# Charles University in Prague Faculty of Mathematics and Physics 



Habilitation Thesis

# Graphs and mappings 

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## Preface

This habilitation thesis consists of eight papers listed below, together with an introduction that puts them in context of the current research.
[1] Matt DeVos, Bojan Mohar, Robert Šámal: Unexpected behaviour of crossing sequences, Journal of Combinatorial Theory Series B 101 (2011), no.6, 448-463.
[2] Matt DeVos, Agelos Georgakopoulos, Bojan Mohar, Robert Šámal: An EberhardLike Theorem for Pentagons and Heptagons, Discrete \& Computational Geometry 44 (2010), no. 4, 931-945.
[3] Matt DeVos, Luis Goddyn, Bojan Mohar, Robert Šámal: Cayley sum graphs and eigenvalues of (3,6)-fullerenes, Journal of Combinatorial Theory Series B 99 (2009), no. 2, 358-369.
[4] Matt DeVos, Bojan Mohar, Robert Šámal: Highly arc-transitive digraphs - structure and counterexamples, Combinatorica 34 (2014), no. 4, 1-19.
[5] Chris Godsil, David E. Roberson, Robert Šámal, Simone Severini: Sabidussi Versus Hedetniemi for Three Variations of the Chromatic Number, accepted to Combinatorica (Jan 31, 2014)
[6] Robert Šámal: Cubical coloring - fractional covering by cuts \& Semidefinite programming (submitted)
[7] Jaroslav Nešetřil, Robert Šámal: Flow-continuous mappings - The influence of the group, European Journal of Combinatorics 36 (2014), 342-347.
[8] Robert Šámal: Cycle-continuous mappings - order structure, Publications of the Scuola Normale Superiore, Vol. 16, CRM Series. Eurocomb 2013 - Pisa (journal version submitted)

The papers represent my research from various periods. Papers [1-4] have been conceived during my postdoctoral stay at Simon Fraser University, [6-7] are an extension of topics from my Ph.D. thesis obtained when I regained my interest in flow-related questions
after returning to Prague. Finally, [5] and [8] are recent and represent two directions that I plan to pursue further in the years to come.

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But most of all, the success of every research is very much affected by people. I was happy I had met colleagues with whom it was a pleasure to collaborate, their contribution is gratefully acknowledged. Finally, I thank to my family for their extraordinary understanding for the peculiarities that living with a research mathematician brings.

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## Introduction

Discrete mathematics, and in particular graph theory, is for computer science what set theory is for mathematics and mathematics for physics - the underlying language on which everyone relies without thinking. This thesis is about several aspects of graph theory and, in particular, of mappings of graphs. Some results are directly relevant to computer science: using semidefinite programming to analyze/define variants of graph coloring, see Appendix E and F. Other develop structural view on graphs by means of various mappings between graphs, either homomorphisms (Appendix D ) or continuous mappings (Appendix G and H). Finally, some results contribute to the depth of the subject by studying in detail some structural aspects of graph theory - Appendices A, B and C deal with geometric structure of graphs. Frequently, the motivation is multidisciplinary, for instance the result in Appendix C is motivated by a question from mathematical chemistry, while Appendix D is relevant for group theory.

As all of this thesis is concerned with graphs, we start with the basic definitions to fix terminology. More specialized definitions will be introduced later on as needed. A graph consists of a finite set $V$ (whose elements are called vertices) and another set $E$ of edges, each of the edges connecting two vertices. Most of the time we avoid loops (edges connecting a vertex to itself) and parallel edges (multiple edges connecting the same pair of vertices. In this (typical) case, we may consider $E$ to be some set of two-element subsets of $V$, that is $E$ is a subset of $\binom{V}{2}$.

While there are many alleys in the realm of graph theory, our path will be guided by the notion of mappings of graphs - in several, seemingly different, guises.

Many questions we will consider deal with graph coloring of some sort. A coloring of a graph $(V, E)$ is simply a mapping from $V$ to a finite set (elements of which are traditionally called colors), where vertices connected by an edge are required to have different image - color. The chromatic number of a graph $G$ is the minimal number of colors for which a coloring exists, it is denoted $\chi(G)$. A more general way to look at this is by means of graph homomorphisms: a homomorphism from a graph $G$ to a graph $H$ is a mapping $f$ from $V(G)$ to $V(H)$ that preserves edges: if $\{u, v\}$ is an edge of $G$, then $\{f(u), f(v)\}$ is an edge of $H$. This obviously generalizes the concept of graph colorings (we let $H$ be the complete graph $K_{n}$, graph with $n$ vertices and all possible edges). However, it also includes graph theory in more general framework of category theory; we will see some
benefits of this later. Another use of graph homomorphisms that we will see is that it captures the notion of a symmetry of a graph, when we consider invertible homorphisms from $G$ to itself.

## 1 Geometric and structural aspects

Another important point of view is to consider geometric representations of a graph. Probably the oldest notion of this type is that of plane embedding: vertices of the graph are mapped to points in a plane, edges to continuous curves between the corresponding vertices. No curve corresponding to an edge contains interior point of another curve. A graph for which such embedding is possible is called a planar graph. Given a particular embedding, we call a face a connected component of the plane with the embedded edges removed. (This alludes to the original motivation to study planar graphs, as a way to represent 3D polytopes.) The boundary (in the topological sense of the word) of a face is formed by edges, that we also call the boundary of the face. When tracing a closed curve inside the face near its boundary, we are approaching the boundary walk. The number of edges in this walk is the length of the face.

Various modifications of this are possible, we mention two natural ones. We may be looking for embedding into surfaces different from a plane (usually compact twodimensional manifolds without border, for our purposes we mainly need torus (orientable surface of genus 1) and double torus (orientable surface of genus 2). Another notion relaxes the condition that curves do not cross: instead, we try to minimize the number of crossings. Minimal such count shall be called the crossing number of the graph $G$ and denoted by $\operatorname{cr}(G)$.

Both these notions (crossing numbers and embedding into surfaces) have been studied to a great extent. Surprisingly little is known about the combination of these two approaches: We will draw a graph on various surfaces and try to minimize the number of crossings. This idea was first considered by Siráň [21]. He showed that for any decreasing convex sequence $\left(a_{i}\right)$ of integers that is eventually zero exists a graph $G$ such that $a_{i}$ is the minimal number of crossings needed to draw $G$ on an orientable surface of genus $i$ (we will write shortly $c r_{i}(G)=a_{i}$ ).

The convexity requirement says that the differences $a_{i}-a_{i+1}$ are non-increasing. Thus, the improvements in the number of crossings that we get from increasing the genus by 1 is a non-increasing sequence - this enables a natural construction, where each of these improvements are done locally in separate parts of the graph.

It was, and still is, very much open, what is the general characterization: for what sequences $a_{0}>a_{1}>\cdots>a_{k}=0$ is there a graph $G$ such that $c r_{i}=a_{i}$. Archdeacon et al. [1] conjectured that we can find such graph $G$ for every decreasing sequence $\left(a_{i}\right)$.

On the other hand, Salazar [18] considered this unlikely, for the following (heuristic) reason due to Dan Archdeacon: Take $a_{0}=N$ (large), $a_{1}=N-1, a_{2}=0$. We get
the double-torus from plane by adding two handles. (Adding a handle means cutting the surface by two circles, removing their interior and adding a tube connecting them.) And, somehow, adding the first handle saved us from just a single crossing, while adding the second one saved $N-1$ of them. So why not just add the second one first?

Of course, this line of reasoning is flawed, one can construct a graph, where the two handles work together very efficiently, while adding just a single one is not worth much. We did exactly this in the paper

> Matt DeVos, Bojan Mohar, Robert Šámal: Unexpected behaviour of crossing sequences, Journal of Combinatorial Theory Series B 101 (2011), no.6, 448463.

The key to the proof is of course a clever (albeit simple) construction. The proof itself, however, is by no means trivial, as one needs to discuss all possibilities to draw our graph on the torus. This involves using several standard gadgets to tame the possibilities how to draw the graph and then discussing homotopy types of closed curves in a surface. It is worthwhile to note, that it is an open problem to determine the crossing number of very simple graphs (such as the complete graph $K_{13}$ ). Thus, we need a specially crafted graph that is easier to analyze.

Recently, McConvey [16] has extended our research in his Ph.D. thesis. He is able to create graphs with long non-convex crossing sequence, i.e., he can control the number of crossings on surfaces of higher genus. However, the complete solution is still elusive.

An obvious collection of numbers associated with a graph $G$ is the degree sequence: sequence $(\operatorname{deg}(v))_{v \in V}$, where $\operatorname{deg}(v)$ is the degree of $v$, i.e., the number of edges containing $v$. It has been known for decades (Havel-Hakimi algorithm, or Erdős-Gallai theorem) how to tell whether a sequence of integers is the degree sequence of some graph. More complicated considerations characterize degree sequences of split graphs, $C_{4}$-minor free graphs, unicyclic graphs, cacti graphs, and Halin graphs. Despite much effort, no such characterization is known for planar graphs. Our next paper can be understood as a partial attempt to address this question.

To be consistent with the literature, we address an important case of the dual question: we get a sequence $\left(p_{k}\right)_{k \geq 1}$ of integers and ask whether there exists a connected 3-regular plane graph with $p_{k}$ faces of length $k$. There is an obvious restriction: if such graph exists then $\sum_{k \geq 1} k p_{k}$ is twice the number of edges, thus even. Furthermore, Euler formula implies that

$$
\begin{equation*}
\sum_{k \geq 1}(6-k) p_{k}=12 \tag{1}
\end{equation*}
$$

otherwise such graph does not exist. We say a sequence $\left(p_{k}\right)$ is plausible if it satisfies (1). A natural question is whether all plausible sequences correspond to a 3-regular plane graph

- or how to strengthen the condition to make it a sufficient condition. It turns out the question is not so simple, in fact it is still open after many decades of work. The beginning of this research goes back to Eberhard in 1891 [4]; he proved that every plausible sequence can be realized by a 3-regular plane graph, provided we are allowed to change the value of $p_{6}$ (number of hexagonal faces). To date, this is still the most general result in this direction, recent progress is summarized in [9]. We contribute to this line of research in our paper

> Matt DeVos, Agelos Georgakopoulos, Bojan Mohar, Robert Šámal: An EberhardLike Theorem for Pentagons and Heptagons, Discrete \& Computational Geometry 44 (2010), no. 4, 931-945.

We did show an extension of Eberhard's result: any plausible sequence can be realized if we replace $p_{5}$ by $p_{5}+x$ and $p_{7}$ by $p_{7}+x$ for some integer $x$. We conjecture that this holds more generally, namely for any sequence of non-negative integers $\left(q_{k}\right)$ satisfying $\sum_{k \geq 1}(6-k) q_{k}=0$ and any plausible sequence $p_{k}$, the sum $\left(p_{k}+x q_{k}\right)$ can be realized; this is left for further research, though. Eberhard's result is a special case of this for sequence $\left(q_{k}\right)$ satisfying $q_{6}=1$ (and zero elsewhere), our result deals with the case $q_{5}=q_{7}=1$ (and zero elsewhere). It may perhaps seem that this extension is a mild one; it was, however, an important test case of the above more general conjecture. It also shows the flexibility of the construction; in particular, it applies to any surface besides the plane (unlike the Eberhard's result that is not valid for torus).

Matt DeVos, Luis Goddyn, Bojan Mohar, Robert Šámal: Cayley sum graphs and eigenvalues of (3,6)-fullerenes, Journal of Combinatorial Theory Series B 99 (2009), no. 2, 358-369.

In this paper we solve a question of Fowler from 1995 [6] that belongs to the field of mathematical chemistry. Hückel's model of hydrocarbons (LCAO-MO theory) is a tool to estimate stability of various hydrocarbon molecules. The mathematical part of this leads to the need of computing the eigenvalues and eigenvectors of the hydrogen-suppressed molecular graph; they correspond to the energy levels and orbitals of the molecule, respectively. Therefore, a (simple) method that enables these quantities to be determined is required.

A particularly important class of such graphs is that corresponding to fullerenes. From mathematical point of view, these are planar 3-regular graphs with four triangles and arbitrary many hexagons. We need to compute the eigenvalues and eigenvectors of such a graph, that is, of its adjacency matrix (adjacency matrix of a graph is a square matrix with columns and rows indexed by the vertices and with all entries 0 or 1 , the 1 's are in positions that correspond to edges of the graph). Fowler observed by numerical experiments, that
adjacency matrix of a fullerene has always a special type of spectrum (set of eigenvalues): There are special eigenvalues $\{3,-1,-1,-1\}$ and all other eigenvalues come in pairs $\pm \lambda$. He asked [6] whether this is, indeed, always true. We answered this affirmatively.

The proof consists of nice structural and geometric considerations: we construct a universal cover of our fullerene (which is the infinite hexagonal lattice), consider its embedding in the plane and then discuss how our graph is obtained by a suitable factorization of this hexagonal grid. This factorization allows us to realize every fullerene as a Cayley sum graph for an abelian group which can be generated by two elements. Using characters of the underlying group, we can then exactly describe the eigenvalues of the graph. This is possible to do efficiently for a given graph using the Smith normal form of a matrix. As a result, we not only prove Fowler's conjecture, but also describe all possible spectra of fullerenes and give an easy algorithm to compute the spectrum.

The underlying idea of the previous paper was to understand and use symmetries of a given graph. In the next paper we take a different view on the topic, discussing graphs that posses as much symmetry as possible. For a digraph (graph with directed edges, usually called arcs), an $s$-arc is an $s$-tuple of arcs that are consecutive in a directed walk (in the interesting cases, a directed path of length $s$ ). A digraph $D$ is $s$-arc transitive, if any two $s$-arcs are the same. Formally, for any pair of $s$-arcs there is an automorphism of $D$ that maps one $s$-arc to the other. Finally, a digraph is highly arc transitive, if it is $s$-arc transitive for every $s \geq 0$. To exclude trivialities, we assume that each vertex has at least one incoming and at least one outgoing arc. With the exception of (disjoint unions of) cycles, every highly arc transitive digraph (or a HAT for short) is infinite.

The study of HATs is an enticing endeavor, both from the graph theory and group theory perspective. Perhaps the main reason is that the notion is rather strict (indeed, it is hard to imagine a nontrivial example of a HAT at first), which gives hope for a complete description. At the same time, there are surprising constructions of HATs that use interesting techniques on the graph and group theory side (such as horocyclic product of graphs and lamplighter product of groups).

The notion was defined by Cameron, Praeger, and Wormald [2] in 1993. The authors provide several nontrivial constructions and structural properties. They also posed several questions aimed at testing from various points of view, whether every HAT is (close to) one of the presented constructions. After an intensive research (over 25 citations in WoS) the structure of HATs was much better understood, but some of the original questions still remained without an answer. In our paper

Matt DeVos, Bojan Mohar, Robert Šámal: Highly arc-transitive digraphs structure and counterexamples, Combinatorica 34 (4) (2014), 1-19.
we resolve two of these questions. The first of them concerns the notion of reachability - an equivalence relation on arcs, defined by declaring two arcs equivalent if they share a
head or a tail and taking the transitive closure. If a HAT has more than one equivalence class of the reachability relation then this can be used to describe its structure. It was known previously, that if a HAT has only one reachability class, then there is a $d$ such that every vertex has in-degree and out-degree $d$, and $d$ is not a prime. However, such graphs were only known to exist for $d$ infinite. We settle the question completely by constructing a HAT with in- and out-degree $d$ and universal reachability relation for every $d$ that is not a prime.

The second question concerns HATs with two ends (roughly speaking, with a linear structure). It was conjectured, that each such digraph can be built in a controlled way from complete bipartite digraphs. More precisely, each class of the reachability relation was conjectured to be formed by arcs of a complete bipartite digraph. We disprove this by providing two more general constructions. We also work towards describing all HATs with two ends, but the complete characterization remains out of reach.

## 2 Variants of coloring

For our next results we need to define variants of graph coloring, so-called vector coloring and strict vector coloring, as defined by Karger, Motwani, and Sudan [13]. Given a graph $G$, a mapping $f: V(G) \rightarrow \mathbb{R}^{n}$ (for $n=|V(G)|$, say) is called a vector $k$-coloring (for a real $k \geq 2$ ) if $\|f(v)\|=1$ for every vertex $v$ and $\langle f(u), f(v)\rangle \leq-\frac{1}{k-1}$ for every edge $u v$. For strict vector $k$-coloring we demand equality instead of inequality here. It is easy to see that $K_{n}$ (a graph with $n$ vertices and all possible edges) has a vector (and also strict vector) $n$-coloring using vertices of a simplex; it is not hard to show that no better solution exists. Thus, one may be inclined to define the (strict) vector chromatic number of $G$ to be the minimal $k$ such that (strict) vector $k$-coloring exists. Compactness of the unit sphere implies that this minimum always exists, we will use $\chi_{v}(G)$ (or $\chi_{s v}(G)$, respectively) to denote these numbers. It is not hard to show that

$$
\omega(G) \leq \chi_{v}(G) \leq \chi_{s v}(G) \leq \chi(G)
$$

where $\omega(G)$ is the maximum number of pairwise adjacent points in $G$.
These parameters, in particular $\chi_{s v}$ have a long history behind them. Under the name theta function of the complement graph, $\vartheta(\bar{G})$, it was defined by Lovász [15] as a tool to solve the problem of computing the Shannon capacity of $C_{5}$. Later it played an important role in research concerning perfect graphs and semidefinite optimization. Finally, Karger, Motwani, and Sudan [13] (independently of Lovász) defined $\chi_{v}$ to apply semidefinite optimization approach of Goemans and Williamson [8] to graph coloring; later they noticed their parameter is closely related to Lovász's $\vartheta$. In fact, the strict vector chromatic number is exactly equal to the Lovász's $\vartheta$ of the complement graph, while vector chromatic number is equal to less well-known number $\vartheta^{\prime}$ defined by Schrijver and (independently)
by McEliece, Rodemich, and Rumsey; illustrating the fact that important notions tend to be discovered over and over again.

In many instances one can use $\chi_{s v}$ or $\chi_{v}$ to find estimates of $\chi$ or $\omega$. Somewhat surprisingly, it is much easier to compute (estimate) these parameters than the chromatic number $\chi$ or the clique number $\omega$. For any $\varepsilon>0$ one can approximate $\chi_{v}$ or $\chi_{s v}$ in time polynomial in the size of the graph and $\log 1 / \varepsilon$ with additive error at most $\varepsilon$. On the other hand, it is impossible to approximate $\chi$ or $\omega$ with multiplicative error smaller then $n^{1-\varepsilon}$, unless $P=N P$.

Unlike previous researchers, who concentrated on the utilitarian function of $\chi_{v}$ and $\chi_{s v}$, we study them more as a graph parameter per se. To describe our results we need to pause to define the notion of categorical graph product. Given two graphs $G, H$, there are many ways to create their product. We will discuss the definition that is also natural from point of view of category theory (indeed, it is the product in the category of graphs with graph homomorphisms as morphisms).

The vertex set of the product $G \times H$ is the Cartesian product of vertex sets of the factors, $V(G \times H)=V(G) \times V(H)$. The edges correspond to pairs that form an edge in both coordinates. That is, $\left(u_{1}, v_{1}\right)$ is connected by an edge to $\left(u_{2}, v_{2}\right)$ whenever $u_{1} u_{2} \in$ $E(G)$ and $v_{1} v_{2} \in E(H)$. It is immediate that any graph coloring of $G$ using $k$ colors yields a coloring of $G \times H$ using $k$ colors (color vertex $(u, v)$ by the color of $u$ in the coloring of $G$ ). This implies $\chi(G \times H) \leq \chi(G)$. In the same way, $\chi(G \times H) \leq \chi(H)$. Hedetniemi [10] conjectured that one of these bounds always applies, that is, $\chi(G \times H)=$ $\min \{\chi(G), \chi(H)\}$. Despite many attempts (see [23] and the references within), the only affirmative answer is for $\chi(G), \chi(H) \leq 4$.

> Chris Godsil, David E. Roberson, Robert Šámal, Simone Severini: Sabidussi Versus Hedetniemi for Three Variations of the Chromatic Number, accepted to Combinatorica (Jan 31, 2014)

In this paper we proved the result for a related graph parameter, namely for the strict vector coloring $\chi_{s v}$. Perhaps surprisingly, the proof involves two other notions of graph product. Less surprising is the occurrence of tensor products (of vectors), as these correspond naturally to the categorical product of graphs. We also discuss connections with quantum coloring and spectrum of 1-homogeneous graphs. In a follow-up work (still in progress) we extended the result also for $\chi_{v}$, and resolved the relation of the two variants of vector coloring: previously it was open, whether the fraction $\chi_{s v}(G) / \chi_{v}(G)$ is bounded or not; we proved that the latter case is true.

I will conclude this overview by discussing three papers related to continuous mappings between graphs. Unlike graph homomorphisms, continuous mappings operate on edges. We say a mapping $f: E(G) \rightarrow E(H)$ is cut-continuous if for every set $X$ of edges
in $H$ that is a cut, the preimage $f^{-1}(X)$ is a cut. Here, a cut is a set of edges of form $\delta(U)$ - all edges leaving some set of vertices of the graph.

Similarly, $f$ is cycle-continuous, if the preimage of every cycle is a cycle. A cycle means simply a set of edges with all degrees even - some authors call this an even subgraph. In the defined sense, cuts and cycles are dual notions (actually in two different ways: by means of dual plane graphs and as orthogonal vector spaces), giving the subject much of its richness.

Similarly as for homomorphisms, both cut- and cycle-continuous mappings form a category, but unlike graphs with homomorphisms, this is a category without products [19]. Back in my Ph.D. thesis I thoroughly analyzed cut-continuous mappings and their relation to graph homomorphisms. Surprisingly, there is a close connection between these.

