phi(z)$, and so Φ depends only on $\{e_\gamma^*\}_{\gamma\in F}$ on $U(x, \frac{\eta}{2})$.

We extend f to the whole of $c_0(\Gamma)$ by $f(x) = 0$ for $x \in c_0(\Gamma) \setminus U$ and put $h = f \circ \Phi$. Clearly, the mapping $h\colon c_0(\Gamma) \to Y$ is $U(0, \frac{\eta}{2})$-ULFC-$\{e_\gamma^*\}_{\gamma\in\Gamma}$. Put $\Omega = \Phi^{-1}(U)$. Then Ω is open and $\operatorname{dist}(V, c_0(\Gamma) \setminus \Omega) \ge \frac{r}{2}$ (in particular $V \subset \Omega$). Indeed, choose $x \in V$ and $y \in c_0(\Gamma)$ such that $\|x - y\| < \frac{r}{2}$. Then $\|\Phi(y) - x\| \le \|\Phi(y) - y\| + \|y - x\| < \eta + \frac{r}{2} \le r$, which means that $\Phi(y) \in U$ and so $y \in \Omega$. Moreover, h is uniformly continuous on Ω with modulus ω. To see this, choose any $x, y \in \Omega$. Then $\Phi(x), \Phi(y) \in U$ and hence $\|h(x) - h(y)\| \le \omega\bigl(\|\Phi(x) - \Phi(y)\|\bigr) \le \omega\bigl(\|x - y\|\bigr)$.

Finally, $\|f - h\|_V \le \sup_{x\in V} \omega\bigl(\|x - \Phi(x)\|\bigr) \le \omega(\eta) < \frac{\varepsilon}{2}$, and Lemma 65 used on h together with Lemma 60 and Lemma 59 finishes the proof.

□

Let (X, ν) be a normed linear space. The norm ν is said to be UG-smooth (or just UG) if it is Gâteaux differentiable on $X \setminus \{0\}$ and for each fixed $h \in S_X$ the limit defining $\frac{\partial \nu}{\partial h}(x)$ is uniform for $x \in S_X$.

PROOF OF THEOREM 5.126. Define a function $f : c_0(\Gamma) \to \mathbb{R}$ by $f(x) = \max\{0, \|x\| - 1\}$. Then f is a 1-Lipschitz convex even function which is $U(0, \frac{1}{2})$-ULFC-$\{e_\gamma^*\}_{\gamma \in \Gamma}$. (Notice that $f = \|\cdot\| \circ \Phi$ as in the proof of Theorem 66 for $\eta = 1$.)

Let $g \in C^\infty(c_0(\Gamma))$ be a 1-Lipschitz convex even function with uniformly continuous directional derivatives produced by Lemma 65 combined with Lemma 60, such that $|g(x) - f(x)| \leq 1$ for all $x \in c_0(\Gamma)$. Then g is separating, as $g(0) \leq 1$ and $g(x) \geq 2$ on $4S_{c_0(\Gamma)}$. The function g is also UG by Lemma 59, and so we can finish by using the next lemma.

□

LEMMA 67. *Let X be a normed linear space, $k \in \mathbb{N} \cup \{\infty\}$, and let $g : X \to \mathbb{R}$ be a C^k-smooth, UG, Lipschitz, even, and convex separating function. Then X admits an equivalent C^k-smooth UG norm.*

PROOF. As shown in [HJ2], the Minkowski functional of a sub-level set of a convex separating UG function need not be UG. To be able to use the Minkowski functional we need to gain more control over $Dg(x)[x]$. To this end we introduce an additional transformation. Basically, we construct a function that is "directionally primitive" to g in a sense, so that its directional derivative is g back again (more or less), hence Lipschitz. So, define $f : X \to \mathbb{R}$ by

$$f(x) = \int_{[0,1]} g(tx)\, \mathrm{d}\lambda(t).$$

Let L be the Lipschitz constant of g. It is easy to check that f is $\frac{L}{2}$-Lipschitz, even, and convex.

Without loss of generality we may assume that $g(0) = 0$. By the convexity of g and the fact that g is even, $g(x) \geq 0$ for $x \in X$. Since g is separating, there are $r > 0$ and $a > 0$ such that $g(x) \geq a$ for all $x \in rS_X$. Hence $g(tx) \geq a - Lr(1-t)$ whenever $t \in [0,1]$ and $\|x\| = r$. It follows that

$$f(x) \geq \int_{1-a/(Lr)}^{1} \big(a - Lr(1-t)\big)\, \mathrm{d}\lambda(t) = \frac{a^2}{2Lr} = b \quad \text{for any } x \in rS_X.$$

By Corollary 1.91 the function f is C^k-smooth and

$$Df(x)[h] = \int_{[0,1]} Dg(tx)[th]\, \mathrm{d}\lambda(t). \tag{20}$$

By the proof of Lemma 5.23 there is an equivalent C^k-smooth norm ν on X satisfying $\nu(x) = 1$ if and only if $f(x) = b$, and

$$D\nu(x) = \frac{1}{Df\big(\frac{x}{\nu(x)}\big)\big[\frac{x}{\nu(x)}\big]} Df\Big(\frac{x}{\nu(x)}\Big).$$

Using Lemma 59 and (20) we can see that the function $x \mapsto Df(x)[h]$ is uniformly continuous on X for any $h \in X$. Moreover, the function $x \mapsto Df(x)[x]$ is Lipschitz on X. Indeed, using the substitution $t(1+\tau) = s$ we get $f(x + \tau x) = \int_{[0,1]} g(tx + t\tau x)\, \mathrm{d}\lambda(t) =$

$\frac{1}{1+\tau}\int_{[0,1+\tau]} g(sx)\,\mathrm{d}\lambda(s)$. Thus, using the continuity of g along the way,

$$\begin{aligned}
Df(x)[x] &= \lim_{\tau\to 0}\frac{1}{\tau}\big(f(x+\tau x)-f(x)\big)\\
&= \lim_{\tau\to 0}\frac{1}{\tau}\left(\left(\frac{1}{1+\tau}-1\right)\int_0^{1+\tau} g(tx)\,\mathrm{d}\lambda(t)+\int_1^{1+\tau} g(tx)\,\mathrm{d}\lambda(t)\right)\\
&= \lim_{\tau\to 0}\frac{-1}{1+\tau}\int_0^{1+\tau} g(tx)\,\mathrm{d}\lambda(t)+\lim_{\tau\to 0}\frac{1}{\tau}\int_1^{1+\tau} g(tx)\,\mathrm{d}\lambda(t)\\
&= g(x)-\int_0^1 g(tx)\,\mathrm{d}\lambda(t)=g(x)-f(x).
\end{aligned}$$

Since both f and g are L-Lipschitz, the function $x\mapsto Df(x)[x]$ is $2L$-Lipschitz. Clearly, $f(0)=0$. So, the convexity of f implies that $Df(x)[x]\geq f(x)$ for any $x\in X$, and in particular $Df(x)[x]\geq b$ for $x\in X$, $\nu(x)=1$.

Finally, we claim that the function $x\mapsto D\nu(x)[h]$ is uniformly continuous on $A_R=X\setminus B(0,R)$ for any $h\in X$ and any $R>0$, which according to Lemma 59 means that the norm ν is UG. Fix any $R>0$ and $h\in X$. Denote $S=\{x\in X;\ \nu(x)=1\}$. As the mapping $\psi\colon A_R\to S$, $\psi(x)=x/\nu(x)$ is Lipschitz and $D\nu(x)=D\nu(\psi(x))$, it is enough to show that $x\mapsto D\nu(x)[h]$ is uniformly continuous on S. Let $\varepsilon>0$. Find $0<\delta<\varepsilon$ such that $|Df(x)[h]-Df(y)[h]|<\varepsilon$ whenever $\|x-y\|<\delta$. Then for any $x,y\in S$, $\|x-y\|<\delta$ we have

$$\begin{aligned}
\big|D\nu(x)[h]-D\nu(y)[h]\big| &= \left|\frac{Df(x)[h]}{Df(x)[x]}-\frac{Df(y)[h]}{Df(y)[y]}\right|\\
&\leq \frac{1}{|Df(x)[x]|}\big|Df(x)[h]-Df(y)[h]\big|\\
&\qquad + |Df(y)[h]|\left|\frac{1}{Df(x)[x]}-\frac{1}{Df(y)[y]}\right|\\
&\leq \frac{\varepsilon}{b}+\frac{L}{2}\|h\|\frac{\big|Df(x)[x]-Df(y)[y]\big|}{|Df(x)[x]|\cdot|Df(y)[y]|}\leq\frac{\varepsilon}{b}+\frac{L}{2}\|h\|\frac{2L\|x-y\|}{b^2}\\
&< \varepsilon\left(\frac{1}{b}+\frac{L^2}{b^2}\|h\|\right).
\end{aligned}$$

□

Assume that a normed linear space X can be embedded in $c_0(\Gamma)$ by a bi-Lipschitz homeomorphism Φ. Then we can use Theorem 66 to approximate mappings on X provided that we can extend the approximated mapping from $\Phi(X)$ onto some uniform neighbourhood of $\Phi(X)$. Since the extensions are intimately tied with retractions (see e.g. [BL]), we recall the following notions.

A retraction of a set A onto $B\subset A$ is a mapping $r\colon A\to B$ such that $r\restriction_B=Id$. A metric space P is called an absolute Lipschitz (resp. uniform) retract if for every metric space Q containing P as a subspace there is a Lipschitz (resp. uniformly continuous) retraction of Q onto P. The space P is called an absolute Lipschitz (resp. uniform) uniform neighbourhood retract if for every metric space Q containing P as a subspace there is a uniform neighbourhood U of P in Q (i.e. $\operatorname{dist}(P,Q\setminus U)>0$) and a Lipschitz (resp. uniformly continuous) retraction of U onto P.

FACT 68. *Let Γ be an arbitrary set, (P, ρ) a metric space, $Q \subset P$, and $f : Q \to \ell_\infty(\Gamma)$ a uniformly continuous mapping with modulus $\omega \in \mathcal{M}_s$. Then f can be extended to a uniformly continuous mapping $g : P \to \ell_\infty(\Gamma)$ with modulus ω. In particular, an L-Lipschitz mapping can be extended to an L-Lipschitz mapping.*

PROOF. For each $\gamma \in \Gamma$ the function $f_\gamma = e^*_\gamma \circ f$ is a uniformly continuous function on Q with modulus ω. By Lemma 30 there are extensions $g_\gamma : P \to \mathbb{R}$ of f_γ which are uniformly continuous with modulus ω. Fix $a \in Q$. For any $x \in P$ we have $|g_\gamma(x)| \leq |g_\gamma(a)| + |g_\gamma(x) - g_\gamma(a)| \leq |g_\gamma(a)| + \omega(\rho(x, a)) = |e^*_\gamma(f(a))| + \omega(\rho(x, a)) \leq \|f(a)\| + \omega(\rho(x, a))$. It follows that $(g_\gamma(x))_{\gamma\in\Gamma} \in \ell_\infty(\Gamma)$ and the mapping g can be defined as $g(x) = (g_\gamma(x))_{\gamma\in\Gamma}$. Further, $\omega_g \leq \omega$, as $\|g(x) - g(y)\| = \sup_{\gamma\in\Gamma} |g_\gamma(x) - g_\gamma(y)| \leq \omega(\rho(x, y))$.