## Robert Šámal: Cubical coloring - fractional covering by cuts \& Semidefinite programming (submitted)

In this paper I investigated another type of graph coloring, the cubical coloring, that measures how efficiently can we cover all edges of a graph by cuts. More precisely: we want to find a collection of $n$ cuts (in the sense defined above) such that every edge is in at least $k$ of them; we want to do this so that we minimize the ratio $k / n$ and denote this $x(G)$. The rescaling $\chi_{q}(G)=\frac{2}{2-x(G)}$ is called the cubical chromatic number of $G$; it shares many properties with the usual chromatic number. Besides being an interesting parameter in its own right, it is useful as an invariant that is monotone with respect to cut-continuous mappings. With this intention, I started the study of $\chi_{q}$ in my Ph.D. thesis, the paper contains new developments:

- In my Ph.D. thesis I stated a conjecture about the behaviour of cubical chromatic number on its "natural scale": The property of having a $k$-fold cover by $n$ cuts can be equivalently characterized by existence of a homomorphism to a graph $Q_{n / k}$ - the vertices are all binary strings of length $n$, edge corresponds to Hamming distance at least $k$. In parallel with the properties of the normal chromatic number (it can be defined by homomorphisms to complete graphs $K_{n}$ and $\chi\left(K_{n}\right)=n$ ), of fractional chromatic number (defined by homomorphisms to Kneser graphs $K(n, k)$ and $\left.\chi_{f}(K(n, k))=n / k\right)$, etc., we demand analogous property here: namely that $x\left(Q_{n / k}\right)=n / k$. In cooperation with Engström et al. [5, 7]) this is now resolved for all cases that satisfy an obvious necessary condition.
- Further, it is possible to use semidefinite programming to find a relation between cubical chromatic number and vector chromatic number (as defined earlier). This allows for a polynomial-time algorithm to approximate the cubical chromatic number up to a constant factor.

As already hinted, the theory of continuous mappings between graphs is a rich one with promising developments. However, there are more reasons to study these mappings, in particular the cycle-continuous ones. Many central conjectures in graph theory are (equivalent to) questions about covering the edges by cycles. As an example we mention the Cycle double cover conjecture [20,22]: for every bridgeless graph there is a collection of cycles such that every edge is in precisely two of them. If the graph is planar, one may use the cycles that form the face boundaries. However, for non-planar graphs this idea does not help: even though one may embed every graph in some surface, in general the face boundaries will not be cycles. While this suggests an approach (choose a better embedding), it is by no means clear whether it is a viable one. Another approach (suggested by Jaeger in [12], by Linial, Meshulam, and Tarsi in [14] and further pursued by DeVos, Nešetřil, and Raspaud [3]) is to use cycle-continuous mappings. A cycle-continuous mapping from $G$ to $H$ enables us to "reduce" the problem to find double cover by cycles in $G$ to doing the same in $H$ : as we then can take preimages of the cycles and have the problem solved for $G$ as well. Because of this, the following conjecture of Jaeger is of utmost interest: For every bridgeless graph $G$ there is a cycle-continuous mapping from $G$ to the Petersen graph. (The Petersen graph - see Fig. 1 - is the prototypical testing example for conjectures in the area, Cycle double cover conjecture is true for it.)


Figure 1: The Petersen graph.
In [3] the concept of cycle-continuous mappings was in fact studied in greater generality as group-flow continuous mappings. The underlying groups were mostly either $\mathbb{Z}_{2}$ and $\mathbb{Z}$. The case of $\mathbb{Z}_{2}$ corresponds naturally to the cycle-continuous mappings, while group $\mathbb{Z}$ allows to study questions related to orientations.

In the paper
Jaroslav Nešetřil, Robert Šámal: Flow-continuous mappings - influence of the group European Journal of Combinatorics 36 (2014), 342-347.
we study what happens in other cases. In general, we find that there is a nice algebraic structure describing for what groups $A$ an $A$-flow continuous mapping between two given
graphs exists. On the other hand, for cubic graphs, which are most relevant to the original motivation, we show that one can restrict to the case when $A$ is one of $\mathbb{Z}_{2}, \mathbb{Z}_{3}$ and $\mathbb{Z}$.

Robert Šámal: Cycle-continuous mappings - order structure, Publications of the Scuola Normale Superiore, Vol. 16, CRM Series. Eurocomb 2013 - Pisa. (journal version submitted)

In this paper I prove two results that give further evidence to Jaeger's conjecture being hard (that is, besides the fact that a positive answer would imply many other long-standing conjectures). I study the structure of cycle-continuous mappings. If the structure was easy, say if all antichains were finite, then this would make the conjecture a finite problem (possibly large, though) and thus arguably a simple one. Indeed, we want to show that the underlying partially ordered set has only one maximal element, the Petersen graph. Equivalently, we want to show, that the antichain formed by all maximal elements is of size one. Showing this antichain is finite would be a good first step.

However, all such hopes are dashed by this paper. First, answering a question by DeVos et al., I prove that there is an infinite antichain: an infinite set of graphs with no cycle-continuous mapping between any two of them. Second, any countable partially ordered set can be represented by a collection of finite 3-regular graphs, with existence of a cycle-continuous mapping as the ordering relation.

This mimics the development done in the study of graph homomorphisms. Back in 1980's, Pultr and Trnková [17] proved that any concrete category can be represented using graphs and their homomorphisms. Later, people started to study whether smaller classes of graphs suffice to represent any category, or a poset. In this paper I use a recent result of Hubička et al. [11] on representing posets by homomorphisms of finite directed paths. However, the application is by no means straightforward. Indeed, on the first sight, cycle-continuous mappings behave rather differently than homomorphisms (nonexistence of products, lack of locality) and it takes special constructions using particular properties of snarks to control the behaviour of these mappings. This paper really defines a new line of research: as for the homomorphisms, it makes very good sense to study the same question for restricted class of graphs. This time, a reasonable class would be cubic graphs with no nontrivial 3-edge cut, as many conjectures in the area can be restricted to this class. As usual in mathematical research, answering a question opens several new ones, which is a good thought to close the introduction with.

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## Appendix A

Crossing sequences

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# Unexpected behaviour of crossing sequences 

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#### Abstract

The $n$th crossing number of a graph $G$, denoted $c r_{n}(G)$, is the minimum number of crossings in a drawing of $G$ on an orientable surface of genus $n$. We prove that for every $a>b>0$, there exists a graph $G$ for which $c r_{0}(G)=a, c r_{1}(G)=b$, and $c r_{2}(G)=0$. This provides support for a conjecture of Archdeacon et al. and resolves a problem of Salazar.


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## 1. Introduction

Planarity is ubiquitous in the world of structural graph theory, and perhaps the two most obvious generalizations of this concept-crossing number, and embeddings in more complicated surfaces-are topics which have been thoroughly researched. Despite this, relatively little work has been done on the common generalization of these two: crossing numbers of graphs drawn on surfaces. This subject seems to have been introduced in [6], and studied further in [1]. Following these authors, we define for every nonnegative integer $i$ and every graph $G$, the $i$ th crossing number, $c r_{i}(G)$, (and also the $i$ th nonorientable crossing number, $\tilde{c}_{i}(G)$ ) to be the minimum number of crossings in a drawing of $G$ on the orientable (nonorientable, respectively) surface of genus $i$. We consider drawings where each vertex $x$ of $G$ is represented by a point $\phi(x)$ of the surface, each edge $u v$ by a curve with ends at points $\phi(u)$ and $\phi(v)$ and with interior avoiding all points $\phi(x)$ for $x \in V(G)$. Moreover, we assume that no three edges are drawn so that they have an interior point in common. Observe that $c r_{i}(G)=0$ (respectively, $\tilde{c} r_{i}(G)=0$ ) if and only if $i$ is greater or equal to the genus

[^0](resp., nonorientable genus) of $G$. This gives, for every graph $G$, two finite sequences of integers, $\left(c r_{0}(G), c r_{1}(G), \ldots, 0\right)$ and ( $\left.\tilde{c}_{0}(G), \tilde{c} r_{1}(G), \ldots, 0\right)$, both of which terminate with a single zero. The first of these is the orientable crossing sequence of $G$, the second the nonorientable crossing sequence of $G$.

A natural question is to characterize crossing sequences of graphs. This is the focus of both [6] and [1]. If we are given a drawing of a graph in a surface $\mathcal{S}$ with at least one crossing, then modifying our surface in the neighborhood of this crossing by either adding a crosscap or a handle gives rise to a drawing of $G$ in a higher genus surface with one crossing less. It follows from this that every orientable and nonorientable crossing sequence is strictly decreasing until it hits 0 . This necessary condition was conjectured to be sufficient in [1].

Conjecture 1.1 (Archdeacon, Bonnington, and Širáň). If $\left(a_{1}, a_{2}, \ldots, 0\right)$ is a sequence of integers which strictly decreases until 0 , then there is a graph whose crossing sequence (nonorientable crossing sequence) is ( $a_{1}, a_{2}, \ldots, 0$ ).

To date, there has been very little progress on this appealing conjecture. For the special case of sequences of the form ( $a, b, 0$ ), Archdeacon, Bonnington, and Širáň [1] constructed some interesting examples for both the orientable and nonorientable cases. We shall postpone discussion of their examples for the oriented case until later, but let us highlight their result for the nonorientable case here.

Theorem 1.2 (Archdeacon, Bonnington, and Šíáň). If $a$ and $b$ are integers with $a>b>0$, then there exists $a$ graph $G$ with nonorientable crossing sequence ( $a, b, 0$ ).

It has been believed by some that such a result cannot hold for the orientable case. For the most extreme special case ( $N, N-1,0$ ), where $N$ is a large integer, Salazar asked [5] if this sequence could really be the crossing sequence of a graph. The following quote of Dan Archdeacon illustrates why such crossing sequences are counterintuitive:

If $G$ has crossing sequence ( $N, N-1,0$ ), then adding one handle enables us to get rid of no more than a single crossing, but by adding the second handle, we get rid of many. So, why would we not rather add the second handle first?

Our main theorem is an analogue of Theorem 1.2 for the orientable case, and its special case $a=N$, $b=N-1$ resolves Salazar's question [5].

Theorem 1.3. If $a$ and $b$ are integers with $a>b>0$, then there exists a graph $G$ whose orientable crossing sequence is $(a, b, 0)$.

Quite little is known about constructions of graphs for more general crossing sequences. Next we shall discuss the only such construction we know of. Consider a sequence $\mathbf{a}=\left(a_{0}, a_{1}, \ldots, a_{g}\right)$ and define the sequence $\left(d_{1}, \ldots, d_{g}\right)$ by the rule $d_{i}=a_{i-1}-a_{i}$. If $\mathbf{a}$ is the crossing sequence of a graph, then, roughly speaking, $d_{i}$ is the number of crossings which can be saved by adding the $i$ th handle. It seems intuitively clear that sequences for which $d_{1} \geqslant d_{2} \geqslant \cdots \geqslant d_{g}$ should be crossing sequences, since here we receive diminishing returns for each extra handle we use. Indeed, Širáň [6] constructed a graph with crossing sequence a whenever $d_{1} \geqslant d_{2} \geqslant \cdots \geqslant d_{g}$.

Constructing graphs for sequences which violate the above condition is rather more difficult. For instance, it was previously open whether there exist graphs with crossing sequence ( $a, b, 0$ ) where $a / b$ is arbitrarily close to 1 . The most extreme examples are due to Archdeacon, Bonnington and Širáñ [1] and have $a / b$ approximately equal to $6 / 5$. Although our main theorem gives us a graph with every possible crossing sequence of the form ( $a, b, 0$ ), we don't know what happens for longer sequences. In particular, it would be nice to resolve the following problem which asks for graphs where the first $s$ handles save only an epsilon fraction of what is saved by the $s+1$ st handle.

Problem 1.4. For every positive integer $s$ and every $\varepsilon>0$, construct a graph $G$ for which $c r_{0}(G)-$ $c r_{s}(G) \leqslant \varepsilon\left(c r_{s}(G)-c r_{s+1}(G)\right)$.

For graph embeddings, the genus of a disconnected graph is the sum of the genera of its connected components. For drawing, this situation is presently unclear. If we have a graph which is a disjoint union of $G_{1}$ and $G_{2}$, then we can always "use part of the surface for $G_{1}$ and the other part for $G_{2}$ ", leading to

$$
c r_{i}\left(G_{1} \cup G_{2}\right) \leqslant \min _{j}\left(c r_{j}\left(G_{1}\right)+c r_{i-j}\left(G_{2}\right)\right) .
$$

To the best of our knowledge, this inequality might always be an equality. More generally we shall pose the following problem.

Problem 1.5. Let $G$ be a disjoint union of the graphs $G_{1}$ and $G_{2}$, and let $\mathcal{S}$ be a (possibly nonorientable) surface. Is there an optimal drawing of $G$ on $\mathcal{S}$, such that no edge of $G_{1}$ crosses an edge of $G_{2}$ ?

This problem is trivially true when $\mathcal{S}$ is the plane, but it also holds when $\mathcal{S}$ is the projective plane:
Proposition 1.6. Let $G$ be a disjoint union of the graphs $G_{1}$ and $G_{2}$. Then

$$
\tilde{c}_{1}(G)=\min \left\{\tilde{c}_{1}\left(G_{1}\right)+c r_{0}\left(G_{2}\right), c r_{0}\left(G_{1}\right)+\tilde{c}_{1}\left(G_{2}\right)\right\} .
$$

In other words, there is an optimal drawing of $G$ where planar drawing of $G_{2}$ is put into one of the regions defined by the drawing of $G_{1}$; or vice versa.

Proof. To see this, consider an optimal drawing of $G$ on the projective plane, and suppose (for a contradiction) that some edge of $G_{1}$ crosses an edge of $G_{2}$. If there is a crossing involving two edges in $G_{1}$, then by creating a new vertex at this crossing point, we obtain an optimal drawing of this new graph. Continuing in this manner, we may assume that both $G_{1}$ and $G_{2}$ are individually embedded in the projective plane. For $i=1,2$, let $a_{i}$ be the length of a shortest noncontractible cycle in the dual graph of the embedding of $G_{i}$. Note that $a_{i} \geqslant 2$ as otherwise $G_{i}$ embeds in the plane, so $G$ embeds in the projective plane. Assume (without loss) that $a_{1} \leqslant a_{2}$. Now, it follows from a theorem of Lins [2] that there exists a half-integral packing of noncontractible cycles in $G_{i}$ with total weight $a_{i}$ for $i=1,2$. Since any two noncontractible curves in the projective plane meet, it follows that the total number of crossings in this drawing is at least $a_{1} a_{2}$. However, we can draw $G$ in the projective plane by embedding $G_{2}$ and then drawing $G_{1}$ in a face of this embedding with a total of $\binom{a_{1}}{2}=$ $\frac{1}{2} a_{1}\left(a_{1}-1\right)<a_{1} a_{2}$ crossings, a contradiction.

Our primary family of graphs used in proving Theorem 1.3 can be constructed with relatively little machinery, so we shall introduce them here. We will however use a couple of gadgets which are common in the study of crossing numbers [1,4]. Let us pause here to define them precisely. A special graph is a graph $G$ together with a distinguished subset $T \subseteq E(G)$ of thick edges, a subset $U \subseteq V(G)$ of rigid vertices and a family $\left\{\pi_{u}\right\}_{u \in U}$ of prescribed local rotations for the rigid vertices. Here, $\pi_{u}$ describes the cyclic ordering of the ends of edges incident with $u$. A drawing of a special graph $G$ in a surface $\Sigma$ is a drawing of the underlying graph $G$ with the added property that for every $u \in U$, the local rotation of the edges incident with $u$ given by this drawing either in the local clockwise or counterclockwise order matches $\pi_{u}$. The crossing number of a drawing of the special graph $G$ is $\infty$ if there is an edge in $T$ which contains a crossing, and otherwise it is the same as the crossing number of the drawing of the underlying graph. We define the crossing number of a special graph $G$ in a surface $\Sigma$ to be the minimum crossing number of a drawing of $G$ in $\Sigma$, and $c r_{i}(G)$ to be the crossing number of $G$ in a surface of genus $i$. In the next section, we shall prove the following result.

Lemma 1.7. If $G$ is a special graph with crossing sequence $\mathbf{a}$ consisting of real numbers, then there exists an (ordinary) simple graph with crossing sequence $\mathbf{a}$.


Fig. 1. The graph $H_{n}$ (for $n=6$ ).
This result permits us to use special graphs in our constructions. Indeed, starting in the third section, we shall consider special graphs on par with ordinary ones, and we shall drop the term special. When defining a (special) graph with a diagram, we shall use the convention that thick edges are drawn thicker, and vertices which are marked with a box instead of a circle have the distinguished rotation scheme as given by the figure. With this terminology, we can now introduce our principal family of graphs.

The $n$th hamburger graph $H_{n}$ is a special graph with $3 n+8$ vertices. Its thick edges form a cycle $C=q v_{1} \ldots v_{n} r r^{\prime} s^{\prime} s u_{n} \ldots u_{1} t t^{\prime} q^{\prime} q$ of length $2 n+8$ together with two additional thick edges $\tau_{0}=q r$ and $\tau_{1}=s t$. See Fig. 1. In addition to these, $H_{n}$ has $n$ special vertices $u_{i}^{\prime}$ (for odd values of $i$ ) and $v_{i}^{\prime}$ (for even values of $i$ ) with rotation as shown in the figure. These vertices are of degree 4 and they lie on paths $r_{1}=q^{\prime} v_{2}^{\prime} v_{4}^{\prime} \ldots v_{m}^{\prime} r^{\prime}$ (where $m=n$ if $n$ is even and $m=n-1$ otherwise) and $r_{2}=t^{\prime} u_{1}^{\prime} u_{3}^{\prime} \ldots u_{1}^{\prime} s^{\prime}$ (where $l=n$ if $n$ is odd and $m=n-1$ otherwise). These two paths will be referred to as the rows of $H_{n}$. Each $u_{i}^{\prime}$ and each $v_{i}^{\prime}$ is adjacent to $u_{i}$ and $v_{i}$, and the 2-path $c_{i}=u_{i} u_{i}^{\prime} v_{i}$ (or $c_{i}=u_{i} v_{i}^{\prime} v_{i}$, depending on the parity of $i$ ) is called a column of $H_{n}, i=1, \ldots, n$.

We claim that the hamburger graph $H_{n}$ has crossing sequence ( $n, n-1,0$ ) whenever $n \geqslant 5$ (or $n=3$ ). Although this does not handle all possible sequences of the form ( $a, b, 0$ ), as discussed above, these are in some sense the most difficult and counterintuitive cases. Indeed, a rather trivial modification of these will be used to get all possible sequences.

Since it is quite easy to sketch proofs of $c r_{0}\left(H_{n}\right)=n$ and $c r_{2}\left(H_{n}\right)=0$, let us pause to do so here (rigorous arguments will be given later). The first of these equalities follows from the observation that every row must meet every column in any planar drawing in which thick edges are crossing-free. The second equality follows from the observation that $H_{n}$ minus the thick edges $\tau_{0}, \tau_{1}$ is a graph which can be embedded in the sphere. Using an extra handle for each of $\tau_{0}, \tau_{1}$ gives an embedding of the whole graph in a surface of genus 2 . Of course, it is possible to draw $H_{n}$ in the torus with only $n-1$ crossings by starting with the drawing in the figure and then adding a handle to remove one crossing. In the third section we shall show that these are indeed optimal drawings (for $n=3$ and $n \geqslant 5$ ).

## 2. Gadgets

The goal of this section is to establish Lemma 1.7 which permits us to use special graphs in our constructions. Similar gadgets as used in our proof have been used previously, cf., e.g., Pelsmajer et al. [4] or Archdeacon et al. [1]. We include the constructions and proofs for reader's convenience.

### 2.1. Thick edges

For every $e \in E(G)$ choose positive integer $w(e)$ and replace $e$ by a graph $L_{e}$ (see Fig. 2) with $w(e)$ new vertices. Let $G^{\prime}$ be the resulting graph. We claim, that the crossing number of $G^{\prime}$ is the same


Fig. 2. Putting weights on the edges (here $w(e)=5$ ).


Fig. 3. Controlling the prescribed local rotations.
as the "weighted crossing number" of $G$ : each crossing of edges $e_{1}, e_{2}$ is counted $w\left(e_{1}\right) w\left(e_{2}\right)$-times. Obviously, $\operatorname{cr}\left(G^{\prime}\right)$ is at most that, as we can draw each $L_{e}$ sufficiently close to where $e$ was drawn. Moreover, there is an optimal drawing of this form (which proves the converse inequality): Given an optimal embedding of $G^{\prime}$, consider the subgraph $L_{e}$ and from the $w(e)$ paths of length 2 between its "end-points" pick the one, that is crossed the least number of times. We can draw the whole subgraph $L_{e}$ close to this path without increasing the number of crossings.

This shows that we can "simulate weighted crossing number" by crossing number of a modified graph. In particular, we can let $w(e)=1$ for each ordinary edge and $w(e)>\operatorname{cr}(G)$ for each thick edge $e$ of $G$. This proves Lemma 1.7 for graphs with thick edges.

### 2.2. Rigid vertices

Suppose that we are considering drawings in surfaces of Euler genus $\leqslant g$; put $n=3 g+2$. Let $G$ be a special graph with rigid vertices. We replace each rigid vertex $v$ by a copy of $V_{n, \operatorname{deg}(v)}$. That is, we add $n$ nested thick cycles of length $d=\operatorname{deg}(v)$ around $v$ as shown in Fig. 3 for $d=6$ and $n=5$. When doing this, the cycles meet the edges incident with $v$ in the same order as requested by the local rotation $\pi_{v}$ around $v$. If an edge incident with $v$ is thick, then all edges in $G^{\prime}$ arising from it are thick too (as indicated in the figure for one of the edges). Call the resulting graph $G^{\prime}$.

We claim that the crossing number of $G^{\prime}$ (graph with thick edges but no rigid vertices) is the same as that of $G$. Any drawing of $G$ that respects the rotations at each rigid vertex can be extended to a drawing of $G^{\prime}$ without any new crossing; in this drawing all $n$ thick cycles in each $V_{v}$ are contractible and $v$ is contained in the disc that any of them is bounding. We will show, that there is an optimal drawing of $G^{\prime}$ of this "canonical" type.

Let us consider an optimal drawing (respecting thick edges) of $G^{\prime}$ in $S$ (of genus $\leqslant g$ ). Let $v$ be a rigid vertex of $G$, and consider the inner $n-1$ out of the $n$ thick cycles in $V_{v}$. No edge of these cycles is crossed; so by [3, Proposition 4.2.6], either one of these cycles is contractible in $S$, or two of them are homotopic.

Suppose first, that one of the cycles, $Q$, is contractible. Since $Q$ separates the graph into two connected components, either the disk $D$ bounded by $Q$ or its exterior contains no vertex or edge of $G^{\prime}$ apart from some cycles and edges of $V_{v}$. Let us assume that this is the interior of $D$. Now delete the drawing of all thick cycles in $V_{v}$ except $Q$, and delete the drawing of all $\operatorname{deg}(v)$ paths from $Q$ to $v$. Now think of $Q$ as the outermost cycle of $V_{v}$ and draw the rest on $V_{v}$ inside $D$ without crossings.

Suppose next, that two of the cycles, $Q_{1}$ and $Q_{2}$ are homotopic (and that $Q_{1}$ is closer to $v$ in $G^{\prime}$ ). We cut $S$ along $Q_{1}$, and patch the two holes with a disc. This simplifies the surface, so if we can draw $G^{\prime}$ on it without new crossings, we get a contradiction. Such drawing of $G^{\prime}$ indeed exists, as we may delete the drawing of all of $V_{v}$ that is "inside" $Q_{1}$ and draw it in one of the new discs.


Fig. 4. Main constituents of the graph $H_{n}($ for $n=5)$.
By performing such a change to each rigid vertex, we obtain an optimum drawing of $G^{\prime}$ which is canonical. Consequently, it gives rise to a legitimate drawing of the special graph $G$, and which is also optimal for $G$. This shows that Lemma 1.7 holds also when there are special vertices.

## 3. Hamburgers

The goal of this section is to prove Theorem 1.3, showing the existence of a graph with crossing sequence $(a, b, 0)$ for every $a>b>0$. The hamburger graphs $H_{n}$ (defined in the introduction) have all of the key features of interest. These are actually special graphs, but thanks to Lemma 1.7 it is enough to consider crossing sequences of special graphs. Indeed, in the remainder of the paper we will omit the term 'special'.

We have redrawn $H_{n}$ (for $n=5$ ) again in Fig. 4 where we have given names to numerous subgraphs of it. We have previously defined the rows $r_{1}, r_{2}$ and columns $c_{1}, \ldots, c_{n}$. For convenience we add rows $r_{0}$ and $r_{3}$ and columns $c_{0}$ and $c_{n+1}$ (see Fig. 4). The cycle $C$ (consisting of $c_{0}, r_{0}, c_{n+1}$, and $r_{3}$ ) has two trivial bridges (the thick edges $\tau_{0}$ and $\tau_{1}$ ) and two other bridges. The first, denoted by $B_{1}$, consists of the row $r_{1}$ together with all columns $c_{i}$ with $i$ even (and, of course, $1 \leqslant i \leqslant n$ ). The second one is denoted by $B_{2}$ and consists of the row $r_{2}$ and columns $c_{i}$ with $i$ odd (and, again, $1 \leqslant i \leqslant n$ ).

To get every possible crossing sequence ( $a, b, 0$ ), we will also require a slightly more general class of graphs. For every $n, k \in \mathbb{N}$ with $n \geqslant 3$, we define the graph $H_{n, k}$, which is obtained from $H_{n}$ by adding $k$ duplicates of the second column $c_{2}$ as shown in Fig. 5 for the case of $n=4$ and $k=3$. Note that $H_{n} \cong H_{n, 0}$.

We shall denote by $\mathbb{S}_{g}(g \geqslant 0)$ the orientable surface of genus $g$.
Lemma 3.1. $c r_{2}\left(H_{n, k}\right)=0$ for every $n, k \in \mathbb{N}$ with $n \geqslant 3$.
Proof. To draw $H_{n}$ in the double torus $\mathbb{S}_{2}$, start by embedding $H_{n}-\tau_{0}-\tau_{1}$ in the sphere $\mathbb{S}_{0}$. Now, use one handle to route the edge $\tau_{0}$, and another handle for $\tau_{1}$.

Lemma 3.2. $c r_{0}\left(H_{n, k}\right)=n+k$ for every $n, k \in \mathbb{N}$ with $n \geqslant 3$.
Proof. Consider a drawing of $H_{n, k}$ in the sphere. If this drawing has finite crossing number, the cycle $C$ must be embedded as a simple closed curve which separates the surface into two discs $D_{1}, D_{2}$ and is not crossed by any edge. Moreover, both thick edges $\tau_{0}$ and $\tau_{1}$ are drawn in the same disc, say $D_{2}$. Now every column of $B_{1}$ crosses the row $r_{2}$ and every column of $B_{2}$ crosses the row $r_{1}$, so we have


Fig. 5. The graph $H_{n, k}$ (for $n=4$ and $k=3$ ).
at least $n+k$ crossings. Since $H_{n, k}$ is drawn in $\mathbb{S}_{0}$ with $n+k$ crossings in Fig. 5, we conclude that $c r_{0}\left(H_{n, k}\right)=n+k$ as required.

Not surprisingly, the situation when drawing our graphs $H_{n}$ on the torus is considerably more complicated to analyze. By drawing $H_{n}$ in the plane with $n$ crossings and then using a handle to remove one crossing, we see that $c r_{1}\left(H_{n}\right) \leqslant n-1$ for all $n \geqslant 3$ (even $c r_{1}\left(H_{n, k}\right) \leqslant n-1$ for all $n \geqslant 3$ and $k \geqslant 0$ ). For $n \geqslant 5$, we shall prove that this is the best which can be achieved. For $n \leqslant 4$, however, there is some exceptional behavior (cf. Lemma 3.7).

Lemma 3.3. For every optimal drawing of $H_{n}$ (in some surface), each column $c_{i}(1 \leqslant i \leqslant n)$ is a simple curve.
Proof. It is easy to see that in every optimal drawing, every edge is represented by a simple curve. Let us now consider a column $c_{i}=v_{i} v_{i}^{\prime} u_{i}$ (or similarly for $v_{i} u_{i}^{\prime} u_{i}$ ) and suppose that the edges $e=v_{i} v_{i}^{\prime}$ and $f=u_{i} v_{i}^{\prime}$ cross. Suppose that $e$ is represented by the simple curve $\alpha(t), 0 \leqslant t \leqslant 1$, where $\alpha(0)=v_{i}$ and $\alpha(1)=v_{i}^{\prime}$. Similarly, let $f$ be represented by the simple curve $\beta(t), 0 \leqslant t \leqslant 1$, where $\beta(0)=u_{i}$ and $\beta(1)=v_{i}^{\prime}$. Let $\alpha\left(t^{\prime}\right)=\beta\left(t^{\prime}\right)\left(0<t^{\prime}<1\right)$ be where they cross. Now let $\tilde{\alpha}(t)=\alpha(t)$ for $t \leqslant t^{\prime}$ and $\tilde{\alpha}(t)=\beta(t)$ for $t \geqslant t^{\prime}$. Change similarly $\beta$ to $\tilde{\beta}$. Then the crossing becomes a touching of the two curves, which can be eliminated yielding a drawing with fewer crossings. Observe that the local rotation at the special vertex $v_{i}^{\prime}$ changes from clockwise to anticlockwise but this is still consistent with the requirement for this special vertex. Therefore the new drawing contradicts the optimality of the original one.

At several occasions in the proof we will use the following well-known fact about closed curves on the torus.

Lemma 3.4. (See [3, Proposition 4.2.6].) Let $\varphi, \psi$ be two simple closed noncontractible curves on the torus that are not freely homotopic. Then $\varphi$ and $\psi$ cross each other.

The following is well known (cf., e.g., [7]).
Lemma 3.5. Let $\varphi, \psi$ be two closed curves on some surface; assume $\psi$ is contractible. The curves may intersect themselves and each other, but we assume that

1. the total number of intersections is finite, and
2. each point of intersection is a crossing (the curves do not touch and there are no more than two arcs that run through the point).

Then, the number of intersections of $\varphi$ with $\psi$ is even.


Fig. 6. Illustration for the proof of Lemma 3.5.


Fig. 7. Nine special types of embedding of the thick subgraph $C+\tau_{0}+\tau_{1}$ in the torus. In types $B-E^{\prime \prime \prime}$, the cycle $C$ is drawn on the top and bottom sides of the square.

Proof (hint). Let us transform $\psi$ continuously to a trivial curve. The number of intersections of $\varphi$ with $\psi$ stays the same, or changes by 2 when we modify $\psi$ as in Fig. 6.

It will be convenient for us to classify different types of drawings of $H_{n}$ in the torus depending on the drawing of the thick subgraph $C+\tau_{0}+\tau_{1}$. In Fig. 7 we have listed nine possible embeddings of $\mathcal{C}+\tau_{0}+\tau_{1}$ in $\mathbb{S}_{1}$, where $\tau_{0}$ and $\tau_{1}$ are drawn with dashed lines. We shall say that a drawing of $H_{n}$ is of type $A, B, C, C^{\prime}, D, E, E^{\prime}, E^{\prime \prime}$, or $E^{\prime \prime \prime}$ if the induced drawing of $C+\tau_{0}+\tau_{1}$ is as in the corresponding part of Fig. 7. Although there are other possible drawings of $C+\tau_{0}+\tau_{1}$ in the torus, our next lemma shows that the only ones which extend to finite crossing number drawings of $H_{n}$ have one of these types.

Lemma 3.6. Every drawing of $H_{n}$ for $n \geqslant 3$ on a torus $\mathcal{S}$ with crossing number less than $n$ has type $A, B, C$, $C^{\prime}, D, E, E^{\prime}, E^{\prime \prime}$, or $E^{\prime \prime \prime}$.

Proof. Let $\mathcal{S}^{\prime}$ be the bordered surface obtained from $\mathcal{S}$ by cutting along the cycle $C$. First suppose that $C$ is contractible. Then $\mathcal{S}^{\prime}$ is disconnected, with one component a disc $D$, and the other component $\mathcal{S}^{\prime \prime}$ homeomorphic to $\mathbb{S}_{1}$ minus a disc. If both $B_{1}$ and $B_{2}$ are drawn in $D$, then we have at least $n$ crossings (as in Lemma 3.2). If only one of $B_{1}$ or $B_{2}$, say $B_{1}$ is drawn in $D$, then $B_{2}$ and the edges $\tau_{0}$ and $\tau_{1}$ are drawn in $\mathcal{S}^{\prime \prime}$ (else the crossing number is infinite). Consider the curves $\tau_{0} \cup r_{0}$ and $\tau_{1} \cup r_{3}$ in $\mathcal{S}^{\prime \prime}$. If either of these is contractible, then $B_{2}$ must cross it (yielding infinite crossing number).


Fig. 8. Exceptional drawings of $\mathrm{H}_{3}$.


Fig. 9. Exceptional type $B$ drawing of $H_{4}$.
Otherwise (using the Lemma 3.4) they must be freely homotopic noncontractible curves in $\mathcal{S}^{\prime \prime}$, so $\tau_{0} \cup c_{0} \cup \tau_{1} \cup c_{n+1}$ is a contractible curve. Therefore $B_{2}$ must cross it, yielding again infinitely many crossings. Thus, we may assume that both $\tau_{0}$ and $\tau_{1}$ are drawn in the disc $D$ and $B_{1}$ and $B_{2}$ are drawn in $\mathcal{S}^{\prime \prime}$ so our drawing is of type $A$.

Next suppose that $C$ is not contractible. In this case, the surface $\mathcal{S}^{\prime}$ is a cylinder bounded by two copies of the cycle $C$. If both $\tau_{0}$ and $\tau_{1}$ have all of their ends on the same copy of $C$, we must have a drawing of type $B, C$, or $C^{\prime}$. If one has both ends on one copy of $C$, and the other has both ends on the other copy of $C$, then there are infinitely many crossings, unless the drawing is of type $D$. Finally, if one of $\tau_{0}$, $\tau_{1}$, has its ends on distinct copies of $C$, then the crossing number will be infinite unless the other one of $\tau_{0}, \tau_{1}$, has both ends on the same copy of $C$ giving us a drawing of type $E, E^{\prime}, E^{\prime \prime}$, or $E^{\prime \prime \prime}$.

If $G$ is a graph drawn on a surface and $A, B \subseteq G$, then we shall denote by $\operatorname{Cr}(A \mid B)$ the total number of crossings of an edge from $A$ with an edge from $B$, where crossings of an edge $e \in E(A \cap B)$ with another edge $f \in E(A \cap B)$ are counted only once. In particular, the total number of crossings of graph $G$ is equal to $\operatorname{Cr}(G \mid G)$.

Lemma 3.7. $c r_{1}\left(H_{n}\right)=n-1$ if $n=3$ or $n \geqslant 5$, while $c r_{1}\left(H_{4}\right)=2$. Furthermore, Fig. 8(a)-( $\left.c^{\prime}\right)$ shows the only drawings of $\mathrm{H}_{3}$ in the torus with two crossings and the added property that $\mathrm{Cr}\left(r_{2} \mid \mathrm{G}\right)=0$. Fig. 9 displays the unique drawing of $\mathrm{H}_{4}$ in the torus with two crossings.

Proof. We proceed by induction on $n$. Consider a drawing $\mathcal{D}$ of $H_{n}$ in a surface $\mathcal{S}$ homeomorphic to the torus, such that $\mathcal{D}$ yields minimum crossing number. We shall frequently use the inductive assumption for $n-1$ and $n-2$, since by deleting the edges of the column $c_{1}$, the column $c_{n}$, or two consecutive columns $c_{i}$ and $c_{i+1}$ we obtain a new graph which is a subdivision of $H_{n-1}$ or $H_{n-2}$ (assuming $n \geqslant 3$ ). This technique will be used throughout the proof. It is also worth noting that after applying this operation to $\mathcal{D}$, the drawing of the smaller hamburger graph is of the same type as the drawing $\mathcal{D}$.

The cycle $C$ is not crossed in $\mathcal{D}$, so we may cut our surface along this curve. This leaves us with a drawing of $H_{n}$ in a closed bordered surface-which we shall denote $\mathcal{S}^{\prime}$-where each edge of $C$ appears twice on the boundary. We shall use $C^{1}$ and $C^{2}$ to denote these copies.

Essential to our proof is an analysis of the homotopy behavior of the rows and columns. To make this precise, let us now choose a point $N$ in the interior of the row $r_{0}, S$ in the interior of $r_{3}$, W in the interior of $c_{0}$ and $E$ in the interior of $c_{n+1}$. (Actually, for each of these points we have two copies: $N^{1}$ and $N^{2}$, etc. But we will avoid distinguishing these if there is no danger of confusion.) For each column $c_{i}(0 \leqslant i \leqslant n+1)$ let $c_{i}^{+}$be a simple curve in $\mathcal{S}^{\prime}$ obtained by extending $c_{i}$ along the appropriate copies of the rows $r_{0}$ and $r_{3}$ so that it has ends $N$ and $S$. Similarly, for each row $r_{i}(0 \leqslant i \leqslant 3)$ let $r_{i}^{+}$be a curve in $\mathcal{S}^{\prime}$ obtained by extending $r_{i}$ along the appropriate copies of the columns $c_{0}$ and $c_{n+1}$ so that it has ends $E$ and $W$. We shall focus our attention on the homotopy types in $\mathcal{S}^{\prime}$ of the curves $c_{i}^{+}$where $N$ and $S$ are the fixed end points (and similarly $r_{i}^{+}$where $E$ and $W$ are fixed): we say that $c_{i}^{+}$and $c_{j}^{+}$are homotopic if $c_{i}^{+}$may be continuously deformed to $c_{j}^{+}$in the surface $\mathcal{S}^{\prime}$, while keeping their endpoints fixed. Note that $c_{i}^{+}$and $c_{j}^{+}$can only be homotopic if $c_{i}$ and $c_{j}$ are connecting the same copies of $N$ and $S$-that is they attach on the same side of $C$ in the original surface $\mathcal{S}$. Also note, that for $i=0$ or $i=n+1$ we actually have two copies of $c_{i}$, so we should be speaking of, e.g., $c_{0}^{+1}$ and $c_{0}^{+2}$. We will refrain from this distinction whenever possible to keep the notation clearer-so when saying $c_{0}^{+}$and $c_{1}^{+}$are homotopic we will actually mean that $c_{1}^{+}$is homotopic to $c_{0}^{+s}$ for some $s \in\{1,2\}$.

We will use frequently the following fact that connects the homotopy types of columns and their crossing behaviour with respect to the rows (and vice versa). We will refer to this statement as to "the Claim".

Claim. If $c_{i}^{+}$and $c_{i+1}^{+}$are homotopic $(1 \leqslant i<n)$, then $\operatorname{Cr}\left(r_{j} \mid c_{i} \cup c_{i+1}\right) \geqslant 1$ for $j=1$, 2. Similarly, if $r_{1}^{+}$and $r_{2}^{+}$are homotopic, then $\operatorname{Cr}\left(r_{1} \cup r_{2} \mid c_{i}\right) \geqslant 1$ for every $1 \leqslant i \leqslant n$.

To see this, let us observe that the closed curve obtained by following $c_{i}^{+}$from $S$ to $N$ and then $c_{i+1}^{+}$from $N$ to $S$ is contractible, after deleting part of its intersection with the cycle $C$, we get a contractible curve $\psi$ that intersects itself only at finitely many points. The row $r_{j}$ must cross either $c_{i}^{+}$or $c_{i+1}^{+}$(depending on the parity) in their common vertex (it cannot only touch it as their common vertex has prescribed local rotation). We may extend $r_{j}^{+}$into a closed curve $\varphi$ by following closely along the cycle $C$. This way we are adding two (or zero) intersections with $\psi$. By Lemma 3.5 curves $\varphi$ and $\psi$ have an even number of intersection, thus $r_{j}$ must have another crossing with $\psi$ and we are done. The same argument holds when the rows and columns exchange their roles.

Corollary. If $r_{1}^{+}$and $r_{2}^{+}$are homotopic, we are done, as there are at least $n$ intersections.
In light of Lemma 3.6 we may assume that our drawing is of type $A, B, C, C^{\prime}, D, E, E^{\prime}, E^{\prime \prime}$, or $E^{\prime \prime \prime}$, and we now split our argument into these nine cases.

## Case 1. Type A.

Let us first suppose that $n \geqslant 4$. If there exists $1 \leqslant i \leqslant n$ so that $c_{i}^{+}$is homotopic to $c_{0}^{+}$, then either $c_{1}$ crosses $c_{i}$, or $c_{1}^{+}$is homotopic to $c_{0}^{+}$. In the latter case, $c_{1}$ crosses $r_{1}$. So, in short, $\operatorname{Cr}\left(c_{1} \mid H_{n}\right) \geqslant 1$ and by removing this column and applying induction, we deduce that there are at least $n-1$ crossings in our drawing. Note here that the resulting drawing of $H_{n-1}$ is still of type $A$, so it must have at
least ( $n-1$ ) -1 crossings, even if $n=5$. Thus, we may assume that $c_{i}^{+}$is not homotopic to $c_{0}^{+}$for any $1 \leqslant i \leqslant n$. By a similar argument, $c_{i}^{+}$is not homotopic to $c_{n+1}^{+}$. If there exist $i, j \in\{1, \ldots, n\}$ with $c_{i}^{+}$not homotopic to $c_{j}^{+}$, then $c_{i}^{+}$and $c_{j}^{+}$cross (Lemma 3.4), and further, $\operatorname{Cr}\left(c_{k} \mid c_{i} \cup c_{j}\right) \geqslant 1$ for every $k \in\{1, \ldots, n\}$ with $k \neq i, j$. This implies that we have at least $n-1$ crossings, as desired. The only other possibility is that $c_{i}^{+}$and $c_{j}^{+}$are homotopic for every $i, j \in\{1, \ldots, n\}$. In this case, it follows from the Claim (applied to $c_{1}^{+}$and $c_{2}^{+}, c_{3}^{+}$and $c_{4}^{+}, \ldots$ ) that there are at least $n-1$ crossings.

Suppose now that $n=3$. If $c_{2}^{+}$is homotopic to $c_{1}^{+}$or $c_{3}^{+}$, then it follows from the Claim that each row has at least one crossing, and we are done. Thus, we may assume that $c_{2}^{+}$has distinct homotopy type from that of $c_{1}^{+}$and from that of $c_{3}^{+}$. If $c_{2}^{+}$is homotopic to $c_{0}^{+}$, then $\operatorname{Cr}\left(c_{2} \mid r_{2}\right) \geqslant 1$ and $\operatorname{Cr}\left(c_{2} \mid c_{1}\right) \geqslant 2$ (since $c_{1}^{+}$is not homotopic to $c_{2}^{+}$) giving us too many crossings. Thus, $c_{2}^{+}$is not homotopic to $c_{0}^{+}$, and by a similar argument, we find that $c_{2}^{+}$is not homotopic to $c_{4}^{+}$. Now, either $c_{1}^{+}$is homotopic to $c_{0}^{+}$(in which case $\operatorname{Cr}\left(c_{1} \mid r_{1}\right) \geqslant 1$ ) or $c_{1}^{+}$is not homotopic to $c_{0}^{+}$(in which case $\left.\operatorname{Cr}\left(c_{1} \mid c_{2}\right) \geqslant 1\right)$. So, in short $\operatorname{Cr}\left(c_{1} \mid r_{1} \cup c_{2}\right) \geqslant 1$. By a similar argument, $\operatorname{Cr}\left(c_{3} \mid r_{1} \cup c_{2}\right) \geqslant 1$. Since there are at most two crossings, we must have $\operatorname{Cr}\left(c_{1} \cup c_{3} \mid r_{1} \cup c_{2}\right)=2$ and this accounts for all of our crossings. In particular, this implies that $r_{1}$ and $r_{2}$ are simple curves. Since $\operatorname{Cr}\left(r_{2} \mid G\right)=0$, it follows that $r_{2}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$. By the Claim, $r_{1}^{+}$is not homotopic to $r_{2}^{+}$, and this together with $\operatorname{Cr}\left(r_{1} \mid r_{2}\right)=0$ implies that $r_{1}^{+}$is homotopic to $r_{0}^{+}$. It follows from this that $\operatorname{Cr}\left(r_{1} \mid c_{i}\right)=1$ for $i=1,3$ and this accounts for all of the crossings. Such a drawing is possible, but must be equivalent with that in Fig. 8(a).

In all the remaining cases, we have that $\mathcal{S}^{\prime}$ is a cylinder, and in our figures we have drawn $\mathcal{S}^{\prime}$ with the boundary component $C^{1}$ on the top and $C^{2}$ on the bottom.