□

FACT 69. *Every metric space X is isometric to a subset of $\ell_\infty(X)$.*

PROOF. Let ρ be the metric on X. Fix $a \in X$ and define the isometric embedding $\Phi : X \to \ell_\infty(X)$ by $\Phi(x)(\gamma) = \rho(x, \gamma) - \rho(a, \gamma)$ for $x, \gamma \in X$.

□

PROPOSITION 70. *Let X be a metric space. Then X is an absolute Lipschitz uniform neighbourhood retract if and only if there are $K > 0$ and $\delta > 0$ such that for any two metric spaces $Q \subset P$ and every L-Lipschitz mapping $f : Q \to X$ there are a uniform neighbourhood $U \subset P$ of Q with $\operatorname{dist}(Q, P \setminus U) \geq \frac{\delta}{L}$ and a KL-Lipschitz mapping $g : U \to X$ which extends f.*

Similarly, X is an absolute uniform uniform neighbourhood retract if and only if there are $\omega_0 \in \mathcal{M}$ and $\delta > 0$ such that for any two metric spaces $Q \subset P$ and every uniformly continuous mapping $f : Q \to X$ with modulus $\omega \in \mathcal{M}_s$ there are a uniform neighbourhood $U \subset P$ of Q with $\operatorname{dist}(Q, P \setminus U) \geq \eta$, where η is any number satisfying $\omega(\eta) < \delta$, and a uniformly continuous mapping $g : U \to X$ with modulus $\omega_0 \circ \omega$ which extends f.

PROOF. $\Rightarrow$ Embed X isometrically into $\ell_\infty(\Gamma)$. Let $V \subset \ell_\infty(\Gamma)$ be a uniform open neighbourhood of X and let $r : V \to X$ be a K-Lipschitz retraction. Let $\delta = \operatorname{dist}(X, \ell_\infty(\Gamma) \setminus V)$. By Fact 68 there is an L-Lipschitz extension $h : P \to \ell_\infty(\Gamma)$ of $f : Q \to X \subset \ell_\infty(\Gamma)$. Put $U = h^{-1}(V)$. Then U is open in P and $\operatorname{dist}(Q, P \setminus U) \geq \frac{\delta}{L}$. Indeed, if $y \in U(z, \delta/L)$ for some $z \in Q$, then $h(y) \in U(h(z), \delta)$, where $h(z) \in X$; hence $h(y) \in V$. Finally, put $g(x) = r(h(x))$ for any $x \in U$.

$\Leftarrow$ Let X be a subspace of a metric space P and put $Q = X$. The Lipschitz extension of the identity mapping $Id : X \to X$ to a uniform neighbourhood of X in P serves as the desired retraction.

The proof for the uniformly continuous version is analogous.

□

Now we are ready to prove the approximation theorem.

THEOREM 71. *Let Y be a Banach space, $k \in \mathbb{N} \cup \{\infty\}$, and let X be a normed linear space such that there are a set Γ and a bi-Lipschitz homeomorphism $\Phi : X \to c_0(\Gamma)$ such that the component functions $e^*_\gamma \circ \Phi \in C^k(X)$ for every $\gamma \in \Gamma$. Assume further that X or Y is an absolute Lipschitz uniform neighbourhood retract. There is a constant $C \in \mathbb{R}$ such that if $f : X \to Y$ is L-Lipschitz and $\varepsilon > 0$, then there is a CL-Lipschitz mapping $g \in C^k(X; Y)$ such that $\|f - g\|_X \leq \varepsilon$.*

Moreover, if $C_1, C_2 \in \mathbb{R}$ are such that Φ is C_1-Lipschitz and Φ^{-1} is C_2-Lipschitz, and if K is the constant from Proposition 70, then $C = C_1 C_2 K$.

This theorem immediately follows from the following more general version.

THEOREM 72. *Let Y be a Banach space, $k \in \mathbb{N} \cup \{\infty\}$, and let X be a normed linear space such that there are a set Γ and a uniform homeomorphism $\Phi\colon X \to c_0(\Gamma)$ such that $\omega_{\Phi^{-1}} \leq \omega_1 \in \mathcal{M}_s$ and the component functions $e^*_\gamma \circ \Phi \in C^k(X)$ for every $\gamma \in \Gamma$. Assume further that X or Y is an absolute uniform uniform neighbourhood retract. If $f\colon X \to Y$ is uniformly continuous and $\varepsilon > 0$, then there is a uniformly continuous mapping $g \in C^k(X;Y)$ such that $\|f - g\|_X \leq \varepsilon$.*

Moreover, if $\omega_0 \in \mathcal{M}$ is the modulus from Proposition 70 for the space X, then $\omega_g \leq \omega_f \circ \omega_0 \circ \omega_1 \circ \omega_\Phi$. If $\omega_0 \in \mathcal{M}$ is the modulus from Proposition 70 for the space Y, then $\omega_g \leq \omega_0 \circ \omega_f \circ \omega_1 \circ \omega_\Phi$.

PROOF. Define $\hat{f}\colon \Phi(X) \to Y$ by $\hat{f}(z) = f(\Phi^{-1}(z))$ for any $z \in \Phi(X)$. The mapping $\hat{f}$ is uniformly continuous with modulus $\omega_f \circ \omega_1 \in \mathcal{M}_s$. If Y is an absolute uniform uniform neighbourhood retract, then by Proposition 70 there are a uniform open neighbourhood U of $\Phi(X)$ in $c_0(\Gamma)$ and an extension $\bar{f}\colon U \to Y$ of $\hat{f}$ such that $\omega_{\bar{f}} \leq \omega_2 = \omega_0 \circ \omega_f \circ \omega_1$.

In case that X is an absolute uniform uniform neighbourhood retract, we use the Proposition 70 to a mapping Φ^{-1} to obtain a uniform open neighbourhood U of $\Phi(X)$ in $c_0(\Gamma)$ and an extension $q\colon U \to X$ of Φ^{-1} such that $\omega_q \leq \omega_0 \circ \omega_1$. Now put $\bar{f} = f \circ q$. Then $\omega_{\bar{f}} \leq \omega_2 = \omega_f \circ \omega_0 \circ \omega_1$ and $\bar{f}$ is an extension of $\hat{f}$.

By Theorem 66 there is a mapping $\bar{g} \in C^\infty(c_0(\Gamma); Y)$ locally dependent on finitely many coordinates and such that $\|\bar{g}(z) - \bar{f}(z)\| \leq \varepsilon$ for any $z \in \Phi(X)$ and $\bar{g}$ is uniformly continuous on $\Phi(X)$ with modulus ω_2. We define the mapping $g\colon X \to Y$ by $g = \bar{g} \circ \Phi$. By Lemma 5.81, $g \in C^k(X;Y)$. Clearly, $\omega_g \leq \omega_2 \circ \omega_\Phi$. To see that g approximates f, choose any $x \in X$. Then

$$\|g(x) - f(x)\| = \|\bar{g}(\Phi(x)) - \hat{f}(\Phi(x))\| = \|\bar{g}(\Phi(x)) - \bar{f}(\Phi(x))\| \leq \varepsilon.$$

□

Let V be a topological space and $v_0 \in V$. By $B_0(V)$ we denote the space of all bounded real-valued functions f on V for which $f(v) \to 0$ whenever $v \to v_0$, considered with the supremum norm. Given a metric space P we denote by $C_{\mathrm{ub}}(P)$ the space of all bounded, uniformly continuous real-valued functions on P with the supremum norm. By the result of Joram Lindenstrauss [L, Theorem 6] (see also [BL]), both $B_0(V)$ and $C_{\mathrm{ub}}(P)$ are absolute Lipschitz retracts. Therefore using Corollary 57 and Theorem 71 we obtain the following result:

COROLLARY 73. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. If at least one of the spaces X or Y is equal to either $B_0(V)$ for some topological space V, or $C_{\mathrm{ub}}(P)$ for some metric space P, then there is a constant $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f\colon X \to Y$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(X;Y)$ for which $\|f - g\|_X \leq \varepsilon$.*

Further, by another result of J. Lindenstrauss, [L, Theorem 8] (see also [BL, Corollary 1.26]), super-reflexive Banach spaces are absolute uniform uniform neighbourhood retracts. Hence using Corollary 57 and Theorem 72 we obtain the following result:

COROLLARY 74. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. If X or Y is a super-reflexive*

Banach space, then there are a constant $C \in \mathbb{R}$ and a modulus $\omega_0 \in \mathcal{M}$ such that for any uniformly continuous mapping $f : X \to Y$ and any $\varepsilon > 0$ there is a uniformly continuous mapping $g \in C^k(X;Y)$ for which $\|f-g\|_X \le \varepsilon$ and $\omega_g(\delta) \le \omega_f(\omega_0(C\delta))$ (if X is super-reflexive) or $\omega_g(\delta) \le \omega_0(\omega_f(C\delta))$ (if Y is super-reflexive) for $\delta \in [0,+\infty)$.

Next, we show another way how to glue together the finite-dimensional approximations. In what follows, the unconditional basis is instrumental so that we can arbitrarily perturb the coordinates of a given vector with a control over the norm.

LEMMA 75 ([Mo]). *Let X be a Banach space with an unconditional Schauder basis $\{e_j\}_{j=1}^\infty$ that admits a C^k-smooth Lipschitz bump function. Denote $X_n = \operatorname{span}\{e_j\}_{j=1}^n$, $X_\infty = \operatorname{span}\{e_j\}_{j=1}^\infty = \bigcup_{n=1}^\infty X_n$. Then there is a constant $K > 0$ such that for any $\varepsilon > 0$ there is a K-Lipschitz mapping $\Psi \in C^k(X;X_\infty)$ such that for each $x \in X$ there are a neighbourhood U of x and $n \in \mathbb{N}$ such that $\Psi(U) \subset X_n$ and $\|x - \Psi(x)\| < \varepsilon$.*

PROOF. Let $\{f_n\}$ be the biorthogonal functionals to $\{e_n\}$. Let A be the unconditional basis constant of $\{e_n\}$ and B the basis constant of $\{e_n\}$. Let $\varphi \in C^k(X;[0,1])$ be a Lipschitz function such that $\varphi(x) = 1$ whenever $\|x\| \ge 1/A$ and $\varphi(x) = 0$ whenever $\|x\| \le r$ for some $r > 0$. Such a function can be constructed from the C^k-smooth Lipschitz bump function, which we have at our disposal, by translating, scaling, and composing with a suitable real function. Let M be the Lipschitz constant of φ and $K = A + M(1+B)$. Denote $R_n = Id - P_n$ for $n \in \mathbb{N}$, where P_n are the projections associated with the basis, $P_0 = 0$, and $R_0 = Id$.