## Case 2. Type $B$.

Here all of the column curves $c_{i}^{+}$have ends $N^{2}$ and $S^{2}$. Recall that these are copies of $N$ and $S$ drawn at the "bottom copy" $C^{2}$ of $C$. Since all of these curves are simple, it follows that for every $1 \leqslant i \leqslant n$, the curve $c_{i}^{+}$is either homotopic to the simple curve $N^{2}-W^{2}-S^{2}$ in $C^{2}$ (we shall call this homotopy type $\ell$ ), or to the simple curve $N^{2}-E^{2}-S^{2}$ in $C^{2}$ (homotopy type $r$ ). Let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$ be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. We now have the following simple crossing property.

P1. If $a_{i}=r$ and $a_{j}=\ell$ where $1 \leqslant i<j \leqslant n$, then $\operatorname{Cr}\left(c_{i} \mid c_{j}\right) \geqslant 2$.
If there exists an $i(1 \leqslant i \leqslant n)$ so that $\operatorname{Cr}\left(c_{i} \mid H_{n}\right) \geqslant 4$, then $n \geqslant 5$ (otherwise the drawing is not optimal), and by removing $c_{i}$ and either $c_{i-1}$ or $c_{i+1}$ and applying the theorem inductively to the resulting graph, we deduce that there are at least $4+c r_{1}\left(H_{n-2}\right) \geqslant n$ crossings in our drawing, a contradiction. It follows from this and P1, that either $\mathbf{a}=\ell^{i} r^{n-i}$ or $\mathbf{a}=\ell^{i} r \ell r^{n-i-2}$. We now split into subcases depending on $n$.

Suppose first that $n=3$. If $a_{1}=a_{2}=\ell$ or $a_{2}=a_{3}=r$, then it follows from the Claim that $\operatorname{Cr}\left(r_{j} \mid\right.$ $\left.c_{1} \cup c_{2} \cup c_{3}\right) \geqslant 1$ for $j=1,2$ and we are finished. Otherwise, a must be $\ell r \ell$ or $r \ell r$ and $\operatorname{Cr}\left(c_{2} \mid c_{1} \cup\right.$ $\left.c_{3}\right) \geqslant 2$. These configurations are possible, but require that our drawing is equivalent with the one in Fig. 8(b)-this comes from $\mathbf{a}=\ell r \ell$, if $\mathbf{a}=r \ell r$ we get a mirror image.

Next we consider the case when $n=4$ and $\mathbf{a}=\ell^{i} r^{4-i}$. Applying the Claim for the columns $c_{1}, c_{2}$ and $c_{3}, c_{4}$ resolves the cases when a is one of $\ell^{4}, r^{4}$, or $\ell^{2} r^{2}$ (each gives at least four crossings-a contradiction). Suppose that $\mathbf{a}=\ell^{3} r$ (or, with the same argument, $\mathbf{a}=\ell r^{3}$ ). It follows from the Claim that $\operatorname{Cr}\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geqslant 2$ and $\operatorname{Cr}\left(c_{2} \cup c_{3} \mid r_{1} \cup r_{2}\right) \geqslant 2$, so the only possibility for fewer than three crossings is that our drawing has 2 crossings, both of which are between $c_{2}$ and the rows $r_{1}$ and $r_{2}$. But then $c_{2}$ does not cross $c_{1}$ or $c_{3}$, so $c_{2}$ is separated from $c_{0}$ by $c_{1}^{+} \cup c_{3}^{+}$, so $\operatorname{Cr}\left(r_{1} \mid c_{1} \cup c_{3}\right)>0$, a contradiction.

Next suppose that $n=4$ and $\mathbf{a}=\ell^{i} r \ell r^{2-i}$. If $\mathbf{a}=\ell^{2} r \ell$, then it follows from P1 that $\operatorname{Cr}\left(c_{3} \mid c_{4}\right) \geqslant 2$ and from the Claim that $\operatorname{Cr}\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geqslant 2$, so we have at least four crossings-a contradiction. Similarly $\mathbf{a}=r \ell r^{2}$ is impossible. The only remaining possibility is $\mathbf{a}=\ell r \ell r$. In this case, we have
$\operatorname{Cr}\left(c_{2} \mid c_{3}\right) \geqslant 2$, so the only possibility is that there are exactly two crossings, both between $c_{2}$ and $c_{3}$. This case can be realized, but requires that our drawing is equivalent to that of Fig. 9.

Lastly, suppose that $n \geqslant 5$. Since $\mathbf{a} \in\left\{\ell^{i} r^{n-i}, \ell^{i} r \ell r^{n-i-2}\right\}$, either $a_{1}=a_{2}=\ell$ or $a_{n-1}=a_{n}=r$. As these arguments are similar, we shall consider only the former case. Now, it follows from the Claim that $\operatorname{Cr}\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geqslant 2$, so removing the first two columns gives us a drawing of $H_{n-2}$ with at least two crossings less than in our present drawing of $H_{n}$. By applying our theorem inductively to this new drawing, we find that the only possibility for less than $n-1$ crossings is that $n=6$ and $\mathbf{a}=\ell^{3} r \ell r$. In this case, we have $\operatorname{Cr}\left(c_{4} \mid c_{5}\right) \geqslant 2$, so we may eliminate two crossings by removing columns 4 and 5 . This leaves us with a drawing of a graph isomorphic to $H_{4}$ as above with the pattern $\ell^{3} r$. It follows from our earlier analysis, that this drawing has at least three crossings. This completes the proof of this case.

## Case 3. Type C.

Now each column curve has one end on the segment of $C^{2}$ between $q^{2}$ and $r^{2}$. As above, every curve $c_{i}^{+}$with both ends on $C^{2}$ must be homotopic with either the simple curve $N^{2}-W^{2}-S^{2}$ in $C^{2}$ (denoted by $\ell$ ), or with the simple curve $N^{2}-E^{2}-S^{2}$ in $C^{2}$ (homotopy type $r$ ). Each row has both its ends on $C^{2}$.

The homotopy types of the other column curves will be represented by integers. Since $\mathcal{S}^{\prime}$ is a cylinder, we may choose a continuous deformation $\Psi$ of $\mathcal{S}^{\prime}$ onto the circle $\mathbb{S}^{1}$ with the property that $C^{1}$ and $C^{2}$ map bijectively to $\mathbb{S}^{1}$, and $N^{2}$ and $S^{1}$ map to the same point $x \in \mathbb{S}^{1}$. Now, each curve $c_{i}^{+}$ maps to a closed curve in $\mathbb{S}^{1}$ from $x$ to $x$, and for an integer $\alpha \in \mathbb{Z}$, we say that $c_{i}^{+}$has homotopy type $\alpha$ if the corresponding curve in $\mathbb{S}^{1}$ has (counterclockwise) winding number $\alpha$. It follows that $c_{i}^{+}$and $c_{j}^{+}$are homotopic if and only if they have the same homotopy type. As before, we let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$ be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. We now have the following crossing properties (for the appropriate choice of "clockwise" direction), whenever $1 \leqslant i<j \leqslant n$ :

P1. $\operatorname{Cr}\left(c_{i} \mid c_{j}\right) \geqslant\left|a_{i}-a_{j}-1\right|$ if $a_{i}, a_{j} \in \mathbb{Z}$.
P2. $\operatorname{Cr}\left(c_{i} \mid c_{j}\right) \geqslant 2$ if $a_{i}=r$ and $a_{j}=\ell$.
P3. $\operatorname{Cr}\left(c_{i} \mid c_{j}\right) \geqslant 1$ if either $a_{i}=r$ and $a_{j} \in \mathbb{Z}$ or $a_{i} \in \mathbb{Z}$ and $a_{j}=\ell$.
By choosing $\Psi$ appropriately, we may further assume that the smallest integer $1 \leqslant i \leqslant n$ for which $a_{i} \in \mathbb{Z}$ (if such $i$ exists) satisfies $a_{i}=0$. Again, we split into subcases depending on $n$.

Suppose first that $n=3$. Note that every column of type $r$ or $\ell$ separates the segment $q^{2} t^{2}$ on $C^{2}$ from $r^{2} s^{2}$. Consequently, $\operatorname{Cr}\left(r_{1} \cup r_{2} \mid c_{i}\right) \geqslant 1$ whenever $a_{i} \in\{\ell, r\}$. Next we shall consider the homotopy types of our rows. If $r_{1}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $\operatorname{Cr}\left(r_{1} \mid r_{1}\right) \geqslant 1$ and further $\operatorname{Cr}\left(r_{1} \mid c_{1} \cup c_{3}\right) \geqslant$ 2 (as in this case, $r_{1}$ separates $C^{2}$ from $C^{1}$ and also segment $q^{2} r^{2}$ from $s^{2} t^{2}$ ) which gives us too many crossings. If $r_{2}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $\operatorname{Cr}\left(r_{2} \mid r_{2}\right) \geqslant 1$ and $\operatorname{Cr}\left(r_{2} \mid c_{2}\right) \geqslant 1$, and we have nothing left to prove. Thus, we may assume that $r_{1}^{+}$(and also $r_{2}^{+}$) is homotopic to one of $r_{0}^{+}$, $r_{3}^{+}$. If $r_{1}^{+}$and $r_{2}^{+}$are homotopic, then the Claim implies that there are at least three crossings. Hence, we may assume that $r_{1}^{+}$is homotopic to $r_{0}^{+}$and $r_{2}^{+}$to $r_{3}^{+}$(the other possibility yields two crossings and each row crossed). It now follows from our assumptions that $\operatorname{Cr}\left(r_{1} \mid c_{i}\right) \geqslant 1$ for $i=1,3$, so assuming we have at most two crossings, our only crossings are between $r_{1}$ and $c_{1}$ and between $r_{1}$ and $c_{3}$. If $a_{i} \in \mathbb{Z}$ for $i \in\{1,3\}$, then $c_{i}$ also crosses $r_{2}$ because of the requirements concerning local rotations at the special vertices $u_{1}^{\prime}$ and $u_{3}^{\prime}$. It follows that there are at least three crossings unless $\mathbf{a}=\ell 0 \ell$, $\ell 0 r, r 0 \ell$, or $r 0 r$. Each of these, except $\ell 0 r$ gives at least three crossings by P3. The remaining case is possible, but only as it appears in Fig. 8(c).

Suppose now that $n \geqslant 4$. If either $c_{1}$ or $c_{n}$ is crossed, then we delete it and use the induction hypothesis. If neither has a crossing, then both $a_{1}$ and $a_{n}$ are integers (otherwise $\operatorname{Cr}\left(c_{1} \cup c_{n} \mid r_{1} \cup\right.$ $\left.r_{2}\right) \geqslant 1$ as above). It follows that $a_{1}=0$, and $a_{n}=-1$ (otherwise $c_{1}$ and $c_{n}$ cross). Now there is no value for $a_{2}$ to avoid crossing with either $c_{1}$ or $c_{n}$. Hence one of $c_{1}$, and $c_{n}$ is crossed, after all, and we may use induction. This completes the proof of Case 3.


Fig. 10. Part of a type $D$ drawing of $H_{3}$.

## Case 4. Type $C^{\prime}$.

This case is nearly identical to the previous one. We may define the homotopy types for the columns to be $r$, $\ell$, or an integer, exactly as before, so that the same homotopy properties are satisfied. Then the analysis for $n \geqslant 4$ is identical, and the only difference is the case when $n=3$. As before, if $r_{1}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $\operatorname{Cr}\left(r_{1} \mid r_{1}\right) \geqslant 1$ and $\operatorname{Cr}\left(r_{1} \mid c_{1} \cup c_{3}\right) \geqslant 2$ giving us too many crossings. Similarly, if $r_{2}^{+}$is not homotopic to $r_{0}^{+}$or $r_{3}^{+}$, then $\operatorname{Cr}\left(r_{2} \mid r_{2}\right) \geqslant 1$ and $\operatorname{Cr}\left(r_{2} \mid c_{2}\right) \geqslant 1$ and there is nothing left to prove. Now, using the Claim, we deduce that $r_{1}^{+}$is homotopic to $r_{0}^{+}$and $r_{2}^{+}$is homotopic to $r_{3}^{+}$. It follows from this that $\operatorname{Cr}\left(c_{2} \mid r_{2}\right) \geqslant 1$. If $a_{2} \in \mathbb{Z}$ then, as the vertex $v_{2}^{\prime}$ is rigid, it follows that $\operatorname{Cr}\left(c_{2} \mid r_{1}\right) \geqslant 1$ and we have nothing left to prove. Thus, we may assume that $a_{2} \in\{\ell$, $r\}$. If $a_{i} \in\{\ell, r\}$ for $i=1$ or $i=3$, then $c_{i}$ crosses $r_{1}$ and we are done. Thus, we may assume that $a_{1}, a_{3} \in \mathbb{Z}$. It now follows that $\operatorname{Cr}\left(c_{2} \mid c_{1} \cup c_{3}\right) \geqslant 1$. This can be realized with exactly two crossings, but row $r_{2}$ must be crossed.

## Case 5. Type $D$.

In this case, every column has one end on $r_{0}^{2}$ and one end on $r_{3}^{1}$. We define the homotopy types of curves $c_{i}^{+}$using integers as in the previous case. Again, $c_{i}^{+}$and $c_{j}^{+}$are homotopic if and only if they have the same homotopy type. As before, we let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$ be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. And as before, we have the following useful crossing property:

P1. $\operatorname{Cr}\left(c_{i} \mid c_{j}\right) \geqslant\left|a_{i}-a_{j}-1\right|$ if $1 \leqslant i<j \leqslant n$.
Suppose first that $n \geqslant 4$. If the first column $c_{1}$ does not cross any other columns, then $\mathbf{a}=$ $0(-1)^{n-1}$. Similarly, if the last column does not cross any other columns, then $\mathbf{a}=0^{n-1}(-1)$. Since these cases are mutually exclusive for $n \geqslant 4$, either the first, or the last column contains a crossing. Then we may remove it and apply induction.

If $n=3$, we proceed as follows. Using P1 (and the convention $a_{1}=0$ ) we get that the number of crossings between the columns is at least $\left|a_{2}+1\right|+\left|a_{3}+1\right|+\left|a_{2}-a_{3}-1\right| \geqslant\left|a_{2}+1\right|+\left|a_{2}\right|$ (using the triangle inequality). Symmetrically, we get another lower bound for the number of crossings: $\left|a_{3}+1\right|+\left|a_{3}+2\right|$. If any of these bounds is at least 3 , we are done. It follows that $a_{2} \in\{0,-1\}$ and $a_{3} \in\{-1,-2\}$. Now, if there are two consecutive columns with the same homotopy type, then each row will cross some of these columns, and we are done. Consequently $\mathbf{a}=0,-1,-2$. It follows that $\operatorname{Cr}\left(c_{1} \mid c_{3}\right) \geqslant 1$. If $c_{2}$ crossed either $c_{1}$ or $c_{3}$, then it would have to cross the column twice-which would yield too many crossings. Similarly, if $\operatorname{Cr}\left(c_{1} \mid c_{3}\right)>1$, then $\operatorname{Cr}\left(c_{1} \mid c_{3}\right) \geqslant 3$ and we would have too many crossings. It follows that the three columns $c_{1}, c_{2}, c_{3}$ are drawn as in Fig. 10. Now we have that $c_{1}$ and $c_{3}$ separate $c_{2}$ from $c_{0}^{1}, c_{0}^{2}, c_{n+1}^{1}$, and $c_{n+1}^{2}$. It follows that $\operatorname{Cr}\left(r_{1} \mid c_{1} \cup c_{3}\right) \geqslant 2$ giving us too many crossings.

## Case 6. Type E.

In this case, every curve $c_{i}^{+}$must have one end in $r_{3}^{2}$ and the other end in either $r_{0}^{1}$ or $r_{0}^{2}$. In the first case, we say that $c_{i}^{+}$has homotopy type 0 and in the second we say it has type $\ell$. It is immediate that any two such curves of the same type are homotopic. As usual, we let $\mathbf{a}=a_{1} a_{2} \ldots a_{n}$


Fig. 11. Towards type $E$ drawings of $\mathrm{H}_{3}$.
be the word given by the rule that $a_{i}$ is the homotopy type of $c_{i}^{+}$. The following rule indicates some forced crossing behavior.

## P1. $\operatorname{Cr}\left(c_{i} \mid c_{j}\right) \geqslant 1$ if $a_{i}=0$ and $1 \leqslant i<j \leqslant n$.

Let us first treat the case when $n \geqslant 4$. If the last column $c_{n}$ contains at least one crossing, then we may remove it and apply induction. Otherwise, P1 implies that $\mathbf{a}=\ell^{n}$ or $\mathbf{a}=\ell^{n-1} 0$. It follows from the Claim that $\operatorname{Cr}\left(c_{1} \cup c_{2} \mid r_{1} \cup r_{2}\right) \geqslant 2$. Thus, if $n \geqslant 5$, we may remove the first two columns and apply induction. If $n=4$ and $\mathbf{a}=\ell^{4}$, then the Claim gives us at least four crossings-a contradiction with the minimality of our drawing. It remains to check $\mathbf{a}=\ell^{3} 0$. If there are fewer than three crossings, then (again by applying the Claim twice) there are exactly two, and both occur on $c_{2}$. However, in this case $\operatorname{Cr}\left(r_{1} \mid c_{3}\right)=0$. As $c_{3}$ separates $c_{2}$ from both $r^{1} s^{1}$ and $r^{2} s^{2}$ and $r_{1}$ has a common vertex with $c_{2}$, we get a contradiction.

Finally, suppose that $n=3$. If there are two consecutive columns with the same homotopy type, then we are finished (by the Claim), so we may assume $\mathbf{a}=0 \ell 0$ or $\mathbf{a}=\ell 0 \ell$. In the former case, we have $\operatorname{Cr}\left(c_{1} \mid c_{2} \cup c_{3}\right) \geqslant 2$, so we may assume that there are exactly two crossings, and the columns must be drawn as in Fig. 11(a). However, it is impossible to complete this drawing to a drawing of $\mathrm{H}_{3}$ with fewer than three crossings.

In the case $\mathbf{a}=\ell 0 \ell$ we have $\operatorname{Cr}\left(c_{2} \mid c_{3}\right) \geqslant 1$ (see Fig. 11(b)) and the total number of crossings is at most two. If $r_{2}$ is crossed, then the drawing is not exceptional and we are done. There is a unique way to add $r_{2}$ to Fig. 11(b) without creating any new crossing. Then there is no way to add $r_{1}$ without crossing $r_{2}$.

## Case 7. Type $E^{\prime}$.

This case is very close to the previous one. A similar analysis reduces the problem to the case when $n=3$. This case is actually identical to the above: By reflecting both the torus pictured in $E^{\prime}$ and the standard drawing of $\mathrm{H}_{3}$ (as in Fig. 1) about a vertical symmetry axis we find ourselves in this previous case.

## Case 8. Type $E^{\prime \prime}$.

This case is somewhat similar to that of type $E$. We may define the homotopy types for the columns $0, \ell$ exactly as before, so that the crossing property (P1) from type $E$ is satisfied. Then the analysis for $n \geqslant 4$ is identical, and the only difference is the case when $n=3$. As before, if there are two consecutive columns with the same homotopy type, we are finished. Thus we may assume that $\mathbf{a}=0 \ell 0$ or $\mathbf{a}=\ell 0 \ell$. Then we get another drawing of $H_{3}$ with two crossings, but again, in this case $r_{1}$ and $r_{2}$ cross each other.

Case 9. Type $E^{\prime \prime \prime}$.
This case is essentially the same as the previous one, in the same way as type $E^{\prime}$ was related to $E$. This completes the proof of Lemma 3.7.


Fig. 12. The special graph $H_{3}^{+}$.

Next we bootstrap to the following lemma.

Lemma 3.8. The graph $H_{n, k}$ has crossing sequence $(n+k, n-1,0)$ for every $n \geqslant 3$ and $k \geqslant 0$ with the exception of $n=4$ and $k=0$.

Proof. Lemmas 3.1 and 3.2 show that $c r_{0}\left(H_{n, k}\right)=n+k$ and $c r_{2}\left(H_{n, k}\right)=0$. We can draw $H_{n, k}$ in the torus with $n-1$ crossings by adding a handle to the drawing from Fig. 5. It remains to show that $c r_{1}\left(H_{n, k}\right) \geqslant n-1$ (for $n \geqslant 3$, unless $n=4$ and $k=0$ ). Take a drawing of $H_{n, k}$ in the torus. By removing the $k$ extra columns we obtain a drawing of $H_{n, 0}$ in the torus, which (by Lemma 3.7) has $\geqslant n-1$ crossings, unless $n=4$. This completes the proof in all cases except when $n=4$.

If $n=4$, the same argument as above shows that $c r_{1}\left(H_{4, k}\right) \geqslant c r_{1}\left(H_{4,1}\right)$; we shall prove now that $c r_{1}\left(H_{4,1}\right) \geqslant 3$. Suppose this is false, and consider a drawing of $H_{4,1}$ in the torus with at most two crossings. By removing the added column, we obtain a drawing of $H_{4}$ in the torus with at most two crossings. It follows from Lemma 3.7 that this drawing is equivalent to that in Fig. 9. Since this drawing does not extend to a drawing of $H_{4,1}$ with $\leqslant 2$ crossings, this gives us a contradiction.

Thus $H_{n, k}$ (for $(n, k) \neq(4,0)$ ), has crossing sequence $(n+k, n-1,0)$ as claimed.
Next we introduce one additional graph to get the crossing sequence ( $4,3,0$ ). We define the graph $\mathrm{H}_{3}^{+}$in the same way as $\mathrm{H}_{3}$ except that we have three rows instead of two. See Fig. 12.

Lemma 3.9. The graph $H_{3}^{+}$has crossing sequence $(4,3,0)$.
Proof. It follows from an argument as in Lemma 3.2 that $c r_{0}\left(H_{3}^{+}\right)=4$. Since $H_{3}^{+}-\tau_{0}-\tau_{1}$ is planar, it follows that $c r_{2}\left(H_{3}^{+}\right)=0$. It remains to show that $c r_{1}\left(H_{3}^{+}\right)=3$. Since $c r_{1}\left(H_{3}^{+}\right) \leqslant 3$, we need only to show the reverse inequality. Consider an optimal drawing of $\mathrm{H}_{3}^{+}$in the torus, and suppose (for a contradiction) that it has fewer than three crossings. If the first row contains a crossing, then by removing its edges, we obtain a drawing of a subdivision of $\mathrm{H}_{3}$ in the torus with at most one crossing-a contradiction. Thus, the first row must not have a crossing, and by a similar argument, the third row must not have a crossing. Now, we again remove the first row. This leaves us with a drawing of a subdivision of $\mathrm{H}_{3}$ in the torus with at most two crossings, and with the added property that one row ( $r_{2}$ in this $H_{3}$ ) has no crossings. By Lemma 3.7 this must be a drawing as in Fig. 8. A routine check of these drawings shows that none of them can be extended to a drawing of $\mathrm{H}_{3}^{+}$ with fewer than 3 crossings.