Let $\varepsilon > 0$. Define $\Psi : X \to X_\infty$ by $\Psi(x) = \sum_{n=1}^\infty \varphi(R_{n-1}(x)/\varepsilon) f_n(x) e_n$. Suppose that $x \in X$. Then there is an $N \in \mathbb{N}$ such that $\|R_n(x)\| < \varepsilon \frac{r}{2}$ for all $n \ge N$ and thus there is a neighbourhood U of x such that $\|R_n(y)\| < \varepsilon r$ whenever $n \ge N$ and $y \in U$. It follows that $\Psi(y) = \sum_{n=1}^N \varphi(R_{n-1}(y)/\varepsilon) f_n(y) e_n$ and hence $\Psi(U) \subset X_N$. This fact also implies that $\Psi \in C^k(X;X_\infty)$, as it is locally a finite sum of C^k-smooth mappings.

To see that $\|x - \Psi(x)\| < \varepsilon$ for all $x \in X$, fix an arbitrary $x \in X$ and find $n_0 \in \mathbb{N}_0$ such that $\|R_{n_0}(x)\| < \varepsilon/A$ and $\|R_n(x)\| \ge \varepsilon/A$ for all $0 \le n < n_0$. Then

$$\begin{aligned}\|x - \Psi(x)\| &= \left\| \sum_{n=1}^\infty \bigl(1 - \varphi(R_{n-1}(x)/\varepsilon)\bigr) f_n(x) e_n \right\| = \left\| \sum_{n>n_0} \bigl(1 - \varphi(R_{n-1}(x)/\varepsilon)\bigr) f_n(x) e_n \right\| \\ &\le A \left\| \sum_{n>n_0} f_n(x) e_n \right\| = A\|R_{n_0}(x)\| < A\frac{\varepsilon}{A} = \varepsilon.\end{aligned}$$

To show that Ψ is Lipschitz we estimate the norm of the derivative of Ψ. Let $x \in X$. Find $n_0 \in \mathbb{N}_0$ such that $\|R_{n_0}(x)\| \le \varepsilon/A$ and $\|R_n(x)\| > \varepsilon/A$ for all $0 \le n < n_0$. Let $h \in B_X$. Notice that all the sums in the following computation are in fact finite and that

$D\varphi(R_{n-1}(x)/\varepsilon) = 0$ for $1 \le n \le n_0$.

$$\begin{aligned}
\|D\Psi(x)[h]\| &= \left\|\sum_{n=1}^{\infty} D\big(\varphi(R_{n-1}(\cdot)/\varepsilon) f_n e_n\big)(x)[h]\right\| \\
&= \left\|\sum_{n=1}^{\infty} \varphi(R_{n-1}(x)/\varepsilon) f_n(h) e_n + f_n(x) e_n \cdot D\varphi(R_{n-1}(x)/\varepsilon)[R_{n-1}(h)/\varepsilon]\right\| \\
&\le \left\|\sum_{n=1}^{\infty} \varphi(R_{n-1}(x)/\varepsilon) f_n(h) e_n\right\| + \left\|\sum_{n=1}^{\infty} f_n(x) e_n \cdot D\varphi(R_{n-1}(x)/\varepsilon)[R_{n-1}(h)/\varepsilon]\right\| \\
&\le A\|h\| + \left\|\sum_{n>n_0} f_n(x) e_n \cdot D\varphi(R_{n-1}(x)/\varepsilon)[R_{n-1}(h)/\varepsilon]\right\| \\
&\le A + A \sup_{n>n_0} \big|D\varphi(R_{n-1}(x)/\varepsilon)[R_{n-1}(h)/\varepsilon]\big| \left\|\sum_{n>n_0} f_n(x) e_n\right\| \\
&\le A + A \sup_{n>n_0} \big\|D\varphi(R_{n-1}(x)/\varepsilon)\big\|(1 + B)\|h/\varepsilon\| \cdot \|R_{n_0}(x)\| \\
&\le A + AM(1 + B)\frac{1}{\varepsilon}\frac{\varepsilon}{A} = A + M(1 + B) = K.
\end{aligned}$$

□

THEOREM 76 (Nicole Moulis, [Mo]). *Let X be a Banach space with a monotone unconditional Schauder basis $\{e_j\}_{j=1}^{\infty}$ that admits a C^k-smooth Lipschitz bump function. Denote $X_n = \operatorname{span}\{e_j\}_{j=1}^{n}$. There is a constant $C > 0$ such that if Y is a Banach space, $M \subset X$ is such that $P_n(M) \subset M$ for all $n \in \mathbb{N}$, Ω is a uniform open neighbourhood of M, $f : \Omega \to Y$ is an L-Lipschitz mapping such that $f \restriction_{\Omega\cap X_n}$ is $C^{1,+}$-smooth for each $n \in \mathbb{N}$, $V \subset M$ is open such that $\operatorname{dist}(V, X \setminus M) > 0$, and $\varepsilon > 0$, then there is $g \in C^k(X; Y)$ such that $\|Dg\|_V \le C(1+\varepsilon)L$ and $\|f - g\|_V \le \varepsilon$.*

PROOF. Without loss of generality we may assume that $\{e_j\}$ is normalised. Denote $X_\infty = \operatorname{span}\{e_j\}_{j=1}^{\infty} = \bigcup_{n=1}^{\infty} X_n$. Let us extend the mapping f to the whole of X by $f(x) = 0$ for $x \in X \setminus \Omega$ and denote $f_n = f \restriction_{X_n}$. For each $n \in \mathbb{N}$ denote by T_n the isomorphism $T_n \colon \mathbb{R}^n \to X_n$, $T_n(y) = \sum_{j=1}^{n} y_j e_j$, and define a mapping $g_n \colon X_n \to Y$ by the Bochner integral

$$g_n(x) = \int_{\mathbb{R}^n} f_n\big(x - T_n(y)\big)\varphi_n(y)\, \mathrm{d}\lambda_n(y),$$

where $\varphi_n \in C^\infty(\mathbb{R}^n)$ are smooth functions with sufficiently small compact supports chosen so that

$$\|g_n(x) - f_n(x)\| < \frac{\varepsilon}{4}\frac{1}{2^n} \quad \text{for every } x \in M \cap X_n, \tag{21}$$

$$\|Dg_n(x) - Df_n(x)\| < L\frac{\varepsilon}{2}\frac{1}{2^n} \quad \text{for every } x \in M \cap X_n. \tag{22}$$

This is possible, since the mappings f_n and Df_n are uniformly continuous on $\Omega \cap X_n$ (see the proof of Lemma 1). Using Corollary 1.91 and substitution (recall that a uniformly continuous mapping is bounded on totally bounded sets) it is easy to see that $g_n \in C^\infty(X_n; Y)$.

Now let us define inductively a sequence of mappings $\bar{g}_n\colon X_n \to Y$. Let $\bar{g}_1 = g_1$. Suppose that $n \in \mathbb{N}$, $n > 1$, and the mapping $\bar{g}_{n-1}$ is already defined. Then we put

$$\bar{g}_n(x) = g_n(x) + \bar{g}_{n-1}(P_{n-1}(x)) - g_n(P_{n-1}(x)) \quad \text{for all } x \in X_n.$$

Notice that $\bar{g}_n \restriction_{X_{n-1}} = \bar{g}_{n-1}$, that is $\bar{g}_n$ is an extension of $\bar{g}_{n-1}$. Furthermore, by induction we can show that

$$\bar{g}_n \in C^\infty(X_n; Y) \quad \text{for each } n \in \mathbb{N}, \tag{23}$$

$$\|\bar{g}_n(x) - f_n(x)\| < \frac{\varepsilon}{2}\left(1 - \frac{1}{2^n}\right) \quad \text{for every } x \in M \cap X_n, n \in \mathbb{N}, \tag{24}$$

$$\|D\bar{g}_n(x) - Df_n(x)\| < L\varepsilon\left(1 - \frac{1}{2^n}\right) \quad \text{for every } x \in M \cap X_n, n \in \mathbb{N}. \tag{25}$$

Indeed, (23) is obvious. For $n = 1$ the inequality (24) follows from (21). Let $n \in \mathbb{N}$, $n > 1$, and suppose the inequality (24) holds for $n - 1$. Then, using (21),

$$\begin{aligned}
\|\bar{g}_n(x) - f_n(x)\| &\le \|g_n(x) - f_n(x)\| + \left\|\bar{g}_{n-1}(P_{n-1}(x)) - g_n(P_{n-1}(x))\right\| \\
&< \frac{\varepsilon}{4}\frac{1}{2^n} + \left\|\bar{g}_{n-1}(P_{n-1}(x)) - f_{n-1}(P_{n-1}(x))\right\| \\
&\qquad + \left\|f_n(P_{n-1}(x)) - g_n(P_{n-1}(x))\right\| \\
&< \frac{\varepsilon}{4}\frac{1}{2^n} + \frac{\varepsilon}{2}\left(1 - \frac{1}{2^{n-1}}\right) + \frac{\varepsilon}{4}\frac{1}{2^n} = \frac{\varepsilon}{2}\left(1 - \frac{1}{2^n}\right)
\end{aligned}$$

for any $x \in M \cap X_n$. Notice, that here we used the fact that $P_{n-1}(M) \subset M$.

The inequality (25) for $n = 1$ follows from (22). Let $n \in \mathbb{N}$, $n > 1$, and suppose the inequality (25) holds for $n - 1$. Then, using (22),

$$\begin{aligned}
\|D\bar{g}_n(x) - Df_n(x)\| &\le \|Dg_n(x) - Df_n(x)\| + \left\|D(\bar{g}_{n-1} \circ P_{n-1})(x) - D(g_n \circ P_{n-1})(x)\right\| \\
&< L\frac{\varepsilon}{2}\frac{1}{2^n} + \left\|D(\bar{g}_{n-1} \circ P_{n-1})(x) - D(f_n \circ P_{n-1})(x)\right\| \\
&\quad + \left\|D(f_n \circ P_{n-1})(x) - D(g_n \circ P_{n-1})(x)\right\| \\
&= L\frac{\varepsilon}{2}\frac{1}{2^n} + \left\|D\bar{g}_{n-1}(P_{n-1}(x)) \circ P_{n-1} - Df_{n-1}(P_{n-1}(x)) \circ P_{n-1}\right\| \\
&\quad + \left\|Df_n(P_{n-1}(x)) \circ P_{n-1} - Dg_n(P_{n-1}(x)) \circ P_{n-1}\right\| \\
&< L\frac{\varepsilon}{2}\frac{1}{2^n} + L\varepsilon\left(1 - \frac{1}{2^{n-1}}\right) + L\frac{\varepsilon}{2}\frac{1}{2^n} = L\varepsilon\left(1 - \frac{1}{2^n}\right)
\end{aligned}$$

for any $x \in M \cap X_n$. Here we used the fact that $P_{n-1}(M) \subset M$ and also the fact that $\|P_{n-1}\| = 1$.

Next, we define the mapping $\bar{g}\colon X_\infty \to Y$ by $\bar{g}(x) = \lim_{n\to\infty} \bar{g}_n(x)$ for all $x \in X_\infty$. Recall that $\bar{g}_n(x) = \bar{g}_m(x)$ for all $n \ge m$ whenever $x \in X_m$ and thus the mapping $\bar{g}$ is well-defined. From (24) it readily follows that

$$\|\bar{g}(x) - f(x)\| < \frac{\varepsilon}{2} \quad \text{for every } x \in M \cap X_\infty. \tag{26}$$

Now, let $\Psi \in C^k(X; X_\infty)$ be the mapping from Lemma 75 such that $\|\Psi(x) - x\| < \min\{\frac{\varepsilon}{2L}, \varepsilon, \operatorname{dist}(V, X \setminus M)\}$ for all $x \in X$. Let $C > 0$ be the Lipschitz constant of Ψ. We define the mapping $g\colon X \to Y$ by $g = \bar{g} \circ \Psi$. For each $x \in X$ there are a neighbourhood U of x and

$n \in \mathbb{N}$ such that $\Psi(U) \subset X_n$. Hence $g = \bar{g}_n \circ \Psi$ on U which together with (23) implies that $g \in C^k(X;Y)$.

To see that g approximates f, choose an arbitrary $x \in V$. Then $\Psi(x) \in M \cap X_\infty$, which used together with (26) gives

$$\|g(x) - f(x)\| \le \|\bar{g}(\Psi(x)) - f(\Psi(x))\| + \|f(\Psi(x)) - f(x)\| < \frac{\varepsilon}{2} + L\frac{\varepsilon}{2L} = \varepsilon.$$

Finally, we estimate the derivative of g on V. Fix any $x \in V$. There are a neighbourhood U of x and $n \in \mathbb{N}$ such that $\Psi(U) \subset X_n$ and so $g = \bar{g}_n \circ \Psi$ on U. Also, $\Psi(x) \in M \cap X_n$ and therefore we can use (25) to obtain

$$\begin{aligned}\|Dg(x)\| &= \|D(\bar{g}_n \circ \Psi)(x)\| = \|D\bar{g}_n(\Psi(x)) \circ D\Psi(x)\| \le \|D\bar{g}_n(\Psi(x))\|\|D\Psi(x)\| \\ &\le C\Big(\big\|D\bar{g}_n(\Psi(x)) - Df_n(\Psi(x))\big\| + \|Df_n(\Psi(x))\|\Big) < C(L\varepsilon + L) = C(1+\varepsilon)L.\end{aligned}$$

□

Combining Lemma 61 and Theorem 76 we would obtain a uniform approximation result on spaces with unconditional basis. However, we postpone the precise formulation until Corollary 79, where we obtain even stronger statement.

Next, we prove a result that allows us to pass from uniform approximations to fine approximations. We start with the existence of smooth and Lipschitz σ-discrete partitions of unity.

LEMMA 77. *Let X, Y be normed linear spaces and $k \in \mathbb{N} \cup \{\infty\}$. Suppose that for each 1-Lipschitz mapping $f : 2U_X \to Y$ and $\varepsilon > 0$ there is a Lipschitz mapping $g \in C^k(U_X;Y)$ satisfying $\|f - g\|_{U_X} \le \varepsilon$. Let $\Omega \subset X$ be open. Then for any open covering $\mathcal{U}$ of Ω there is a Lipschitz and C^k-smooth locally finite and σ-uniformly discrete partition of unity on Ω subordinated to $\mathcal{U}$.*

PROOF. Let $S \subset C^k(\Omega)$ be the subset consisting of bounded Lipschitz functions. Analogously as in the proof of Theorem 48 it can be shown that S is a partition ring. Further, notice that approximation of mappings into Y gives us also approximations of functions. Indeed, if $f : 2U_X \to \mathbb{R}$ is 1-Lipschitz, then choose some $y \in S_Y$ and consider the mapping $\bar{f} : 2U_X \to Y$, $\bar{f}(x) = f(x) \cdot y$. Let $\bar{g} \in C^k(U_X;Y)$ be an approximation of $\bar{f}$ provided by our assumption and $F \in Y^*$ be a Hahn-Banach extension of the norm-one functional $ty \mapsto t$ defined on $\operatorname{span}\{y\}$. Then $g = F \circ \bar{g}$ is the desired approximation of the function f. Thus by approximating the function $x \mapsto \operatorname{dist}(x, \Omega \setminus W)$ we can show that (ii) in Lemma 40 is satisfied, which finishes the proof.

□

THEOREM 78. *Let X, Y be normed linear spaces and $k \in \mathbb{N} \cup \{\infty\}$. Suppose that there is a $C \ge 1$ such that for each L-Lipschitz mapping $f : 2U_X \to Y$ and $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X;Y)$ satisfying $\|f - g\|_{U_X} \le \varepsilon$. Let $\Omega \subset X$ be open. Then for any L-Lipschitz mapping $f : \Omega \to Y$, any continuous function $\varepsilon\colon \Omega \to \mathbb{R}^+$, and any $\eta > 1$ there is an ηCL-Lipschitz mapping $g \in C^k(\Omega;Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

PROOF. First notice that from approximations on U_X by translating and scaling we immediately obtain approximations on any open ball in X. For each $x \in \Omega$ find $r(x) > 0$ such that $U(x, 4r(x)) \subset \Omega$ and

$$\varepsilon(y) > \frac{\varepsilon(x)}{3} \quad \text{for each } y \in U(x, r(x)). \tag{27}$$

By Lemma 77 there is a locally finite and σ-discrete C^k-smooth Lipschitz partition of unity on Ω subordinated to $\{U(x, r(x));\ x \in \Omega\}$. We may assume that the partition of unity is of the form $\{\psi_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, where for each $n \in \mathbb{N}$ the family $\{\operatorname{supp}_\circ \psi_{n\alpha}\}_{\alpha\in\Lambda}$ is discrete in Ω. For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ let $U_{n\alpha} = U(x_{n\alpha}, r(x_{n\alpha}))$ be such that $\operatorname{supp}_\circ \psi_{n\alpha} \subset U_{n\alpha}$. Let $L_{n\alpha}$ be the Lipschitz constant of $\psi_{n\alpha}$, and without loss of generality assume that $L_{n\alpha} \geq 1$. Further, denote $V_{n\alpha} = U(x_{n\alpha}, 2r(x_{n\alpha}))$.

For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ we approximate f on $V_{n\alpha}$ by CL-Lipschitz mapping $g_{n\alpha} \in C^k(V_{n\alpha}; Y)$ such that

$$\|f(x) - g_{n\alpha}(x)\| \leq \min\left\{\frac{(\eta-1)CL}{2^n L_{n\alpha}}, \frac{\varepsilon(x_{n\alpha})}{3}\right\} < \varepsilon(x) \quad \text{for each } x \in U_{n\alpha}. \tag{28}$$

(The second inequality follows from (27).) Define the mapping $\bar{g}_{n\alpha}\colon \Omega \to Y$ by $\bar{g}_{n\alpha}(x) = g_{n\alpha}(x)$ for $x \in V_{n\alpha}$, $\bar{g}_{n\alpha}(x) = 0$ otherwise.

Finally, we define the mapping $g\colon \Omega \to Y$ by

$$g(x) = \sum_{n\in\mathbb{N},\alpha\in\Lambda} \psi_{n\alpha}(x)\bar{g}_{n\alpha}(x).$$

Since $\operatorname{supp}_\circ \psi_{n\alpha} \subset U_{n\alpha}$, $g_{n\alpha} \in C^k(V_{n\alpha}; Y)$, and the sum is locally finite, the mapping g is well-defined and moreover $g \in C^k(\Omega; Y)$.

Choose $x \in \Omega$ and let us compute how far $g(x)$ is from $f(x)$:

$$\begin{aligned}\|f(x) - g(x)\| &= \Bigg\|\sum_{n\in\mathbb{N},\alpha\in\Lambda} \psi_{n\alpha}(x)\big(f(x) - \bar{g}_{n\alpha}(x)\big)\Bigg\| \leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \psi_{n\alpha}(x)\|f(x) - g_{n\alpha}(x)\| \\ &< \varepsilon(x) \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda\colon x\in U_{n\alpha}}} \psi_{n\alpha}(x) = \varepsilon(x),\end{aligned}$$

where the last inequality follows from (28).

To estimate the derivative of g at some fixed $x \in \Omega$, notice that by the discreteness of $\{\operatorname{supp}_\circ \psi_{n\alpha}\}_{\alpha\in\Lambda}$, for each $n \in \mathbb{N}$ there is at most one $\alpha \in \Lambda$ such that $D\psi_{n\alpha}(x) \neq 0$. Put $M = \{n \in \mathbb{N};\ \exists\alpha \in \Lambda\colon D\psi_{n\alpha}(x) \neq 0\}$. Then there is a mapping $\beta\colon M \to \Lambda$ such that for each $n \in M$, $D\psi_{n\alpha}(x) = 0$ whenever $\alpha \neq \beta(n)$ and moreover $x \in U_{n\beta(n)}$. (Notice that if $D\psi_{n\alpha}(x) \neq 0$, then necessarily $x \in U_{n\alpha}$.) Further, since $\sum \psi_{n\alpha} = 1$, it follows that

$\sum D\psi_{n\alpha} = 0$. Hence

$$\begin{aligned}
\|Dg(x)\| &= \Bigg\| \sum_{n\in\mathbb{N},\alpha\in\Lambda} D(\psi_{n\alpha}\bar{g}_{n\alpha})(x) \Bigg\| = \Bigg\| \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} D(\psi_{n\alpha}\bar{g}_{n\alpha})(x) \Bigg\| \\
&= \Bigg\| \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} \psi_{n\alpha}(x) Dg_{n\alpha}(x) + \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} D\psi_{n\alpha}(x)\cdot g_{n\alpha}(x) \Bigg\| \\
&= \Bigg\| \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} \psi_{n\alpha}(x) Dg_{n\alpha}(x) + \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} D\psi_{n\alpha}(x)\cdot \big(g_{n\alpha}(x) - f(x)\big) \Bigg\| \\
&\le \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} \psi_{n\alpha}(x)\|Dg_{n\alpha}(x)\| + \sum_{n\in M} \|D\psi_{n\beta(n)}(x)\| \|g_{n\beta(n)}(x) - f(x)\| \\
&\le \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} CL\psi_{n\alpha}(x) + \sum_{n\in M} L_{n\beta(n)} \|g_{n\beta(n)}(x) - f(x)\| \\
&\le CL + \sum_{n\in M} L_{n\beta(n)} \frac{(\eta-1)CL}{2^n L_{n\beta(n)}} \le \eta CL,
\end{aligned}$$

where the last but one inequality follows from (28).