We require one added lemma for some simple crossing sequences.

Lemma 3.10. For every $a>1$ there is a graph with crossing sequence $(a, 1,0)$.

Proof. Let $G_{1}$ be a copy of $K_{5}$, let $G_{2}$ be the graph obtained from a copy of $K_{5}$ by replacing each edge, except for one of them, with $a-1$ parallel edges joining the same pair of vertices. Let $G$ be the disjoint union of $G_{1}$ and $G_{2}$. It is immediate that $c r_{0}(G)=a, c r_{2}(G)=0$, and $c r_{1}(G) \geqslant 1$. A drawing of $G$ in $\mathbb{S}_{1}$ with this crossing number is easy to obtain by embedding $G_{2}$ in the torus, and then drawing $G_{1}$ disjoint from $G_{2}$ with one crossing. Thus, $G$ has crossing sequence ( $a, 1,0$ ) as required.

Proof of Theorem 1.3. Let $(a, b, 0)$ be given with integers $a>b>0$. If $b=1$, then the previous lemma shows that there is a graph with crossing sequence $(a, b, 0)$. If $(a, b, 0)=(4,3,0)$ then Lemma 3.9 provides such a graph. Otherwise, Lemma 3.8 shows that the graph $H_{b+1, a-b-1}$ has crossing sequence ( $a, b, 0$ ).

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## Appendix B

## An Eberhard-Like Theorem for Pentagons and Heptagons

# An Eberhard-Like Theorem for Pentagons and Heptagons 

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#### Abstract

Eberhard proved that for every sequence $\left(p_{k}\right), 3 \leq k \leq r, k \neq 6$, of nonnegative integers satisfying Euler's formula $\sum_{k \geq 3}(6-k) p_{k}=12$, there are infinitely many values $p_{6}$ such that there exists a simple convex polyhedron having precisely $p_{k}$ faces of size $k$ for every $k \geq 3$, where $p_{k}=0$ if $k>r$. In this paper we prove a similar statement when nonnegative integers $p_{k}$ are given for $3 \leq k \leq r$, except for $k=5$ and $k=7$ (but including $p_{6}$ ). We prove that there are infinitely many values $p_{5}, p_{7}$ such that there exists a simple convex polyhedron having precisely $p_{k}$ faces of size $k$ for every $k \geq 3$. We derive an extension to arbitrary closed surfaces, yielding maps of


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arbitrarily high face-width. Our proof suggests a general method for obtaining results of this kind.

Keywords Eberhard theorem • Simple polyhedron • Planar graph • Face-width

## 1 Introduction

Consider a cubic (i.e., 3-regular) plane graph, and let $p_{k}(k \geq 1)$ denote the number of its $k$-gonal faces. It is a simple corollary of Euler's formula that

$$
\begin{equation*}
\sum_{k \geq 1}(6-k) p_{k}=12 \tag{1}
\end{equation*}
$$

It is natural to ask for which sequences $\left(p_{k}\right)_{k \geq 1}$ satisfying (1) there exists a cubic plane graph whose face sizes comply with the sequence $\left(p_{k}\right)$. This question is even more interesting when additional restrictions on the graph are given. The most important case is to consider graphs of three-dimensional convex polyhedra, so-called polyhedral graphs. By Steinitz's theorem, this is the same as requiring the graphs to be 3-connected. An important subcase is where the polyhedra are simple, in other words, where the corresponding graphs are cubic.

The general problem about the existence of polyhedral graphs with given face sizes is still wide open. However, there are many special cases that have been solved. For example [9, Theorem 13.4.1], it is known that there exists a simple polyhedron with six quadrangular faces and $p_{6}$ faces of size six if and only if $p_{6} \neq 1$; and there exists a simple polyhedron with twelve pentagonal faces and $p_{6}$ faces of size six (a "fullerene" graph) if and only if $p_{6} \neq 1$. A similar case of four triangular faces and $p_{6}$ faces of size 6 has infinitely many exceptions: such a polyhedron exists if and only if $p_{6}$ is even. We refer to [9] for a complete overview. The most fundamental result in this area is the following classical theorem of Eberhard [3], stating that there is always a solution, provided that we are allowed to replace $p_{6}$ (whose value does not affect the satisfaction of (1)) by a large enough integer.

Theorem 1.1 (Eberhard [3]) For every sequence $\left(p_{k}\right), 3 \leq k \leq r, k \neq 6$, of nonnegative integers satisfying (1), there are infinitely many values $p_{6}$ such that there exists a simple convex polyhedron having precisely $p_{k}$ faces of size $k$ for every $k \geq 3$, where $p_{k}=0$ if $k>r$.

Eberhard's proof is not only long and messy, but also some of its parts may not satisfy today's standards of rigor. Grünbaum [9] gave a simpler complete proof utilizing graphs and Steinitz's Theorem. This result was strengthened by Fisher [5], who proved that there is always a value of $p_{6}$ that satisfies $p_{6} \leq p_{3}+p_{4}+p_{5}+\sum_{k \geq 7} p_{k}$.

Grünbaum also considered a 4-valent analogue of Eberhard's theorem. Fisher [6] proved a similar result for 5-valent polyhedra, establishing existence for all sequences of face sizes with $p_{4} \geq 6$ that comply with Euler's formula.

Various other generalizations of Eberhard's theorem have been discovered. Papers by Jendrol' $[10,11]$ give a good overview and bring some of today's most general
results in this area. Some other relevant works include [1, 2, 4, 8, 13]. Several papers treat extensions of Eberhard's theorem to the torus [7, 12, 15, 16] and more general surfaces [10]. It is worth pointing out that on the torus there is precisely one admissible sequence (namely $p_{5}=p_{7}=1$ and $p_{i}=0$ for $i \notin\{5,7\}$ ), for which an Eberhard-type result with added hexagons does not hold [12].

In this paper we consider a similar problem that is also motivated by (1). Let us suppose that we are given face sizes as before, but we are only allowed to change $p_{5}$ and $p_{7}$ (or $p_{6-t}$ and $p_{6+t}$ for some $t, 1 \leq t \leq 3$ ). In this case, we think of $p_{k}$ (for $k \geq 3, k \neq 5,7$ ) as being fixed and $p_{5}, p_{7}$ as being free to choose. Equation (1) determines the difference $s=p_{7}-p_{5}$, and we are asking if there exist $p_{5}$ and $p_{7}=p_{5}+s$ with a polyhedral realization. We give an affirmative answer to this question and derive an extension solving the corresponding problem on an arbitrary closed surface. Our construction gives simple (i.e., 3-regular) polyhedral maps on a surface, and one can impose the additional conditions that these maps have large facewidth and that their graphs are 3-connected. More precisely, we prove the following:

Theorem 1.2 Let $\left(p_{k}\right), 3 \leq k \leq r, k \neq 5,7$, be a sequence of nonnegative integers, let $S$ be a closed surface, and let $w$ be a positive integer. Then there exist infinitely many pairs of integers $p_{5}$ and $p_{7}$ such that there is a 3-connected cubic map realizing $S$, with face-width at least $w$, having precisely $p_{k}$ faces of size $k$ for every $k \in\{3, \ldots, r\}$.

It is worth observing that we also fix the number $p_{6}$ of hexagonal faces. Secondly, observe that the extension of Eberhard's Theorem to a surface $S$ other than the sphere needs an adjustment in (1); the right-hand side has to be replaced by $6 \chi(S)$ where $\chi(S)$ is the Euler characteristic of $S$. However, in our setting the formula adjusts itself by using an appropriate number of pentagons and heptagons.

Finally, as we point out in Sect. 4, our proof suggests a general method for obtaining results of this kind.

## 2 Definitions

A finite sequence $p=\left(p_{3}, p_{4}, \ldots, p_{r}\right)$ is plausible for a closed surface $S$ if

$$
\begin{equation*}
\sum_{3 \leq k \leq r}(6-k) p_{k}=6 \chi(S) \tag{2}
\end{equation*}
$$

where $\chi(S)$ is the Euler characteristic of $S$. By Euler's formula, (2) is a necessary condition for the existence of a cubic graph embeddable in $S$ with precisely $p_{k} k$-gons for $3 \leq k \leq r$ and no other faces. If there exists a cubic graph which is 2-cell embeddable in $S$ with precisely $p_{k}$ faces of size $k$ for $3 \leq k \leq r$ and no other faces, then we say that $p$ is realizable in $S$. If $\sum_{3 \leq k \leq r}(6-k) p_{k}=0$, then we call $p$ a neutral sequence. For any two such sequences, one can consider their sum which is defined in the obvious way. Let us observe that the sum of a neutral sequence and a plausible sequence is a plausible sequence. We would like to understand in this context which plausible sequences are realizable and try to do so by asking when a sum of a
plausible sequence with an appropriate neutral sequence is realizable. For the neutral sequence $(0,0,0,1)$, this is Eberhard's theorem.

The most important building block in both Eberhard's and our proofs is a construction called a triarc. A triarc is a plane graph $T$ such that the boundary $C$ of the outer face of $T$ is a cycle, and moreover the following conditions are satisfied (examples are the graphs in Fig. 4 with the half-edges in the outer face removed):

- every vertex of $T-C$ has degree 3 in $T$;
- $C$ contains distinct vertices $x, y, z$ of degree 2 (called the corners of the triarc) such that the degrees (in $T$ ) of the vertices on each of the three paths in $C-\{x, y, z\}$ alternate between 2 and 3, starting and ending with a vertex of degree 2 .
A side of a triarc $T$ as above is a subpath of $C$ that starts and ends at distinct corners of $T$ and does not contain the third corner. The length of a side $P$ of $T$ is the number of inner vertices of degree 2 on $P$; note that although the corners of a triarc have degree 2 , they are not counted when calculating the lengths of its sides. A triarc with sides of lengths $a, b, c$ is called an $(a, b, c)$-triarc. Of course, we can flip or rotate such a triarc and consider it, for example, as a ( $b, a, c$ )-triarc.

Triarcs are very versatile tools. Firstly, if the length of some side of a triarc $T$ equals the length of some side of another triarc $R$, then $T$ and $R$ can be glued together along those sides to yield a new plane graph with all inner vertices having degree 3 ; see, for example, Fig. 1. Secondly, every triarc $T$ has zero total curvature; to see this, take two copies of $T$, turn one of them upside down, glue them along a common side to obtain a "parallelogram" (see Fig. 1 again), and identify opposite sides of this parallelogram to obtain a graph embeddable in the torus. But perhaps the most important property of triarcs is the possibility to "glue" them together to obtain larger triarcs; we describe this operation below.

Suppose we have an ( $a_{1}, b_{1}, c_{1}$ )-triarc and an ( $a_{2}, b_{2}, c_{2}$ )-triarc such that $b_{1}=2 m$ and $c_{2}=2 l$ are even. Then, we may combine these triarcs (and several pentagons and heptagons) to construct an $\left(a_{1}+a_{2}, b_{1}+b_{2}, c_{1}+c_{2}\right)$-triarc. To do this, we identify a corner (and an incident edge) of the first triarc with a corner (and an edge) of the second triarc-see Fig. 2-so that the two identified corners yield a vertex of degree 3 on a side of length $a_{1}+a_{2}$ in a new triarc. Then, we can add a "parallelogram" consisting of hexagons to obtain an $\left(a_{1}+a_{2}, b_{1}+b_{2}, c_{1}+c_{2}\right)$-triarc. However, we do not want to add hexagons. Instead, we decompose the parallelogram into tiles, each consisting of four hexagons as depicted in Fig. 2, and replace each of these tiles by two pentagons and two heptagons as indicated in Fig. 3. The "tile" on the right of Fig. 3 will be used several times in the sequel, and we shall refer to it as a H -tile.

We are going to use this operation of glueing two triarcs into a larger one several times in the following section.

Fig. 1 Glueing two triarcs along sides of equal length. The dots represent the corners of the triarcs


Fig. 2 Glueing two triarcs with two sides of even length together using the tile of Fig. 3



Fig. 3 In a configuration of 4 hexagons we may contract the central edge and then "uncontract" it in the other direction. A "tile" consisting of two pentagons and two heptagons results; we use such tiles in Fig. 2

## 3 Proof of Theorem 1.2

We are ready to state and prove our main result. Let us observe that, unlike Eberhard's theorem, we do not need to assume that the given face-sizes form a plausible sequence (although we make this assumption in the formulation of the theorem) because given a sequence $\left(p_{k}\right), 3 \leq k \leq r, k \neq 5,7$, the sequence can always be appended by appropriate values $p_{5}$ and $p_{7}$ to become plausible.

Theorem 3.1 Let $p=\left(p_{3}, p_{4}, \ldots, p_{r}\right)$ be a plausible sequence for the sphere. Then there exist infinitely many integers $n \in \mathbb{N}$ such that the sequence $p+n \cdot(0,0,1,0,1)$ is realizable in the sphere.

Proof We will give an explicit construction of a cubic graph embeddable in the sphere whose face sequence is of the form $p+n \cdot(0,0,1,0,1)$. The rough plan for this is as follows. For each face imposed by the sequence $p$, we create a basic triarc containing this face as well as some pentagons and heptagons. Then, we glue all these triarcs together and extend to a triarc with sides of suitable lengths. Finally, we construct a new triarc having the same side lengths and glue these two triarcs together (as explained later) to obtain the desired graph embedded in the sphere.

To construct a basic triarc for a $k$-gon (we will make $p_{k}$ copies of it), we surround the $k$-gon by three heptagons and $k-3$ pentagons as shown in the right half of Fig. 4


Fig. $4 \mathrm{~A}(4,4,3)$-triarc and a (2, 2, 4)-triarc
(where the $k$-gon we are surrounding happens to be a pentagon). Note that we can always make the basic triarc isosceles with the equal sides having even length. We call the $k$-gon we started with the nucleus of this triarc.

Having constructed all basic triarcs, our next step is to glue them all together to obtain a single triarc $T$ containing them all. We do so recursively, attaching one basic triarc at a time as shown in Fig. 2, where we use many copies of the H -tile in order to build the parallelogram needed. Each time we use this gluing operation, we are assuming that both triarcs in Fig. 2 are isosceles, with the equal sides having even length, and align them so that the two equal even sides are the upper left and upper right sides. Note that the resulting triarc is also isosceles with two equal sides of even length. Thus, we can continue recursively to glue all basic triarcs into one isosceles triarc $T$.

Our next aim is to enlarge $T$ into an equilateral triarc $T^{\prime}$ with sides of length $n$, where $n$ is a multiple of 8 and satisfies $n \equiv 2(\bmod 3)$, using only pentagons and heptagons. To this end, we will use the gluing operation of Fig. 2 and many copies of a (4, 4, 3)-triarc and a (2,2, 4)-triarc. Figure 4 shows how to construct those triarcs with pentagons and heptagons only.

Note that gluing $T^{\prime}$ with a (4, 4, 3)-triarc (as in Fig. 2) keeps it isosceles and decreases the difference of lengths between the "base" and the other two sides by 1 , while gluing with a ( $2,2,4$ )-triarc increases that difference by 2 . Thus, recursively gluing with such triarcs we can enlarge $T$ into an equilateral triarc $S$ with sides of even length.

Moreover, using the gluing operation of Fig. 2 three times, once with a (2, 2, 4)triarc and twice with a $(4,4,3)$-triarc, we can increase the side-lengths by $(2,2,4)+$ $(4,4,3)+(4,4,3)=(10,10,10)$. Thus we can increase the length of each side of $S$ by 10 . Since $10 \equiv 1(\bmod 3)$, we can use this operation to enlarge $S$ into an equilateral triarc $S^{\prime}$ with even sides of length $2(\bmod 3)$. Moreover, since performing this operation three times increases the length of each side by 30 , and $30 \equiv 6(\bmod 8)$, we can enlarge $S^{\prime}$ into an equilateral triarc $T^{\prime}$ with the length of each side being a multiple of 8 and congruent to 2 modulo 3 .

Fig. 5 Gluing $R$ and $T^{\prime}$
together along a "ring" consisting of pentagons and heptagons. This operation is possible because we made sure that every side of $T^{\prime}$, and thus also of $R$, has length congruent to $2(\bmod 3)$


Next, we are going to construct a triarc $R$ that has the same side lengths as $T^{\prime}$ but consists of pentagons and heptagons only. By gluing together a $(2,2,4)$-triarc, a $(2,4,2)$-triarc, and a $(4,2,2)$-triarc (that is, the same triarc in three different rotations), we get an $(8,8,8)$-triarc, which we will call $D$. Since the sides of $T^{\prime}$ have length a multiple of 8 , by gluing copies of $D$ together recursively as in Fig. 2 we can indeed construct a triarc $R$ that has the same dimensions as $T^{\prime}$.

We can now combine $R$ and $T^{\prime}$ together to produce a cubic graph tiling the sphere as shown in Fig. 5. By construction, this graph has for every $k \in \mathbb{N} \backslash\{0,1,2,5,7\}$, precisely $p_{k}$ faces of size $k$, and moreover it has at least $p_{5}$ pentagons and at least $p_{7}$ heptagons. Thus its face sequence is of the form $p+(0,0, n, 0, m)$ for some $n, m \in \mathbb{N}_{+}$. Since both $p$ and $p+(0,0, n, 0, m)$ satisfy Euler's formula (the former by assumption, the latter because the plane graph we just constructed implements it), we have $n=m$.

This completes the construction and shows the existence of one particular value of $n$ as desired. However, observe that the construction of $T^{\prime}$ and $R$ allows us to make the side lengths of these triarcs arbitrarily large. This shows that we can get examples for infinitely many values of $n$ and thus completes the proof.

We now turn from planar graphs to maps on arbitrary (compact) surfaces. A map on a surface $S$ is a graph together with a 2 -cell embedding in $S$. A map is polyhedral if all faces are closed disks in the surface and the intersection of any two faces is either empty, a common vertex, or a common edge. If the graph of the map is cubic, then we say that the map is simple.

A cycle contained in the graph of a map is contractible if it bounds a disk on the surface. The edge-width of a map $M$ is the length of a shortest noncontractible cycle in $M$. The face-width of $M$ is the minimum number of faces, the union of whose boundaries contains a noncontractible cycle. We refer the reader to [14] for more about the basic properties and the importance of these parameters of maps. At this point we only note that a map is polyhedral if and only if its graph is 3-connected and its face-width is at least three, see [14, Proposition 5.5.12]. We also note that if $r$ is the largest size of a face of $M$, then the edge-width of $M$ cannot exceed $\frac{r}{2}$ times the face-width of $M$.

We now restate and prove our main result, Theorem 1.2.

Corollary 3.2 Let $\left(p_{k}\right), 3 \leq k \leq r, k \neq 5,7$, be a sequence of nonnegative integers, let $S$ be a closed surface, and let $w$ be a positive integer. Then there exist infinitely many pairs of integers $p_{5}$ and $p_{7}$ such that the sequence $\left(p_{3}, p_{4}, p_{5}, \ldots, p_{r}\right)$ is realizable in $S$ and there is a 3-connected realizing cubic map of face-width at least $w$.

Proof Let us first describe a construction that does not necessarily achieve the desired face-width; we will later explain how to modify this construction in order to get large face-width.

The rough sketch of this construction is as follows. Firstly, we increase the number of hexagons in the sequence $\left(p_{k}\right)$ to $p_{6}^{\prime}:=p_{6}+2 h+c$, where $h$ is the number of handles of $S$ and $c$ the number of its crosscaps (by the surface classification theorem we may assume that one of $h, c$ is zero, but we do not have to). It follows from Theorem 3.1 that we can increase the numbers $p_{5}$ and $p_{7}$ of this sequence to some appropriate values so that the resulting sequence $p^{\prime}$ is realized by a map on the sphere. We will then use the $2 h+c$ auxiliary hexagons of this map we added above to introduce some handles and/or crosscaps. After doing so, all auxiliary hexagons will have disappeared, and we obtain a map on $S$ whose sequence of faces differs from $\left(p_{k}\right)$ by some pentagons and heptagons only.

More precisely, similarly to the proof of Theorem 3.1, we construct a basic triarc for each face in $p^{\prime}$, but with one modification: for each hexagon, we construct a triarc like the one in Fig. 6 (on the left) rather than one with two even sides of equal lengths (in fact, we need this modification for the auxiliary hexagons only, but we might as well use it for the original hexagons in $p$ as well).

Next, we proceed as in Theorem 3.1 to glue all basic triarcs together into one triarc $T$. However, since we now have basic triarcs with all sides odd (the ones of Fig. 6), the gluing operation of Fig. 2 will not work for these triarcs. For this reason, we first extend each such triarc into an equilateral triarc with even sides using three copies of the (2, 2, 4)-triarc of Fig. 4 as shown in Fig. 6 (right).

We continue imitating the proof of Theorem 3.1 to obtain a cubic graph $G$ embedded in a homeomorphic copy $S^{\prime}$ of the sphere that contains all basic triarcs. We will


Fig. 6 On the left: the new basic triarc for a hexagon. On the right: extending the triarc from the left into an equilateral triarc with even sides

Fig. 7 The situation arising after introducing a handle. The 12-cycle $C$ consists of the dashed and the thick edges

now perform some cutting and gluing operations on both $S^{\prime}$ and $G$ to obtain a new surface, homeomorphic to $S$, with a cubic graph $G^{\prime}$ embedded in it.