To finish the proof we show that g is ηCL-Lipschitz on the set Ω. Without loss of generality we may assume that $\varepsilon(x) \le (\eta C - 1)L \operatorname{dist}(x, X\setminus\Omega)$ for every $x\in\Omega$. Now fix $x, y\in\Omega$. If the line segment l with end points x and y lies in Ω, then $\|g(x)-g(y)\| \le \eta CL\|x-y\|$ by Proposition 1.71. Otherwise there is $z\in l\cap(X\setminus\Omega)$. Then

$$\begin{aligned}
\|g(x)-g(y)\| &\le \|g(x)-f(x)\| + \|f(x)-f(y)\| + \|f(y)-g(y)\| \\
&< \varepsilon(x) + L\|x-y\| + \varepsilon(y) \\
&\le (\eta C-1)L\|x-z\| + L\|x-y\| + (\eta C-1)L\|y-z\| = \eta CL\|x-y\|.
\end{aligned}$$

□

COROLLARY 79. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k\in\mathbb{N}\cup\{\infty\}$. Let Y be a Banach space. Suppose further that one of the following conditions is satisfied:*

- *X is a Banach space with an unconditional Schauder basis, or*
- *at least one of the spaces X or Y is equal to $B_0(V)$ for some topological space V, or*
- *at least one of the spaces X or Y is equal to $C_{\mathrm{ub}}(P)$ for some metric space P.*

Then there is a constant $C\in\mathbb{R}$ such that for any open $\Omega\subset X$, any L-Lipschitz mapping $f\colon\Omega\to Y$, and any continuous function $\varepsilon\colon\Omega\to\mathbb{R}^+$ there is a CL-Lipschitz mapping $g\in C^k(\Omega;Y)$ for which $\|f(x)-g(x)\| < \varepsilon(x)$ for all $x\in\Omega$.

PROOF. It suffices to notice that under our assumptions the hypothesis of Theorem 78 is satisfied. Indeed, since B_X is a 2-Lipschitz retract of X, every L-Lipschitz mapping defined on B_X can be extended to a $2L$-Lipschitz mapping defined on X. Thus we may apply either Corollary 73, or (in the case of the unconditional basis) we combine Lemma 61 and Theorem 76.

□

Further, Theorem 78 together with Theorem 32 gives us the next corollary.

COROLLARY 80. *Let X be a Hilbert space and $\Omega \subset X$ an open set. Then for any L-Lipschitz function $f : \Omega \to \mathbb{R}$, any continuous function $\varepsilon\colon \Omega \to \mathbb{R}^+$, and any $\eta > 1$ there is an ηL-Lipschitz function $g \in C^1(\Omega)$ such that $|f(x) - g(x)| < \varepsilon(x)$ for all $x \in \Omega$.*

Similarly, Theorem 78 together with Theorem 66 produces the following corollary.

COROLLARY 81. *Let Γ be an arbitrary set, $\Omega \subset c_0(\Gamma)$ an open set, and Y a Banach space. Then for any L-Lipschitz mapping $f : \Omega \to Y$, any continuous function $\varepsilon\colon \Omega \to \mathbb{R}^+$, and any $\eta > 1$ there is an ηL-Lipschitz mapping $g \in C^\infty(\Omega; Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

In the rest of the section we deal with the approximation of Lipschitz functions by real analytic Lipschitz functions.

THEOREM 82. *Let X be a separable Banach space that admits a Lipschitz separating real-analytic function with uniform radii of convergence as in property (K). Then there is a constant $K \in \mathbb{R}$ such that for each $\varepsilon > 0$ and any L-Lipschitz function $f : X \to \mathbb{R}$ there is a KL-Lipschitz function $g \in C^\omega(X)$ satisfying $|f - g|_X \le \varepsilon$.*

We remark that the assumption is in particular satisfied if X admits a separating polynomial. Indeed, by Fact 4.45 there is $P \in \mathcal{P}(^nX)$ for some $n \in \mathbb{N}$ even such that $P(x) \ge 1$ for $x \in S_X$. Let $0 < r \le 1$ be such that $r^2\|\check{P}\|(1+t)^n < 1 + t^n$ for each $t \in [0, +\infty)$. Consider the function $q(z) = \left(1 + \tilde{P}(z)\right)^{\frac{1}{n}}$. By Proposition 1.61 we have $\operatorname{Re} \tilde{P}(x + iy) \ge P(x) - \|\check{P}\| \sum_{k=1}^{[\frac{n}{2}]} \binom{n}{2k} \|x\|^{n-2k}\|y\|^{2k} \ge \|x\|^n - \|\check{P}\| r^2(1 + \|x\|)^n > -1$ for all $x, y \in X$, $\|y\| \le r$. Thus $q \in H(G)$, where $G = \{z \in \tilde{X};\ \|\operatorname{Im} z\| < r\}$, and by Corollary 1.165 the radius of norm convergence of the Taylor series of q at each $x \in X$ is at least r. Further, $q\restriction_X$ is clearly separating. Finally, using the fact that $P(x) \ge \|x\|^n$ for $x \in X$ we obtain $\|Dq\restriction_X(x)\| = \frac{1}{n}(1 + P(x))^{\frac{1}{n}-1}\|DP(x)\| \le \frac{1}{n}(1 + \|x\|^n)^{\frac{1}{n}-1}\|DP\|\|x\|^{n-1} \le \frac{1}{n}\|DP\|$, and so $q\restriction_X$ is Lipschitz.

The proof of Theorem 82 is divided into a few steps (Proposition 83, Proposition 84, and Lemma 85). We begin by introducing an auxiliary notion. Let X be a normed linear space and let $\mathcal{U} = \{U_x;\ x \in U_x \subset \tilde{X}, x \in X\}$ be a collection of open neighbourhoods in $\tilde{X}$. Let $A \subset X$. We say that a function $h\colon \bigcup \mathcal{U} \to \mathbb{C}$ *separates A with respect to $\mathcal{U}$* if

(S1) $h\restriction_X$ maps into $\mathbb{R}$,
(S2) $h(x) \ge 1$ whenever $x \in A$,
(S3) $|h(z)| \le \frac{1}{4}$ whenever $z \in U_x$, $x \in X$, $\operatorname{dist}(x, A) \ge 1$.

PROPOSITION 83. *Let X be a Banach space. Assume that there are $\mathcal{U} = \{U_x;\ x \in U_x \subset \tilde{X}, x \in X\}$ a collection of open neighbourhoods in $\tilde{X}$ and $C > 0$ such that for each $A \subset X$ there is a function $h_A \in H\left(\bigcup \mathcal{U}\right)$ which separates A with respect to $\mathcal{U}$ and such that $h_A\restriction_X$ is C-Lipschitz. Then for every $\varepsilon > 0$ and every L-Lipschitz function $f : X \to \mathbb{R}$ there is a $10CL$-Lipschitz function $g \in C^\omega(X)$ satisfying $|f - g|_X \le \varepsilon$.*

PROOF. Let us define a function $\bar{f} : X \to \mathbb{R}$ by $\bar{f}(x) = \frac{4}{\varepsilon} f\left(\frac{\varepsilon}{4L}x\right)$. This function is obviously 1-Lipschitz. Denote $\bar{f}^+ = \max\{\bar{f}, 0\}$ and $\bar{f}^- = \max\{-\bar{f}, 0\}$ and notice that both functions are again 1-Lipschitz. Next, let us define the sets $A_n = \{x \in X;\ \bar{f}^+(x) \ge n\}$ for $n \in \mathbb{N}_0$. Clearly, $A_n \subset A_{n-1}$ for all $n \in \mathbb{N}$, and using the 1-Lipschitz property of $\bar{f}^+$ it is easy to check that

$$\operatorname{dist}(X \setminus A_n, A_{n+1}) \ge 1 \quad \text{for all } n \in \mathbb{N}. \tag{29}$$

Denote $h_n = \theta_n \circ h_{A_n}$ for $n \in \mathbb{N}$, where the functions θ_n come from Lemma 4. For each $n \in \mathbb{N}$, $h_n \in H\big(\bigcup \mathcal{U}\big)$ and $h_n \restriction_X$ is $4C$-Lipschitz. Put $h^+ = \sum_{n=1}^{\infty} h_n$.

Fix an arbitrary $x \in X$. Then there is $m \in \mathbb{N}$ such that $x \in A_{m-1} \setminus A_m$. Hence

$$x \in A_n \text{ for } n < m \quad \text{and} \quad x \in X \setminus A_{n-1} \text{ for } n > m. \tag{30}$$

From this, (29), (S3), and (T3) it follows that $|h_n(z)| \le 2^{-n}$ for all $n > m$ and $z \in U_x$. Hence the sum in the definition of h^+ converges absolutely uniformly on U_x and so $h^+ \in H\big(\bigcup \mathcal{U}\big)$. This together with (S1) and (T1) implies that $h^+ \restriction_X \in C^{\omega}(X)$.

Using (30), (S2) and (T4), (29), (S3) and (T3), and finally $m - 1 + h_m(x) \in [m-1, m]$ and $\bar{f}^+(x) \in [m-1, m)$, we obtain

$$\begin{aligned}\big|h^+(x) - \bar{f}^+(x)\big| &= \left|\sum_{n=1}^{m-1} h_n(x) + h_m(x) + \sum_{n=m+1}^{\infty} h_n(x) - \bar{f}^+(x)\right| \\ &\le \sum_{n=1}^{m-1} |h_n(x) - 1| + \sum_{n=m+1}^{\infty} |h_n(x)| + \big|m - 1 + h_m(x) - \bar{f}^+(x)\big| \\ &< \sum_{n=1}^{m-1} 2^{-n} + \sum_{n=m+1}^{\infty} 2^{-n} + 1 < 2.\end{aligned}$$

Further, (30), (29), and (T5) imply $\|D(h_n \restriction_X)(x)\| = |(\theta_n \restriction_{\mathbb{R}})'(h_{A_n}(x))| \|D(h_{A_n} \restriction_X)(x)\| \le 2^{-n}C$ for $n \in \mathbb{N} \setminus \{m\}$. Hence by Corollary 1.166

$$\|D(h^+ \restriction_X)(x)\| \le \sum_{n=1}^{\infty} \|D(h_n \restriction_X)(x)\| \le \sum_{n \ne m} 2^{-n}C + \|D(h_m \restriction_X)(x)\| < C + 4C = 5C.$$

Similarly we obtain an approximation of $\bar{f}^-$ denoted by h^-. Put $h = h^+ - h^-$. Then $h \restriction_X \in C^{\omega}(X)$, $\big|h(x) - \bar{f}(x)\big| < 4$ for every $x \in X$, and $\|D(h \restriction_X)(x)\| \le \|D(h^+ \restriction_X)(x)\| + \|D(h^- \restriction_X)(x)\| < 10C$ for every $x \in X$.