Suppose that $h>0$. Then, pick $h$ pairs $\left(F_{1}, F_{1}^{\prime}\right), \ldots,\left(F_{h}, F_{h}^{\prime}\right)$ of hexagonal faces of $G$ such that all the faces $F_{i}$ and $F_{i}^{\prime}$ are distinct (there are enough hexagonal faces by our choice of the sequence $p^{\prime}$ ). Now, for each pair ( $F_{i}, F_{i}^{\prime}$ ), perform the following operations. Cut out the two discs of $S^{\prime}$ corresponding to $F_{i}, F_{i}^{\prime}$ and glue their boundaries together with a half-twist; that is, each vertex of the boundary of $F_{i}$ is identified with the midpoint of an edge of $F_{i}^{\prime}$ and vice-versa. This operation creates a handle in $S^{\prime}$, and the embedded graph remains cubic; however, it also gives rise to some unwanted faces: the size of each face that was incident to $F_{i}$ or $F_{i}^{\prime}$ has now been increased by 1. We thus have the situation depicted in Fig. 7, where $C$ is the cycle of length 12 resulting from the boundaries of $F_{i}$ and $F_{i}^{\prime}$. Recall that since every hexagon is put in a basic triarc like the one in Fig. 6, the sizes of the faces on each side of $C$ alternate between 6 and 8 as shown in Fig. 7. But now, contracting and uncontracting each of the three thick edges (in the way explained in Fig. 3) turns each of the faces incident with $C$ into a heptagon.

On the other hand, if $c>0$, then pick $c$ distinct hexagonal faces $F_{1}, \ldots, F_{c}$, and for every $i$, cut out the disc corresponding to $F_{i}$ and glue in its place the outside of the hexagon of Fig. 8 with a half twist. Each such operation gives rise to a new crosscap and also to unwanted faces just like in Fig. 7. But again, contracting and uncontracting each of the three thick edges, we can turn all these unwanted hexagons and octagons into heptagons.

Thus, after all these operations have been completed, we obtain a surface with $h$ handles and $c$ crosscaps with a cubic graph embedded in it whose face sequence is $p+(0,0, n, 0, m)$ for some $n, m \in \mathbb{N}_{+}$. Note that all auxiliary hexagons in $p^{\prime}-p$ have disappeared after the above operations.

It is easy to check that our maps are 3-connected. Indeed, our "building blocks"the basic triarcs and the triarcs of Fig. 4-are 3-connected after suppressing the de-

Fig. 8 The gadget used to create a crosscap inside a hexagon

gree 2 vertices. This property is also true for triarcs in Fig. 6, and it is maintained by the gluing operation of Fig. 2. By gluing two triarcs along a "circumference," using the ring in Fig. 5, we get a 3-connected graph. The gadget we used for introducing cross-caps (Fig. 8) is 3-connected, and it is not hard to see that we maintain 3-connectivity when adding this gadget or when creating a handle as depicted in Fig. 7.

It remains to discuss how to modify this construction to obtain maps with arbitrarily large face-width. By the remark preceding Corollary 3.2, it suffices to construct maps with arbitrarily large edge-width $z$ since the face sizes are bounded from above by $r$. This is achieved as follows.

First of all, we make every basic triarc used in the construction large enough that the distance from its nucleus to the boundary of the triarc is at least $z$, and the length of each side of each triarc is at least $3 z$. To achieve this, we first glue the triarc with several (2, 2, 4)-triarcs (or any other triarcs) both on the left and on the right, to obtain a large triarc with the nucleus in the middle of the bottom side. Then we possibly glue it with a $(4,4,3)$-triarc to create a triarc with all sides even. Finally, we rotate the triarc by $120^{\circ}$ and perform more glueing with $(2,2,4)$-triarcs to get the nucleus away from the boundary. (The notions "left", "right", and " $120^{\circ}$ " in this paragraph refer to the gluing operation of Fig. 2.)

Next, we replace the auxiliary hexagons used in order to add handles and crosscaps with $6 N$-gons, where $N$ is odd and greater than $z / 2$. Of course, this will force us to add some more pentagons to our sequence $p_{k}$ to make it plausible. Note that we can generalize the triarc on the left of Fig. 6 so that the inner 6 -gon is replaced by a 6 N gon surrounded by three heptagons and $6 N-3$ pentagons, arranged in a symmetric way so that any two heptagons separate $2 N-1$ pentagons from the rest. We will make use of the fact that $2 N-1$ is odd. We need to adapt the right half of Fig. 6 as well, since the inner triarc has now grown larger. For this, note that each side of the inner triarc has now length $2 N+1$, and so in order to use the method of the right half of Fig. 6, the three peripheral triarcs must have a base of length $2 N+2$ (in addition to having their other two sides of equal length). Since we chose $N$ to
be odd, it turns out that $2 N+2$ is a multiple of four, and so we can construct the required peripheral triarcs by gluing several (2,2,4)-triarcs together, using Fig. 2, into a $(N+1, N+1,2 N+2)$-triarc.

Moreover, the crosscap gadget shown in Fig. 8 can be generalized so that the inner 6 -gon is replaced by a 6 N -gon that is surrounded by 3 N heptagons and 3 N pentagons, arranged alternatingly around the 6 N -gon (here it is also important that we chose $N$ to be odd).

When the time comes to insert crosscaps or glue pairs of such 6 N -gons together (after a half-twist), we obtain a similar configuration as in Fig. 7, but with $3 N$ thick edges. Some of these thick edges are surrounded by faces of sizes $8,6,8,6$ (as in Fig. 7), while others are surrounded by four hexagons or by one octagon and three hexagons. Note, however, that for parity reasons, we can make sure that every octagon is incident with a thick edge, and still every fourth edge on the dashed cycle is thick. Finally, the contract-uncontract operation of Fig. 3 turns these faces into pentagons and heptagons only.

It is easy to see that these changes did not hurt 3-connectivity. Let us now argue that the resulting map $G$ has edge-width at least $z$. Recall that the surface $S$ is obtained from a plane graph $G^{\prime}$, embedded in the sphere, that is composed of large basic triarcs $T_{1}, \ldots, T_{m}$, some large parallelograms used to glue the basic triarcs together into a large triarc $T$, and a remainder $X$ comprising the material we used to enlarge $T$ into $T^{\prime}$, the ring of Fig. 5, and the triarc $R$. Let $L_{i}$ be the nucleus of $T_{i}$. Then $S$ is obtained from $G^{\prime}$ by gluing the crosscap gadget into some of the $6 N$-gons $L_{i}$ and/or by identifying some pairs $L_{i}, L_{j}$ of the 6 N -gons to create handles.

We claim that for every basic triarc $T_{i}$ such that the nucleus $L_{i}$ of $T_{i}$ is a $6 N$-gon and
for every side $P$ of $T_{i}$, there is a set of $z$ pairwise disjoint $L_{i}-P$ paths.
Indeed, recall that in order to construct $T_{i}$, we first surrounded $L_{i}$ by several pentagons and heptagons, 6 N in total, to obtain a triarc $T_{i}{ }^{1}$, then we performed the operation of the right half of Fig. 6 to obtain a triarc $T_{i}^{2}$, and finally we enlarged this into a larger triarc $T_{i}^{3}=T_{i}$ using the operation of Fig. 2 several times (this final step was described later, in the part of the current proof concerning large face-width). Now given any side $P^{\prime}$ of $T_{i}^{2}$, it is possible to find, within $T_{i}^{2}$, a set of $z$ pairwise disjoint $L_{i}-P^{\prime}$ paths, see Fig. 9. Then, every time we use the operation of Fig. 2 while enlarging $T_{i}^{2}$ into $T_{i}^{3}$, it is possible to recursively propagate those paths to reach the side of $T_{i}^{3}$ corresponding to $P^{\prime}$; if $P^{\prime}$ is included within a side of $T_{i}^{3}$, then nothing needs to be done, and if not, then we can propagate our paths through the parallelogram of Fig. 2 while keeping them disjoint (this is true even after performing the contract-uncontract operations of Fig. 3). This proves our claim (3).

Next, we claim that any two nuclei $L_{i}, L_{j}$ can be joined by $z$ pairwise disjoint paths in $G^{\prime}$. Indeed, this follows easily from (3) and the fact that whenever we glue two triarcs $T, T^{\prime}$ together as in Fig. 2 by a parallelogram $R$ with side-lengths $m, n$, then we can find a set of $m$ pairwise disjoint paths within $R$ joining its two opposite sides of length $m$, as well as a set of $\min (m, n)$ pairwise disjoint paths within $R$ joining the sides of $T$ and $T^{\prime}$ incident with $R$.

Fig. 9 Constructing $z$ disjoint $L_{i}-P^{\prime}$ paths, in the case that the auxiliary hexagons are replaced with 42 -gons ( $6 N$ for $N=7$ ). In light gray are the
(2, 2, 4)-triarcs, in dark gray the H -tiles resulting from the glueing operation of Figs. 2 and 3. The empty triangle at the bottom is a part of the triarc, isomorphic to the top-left and top-right ones. As our paths do not use the bottom part, we don't show the details in the figure. The white triangular shape in the middle represents the nucleus. The 16 thick paths are the ones we need in order to prove that our graphs have large face-width


We distinguish two cases.
Case 1: the surface $S$ is orientable There are three types of noncontractible cycles in $G$. The first one comes from a path $P$ in $G^{\prime}$ that connects two nuclei $L_{i}, L_{j}$ such that these nuclei are glued to create a handle. As the distance from each nucleus to the boundary of the corresponding triarc $T_{i}$ is at least $z$, the length of $P$ is at least $z$ as well (even at least $2 z$ ).

The second type of noncontractible cycle $C$ comes from a cycle $C^{\prime}$ in $G^{\prime}$ such that $\left|C^{\prime}\right| \leq|C|$ and $C^{\prime}$ separates some nucleus $L_{i}$ from some other nucleus $L_{j}$ in $G^{\prime}$. We use the above construction of $z$ pairwise disjoint paths from $L_{i}$ to $L_{j}$ to conclude that $|C| \geq z$ as desired (in fact, we have $|C| \geq 2 z$ because the graph is cubic and so any two paths that have a common inner vertex must have a common edge).

The last type is similar to the second one: it is a cycle $C$ that crosses some cycle $L$ of $G$ obtained by gluing two nuclei $L_{i}, L_{j}$ to introduce a handle. Such a cycle $C$ comes from a $k$-tuple of paths in $G^{\prime}$, where $k$ is the number of times that $C$ crosses $L$, half of which paths have ends on $L_{i}$, and the other half of them on $L_{j}$. We may assume that none of these paths $P$ leaves the triarc containing the endpoints of $P$, for otherwise $|P| \geq z$ holds. We will consider again the $z$ pairwise disjoint paths connecting $L_{i}$ to $L_{j}$. In fact, we only need to consider their parts that are contained in the triarcs $T_{i}, T_{j}$ : Let these parts be $P_{i, 1}, \ldots, P_{i, z}$ (connecting $L_{i}$ to the boundary of $T_{i}$ ) and $P_{j, 1}, \ldots, P_{j, z}$ (connecting $L_{j}$ to the boundary of $T_{j}$ ). As the paths $P_{i, t}$ start regularly along two thirds of the nucleus $L_{i}$ and the same holds for $L_{j}$, we can use them to create $z / 2$ pairwise disjoint paths $Q_{1}, \ldots, Q_{z / 2}$ in $G$ connecting the boundary of $T_{i}$ to the boundary of $T_{j}$. Note that the part of the surface $S$ containing $T_{i}$ and $T_{j}$ is a cylinder, and the cycle $C$ goes around this cylinder. Thus $C$ must intersect all of the paths $Q_{t}$. As the graph is cubic, each intersection with a path has to use at least two vertices, proving again that $|C| \geq z$.

Case 2: $S$ is nonorientable In this case a noncontractible cycle $C$ in $G$ will either yield a cycle $C^{\prime}$ as above, in which case the same argument applies, or it will yield
a path $P^{\prime}$ in $G^{\prime}$ whose endpoints were identified when introducing crosscaps. Recall that we made every basic triarc used in the construction large enough that the distance from its nucleus to the boundary of the triarc is at least $z$; thus $P^{\prime}$ is, without loss of generality, contained within one of the triarcs in which a crosscap was introduced. With the help of Fig. 9 and Fig. 8 (modified with a 6 N -gon replacing the hexagon as described above), it is now not hard to see that $\left|P^{\prime}\right| \geq z$ as desired.

## 4 Other Neutral Sequences

In this paper we concentrated on the neutral sequence $(0,0,1,0,1)$, but we believe that our methods and results apply in a much more general setting-see also Sect. 5and it is the purpose of this section to explain this.

In Sect. 3 we showed that every plausible sequence can be extended into a realizable one by adding pentagons and heptagons only. In what follows we are going to give a rough sketch of a proof that an arbitrary neutral sequence $s$ can be used to extend any plausible sequence into a realizable one under the assumption that a couple of basic building blocks can be constructed using precisely the faces that appear in some multiple of $s$. We expect that our construction will help yield more general results in the future, by showing that these building blocks can indeed be constructed.

So let $p=\left(p_{3}, p_{4}, \ldots, p_{r}\right)$ be a plausible sequence for the sphere or the torus, and let $s=\left(p_{3}^{\prime}, p_{4}^{\prime}, \ldots, p_{t}^{\prime}\right)$ be a neutral sequence. In order to prove that there is some $n$ such that $p+n s$ is realizable, it suffices to find some $k \in \mathbb{N}$ for which it is possible to construct the following building blocks using precisely the faces that appear in some multiple of $s$ :
(i) a $(k, k, k)$-triarc
(ii) a $(k, k, k-1)$-triarc
(iii) for every nonzero entry $p_{l}$ in $p$, a triarc containing a face of size $l$, such that the length of two of the sides of this triarc is a multiple of $k$
(iv) a "ring" like the one in Fig. 5 (using the faces from $s$ in the right proportion rather than pentagons and heptagons) for combining two equilateral triarcs

Indeed, to begin with, construct a parallelogram with all sides of length $k$ out of two ( $k, k, k-1$ )-triarcs (supplied by (ii)) as shown in Fig. 10. (In figures explaining


Fig. 10 Constructing a parallelogram out of two $(k, k, k-1)$-triarcs

Fig. 11 Gluing $R$ and $T^{\prime}$ together. The black dots depict the faces imposed by the sequence $p$

our construction, we shall use triarcs made of hexagonal faces, but this is for illustration purposes only; in fact they have to be made of multiples of $s$.) This also allows us to construct any parallelogram with dimensions $m k, l k$ for every $m, l \in \mathbb{N}$.

Next, similarly to the construction in Theorem 3.1, construct a "basic" triarc as in (iii) for each face-size $l$ for which $p_{l} \neq 0$; in fact, we construct $p_{l}$ copies of this basic triarc for every $l$. Then, using the parallelograms, we constructed earlier, we recursively glue all those triarcs together into a single triarc $T$, in a manner very similar to the operation of Fig. 2.

By recursively gluing the resulting triarc with a ( $k, k, k-1$ )-triarc provided by (ii) using the glueing operation of Fig. 2, we can transform $T$ into an equilateral ( $m k, m k, m k$ )-triarc $T^{\prime}$ for some (large) $m \in \mathbb{N}$.

Using the glueing operation of Fig. 2 it is possible to construct a triarc $R$ with the same side-lengths as $T^{\prime}$, using only ( $k, k, k$ )-triarcs (provided by (i)) and the above parallelograms.

In the case of the sphere, we can combine $R$ and $T^{\prime}$ by using the "ring" provided by (iv) to complete the construction.

If the underlying surface $S$ is the torus, we glue $R$ and $T^{\prime}$ together along one of their sides to obtain a parallelogram and glue two opposite sides of this parallelogram together to obtain a cylinder $C$ both of whose bounding cycles are in-out alternating cycles of length $m k$, see Fig. 11. We then glue the two bounding cycles of $C$ together to obtain a realization of a torus.

If $p$ is plausible for some other surface $S$, then we would need additional gadgets like those used in the proof of Corollary 3.2.

## 5 Outlook

Trying to achieve a better understanding of the implications of Euler's formula, we studied the question of whether, given a plausible sequence $p$ and a neutral sequence $q$, it is possible to combine $p$ and $q$ into a realizable sequence $p+n q$, but we did so in very restricted cases. The general problem remains wide open; in particular, we would be interested to see an answer to the following problem.

Problem 5.1 Given a closed surface $S$, is it true that for every plausible sequence $p$ for $S$ and every neutral sequence $q$, there is an $n \in \mathbb{N}$ such that $p+n q$ is realizable in $S$ with the exception of only finitely many pairs $(p, q)$ ?
(As mentioned in the introduction, if $S$ is the torus, then the list of exceptional pairs ( $p, q$ ) is not empty.)

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## Appendix C

Cayley sum graphs and eigenvalues of (3,6)-fullerenes

# Cayley sum graphs and eigenvalues of (3, 6)-fullerenes 

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#### Abstract

We determine the spectra of cubic plane graphs whose faces have sizes 3 and 6 . Such graphs, " $(3,6)$-fullerenes," have been studied by chemists who are interested in their energy spectra. In particular we prove a conjecture of Fowler, which asserts that all their eigenvalues come in pairs of the form $\{\lambda,-\lambda\}$ except for the four eigenvalues $\{3,-1,-1,-1\}$. We exhibit other families of graphs which are "spectrally nearly bipartite" in the sense that nearly all of their eigenvalues come in pairs $\{\lambda,-\lambda\}$. Our proof utilizes a geometric representation to recognize the algebraic structure of these graphs, which turn out to be examples of Cayley sum graphs. © 2008 Elsevier Inc. All rights reserved.


## 1. Introduction

A (3, 6)-fullerene is a cubic plane graph whose faces have sizes 3 and 6. (In fact, Euler's formula implies that there are exactly four faces of size 3.) These graphs have received recent attention from chemists due to their similarity to ordinary fullerenes. (Such graphs are sometimes called (3,6)-cages in that community, but in graph theory this term already has a different, well-established meaning.) In 1995, Patrick Fowler (see [7]) conjectured the following result, which we prove here. Prior to this work, this result had been established for several subfamilies of ( 3,6 )-fullerenes [5,7,14]. Recall that the spectrum of a graph is the multiset of eigenvalues of its adjacency matrix.

[^2]

Fig. 1. Examples of some small $(0,3,6)$-fullerenes.

Theorem 1.1. If $G$ is $a(3,6)$-fullerene, then the spectrum of $G$ has the form $\{3,-1,-1,-1\} \cup L \cup(-L)$ where $L$ is a multiset of nonnegative real numbers, and $-L$ is the multiset of their negatives.

In fact we prove (as Theorem 3.2) an extended conjecture of Fowler et al. [7]. They propose that a generalized class of graphs called ( $0,3,6$ )-fullerenes also exhibit this "spectrally nearly bipartite" behavior. A semiedge of a graph is an edge with one endpoint, but unlike a loop, a semiedge contributes just one to both the valency of its endpoint ${ }^{6}$ and the corresponding diagonal entry of the adjacency matrix. In a plane embedding, a semiedge $s$ with endpoint $v$ is drawn as an arc with one end at $v$ which sits in a face $f$, and $s$ contributes one to the length of $f$. A $(0,3,6)$-fullerene is a connected 3 -regular graph, possibly with semiedges, embedded in the plane so that each face has length 3 or 6 . (The " 0 " in the above definition comes from the fact, that in physics literature, they treat semiedges as faces of length 0 .) Fig. 1 displays some examples of small $(0,3,6)$-fullerenes. It can be proved that $(0,3,6)$-fullerenes have at most four semiedges, see (1).

The outline of our proof is as follows. We show that every ( $0,3,6$ )-fullerene can be represented as a quotient of a certain lattice-like graph in the plane. This geometric description allows us to prove that these graphs are Cayley sum graphs. Then we call on a theorem which describes the spectral behavior of Cayley sum graphs in terms of the characters of the group.

In fact, the geometric description of $(0,3,6)$-fullerenes which is inherent in our proof is just a slight extension of a construction for ( 3,6 )-fullerenes which has been discovered by several authors [5,7,16], and follows easily from a deep theorem on the intrinsic metric of polygonal surfaces by Alexandrov [1]. In Section 4, we give a proper exposition of this construction, and a proof that it is universal.

With this construction in hand, it is possible to explicitly compute the spectrum of $(0,3,6)$ fullerenes, and in Section 5 we detail precisely how this computation can be carried out. Finally, in Section 6, we generalize this construction to show how a general Cayley sum graph can be obtained from a similar construction.

## 2. Cayley sum graphs

Let $\Gamma$ be a finite additive abelian group, and let $S \subseteq \Gamma$. We define the Cayley sum graph Cay $S(\Gamma, S)$ to be the graph $(V, E)$ with $V=\Gamma$, and $u v \in E$ if and only if $u+v \in S$. If $S$ is a multiset, then $\operatorname{CayS}(\Gamma, S)$ contains multiple edges, and if there exists $u \in \Gamma$ with $2 u \in S$, then the edge $u u$ is a

[^3]semiedge. This definition is a variation of the well-studied Cayley graph Cay $(\Gamma, S)$, in which $u v$ forms an edge if and only if $u-v \in S$.

In contrast with Cayley graphs, there are only a few appearances of Cayley sum graphs in the literature (see [9] and references therein). For this reason we state some of their elementary properties. The graph $G=\operatorname{CayS}(\Gamma, S)$ is $|S|$-regular. While $G$ is not generally vertex-transitive, the map $x \mapsto x+t$ is an isomorphism from $G$ to $\operatorname{CayS}(\Gamma, S+2 t)$, for every $t \in \Gamma$. Finally, the squared graph $G^{(2)}$, which has an edge for each walk of length 2 in $G$, is the ordinary Cayley graph Cay $(\Gamma, S-S$ ) where $S-S$ is the multiset $\left\{s_{1}-s_{2} \mid s_{1}, s_{2} \in S\right\}$.

The spectrum of a (finite abelian) Cayley graph Cay $(\Gamma, S)$ is easy to describe (see [10, Ex. 11.8] or [11], where the nonabelian case is dealt with). Every character $\chi$ of $\Gamma$ is a (complex-valued) eigenvector corresponding to the eigenvalue

$$
\chi(S):=\sum_{s \in S} \chi(s) .
$$

We may assume $\Gamma=\mathbb{Z}_{n_{1}} \times \cdots \times \mathbb{Z}_{n_{u}}$, where $|\Gamma|=\prod_{i} n_{i}$ and $\mathbb{Z}_{k}$ denotes the cyclic group of order $k$. To each $a=\left(a_{1}, \ldots, a_{u}\right) \in \Gamma$ we associate the group character

$$
\chi_{a}:\left(x_{1}, \ldots, x_{u}\right) \mapsto \exp \left(2 \pi i \sum_{j} \frac{a_{j} x_{j}}{n_{j}}\right) .
$$

The characters for $a$ and $-a$ satisfy $\chi_{-a}(x)=\overline{\chi_{a}(x)}$, so $\chi_{a}$ is a real-valued (indeed $\pm 1$-valued) eigenvector of $\operatorname{Cay}(\Gamma, S)$ if and only if $a$ is an involutive group element. If $a$ is not involutive, then the real and imaginary parts of $\chi_{a}$ provide real-valued eigenvectors for the conjugate pair of eigenvalues $\chi_{a}(S), \chi_{-a}(S)$.

Cayley sum graphs exhibit a similar phenomenon. Let $R=\left\{\chi_{a} \mid a+a=0\right\}$ be the real-valued characters of $\Gamma$, and let $C$ be a set containing exactly one character from each conjugate pair $\left\{\chi_{a}, \chi_{-a}\right\}$ (where $a \in \Gamma$ and $a+a \neq 0$ ). So the set of characters of $\Gamma$ is $R \cup\{\chi, \bar{\chi} \mid \chi \in C\}$. Versions of the following result can be found in the literature $[6,2]$.

Theorem 2.1. Let $G=\operatorname{CayS}(\Gamma, S)$ be a Cayley sum graph on a finite abelian group $\Gamma$, and let $R, C$ be as above. The multiset of eigenvalues of $G$ is

$$
\{\chi(S): \chi \in R\} \cup\{ \pm|\chi(S)|: \chi \in C\} .
$$

The corresponding eigenvectors are $\chi$ (for $\chi \in R$ ), and the real and the imaginary parts of $\alpha \chi$ (for $\chi \in C$ with a suitable complex scalar $\alpha$ which depends only on $\chi(S)$ ).

Proof. Let $\chi$ be a character of $\Gamma$ and $u \in \Gamma$ a vertex of $\operatorname{Cay} S(\Gamma, S)$. Then

$$
\sum_{v \in N(u)} \chi(v)=\sum_{s \in S} \chi(s-u)=\chi(S) \overline{\chi(u)} .
$$

This shows that every real-valued character is an eigenvector corresponding to the eigenvalue $\chi(S)$. If $\chi \in C$, then $\chi$ is not an eigenvector. In this case we choose a complex number $\alpha$ such that $|\alpha|=1$ and $\alpha^{2} \chi(S)=|\chi(S)|$ and we define $\chi(v)=\alpha \chi(v)$. It follows that for every $u \in \Gamma$,

$$
\sum_{v \in N(u)} x(v)=\alpha^{2} \chi(S) \cdot \alpha^{-1} \overline{\chi(u)}=|\chi(S)| \cdot \overline{x(u)} .
$$

Consequently, $\operatorname{Re} x$ and $\operatorname{Im} x$ are real eigenvectors corresponding to eigenvalues $|\chi(S)|$ and $-|\chi(S)|$, respectively. Both of these vectors are nonzero, as they generate the same 2-dimensional (complex) vector space as the characters $\{\chi, \bar{\chi}\}$. This, together with the orthogonality of characters, implies that we have described the complete set of eigenvectors, and thus the entire spectrum of $\operatorname{Cay} S(\Gamma, S)$.

## 3. (0, 3, 6)-fullerenes as Cayley sum graphs

The goal of this section is to prove that $(0,3,6)$-fullerenes are Cayley sum graphs, and to subsequently prove Fowler's conjecture regarding their spectra.

The proof of Theorem 3.1 utilizes structural properties of 3-regular hexagonal tilings (hereafter called hexangulations) of the torus. This class of graphs was classified by Altshuler [4] and studied by many others (e.g., Thomassen [15]). In a recent work of Alspach and Dean [3], it is shown that they are indeed Cayley graphs, and a description of the group is given. Although the properties we require of these graphs are similar to those found elsewhere, our approach is novel since it is inherently geometric.

A polygonal surface $\mathcal{H}$ is a connected 2-manifold without boundary which is obtained from a collection of disjoint simple polygons in $\mathbb{E}^{2}$ by identifying them along edges of equal length. Thus we view $\mathcal{H}$ both (combinatorially) as an embedded graph with vertices, edges, and faces, and as a manifold with a (local) metric inherited from $\mathbb{E}^{2}$.

Theorem 3.1. Every $(0,3,6)$-fullerene is isomorphic to a Cayley sum graph for an abelian group which can be generated by two elements.

Proof. Let $G$ be a cubic $(0,3,6)$-fullerene with vertex set $V$. Let $G_{2}=G \times K_{2}$ (the categorical graph product); $G_{2}$ is also known as the Kronecker double cover of $G$. Let ( $V_{\bullet}, V_{\circ}$ ) be the corresponding bipartition of $V\left(G_{2}\right)$, and for every $v \in V$, let $v_{\bullet} \in V_{\bullet}$ and $v_{\circ} \in V_{\circ}$ be the vertices of $G_{2}$ which cover $v$. Every semiedge $v v \in E(G)$ lifts to the edge $v_{\bullet} v_{\circ}$ in $G_{2}$. Each facial walk of $G$ bounding a face of size 6 lifts to two closed walks of length 6 in $G_{2}$, and each facial walk of $G$ bounding a face of size 3 lifts to a closed walk of length 6 in $G_{2}$. Accordingly, we may extend $G_{2}$ to a polygonal surface $\mathcal{H}$ by treating all edges as having equal length and adding a regular hexagon to each closed walk which is the preimage of a facial walk of $G$, with clockwise orientation as given by the clockwise orientation of that face. Now, $\mathcal{H}$ is an orientable polygonal surface, all vertices have degree three, and all faces are regular hexagons, so $\mathcal{H}$ is a regular hexangulation of the flat torus. Let $\widetilde{\mathcal{H}}$ be the universal cover of $\mathcal{H}$ and let $\mathbf{p}: \widetilde{\mathcal{H}} \rightarrow \mathcal{H}$ be the covering map. Then $\widetilde{\mathcal{H}}_{\sim}$ (with the metric inherited from $\mathcal{H}$ ) is the regular hexangulation of the Euclidean plane. We define $\widetilde{V}_{\bullet}=\mathbf{p}^{-1}\left(V_{\bullet}\right), \widetilde{V}_{\circ}=\mathbf{p}^{-1}\left(V_{\circ}\right)$, and $\tilde{x}=\mathbf{p}^{-1}(x)$ for $x \in V_{\bullet} \cup V_{\circ}$.

Fix a vertex $u_{\bullet} \in V_{\bullet}$, and treat $\tilde{\mathcal{H}}$ as a regular hexangulation of $\mathbb{E}^{2}$ with $\mathbf{p}((0,0))=u_{\bullet}$. This equips $\widetilde{\mathcal{H}}$ with an (additive abelian) group structure. The point set $\widetilde{V}_{\bullet}$ is a geometric lattice. The point set $\tilde{u}_{\bullet}$ is a sublattice of $\widetilde{V}_{0}$. Any fundamental parallelogram of $\tilde{u}_{\bullet}$ is a fundamental region of the cover $\mathbf{p}$. We may identify $\mathcal{H}$ with $\mathcal{H} / \tilde{u}_{\bullet}$, and this equips $\mathcal{H}$ with a group structure whose identity is $u_{\text {. }}$.

For every $y \in \mathcal{H}(y \in \widetilde{\mathcal{H}})$ the map $x \mapsto x+y$ is an isometry of $\mathcal{H}(\widetilde{\mathcal{H}}$, respectively). This map may or may not preserve the combinatorial structure of $\mathcal{H}(\widetilde{\mathcal{H}})$. An isometry $\mu: \mathcal{H} \rightarrow \underset{\sim}{\mathcal{H}}$ is respectful if $\mu$ is an automorphism of the embedded graph associated with $\mathcal{H}$. An isometry $\tilde{\mu}: \widetilde{\mathcal{H}} \rightarrow \widetilde{\mathcal{H}}$ is respectful if it is a lift of a respectful isometry of $\mathcal{H}$. Now, for every $y \in \widetilde{V}_{\bullet}$ the map $x \mapsto x+y$ is a respectful isometry of $\widetilde{\mathcal{H}}$. Accordingly, $V_{\bullet}$ is a subgroup of $\mathcal{H}$ with identity $u_{\bullet}$, and for every $y \in V_{\bullet}$ the map $x \mapsto x+y$ is a respectful isometry of $\mathcal{H}$.

Let $\rho$ be the automorphism of the graph $G_{2}$ given by the rule $\rho\left(v_{\circ}\right)=v_{\bullet}$ and $\rho\left(v_{\bullet}\right)=v_{\circ}$ for every $v \in V$. Now, $\rho$ extends naturally to a respectful isometry of $\mathcal{H}$ which preserves the orientation of the hexagons, but interchanges $V_{\circ}$ and $V_{\bullet}$. We choose a respectful isometry $\tilde{\rho}$ of $\mathbb{E}^{2}$ so that $\rho$ lifts to $\tilde{\rho}$. Because $\tilde{\rho}$ preserves the orientation of $\mathbb{E}^{2}$, the isometry $\tilde{\rho}$ is either a rotation or translation. Since $\tilde{\rho}$ is respectful and maps $\widetilde{V}_{\bullet}$ to $\widetilde{V}_{\circ}$, it easily follows that either $\tilde{\rho}$ is a rotation by $\pi$ about the center of an edge or a face, or $\tilde{\rho}$ is a rotation by $\pi / 3$ about the center of a face $F$.

We first consider the latter case. Here, all three vertices of $\widetilde{V}_{\bullet}$ which are on the boundary of $F$, lie in the same orbit of $\tilde{\rho}^{2}$. Since $\rho^{2}$ is the identity, all three vertices cover the same vertex, say $v_{\bullet}$ in $\mathcal{H}$. The other three vertices of $F$ cover $v_{0}$. In this case $G_{2}$ is the theta-graph with vertex set $\left\{v_{\bullet}, v_{0}\right\}$; we have $G \cong \operatorname{CayS}(\{0\},\{0,0,0\})$, the graph with one vertex and three semiedges, and there is nothing left to prove.

We henceforth assume that $\tilde{\rho}$ is a rotation by $\pi$. Let $x, y \in V_{\bullet}$ and choose $\tilde{x}, \tilde{y} \in \widetilde{V}_{\bullet}$ which project (respectively) to $x, y$. Then (using the fact that $\tilde{\rho}$ is a rotation by $\pi$ ) we find that

$$
\begin{aligned}
\rho(\rho(x)+y) & =\mathbf{p}(\tilde{\rho}(\tilde{\rho}(\tilde{x})+\tilde{y})) \\
& =\mathbf{p}(\tilde{x}-\tilde{y}) \\
& =x-y .
\end{aligned}
$$

In other words, for any fixed $y \in V_{\boldsymbol{\bullet}}$, conjugating the map on $\mathcal{H}$ given by $x \mapsto x+y$, by $\rho$ yields the map $x \mapsto x-y$.

We define a labeling $\ell: V_{\bullet} \cup V_{\circ} \rightarrow V_{\bullet}$ by the rule $\ell\left(v_{\mathbf{\bullet}}\right)=\ell\left(v_{0}\right)=v_{\boldsymbol{\bullet}}$. We regard $\ell$ to be a labeling of $V\left(G_{2}\right)$ by elements of the abelian group $V_{\boldsymbol{\bullet}}$. Let $v \in V$ and $y \in V_{\boldsymbol{\bullet}}$. Then we have

$$
\ell\left(v_{\bullet}+y\right)=\ell\left(v_{\mathbf{\bullet}}\right)+y
$$

and

$$
\begin{aligned}
\ell\left(v_{\circ}+y\right) & =\ell\left(\rho\left(v_{\circ}+y\right)\right) \\
& =\ell\left(\rho\left(\rho\left(v_{\bullet}\right)+y\right)\right) \\
& =\ell\left(v_{\bullet}-y\right) \\
& =\ell\left(v_{0}\right)-y .
\end{aligned}
$$

That is, the group $V_{\bullet}$ acts on the labels of points in $V_{\bullet}$ by addition and on the labels of points in $V_{0}$ by subtraction. Let $S$ be the multiset of labels of the three vertices in $V_{\circ}$ which are adjacent to $u_{0}$ (recall that $u_{\boldsymbol{0}}$ is the group identity for $V_{\boldsymbol{\bullet}}$ ). Then, for every $v_{\boldsymbol{\bullet}} \in V_{\boldsymbol{\bullet}}$, the labels of the three neighbors of $v_{0}$ in $G_{2}$ form the multiset $S-v_{0}$. In particular, $v$ and $v^{\prime}$ are adjacent vertices in $G$ if and only if $\ell\left(v_{\mathbf{\bullet}}\right)+\ell\left(v_{0}^{\prime}\right)=v_{\bullet}+v_{\mathbf{\bullet}}^{\prime} \in S$. It follows immediately from this that $G \cong \operatorname{Cay} S\left(V_{\boldsymbol{\bullet}}, S\right)$. Since $\widetilde{V}_{\mathbf{\bullet}}$ can be generated by two elements, $V_{\bullet}=\widetilde{V}_{\bullet} / \widetilde{u}_{\bullet}$ can also be generated by two elements, and this completes the proof.

We need only one quick observation before we resolve Theorem 1.1 and the extended conjecture of Fowler et al. If $G$ is a cubic plane graph with $s$ semiedges, and $f_{i}$ faces of size $i$ for every $i \geqslant 1$, then $3|V(G)|=2|E(G)|-s=\sum_{i \geqslant 1} i f_{i}$. Applying Euler's formula, we find that $\sum_{i \geqslant 1}(6-i) f_{i}=12-3 s$. In particular, every $(0,3,6)$-fullerene satisfies

$$
\begin{equation*}
s+f_{3}=4 \tag{1}
\end{equation*}
$$

Theorem 3.2. If $G$ is a $(0,3,6)$-fullerene with $s$ semiedges, then the spectrum of $G$ may be partitioned as $M \cup L \cup(-L)$ where one of the following holds:
(a) $s=0$ and $M=\{3,-1,-1,-1\}$,
(b) $s=2$ and $M=\{3,-1\}$,
(c) $s=3$ and $M=\{3\}$, or
(d) $s=4$ and $M=\{3,1\}$.

Proof. By the previous theorem, there is an abelian group $\Gamma$ which can be generated by two elements so that $G \cong \operatorname{CayS}(\Gamma, S)$ for some $S \subseteq \Gamma$ with $|S|=3$. By Theorem 2.1, we may partition the eigenvalues of $G$ into multisets $M, L,-L$ where $M=\{\chi(S): \chi \in R\}$ and $R$ is the set of $\pm 1$-valued characters of $\Gamma$. Every eigenvalue in $M$ is the sum of three integers in $\{ \pm 1\}$. The identity character corresponds to $3 \in M$. Since $G$ is not bipartite, ${ }^{7}$ we have $-3 \notin M$, so every other element of $M$ is $\pm 1$. The trace of the adjacency matrix is equal to $s$, and is also equal to the sum of the eigenvalues. Since $L$ and $-L$ sum to 0 , we conclude that $s=\sum M$.

We have $|R| \in\{1,2,4\}$ because $\Gamma$ has $2^{k}$ involutive elements, for some $k \leqslant 2$. If $|R|=1$, then $M=\{3\}$ and $s=3$ as in the statement. If $|R|=2$, then $s=\sum M=3 \pm 1$, so we have either the case

[^4]$s=2$ or $s=4$ of the statement. Finally, we assume $|R|=4$. By Eq. (1) we have $s \leqslant 4$, so $\sum M \in$ $\{0,2,4\}$. If $\sum M=0$, then $s=0$ ( $G$ is a (3, 6)-fullerene), and we have case (a). Finally, if $\sum M \in\{2,4\}$, then $M$ contains both a 1 and a -1 . By transferring these two entries from $M$ to the multisets $L$ and $-L$, we find ourselves again in either the case $s=2$ or the case $s=4$ of the statement. This completes the proof.

We remark that there are infinitely many $(0,3,6)$-fullerenes with $s$ semiedges, for each $s=0,2$, 3 , 4. As shown by Theorem 3.2, there are none with $s=1$, a fact that is non-trivial to prove from the first principles (compare Theorem 2 (with $k=3$ ) in [8, Section 13.4, p. 272]).

## 4. An explicit construction

It is known (see references in the Introduction) that all (3,6)-fullerenes arise from the so-called grid construction. Roughly speaking, the grid construction expresses the dual plane graph, which is a triangulation of the sphere, as a quotient of the regular triangular grid. The grid construction is also used by physicists $[5,14]$ (sometimes without formal justification) since it is a convenient way to classify $(3,6)$-fullerenes and compute their invariants.

We describe an extension of the grid construction and show that it characterizes the ( $0,3,6$ )fullerenes. The construction makes clear how semiedges arise. The group structure of $(0,3,6)$ fullerenes is explicitly determined as a quotient of the group of translations of the triangular grid. With this, we can easily find the Cayley sum graph representation via standard lattice computations, and thereby determine the spectrum and the eigenvectors of every $(0,3,6)$-fullerene.

In the following, $\mathcal{T}$ denotes the infinite triangular grid. Its vertices (called gridpoints) form the so-called $A_{2}$ lattice. The midpoint of any edge in $\mathcal{T}$ is called an edgepoint. The dual $G^{*}$ of a plane graph $G$ with semiedges is defined as an obvious extension of the dual of an ordinary graph; every semiedge in $G$ which is incident with vertex $v$ and face $f$ corresponds to a semiedge in $G^{*}$ which is incident with the dual vertex $f^{*}$ and the dual face $v^{*}$.

Construction 4.1. The following procedure results in a ( $0,3,6$ )-fullerene $G$.

1. Let $\triangle A B C$ be a triangle having no obtuse angle, and whose vertices are gridpoints of $\mathcal{T}$. Let $\bar{A}, \bar{B}, \bar{C}$ be the midpoints of the edges which are opposite to $A, B, C$ (respectively) in $\triangle A B C$.
2. Optionally, translate $\triangle A B C$ so that $A$ coincides with an edgepoint of $\mathcal{T}$.
3. From $\triangle A B C$, we fold an (isosceles) tetrahedron $Q=A \bar{A} \bar{B} \bar{C}$ by identifying the boundary segment $\bar{A} B$ with $\bar{A} C, \bar{B} C$ with $\bar{B} A$, and $\bar{C} A$ with $\bar{C} B$ (so $A, B$, and $C$ are identified into a single vertex in $Q$ ). The portion of $\mathcal{T}$ lying within $\triangle A B C$ becomes a finite graph $G^{*}$, possibly with semiedges, and drawn on the surface of $Q$.
4. Let $G$ be the dual of the plane graph $G^{*}$.

Every gridpoint within or on the boundary of $\triangle A B C$, except $A, \bar{A}, \bar{B}$, and $\bar{C}$, has degree 6 in $G^{*}$, and corresponds to a hexagonal face of $G$. After Step 2, each of $A, \bar{A}, \bar{B}, \bar{C}$ is either a gridpoint or an edgepoint of $\mathcal{T}$. If $X \in\{A, \bar{A}, \bar{B}, \bar{C}\}$ is a gridpoint, then $X$ becomes a vertex of degree 3 in $G^{*}$, and corresponds to a triangular face in $G$. If $X$ is an edgepoint, then $X$ becomes one end of a semiedge in $G^{*}$, which corresponds to a semiedge in $G$. It follows that Construction 4.1 results in a $(0,3,6)$ fullerene.

We remark that Construction 4.1 works even if $\triangle A B C$ has an obtuse angle (although it does not yield a geometric tetrahedron). However, this does not give any new ( $0,3,6$ )-fullerenes, as the following theorem shows. By forbidding obtuse triangles, we lose no generality and gain canonicality.

Theorem 4.2. Every (0, 3, 6)-fullerene arises from Construction 4.1.
Proof. Let $G$ be a $(0,3,6)$-fullerene. If $G$ has just one vertex, then $G$ arises from the construction when $\triangle A B C$ is a triangular face of $\mathcal{T}$. We assume next that $G$ has at least two vertices. The proof of Theorem 3.1 shows that the direct product $G_{2}=G \times K_{2}$ is a bipartite hexangulation $\mathcal{H}$ of the flat
torus. Moreover, $\mathcal{H}$ is the image of a covering map $\mathbf{p}: \widetilde{\mathcal{H}} \rightarrow \mathcal{H}$ from a hexagonal tessellation of the plane.

We further recall that there is an isometry $\rho$ of $\mathcal{H}$ which is respectful of $G_{2}$ and interchanges its partite sets $V_{\bullet}$. and $V_{0}$. This isometry lifts to an isometry of $\widetilde{\mathcal{H}}$ which is a rotation $\tilde{\rho}$ by $\pi$ about a point, say $A \in \widetilde{\mathcal{H}}$, which is either the center of a hexagonal face, or the midpoint of an edge of $\widetilde{\mathcal{H}}$. (More precisely $\tilde{\rho}: x \mapsto 2 A-x$ is the central symmetry through $A$.) The kernel of $\mathbf{p}$ (more precisely, the set $\mathbf{p}^{-1}(\mathbf{p}(A))$ ) is a geometric lattice $\Lambda$ in $\widetilde{\mathcal{H}}$, and rotation by $\pi$ about any point in the scaled lattice $\frac{1}{2} \Lambda$ projects to $\rho$. Let $B, C$ be points in $\widetilde{\mathcal{H}}$ such that the vectors $A B, A C$ form a lattice basis for $\Lambda$. By possibly translating $C$ by a (unique) integer multiple of $A B$, we can assume that $\triangle A B C$ has no obtuse angles. This lattice basis defines a fundamental parallelogram $A B D C$ where $A D=A B+A C$. Scaling the parallelogram by $\frac{1}{2}$ results in a fundamental parallelogram for $\frac{1}{2} \Lambda$ whose vertices we may label $A \bar{C} \bar{A} \bar{B}$ as in Construction 4.1.

Now each vertex $v$ of $G$ lifts to a unique pair of vertices $v_{\bullet}, v_{\circ}$ in the (half-open) parallelogram $A B D C$. If one of the vertices in $\left\{v_{0}, v_{0}\right\}$ is not on the boundary of $\triangle A B C$, then $v_{0}, v_{\mathbf{0}}$ are centrally symmetric about $\bar{A}$; we may represent $v$ by the unique vertex in $\left\{v_{0}, v_{0}\right\}$ which lies in $\triangle A B C$. Otherwise, both vertices in $\left\{v_{0}, v_{0}\right\}$ lie on the same edge of $\triangle A B C$, and they are centrally symmetric about either $\bar{A}, \bar{B}$, or $\bar{C}$, so they will be identified in Step 3 of the construction. In this way we obtain an isomorphic copy of $\mathcal{G}$. Finally, Construction 4.1 is stated in terms of the triangular grid $\mathcal{T}$, which is the plane dual of $\widetilde{\mathcal{H}}$.