Finally, let $g(x) = \frac{\varepsilon}{4} h\left(\frac{4L}{\varepsilon} x\right)$ for $x \in X$. It is straightforward to check that g satisfies the conclusion of our proposition.

□

PROPOSITION 84. *Let X be a Banach space. Suppose that there are an open neighbourhood G of X in $\tilde{X}$ and a collection $\{\psi_n\}_{n \in \mathbb{N}}$ of functions on G with the following properties:*

(P1) *$\{\psi_n \restriction_X\}_{n \in \mathbb{N}}$ is a sup-partition on X,*
(P2) *the mapping $z \mapsto (b_n \psi_n(z))_{n \in \mathbb{N}}$ is a holomorphic mapping from G into $\tilde{c}_0$ for any $(b_n) \in \ell_\infty$,*
(P3) *there is $M > 0$ such that each $\psi_n \restriction_X$ is M-Lipschitz,*
(P4) *for each $n \in \mathbb{N}$ there is $x_n \in X$ such that $\psi_n(x) \le \frac{Q}{8}$ for $x \in X$, $\|x - x_n\| \ge \frac{1}{2}$, where Q is the quantity from the definition of a sup-partition.*

Then there is a collection $\mathcal{U}$ of open neighbourhoods in $\tilde{X}$ such that for each $A \subset X$ there is a function $h_A \in H\big(\bigcup \mathcal{U}\big)$ which separates A with respect to $\mathcal{U}$ and such that $h_A \restriction_X$ is C-Lipschitz, where $C = \sqrt{2}M/Q$.

PROOF. Let W, μ, and Δ_w be as in Proposition 7 for $q = \frac{8}{Q}$. Denote $w(z) = (\psi_n(z))_{n \in \mathbb{N}}$ for $z \in G$. By the continuity of the mapping w (which follows from (P2)), for each $x \in X$ there is an open neighbourhood U_x of x in $\tilde{X}$ such that $U_x \subset G$ and $\|w(z) - w(x)\| < \Delta_{w(x)}/q$ whenever $z \in U_x$. (Notice that $w(x) \in c_0 \setminus \{0\}$.) Put $\mathcal{U} = \{U_x;\ x \in X\}$.

Let $A \subset X$. For each $n \in \mathbb{N}$ put $b_n = q$ if $\operatorname{dist}(x_n, A) \le \frac{1}{2}$ and $b_n = 1$ otherwise. Choose $z \in \bigcup \mathcal{U}$ and let $x \in X$ be such that $z \in U_x$. Then

$$\begin{aligned} \big\|(b_n\psi_n(z)) - (b_n\psi_n(x))\big\| &= \sup_{n\in\mathbb{N}} \big|b_n(\psi_n(z) - \psi_n(x))\big| \le q \sup_{n\in\mathbb{N}} |\psi_n(z) - \psi_n(x)| \\ &= q\|w(z) - w(x)\| < \Delta_{w(x)} \end{aligned} \tag{31}$$

and since $0 \le w(x) \le (b_n\psi_n(x)) \le qw(x)$ in the lattice sense, from (M1) it follows that $(b_n\psi_n(z)) \in W$. Therefore we may define $h_A(z) = \frac{1}{8}\mu\big((b_n\psi_n(z))\big)$ for $z \in \bigcup \mathcal{U}$ and (P2) implies that $h_A \in H\big(\bigcup \mathcal{U}\big)$. Further, $h_A \restriction_X$ is obviously C-Lipschitz.

Next we show that h_A separates A with respect to $\mathcal{U}$. Clearly h_A has property (S1). Pick any $x \in A$. From (P1) and (P4) it follows that $\sup\{\psi_n(x);\ n \in \mathbb{N}, \operatorname{dist}(x_n, A) \le \frac{1}{2}\} \ge Q$. Therefore $\|(b_n\psi_n(x))\| \ge qQ = 8$ and consequently (M2) gives property (S2). Finally, to show (S3) let $x \in X$ be such that $\operatorname{dist}(x, A) \ge 1$. Then, by (P4), $\psi_n(x) \le \frac{Q}{8}$ for those $n \in \mathbb{N}$ for which $\operatorname{dist}(x_n, A) \le \frac{1}{2}$. Thus $\|(b_n\psi_n(x))\| \le \max\{q\frac{Q}{8}, 1\} = 1$. Now (31) together with (M3) implies $|h_A(z)| \le \frac{1}{4}$ for $z \in U_x$.

□

LEMMA 85. *Let X be a separable Banach space and $\{x_n\}_{n=1}^{\infty}$ a dense sequence in X. Suppose that there are $\Delta > 0$ and a function $q \in H(\Omega)$ where $\Omega = \{z \in \tilde{X};\ \|\operatorname{Im} z\| < \Delta\}$, such that $q \restriction_X$ is Lipschitz and maps into $[0, +\infty)$, $q(0) = 0$, $q(x) \ge 2$ for $x \in X$, $\|x\| \ge \frac{1}{2}$, and suppose there is a sequence $\{a_n\}_{n=1}^{\infty}$ of positive real numbers such that for each $x \in X$ the function*

$$z \mapsto \sum_{n=1}^{\infty} a_n\big(\operatorname{Im} q(x - x_n + z)\big)^2 \tag{32}$$

is defined on some neighbourhood of 0 in $\tilde{X}$ and is continuous at 0. Then there are an open neighbourhood G of X in $\tilde{X}$ and a collection of functions $\{\psi_n\}_{n\in\mathbb{N}}$ satisfying the properties (P1)–(P4) in Proposition 84.

PROOF. Put $\varepsilon_n = \min\{2^{-n}, \frac{1}{16}\}$ and let ζ_n be the functions and $\{\delta_n\}$ the sequence from Lemma 5. Put

$$\psi_n(z) = \zeta_n\big(q(z - x_1), \dots, q(z - x_n)\big) \qquad \text{for } z \in \Omega, n \in \mathbb{N}.$$

Then $\psi_n \in H(\Omega)$ and by (Z1) $\psi_n \restriction_X$ maps into $[0, 1]$.

Pick any $x \in X$. Then from the density of $\{x_n\}$ and the fact that $q(0) = 0$ it follows that there is $l \in \mathbb{N}$ such that $q(x - x_l) < 1$. Let $k \in \mathbb{N}$ be the smallest such number. Then property (Z4) implies that $\psi_k(x) \ge \frac{1}{2}$. Thus $\sup_{n\in\mathbb{N}} \psi_n(x) \ge Q$ for each $x \in X$, where $Q = \frac{1}{2}$.

By the continuity of q there is $\rho > 0$ such that $\operatorname{Re} q(z) \le \frac{1}{4}$ whenever $z \in \tilde{X}$, $\|z\| < \rho$. Now fix $x \in X$ and find an index $j \in \mathbb{N}$ such that $\|x_j - x\| < \rho$. Let $\delta_x > 0$ be such that $\|x - x_j + w\| < \rho$ and $\sum_{n=1}^{\infty} a_n\big(\operatorname{Im} q(x - x_n + w)\big)^2 \le \delta_j$ whenever $w \in \tilde{X}$, $\|w\| \le \delta_x$. Then $\operatorname{Re} q(x - x_j + w) \le \frac{1}{4}$ and hence, by (Z3), $|\psi_n(z)| \le 2^{-n}$ for $n > j$ and $z \in U_x = U_{\tilde{X}}(x, \delta_x)$. It follows that for any $(b_n) \in \ell_\infty$, $(b_n\psi_n(z))_{n\in\mathbb{N}} = \sum_{n=1}^{\infty} b_n\psi_n(z)e_n \in \tilde{c_0}$ and the sum converges absolutely uniformly on U_x. As the mappings $z \mapsto b_n\psi_n(z)e_n$ are holomorphic as mappings from Ω into $\tilde{c_0}$, we can conclude that $(b_n\psi_n)$ is a holomorphic mapping from $G = \bigcup_{x\in X} U_x$ into $\tilde{c_0}$, which gives (P2). (P1) then immediately follows.

Property (P3) obviously holds by (Z2). Finally we show that (P4) is satisfied. Indeed, fix $n \in \mathbb{N}$. For $x \in X$, $\|x - x_n\| \ge \frac{1}{2}$ we have $q(x - x_n) \ge 2$, hence, by (Z5), $\psi_n(x) \le \frac{1}{16} = \frac{Q}{8}$.

□

PROOF OF THEOREM 82. There are $d > 0$ and a function $q \in H(G)$, $G = \{z \in \tilde{X};\ \|\operatorname{Im} z\| < d\}$, such that $q\restriction_X$ is Lipschitz and maps into $[0, +\infty)$, $q(0) = 0$, $q(x) \geq 2$ for $x \in X$, $\|x\| \geq \frac{1}{2}$, and the radius of convergence of the Taylor expansion of q at every point $x \in X$ is at least d (Theorem 1.171).

Let $\{x_n\}_{n=1}^{\infty}$ be a dense sequence in X. Put

$$M_n = \sup\Big\{\big|q(x_j - x_n + w)\big|;\ w \in \tilde{X}, \|w\| \leq \tfrac{d}{2}, 1 \leq j \leq n\Big\}$$

and $a_n = 1/(2^n M_n^2)$. (Note that by the assumption on the radius of the Taylor series $M_n < +\infty$.) Fix $x \in X$. There is $k \in \mathbb{N}$ such that $x \in U\big(x_k, \frac{d}{4}\big)$. For $z \in \tilde{X}$ satisfying $\|z\| \leq \frac{d}{4}$ we have $\|x - x_k + z\| \leq \frac{d}{2}$ and hence for $n \geq k$

$$a_n\big(\operatorname{Im} q(x - x_n + z)\big)^2 \leq a_n\big|q(x_k - x_n + x - x_k + z)\big|^2 \leq a_n M_n^2 = \frac{1}{2^n}.$$

Therefore the sum in (32) converges absolutely uniformly on $B_{\tilde{X}}\big(0, \frac{d}{4}\big)$ to a continuous function. Using Lemma 85 together with Proposition 84 and Proposition 83 finishes the proof.

□

8. Approximation of C^1-smooth mappings

In this section we prove our most general result on an approximation of a C^1-smooth mapping together with its first derivative by a C^k-smooth mapping in the fine topology. A concise formulation is in Corollary 88.

In order to avoid repeating the same argument in various contexts, we prove the following somewhat technical proposition. One of the main ideas is based on the same argument as the proof of Theorem 78.

PROPOSITION 86. *Let X, Y be normed linear spaces, $k \in \mathbb{N} \cup \{\infty\}$, and $\Omega \subset X$ open. Suppose that for any open covering $\mathcal{U}$ of Ω there is a C^k-smooth Lipschitz locally finite and σ-discrete partition of unity on Ω subordinated to $\mathcal{U}$. Suppose further that $\{Y_\gamma\}_{\gamma\in\Gamma}$ is a collection of closed subspaces of Y such that for each $\gamma \in \Gamma$ there is a constant $C_\gamma \in \mathbb{R}$ such that for any L-Lipschitz mapping $f \in C^1(2U_X; Y_\gamma)$ and any $\varepsilon > 0$ there is a $C_\gamma L$-Lipschitz mapping $g \in C^k(U_X; Y)$ satisfying $\|f - g\|_{U_X} \leq \varepsilon$. Let $f \in C^1(\Omega; Y)$ be such that it is locally a mapping into some Y_γ, $\gamma \in \Gamma$. Then for any continuous function $\varepsilon\colon \Omega \to \mathbb{R}^+$ there is $g \in C^k(\Omega; Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|Df(x) - Dg(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

PROOF. First notice that from approximations on U_X by translating and scaling we immediately obtain approximations on any open ball in X. For each $x \in \Omega$ find $r(x) > 0$ and $\gamma(x) \in \Gamma$ such that $U(x, 4r(x)) \subset \Omega$, $f\big(U(x, 4r(x))\big) \subset Y_{\gamma(x)}$,

$$\varepsilon(y) > \frac{\varepsilon(x)}{3} \quad \text{for each } y \in U(x, 4r(x)), \text{ and} \tag{33}$$

$$\|Df(x) - Df(y)\| < \frac{\varepsilon(x)}{9C_{\gamma(x)}} \quad \text{for each } y \in U(x, 4r(x)). \tag{34}$$

By our assumption there is a locally finite and σ-discrete C^k-smooth Lipschitz partition of unity on Ω subordinated to $\{U(x, r(x));\ x \in \Omega\}$. We may assume that the partition of unity is of the form $\{\psi_{n\alpha}\}_{n\in\mathbb{N},\alpha\in\Lambda}$, where for each $n \in \mathbb{N}$ the family $\{\operatorname{supp}_{\circ} \psi_{n\alpha}\}_{\alpha\in\Lambda}$ is discrete in Ω. For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ let $U_{n\alpha} = U(x_{n\alpha}, r(x_{n\alpha}))$ be such that $\operatorname{supp}_{\mathrm{o}} \psi_{n\alpha} \subset U_{n\alpha}$. Let $L_{n\alpha}$ be the Lipschitz constant of $\psi_{n\alpha}$. Further, denote $C_{n\alpha} = C_{\gamma(x_{n\alpha})}$ and $V_{n\alpha} = U(x_{n\alpha}, 2r(x_{n\alpha}))$. Without loss of generality assume that $L_{n\alpha} \geq 1$ and $C_{n\alpha} \geq 1$.

For each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ let us define the mapping $f_{n\alpha} \colon U(x_{n\alpha}, 4r(x_{n\alpha})) \to Y_{\gamma(x_{n\alpha})}$ by $f_{n\alpha}(x) = f(x) - Df(x_{n\alpha})[x]$. Then, by (34) and (33),

$$\begin{aligned} \|Df_{n\alpha}(x)\| &= \|Df(x) - Df(x_{n\alpha})\| \\ &< \frac{\varepsilon(x_{n\alpha})}{9C_{n\alpha}} < \frac{\varepsilon(x)}{3C_{n\alpha}} \leq \frac{\varepsilon(x)}{3} \quad \text{for each } x \in U(x_{n\alpha}, 4r(x_{n\alpha})). \end{aligned} \tag{35}$$

According to our assumption, for each $n \in \mathbb{N}$ and $\alpha \in \Lambda$ we can approximate $f_{n\alpha}$ on $V_{n\alpha}$ by $g_{n\alpha} \in C^k(V_{n\alpha}; Y)$ such that

$$\|Dg_{n\alpha}(x)\| \leq \frac{\varepsilon(x_{n\alpha})}{9} < \frac{\varepsilon(x)}{3} \quad \text{for each } x \in V_{n\alpha}, \tag{36}$$

$$\|f_{n\alpha}(x) - g_{n\alpha}(x)\| \leq \frac{\varepsilon(x_{n\alpha})}{9 \cdot 2^n L_{n\alpha}} < \frac{\varepsilon(x)}{3 \cdot 2^n L_{n\alpha}} < \varepsilon(x) \quad \text{for each } x \in V_{n\alpha}. \tag{37}$$

(The second inequalities follow from (33).) Define the mapping $\bar{g}_{n\alpha} \colon \Omega \to Y$ by $\bar{g}_{n\alpha}(x) = g_{n\alpha}(x)$ for $x \in V_{n\alpha}$, $\bar{g}_{n\alpha}(x) = 0$ otherwise. Finally, we define the mapping $g \colon \Omega \to Y$ by

$$g(x) = \sum_{n \in \mathbb{N}, \alpha \in \Lambda} \psi_{n\alpha}(x)\big(\bar{g}_{n\alpha}(x) + Df(x_{n\alpha})[x]\big).$$

Since $\operatorname{supp}_{\mathrm{o}} \psi_{n\alpha} \subset U_{n\alpha}$, $g_{n\alpha} \in C^k(V_{n\alpha}; Y)$, and the sum is locally finite, the mapping g is well-defined and moreover $g \in C^k(\Omega; Y)$.

Choose $x \in \Omega$ and let us compute how far $g(x)$ is from $f(x)$:

$$\begin{aligned} \|f(x) - g(x)\| &= \left\| \sum_{n \in \mathbb{N}, \alpha \in \Lambda} \psi_{n\alpha}(x)\big(f(x) - \bar{g}_{n\alpha}(x) - Df(x_{n\alpha})[x]\big) \right\| \\ &= \left\| \sum_{\substack{n \in \mathbb{N} \\ \alpha \in \Lambda \colon x \in U_{n\alpha}}} \psi_{n\alpha}(x)\big(f_{n\alpha}(x) - g_{n\alpha}(x)\big) \right\| \\ &\leq \sum_{\substack{n \in \mathbb{N} \\ \alpha \in \Lambda \colon x \in U_{n\alpha}}} \psi_{n\alpha}(x) \|f_{n\alpha}(x) - g_{n\alpha}(x)\| < \varepsilon(x) \sum_{\substack{n \in \mathbb{N} \\ \alpha \in \Lambda \colon x \in U_{n\alpha}}} \psi_{n\alpha}(x) = \varepsilon(x), \end{aligned}$$

where the last inequality follows from (37).

To estimate the distance between the derivatives at some fixed $x \in \Omega$, notice that by the discreteness of $\{\operatorname{supp}_{\mathrm{o}} \psi_{n\alpha}\}_{\alpha \in \Lambda}$, for each $n \in \mathbb{N}$ there is at most one $\alpha \in \Lambda$ such that $D\psi_{n\alpha}(x) \neq 0$. Put $M = \{n \in \mathbb{N};\ \exists \alpha \in \Lambda \colon D\psi_{n\alpha}(x) \neq 0\}$. Then there is a mapping $\beta \colon M \to \Lambda$ such that for each $n \in M$, $D\psi_{n\alpha}(x) = 0$ whenever $\alpha \neq \beta(n)$, and moreover

$x \in U_{n\beta(n)}$. (Notice that if $D\psi_{n\alpha}(x) \neq 0$, then necessarily $x \in U_{n\alpha}$.) Hence

$$\begin{aligned}
\|Df(x) - Dg(x)\| = \|D(f-g)(x)\| &= \Bigg\| \sum_{n\in\mathbb{N},\alpha\in\Lambda} D\Big(\psi_{n\alpha}\cdot\big(f - \bar{g}_{n\alpha} - Df(x_{n\alpha})\big)\Big)(x)\Bigg\| \\
&= \Bigg\| \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} D\Big(\psi_{n\alpha}\cdot\big(f - g_{n\alpha} - Df(x_{n\alpha})\big)\Big)(x)\Bigg\| \\
&= \Bigg\| \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} D\big(\psi_{n\alpha}\cdot(f_{n\alpha} - g_{n\alpha})\big)(x)\Bigg\| \\
&= \Bigg\| \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} \psi_{n\alpha}(x) D(f_{n\alpha} - g_{n\alpha})(x) + \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} D\psi_{n\alpha}(x)\cdot\big(f_{n\alpha}(x) - g_{n\alpha}(x)\big)\Bigg\| \\
&\leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} \psi_{n\alpha}(x)\big\|Df_{n\alpha}(x) - Dg_{n\alpha}(x)\big\| + \sum_{n\in M}\|D\psi_{n\beta(n)}(x)\|\,\big\|f_{n\beta(n)}(x) - g_{n\beta(n)}(x)\big\| \\
&\leq \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} \psi_{n\alpha}(x)\Big(\|Df_{n\alpha}(x)\| + \|Dg_{n\alpha}(x)\|\Big) + \sum_{n\in M} L_{n\beta(n)}\,\big\|f_{n\beta(n)}(x) - g_{n\beta(n)}(x)\big\| \\
&< \left(\frac{\varepsilon(x)}{3} + \frac{\varepsilon(x)}{3}\right) \sum_{\substack{n\in\mathbb{N}\\ \alpha\in\Lambda:\, x\in U_{n\alpha}}} \psi_{n\alpha}(x) + \sum_{n\in M} L_{n\beta(n)} \frac{\varepsilon(x)}{3\cdot 2^n L_{n\beta(n)}} \leq \varepsilon(x),
\end{aligned}$$

where the last but one inequality follows from (35), (36), and (37).

□

THEOREM 87. *Let X, Y be normed linear spaces, $k \in \mathbb{N}\cup\{\infty\}$. Consider the following statements:*

(i) *There is $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f: 2U_X \to Y$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X; Y)$ such that $\|f - g\|_{U_X} \leq \varepsilon$.*
(ii) *For any open $\Omega \subset X$ and any open covering $\mathcal{U}$ of Ω there is a C^k-smooth Lipschitz locally finite and σ-discrete partition of unity on Ω subordinated to $\mathcal{U}$. There is $C \in \mathbb{R}$ such that for any L-Lipschitz mapping $f \in C^1(2U_X; Y)$ and any $\varepsilon > 0$ there is a CL-Lipschitz mapping $g \in C^k(U_X; Y)$ such that $\|f - g\|_{U_X} \leq \varepsilon$.*
(iii) *For any open $\Omega \subset X$, any mapping $f \in C^1(\Omega; Y)$, and any continuous function $\varepsilon: \Omega \to \mathbb{R}^+$ there is $g \in C^k(\Omega; Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|Df(x) - Dg(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*
(iv) *For any open $\Omega \subset X$, any L-Lipschitz mapping $f \in C^1(\Omega; Y)$, any continuous function $\varepsilon: \Omega \to \mathbb{R}^+$, and any $\eta > 1$ there is an ηL-Lipschitz mapping $g \in C^k(\Omega; Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

Then (i)⇒(ii)⇒(iii)⇒(iv).

PROOF. (i)⇒(ii) follows from Lemma 77, (ii)⇒(iii) follows from Proposition 86 (consider the collection of subspaces of Y consisting only of the space Y itself), and for (iii)⇒(iv) see the end of the proof of Theorem 78.

□

COROLLARY 88. *Let X be a separable normed linear space that admits a C^k-smooth Lipschitz bump function, $k \in \mathbb{N} \cup \{\infty\}$. Let Y be a Banach space. Suppose further that one of the following conditions is satisfied:*

- *at least one of the spaces X or Y is equal to $B_0(V)$ for some topological space V, or*
- *at least one of the spaces X or Y is equal to $C_{\mathrm{ub}}(P)$ for some metric space P, or*
- *X is a Banach space with an unconditional Schauder basis, or*
- *Y is a Banach space with an unconditional Schauder basis and with a separable dual.*

Then for any open $\Omega \subset X$, any mapping $f \in C^1(\Omega; Y)$, and any continuous function $\varepsilon\colon \Omega \to \mathbb{R}^+$ there is $g \in C^k(\Omega; Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|Df(x) - Dg(x)\| < \varepsilon(x)$ for all $x \in \Omega$.

PROOF. Suppose that one of the first three conditions is satisfied. Then our corollary follows from Theorem 87. It suffices to notice that under our assumptions the statement (i) of Theorem 87 holds. We may either apply Corollary 79, or the less involved Corollary 73 together with the observation that since B_X is a 2-Lipschitz retract of X, every L-Lipschitz mapping defined on B_X can be extended to a $2L$-Lipschitz mapping defined on X.

It remains to prove the case that Y has an unconditional Schauder basis $\{e_n\}$ and has a separable dual (which means that Y admits a C^1-smooth Lipschitz bump function, Theorem 5.2). We will show that statement (ii) in Theorem 87 is satisfied, which will prove our claim. For the first part we use Lemma 77 together with the approximation of Lipschitz functions given by either Corollary 73, or the less involved combination of Theorem 55 and Theorem 56. (Although, since X is separable, it is not overly difficult to construct the required partitions of unity directly.)

To prove the second assertion in statement (ii) of Theorem 87 let K be the constant from Lemma 75 used on the space Y. Put $C = 2K$. Let $f \in C^1(U_X; Y)$ be L-Lipschitz and $\varepsilon > 0$. Denote $Y_n = \operatorname{span}\{e_j\}_{j=1}^n$. By Lemma 75 there is a K-Lipschitz mapping $\Psi \in C^1(Y; Y)$ which locally maps into some Y_n and such that $\|y - \Psi(y)\| < \frac{\varepsilon}{2}$ for every $y \in Y$. Put $h = \Psi \circ f$. Then $h \in C^1(U_X; Y)$ is a KL-Lipschitz mapping which locally maps into some Y_n and such that $\|f - h\|_{U_X} \le \frac{\varepsilon}{2}$. Since the spaces Y_n, $n \in \mathbb{N}$, are finite-dimensional, by Corollary 73 there are constants C_n such that any M-Lipschitz mapping from U_X into Y_n can be approximated by C^k-smooth $C_n M$-Lipschitz mapping. Therefore we can use Proposition 86 to find a CL-Lipschitz mapping $g \in C^k(U_X; Y)$ such that $\|g - h\|_{U_X} \le \frac{\varepsilon}{2}$. As $\|f - g\|_{U_X} \le \varepsilon$, we have just shown that the statement (ii) in Theorem 87 holds.

□

Finally, combining Theorem 66 and Theorem 87 we obtain the following corollary.

COROLLARY 89. *Let Γ be an arbitrary set, Y a Banach space, $\Omega \subset c_0(\Gamma)$ open, $f \in C^1(\Omega; Y)$, and $\varepsilon\colon \Omega \to \mathbb{R}^+$ a continuous function. Then there is $g \in C^\infty(\Omega; Y)$ such that $\|f(x) - g(x)\| < \varepsilon(x)$ and $\|Df(x) - Dg(x)\| < \varepsilon(x)$ for all $x \in \Omega$.*

9. Notes and remarks

In [AS] approximations in the spaces of holomorphic functions under various topologies are studied.

Section 2. Theorem 9 was proved by Karl Weierstraß in 1885 for functions in $C(\mathbb{R})$, by Émile Picard in 1891 for functions in $C(\mathbb{R}^n)$, and by Charles de la Vallée Poussin in 1908–12 for functions in $C^k(\mathbb{R}^n)$. Theorem 12 is due to Stanisław Mazur, Theorem 16 is due to Maurice Fréchet. Theorem 17 was proved by Arkadij Semenovich Nemirovskij.

The origin of some of the polynomial approximation results in infinite-dimensional spaces can be traced to Guillermo Restrepo's work [Re], and also Georgiy Evgenievich Shilov's paper [Sh] where the relevant problem of characterising the closure of the space of continuous polynomials was posed. For results regarding approximations on compact sets together with higher derivatives see e.g. [Pr], [AP], [AS].

Section 3. The real-analytic part of Theorem 19 was proved by Hassler Whitney ([Wh]). Torsten Carleman ([Ca]) proved the approximation of functions in $C(\mathbb{R})$ by entire functions and Stephen Scheinberg generalised it in [Sc] for functions in $C(\mathbb{R}^n)$.

Theorem 20 is a pioneering result of Jaroslav Kurzweil, whose influence on this area cannot be exaggerated. Various versions of Theorem 22 have been proved independently by Robb Fry [Fry1], Manuel Cepedello-Boiso, and Petr Hájek. The last two authors decided to publish a joint paper [CH].

The following is one of the main open problems in this area.

PROBLEM 90. Is every continuous function on c_0 uniformly approximable by real analytic functions?

The method of proof of Theorem 22 is not strong enough to make this conclusion. If the answer is negative, then Theorem 20 would have a converse, via Deville's Theorem 5.67, namely the existence of analytic approximations for all continuous functions on a separable space X would imply that X has a separating polynomial. This fact was also noted, for super-reflexive spaces, in [K2].

Section 4. Lemma 30 is usually attributed to Edward James McShane [McS], but it was known at least two years earlier to H. Whitney [Wh]. For more information on the infimal convolution see e.g. the survey [St]. Extending further the infimal convolution technique M. Cepedello-Boiso obtained the following result.

THEOREM 91 ([Ce]). *Let X be a super-reflexive Banach space. Then there is $\alpha \in (0,1]$ such that for any Lipschitz function $f\colon X \to \mathbb{R}$ and any $\varepsilon > 0$ there is a function $g \in C^1(X)$ with its derivative α-Hölder on bounded sets (and so g is Lipschitz on bounded sets) such that $|f-g|_X \le \varepsilon$.*

Many versions of Proposition 35 exist in the literature, e.g. for compact K it is true for analytic functions on ℓ_p, [Do2]. This is related to the negligibility theory of subsets of Banach spaces, initiated by Czesław Bessaga in [B], and studied in detail e.g. in [Do1], [Do2], [Do3], [Az], [AD], [De], [DH].

Section 5. Most of Lemma 40 was obtained in a seminal work by Henryk Toruńczyk, [T]. The main open problem in this area is the following one.

PROBLEM 92 ([DGZ]). Suppose that a Banach space X admits a C^k-smooth bump function. Does X have C^k-smooth partitions of unity, or equivalently are continuous mappings on X approximable by C^k-smooth mappings?

As we have seen, the answer is positive for some classes of Banach spaces, including separable, WCG, and $C(K)$ spaces. Some earlier results on smooth partitions of unity on $C(K)$ spaces were obtained in [DGZ1]. In [JZ] it is shown that WCG spaces always admit Gâteaux smooth partitions of unity. It is shown in [Fro] that if a Banach space X has an LUR norm and every Lipschitz convex function on X can be approximated by C^k-smooth functions, then X has C^k-smooth partitions of unity.

Recently, several papers focused on the problem of C^k-smooth approximations by functions that lack critical points, or more generally their derivative avoids a prescribed set of values, e.g. [AC], [HJ1], [AJ], [Ji]. This interest was spurred by the result on the existence of bump functions without critical points we referred to earlier.

Section 6. The important technique of approximation of Lipschitz functions using supremal partitions of unity is due to R. Fry [Fry2]. It is closely related, in spirit and technique, to the method of boundaries or generalised boundaries used for obtaining C^k-smooth renormings. It is however a purely scalar method, and so the results concerning C^k-smooth Lipschitz approximations are less satisfactory than the continuous case.

The first part of Fact 54 was shown in [Pe, Proposition 2.3]. Many of the results in Sections 6–8 come from [HJ3].

A natural question is the following.

PROBLEM 93. Let X be a Banach space (e.g. WCG) with a Lipschitz and C^k-smooth bump function. Does X admit C^k-smooth Lipschitz sup-partitions of unity?

Section 7. The method of Lemma 61 originated from [FWZ]. Similar results are for example in [Jo1], [FZ2]. Corollary 64 for $k = 1$ is proved in [FZ1] and its older norm variant (for $k = 1$) is proved in [FWZ]. Theorem 82 is proved in [AFK] under slightly stronger assumptions.

The results given in this section are partial cases of the following general open problem.

PROBLEM 94. Suppose that a Banach space X admits a C^k-smooth and Lipschitz bump function. Are Lipschitz mappings into another Banach space Y approximable by C^k-smooth and Lipschitz mappings?

This problem is open even for a pair of general separable X and Y, or for a general X and $Y = \mathbb{R}$.

Section 8. The implication (i)$\Rightarrow$(iii) in Theorem 87 in the separable case was proved by Nicole Moulis [Mo]. The case of Corollary 88 when X has an unconditional Schauder basis was proved by N. Moulis [Mo], although the result is stated only for c_0 and ℓ_p spaces. This seminal paper has essentially started the line of research into approximations together with derivatives. The motivation for N. Moulis's results apparently comes from the investigation of Banach manifolds, where they find important applications. The only result available for approximations together with higher derivatives comes again from [Mo] and applies only to the Hilbert space. The statement of this result, unlike that of the first order, is nevertheless not very satisfactory, as it requires C^{2k-1}-smoothness for approximation of the kth order derivatives.

The most general form of the approximation problem can be stated as follows.

PROBLEM 95. Let X be a (separable) Banach space admitting a C^k-smooth bump function. Is it true that every C^n-smooth mapping from X into a Banach space Y (or just into $\mathbb{R}$) can be approximated, together with all derivatives of order up to $n < k$, by C^k-smooth mappings?

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