We remark that Construction 4.1 in fact produces a $(0,3,6)$-fullerene $G$ rooted at a triangle or a semiedge labeled with $A$. Two triangles drawn in $\mathcal{T}$ result in isomorphic pairs ( $G, A$ ) if and only if the triangles are congruent. Therefore the map $\triangle A B C \mapsto G$ is at most 4-to-1 up to symmetries of $\mathcal{T}$.

## 5. Computing the spectrum

In this section, we use Construction 4.1 to compute the group and spectrum of any particular ( $0,3,6$ )-fullerene $G$.

The faces of $\mathcal{T}$ consist of up-triangles ( $\Delta$ ) and down-triangles ( $\nabla$ ). Let $\Lambda_{\bullet}$, be the set of (the centers of) the up-triangles in $\mathcal{T}$. We regard $\Lambda_{0}$ to be a lattice (called the $A_{2}$-lattice) generated by unit-length vectors $\mathbf{a}, \mathbf{b}$ with $\angle \mathbf{a b}=\pi / 3$. With $A$ being the gridpoint selected in Step 1 of Construction 4.1, we shall assume that the origin of $\Lambda_{\mathbf{\bullet}}$ is (the center of) the up-triangle $u_{\mathbf{0}}:=\triangle A(A+\mathbf{a})(A+\mathbf{b})$. Note that $\Lambda_{\bullet}$ is a translation of the gridpoints of $\mathcal{T}$ and corresponds to $\widetilde{V}_{\mathbf{0}}$ in the proof of Theorem 3.1. We denote by $\Lambda$ the sublattice of $\Lambda_{\bullet}$ generated by vectors $\overrightarrow{A B}$ and $\overrightarrow{A C}$. (A translation of $\Lambda$ is used in the proof of Theorem 4.2.) In Step 2, we translate $\triangle A B C$ by a vector

$$
\begin{equation*}
\mathbf{c}:=\frac{p_{1}}{2} \mathbf{a}+\frac{p_{2}}{2} \mathbf{b} \tag{2}
\end{equation*}
$$

for integers $p_{1}, p_{2}$. We may assume without loss of generality that $p_{1}, p_{2} \in\{0,1\}$, so, after Step 2 , the point $A$ is either a vertex or an edgepoint on the boundary of $u_{0}$.

Let $p, q, r, s$ be integers satisfying

$$
\begin{equation*}
A B=p \mathbf{a}+q \mathbf{b}, \quad A C=r \mathbf{a}+s \mathbf{b} . \tag{3}
\end{equation*}
$$

(Observe that the construction results in a graph $G$ with no semiedges if and only if each of $p_{1}, p_{2}, p, q, r, s$ is an even integer.) Let $\bar{A}, \bar{B}, \bar{C}$ and $T$ be as in the construction of $G$.

To express $G$ as a Cayley sum graph we label the faces of $\mathcal{T}$ with elements of the finite abelian group presented as $\Gamma=\langle\alpha, \beta \mid p \alpha+q \beta=0, r \alpha+s \beta=0\rangle$. We define $f: \Lambda_{\bullet} \rightarrow \Gamma$ by

$$
\begin{equation*}
f(i \mathbf{a}+j \mathbf{b})=i \alpha+j \beta, \tag{4}
\end{equation*}
$$

and extend $f$ to the down-triangles in such a way that triangles which are centrally symmetric with respect to $A$ receive the same value of $f$. The kernel of $f$ is the lattice $\Lambda$ generated by $A B$ and $A C$. We observe the following properties:

- $f$ assigns the same value to triangles that are identified during the 'folding' stage of the construction. This is because the triangles that are identified are symmetric with respect to one of $\bar{C}, \bar{B}$, and $\bar{A}$; each of these symmetries is a composition of the symmetry through $A$ and a translation by an element of $\Lambda=\operatorname{ker} f$.
- $f$ is a bijection from $V(G)$ to $\Gamma$. By construction, the up-triangles within the fundamental region ABDC correspond to elements of $\Gamma$. The down-triangles within the triangle $A B C$ correspond to up-triangles within DCB.
- If $u_{1}$ and $u_{2}$ are two up-triangles, then $f\left(u_{2}\right)=f\left(u_{1}\right)+f\left(u_{2}-u_{1}\right)$. If $d_{1}$ and $d_{2}$ are two downtriangles then $f\left(d_{2}\right)=f\left(d_{1}\right)-f\left(d_{2}-d_{1}\right)$.

Now let $u$ be any up-triangle and $d_{1}, d_{2}, d_{3}$ its neighbors. We define the sum-set $S=\left\{f(u)+f\left(d_{i}\right) \mid\right.$ $i=1,2,3\}$. From the above-mentioned properties of $f$ it follows that $S$ does not depend on the choice of $u$. The symmetry around $A$ shows that we get the same sum-set if we consider neighbors of a down-triangle to define $S$. It follows that $G \cong \operatorname{CayS}(\Gamma, S)$.

We can explicitly compute $\Gamma$ and $S$ by applying standard lattice computations. We recall that the Smith normal form of a nonsingular integer matrix $M$ is the unique matrix $\operatorname{diag}\left(\delta_{1}, \delta_{2}, \ldots, \delta_{k}\right)=U M V$ where $U$ and $V$ are unimodular and $\delta_{1}\left|\delta_{2}\right| \ldots$. The product $\delta_{1} \delta_{2} \cdots \delta_{i}$ is the g.c.d. of the order $i$ subdeterminants of $M$, whenever $1 \leqslant i \leqslant k$ (see, e.g., [12, Section 4.4]).

Lemma 5.1. Let $G$ be a $(0,3,6)$-fullerene obtained from Construction 4.1, and let $\mathbf{c}, p, q, r$, $s$ be as in (2) and (3). Let $\operatorname{diag}(m, n)=U M V$ be the Smith normal form of the matrix $M=\binom{p r}{q}$. Let $\mathbf{u}, \mathbf{v}$ denote the columns of $U$. Then $G=\operatorname{CayS}(\Gamma, S)$ where $\Gamma=\mathbb{Z}_{m} \times \mathbb{Z}_{n}$ and

$$
S=\left\{\left(p_{1}-1\right) \mathbf{u}+p_{2} \mathbf{v}, p_{1} \mathbf{u}+\left(p_{2}-1\right) \mathbf{v},\left(p_{1}-1\right) \mathbf{u}+\left(p_{2}-1\right) \mathbf{v}\right\} .
$$

Here we interpret each column vector $\binom{x_{1}}{x_{2}} \in S$ to be the group element $\left(x_{1} \bmod m, x_{2} \bmod n\right) \in \Gamma$.
Proof. The columns of the matrix $B:=(\mathbf{a}, \mathbf{b})$ form a lattice basis for $\Lambda_{0}$. whereas those of $B M$ generate the sublattice $\Lambda$. Since $U$ and $V$ are unimodular, the columns of $B^{\prime}:=B U^{-1}$ also generate $\Lambda_{0}$. Accordingly, $\Lambda$ is generated by the columns of $B M V=B^{\prime} \operatorname{diag}(m, n)$. It follows that $\Gamma=$ $\Lambda_{\mathbf{0}} / \Lambda \cong \mathbb{Z}_{m} \times \mathbb{Z}_{n}$. If we index the up-triangles with respect to the basis $B^{\prime}$, then the mapping $f: B^{\prime}\binom{\left(i^{\prime}\right)}{j^{\prime}} \mapsto\left(i^{\prime} \bmod m, j^{\prime} \bmod n\right)$ is the one defined in (4). Changing the basis to $B=B^{\prime} U$, we find that $f(i \mathbf{a}+j \mathbf{b})=i \mathbf{u}+j \mathbf{v}$, where we again interpret $i \mathbf{u}+j \mathbf{v}$ to be an element of $Z_{m} \times Z_{n}$.

After Step 1 of the construction, the three down-triangles which are neighbours of $u_{0}$ reflect through $A$ to the up-triangles at $-\mathbf{a},-\mathbf{b}$ and $-\mathbf{a}-\mathbf{b}$. When $A$ is translated by $\mathbf{c}$ in Step 2 , the three up-triangles are accordingly translated by $2 \mathbf{c}=p_{1} \mathbf{a}+p_{2} \mathbf{b}$. Therefore

$$
S=\left\{f\left(\left(p_{1}-1\right) \mathbf{u}+p_{2} \mathbf{v}\right), f\left(p_{1} \mathbf{u}+\left(p_{2}-1\right) \mathbf{v}\right), f\left(\left(p_{1}-1\right) \mathbf{u}+\left(p_{2}-1\right) \mathbf{v}\right)\right\}
$$

as claimed.
We present a sample computation illustrating the determination of the group and spectrum.
Example 5.2. The example of Fig. 2 corresponds to $\left(p_{1}, p_{2}\right)=(0,0)$ and $(p, q, r, s)=(6,2,-2,6)$. All six integers are even, so the resulting graph $G$ has no semiedges. We compute the Smith normal form to be

$$
U M V=\left(\begin{array}{cc}
0 & 1 \\
-1 & -7
\end{array}\right)\left(\begin{array}{cc}
6 & -2 \\
2 & 6
\end{array}\right)\left(\begin{array}{cc}
-2 & -3 \\
1 & 1
\end{array}\right)=\left(\begin{array}{cc}
2 & 0 \\
0 & 20
\end{array}\right)
$$

Hence $\Gamma=\mathbb{Z}_{2} \times \mathbb{Z}_{20}$. Furthermore, the generating set is

$$
S=\{-\mathbf{u}+0 \mathbf{v}, 0 \mathbf{u}-\mathbf{v},-\mathbf{u}-\mathbf{v}\}=\left\{\binom{0}{1},\binom{1}{7},\binom{1}{8}\right\} .
$$



Fig. 2. An example of Construction 4.1.
This implies $G$ has eigenvalues $3,-1,-1,-1$, and

$$
\left\{ \pm\left|\varepsilon^{b}+(-1)^{a} \varepsilon^{7 b}+(-1)^{a} \varepsilon^{8 b}\right|: 0 \leqslant a \leqslant 1,1 \leqslant b \leqslant 9\right\}
$$

where $\varepsilon=e^{2 \pi i / 20}$.
If we were to translate $\triangle A B C$ by $\left(\frac{1}{2} \mathbf{a}, 0\right)$, then we get a $(0,3,6)$-fullerene $G^{\prime}$ with four semiedges. Here we have $\left(p_{1}, p_{2}\right)=(1,0)$, which has the effect of translating the generating set by $\mathbf{u}$. That is,

$$
G^{\prime}=\operatorname{CayS}\left(\mathbb{Z}_{2} \times \mathbb{Z}_{20},\{(0,0),(1,6),(1,7)\}\right),
$$

and the spectrum of $G^{\prime}$ is

$$
\{3,1,1,-1\} \cup\left\{ \pm\left|1+(-1)^{a} \varepsilon^{6 b}+(-1)^{a} \varepsilon^{7 b}\right|: 0 \leqslant a \leqslant 1,1 \leqslant b \leqslant 9\right\} .
$$

It is worth noting that the symmetric parts of the spectra of $G$ and $G^{\prime}$ coincide. The four semiedges of $G^{\prime}$ are incident with the vertices $(0,0),(1,0),(0,10),(1,10) \in \Gamma$.

## 6. The geometry of Cayley sum graphs

In Section 4 we saw how the geometric description of $(0,3,6)$-fullerenes in terms of the $A_{2}$ lattice implies that they are Cayley sum graphs. Therefore their eigenvectors are easy to calculate, and their spectra are "nearly bipartite." Here we explore the circumstances under which Cayley sum graphs arise from geometric lattices in this manner. In fact we will see that every Cayley sum graph arises as a quotient of two cosets of a geometric lattice. We then exhibit some families of Cayley sum graphs which have a recognizable crystallographic local structure.

It is an easy fact that a graph $G$ is a Cayley graph on a group $\Gamma$ if and only if $\Gamma$ is isomorphic to a subgroup of the automorphism group which acts regularly on $V(G)$. Next we shall describe a similar equivalence for Cayley Sum graphs. Let $G_{2}=G \times K_{2}$ be the Kronecker double cover with bipartition $\left(V_{\bullet}, V_{\circ}\right)$. Note that $G_{2}$ has a natural automorphism, $\rho$,-we call it the inversion map-which transposes the two vertices within each fibre. By following the proof of Theorem 1.1, we find that $G$ is a Cayley sum graph on an abelian group $\Gamma$ if (and only if) $\Gamma$ acts regularly on each of $V_{\bullet}$ and $V_{\circ}$ as a group of $G_{2}$-automorphisms, and this action satisfies

$$
\begin{equation*}
\rho^{-1} g \rho=-g \quad \text { for each } g \in \Gamma \tag{5}
\end{equation*}
$$

Our construction proceeds with a sequence of graphs

$$
\widetilde{G} \mapsto \widetilde{G}_{2} \mapsto G_{2} \mapsto G .
$$



Fig. 3. Constructing finite Cayley sum graph from a lattice. Illustrated with the $D_{2}$-lattice, resulting in a 28 -vertex Cayley sum graph which is also a 4-regular quadrangulation of the tetrahedron (one semiedge appears in each corner of the tetrahedron).

We start with a geometric lattice $\Lambda_{\bullet} \subset \mathbb{E}^{d}$ and a Cayley sum graph $\widetilde{G}=\operatorname{CayS}\left(\Lambda_{\bullet}, S\right)$. When $\widetilde{G}$ is drawn with edges as straight line segments, each generator $s \in S$ corresponds to a set of edges of $\widetilde{G}$ whose midpoints are concurrent at the point $\frac{1}{2} s$. Let $\Lambda_{\circ}$ be any nontrivial coset of $\Lambda_{\bullet}$, and let $A \in \mathbb{R}^{d}$ be such that $\Lambda_{0}=2 A+\Lambda_{0}$. Let $\tilde{\rho}: x \mapsto 2 A-x$ be the inversion map through $A$. Note that (since $\Lambda_{0}$ is a lattice) we have $\tilde{\rho}\left(\Lambda_{\bullet}\right)=\Lambda_{\mathfrak{\sim}}$. As above, we construct $\widetilde{G}_{2}=\widetilde{G} \times K_{2}$ with partite sets $\left(\widetilde{V}_{\mathbf{0}}, \widetilde{V}_{0}\right)=\left(\Lambda_{\bullet}, \Lambda_{0}\right)$, where the fibres of $\widetilde{G}_{2}$ are the orbits of $\tilde{\rho}$. Note that the adjacency rule in $\widetilde{G}_{2}$ is similar to that of Cayley graphs (vertices $u \in \Lambda_{0}$ and $v \in \Lambda_{\circ}$ are adjacent iff $u-v \in S-2 A$ ); the vertex set, however, is not a group.

The graph $\widetilde{G}_{2}$ is drawn in Euclidean $d$-space $\mathbb{E}^{d}$ with straight line segments for edges. Let $\mathbb{E}^{d} / \tilde{\rho}$ denote the quotient space (an orbifold) whose points are the $\tilde{\rho}$-orbits $\{x, \tilde{\rho}(x)\}, x \in \mathbb{E}^{d}$. Geometrically speaking, $\mathbb{E}^{d} / \tilde{\rho}$ is a cone with apex $A$ having the solid angle of a halfspace. By mapping each point in $\mathbb{E}^{d}$ to its $\tilde{\rho}$-orbit, we may view $\widetilde{G} \cong \widetilde{G}_{2} / \tilde{\rho}$ as being naturally embedded in $\mathbb{E}^{d} / \tilde{\rho}$. Every edge of $\widetilde{G}_{2}$ whose midpoint is $A$ folds to a semiedge of $\widetilde{G}$. In the case of $(0,3,6)$-fullerenes, $\widetilde{G}_{2}$ is the plane hexagonal grid, and $\widetilde{G}$ is a grid drawn on a cone where every face is a hexagon except at $A$, where $A$ is either the midpoint of a triangular face, or the end of a semiedge.

Now let $\Lambda$ be any sublattice of $\Lambda_{\mathbf{0}}$, and let $\mathbf{p}$ be the natural projection from $\mathbb{E}^{d}$ to the $d$-torus $\mathbb{E}^{d} / \Lambda$. Then $G_{2}:=\mathbf{p}\left(\widetilde{G}_{2}\right)$ is a finite bipartite graph with partite sets $\left(V_{\bullet}, V_{0}\right):=\left(\mathbf{p}\left(\Lambda_{\bullet}\right), \mathbf{p}\left(\Lambda_{0}\right)\right)$, which is embedded in $\mathbb{E}^{d} / \Lambda$. Then $\tilde{\rho}$ projects to $\rho$, a symmetry of order 2 in the $d$-torus. Evidently $\rho$ is an inversion map for $G_{2}$ satisfying (5) with $\Gamma=\Lambda_{\boldsymbol{\bullet}} / \Lambda$. Therefore $G \cong G_{2} / \rho$ is a finite Cayley sum graph embedded in the orbifold $\left(\mathbb{E}^{d} / \Lambda\right) / \rho$ (hereafter denoted by $\mathbb{E}^{d} / \rho \Lambda$ ). Let $\mathcal{A} \subset \mathbb{E}^{d} / \Lambda$ be the fixed points of $\rho$. Then $\mathcal{A}=\mathbf{p}\left(A+\frac{1}{2} \Lambda\right)$ consists of exactly $2^{d}$ points and $\rho$ acts on $\mathbb{E}^{d} / \Lambda$ as an inversion through any point in $\mathcal{A}$. As an orbifold, $\mathbb{E}^{d} / \rho \Lambda$ is orientable if and only if $d$ is even. To visualize $\mathbb{E}^{d} / \rho \Lambda$, it is convenient to select a fundamental region for $\mathbb{E}^{d} / \Lambda$ whose $2^{d}$ extreme points belong to $A+\Lambda$. Choose a hyperplane $H$, which contains the region's centroid and let $T$ be the part of the region which lies on the positive side of $H$. All points in $\mathcal{A}$ lie on the boundary of $T$ so we obtain $\mathbb{E}^{d} / \rho \Lambda$ by an appropriate gluing of the boundary of $T$. The graph $G$ is embedded in $T$ with each vertex $\{x, \rho(x)\}$ represented by the unique point in $\{x, \rho(x)\} \cap T$. For example, $\mathbb{E}^{2} / \rho \Lambda$ is an isosceles tetrahedron, whose four extreme points comprise $\mathcal{A}$. The grid construction of ( $0,3,6$ )-fullerenes corresponds to selecting $H$ to be a diagonal of a fundamental parallelogram. The Cayley sum graph $G$ has one semiedge for every point of $\mathcal{A}$ which lies on an edge of $G_{2}$. Fig. 3 summarizes the commuting projections and the four embedded graphs.

Since every finite abelian group is the quotient of two geometric lattices, it follows that every finite Cayley sum graph $G$ arises from a quadruple ( $\Lambda_{\bullet}, S, A, \Lambda$ ) as described above. By employing a linear transformation we can even assume that $\Lambda_{\bullet}=\mathbb{Z}^{d}$. We do not make this assumption here, since that would obfuscate the following examples. When the sum set $S$ is a set of lattice points which are close to $2 A$, then each edge of $\widetilde{G}_{2}$ is a short line segment, and $\widetilde{G}_{2}$ is often a recognizable bipartite crystallographic configuration. After selecting $\Lambda$ and applying the above construction, we obtain a finite Cayley sum graph embedded in $T$ with a local geometry that reflects the crystallographic structure of $\widetilde{G}_{2}$. We present some examples.

- For $d=1$, if $\widetilde{G}_{2}$ is the two-way infinite path, then $\widetilde{G}$ is the infinite ray with a semiedge at its origin, and $G_{2}$ is an even cycle. The inversion $\rho$ identifies points reflected through a line which bisects a pair of opposite edges of the cycle (when it is drawn as a regular polygon). Consequently, $G$ is a finite path with a semiedge at each end. It is easy to observe (either directly, or by realizing $G$ as a Cayley sum graph) that the spectrum of $G$ takes the form $M \cup L \cup(-L)$ where $M=\{2\}$ or $M=\{2,0\}$ (depending on the parity of $|V(G)|)$.
- (Grid-like examples.) If $\Lambda_{\bullet}=D_{d}$, the lattice of integer points of even weight, and $\Lambda_{0}=\Lambda_{\bullet}+$ $(1,0,0, \ldots)$, then $\Lambda_{\bullet} \cup \Lambda_{\circ}=\mathbb{Z}^{d}$, and we may (by a suitable choice of $S$ ) take $\widetilde{G}_{2}$ to be the standard cartesian grid. If $A=\left(\frac{1}{2}, 0,0, \ldots\right)$, then applying the construction with any sublattice $\Lambda$ of $\Lambda_{\bullet}$ leads to a Cayley sum graph $G$ having exactly $2^{d}$ semiedges.
If $d=2$, then $G$ is a 4-regular quadrangulation of an isosceles tetrahedron, with a semiedge at each tetrahedral vertex. Such a graph is illustrated in Fig. 3. The set of unmatched eigenvalues of $G$ is either $M=\{4\}$ or $M=\{4,0\}$. Indeed, every 4-regular quadrangulation of a sphere can be expressed in this way. To see this fact, we need only adapt the proof of Theorem 3.1.
Another possibility (for an odd $d>1$ ) is to start with the same $\Lambda_{\bullet}, \Lambda_{\circ}$ and $S$ as above, and to take $A=\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\right)$. Since $A$ is not on an edge of the 'hypercubic' grid, this results in a grid-like Cayley sum graph $G$ having fewer than $2^{d}$ semiedges. Indeed $G$ has no semiedges at all if $\Lambda$ is a sublattice of $2 \Lambda_{\text {. }}$.
- (Diamond-like examples.) Again we take $\Lambda_{\bullet}$ to be the $D_{d}$-lattice, but put $\Lambda_{0}=\Lambda_{\bullet}+\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \ldots\right)$. The set $\Lambda_{\bullet} \cup \Lambda_{\circ}$ is commonly called the generalized diamond packing, and is denoted by $D_{d}^{+}$(see [13, p. 119]). The diamond grid is the graph $\widetilde{G}_{2}$ in which each point in $\Lambda_{0}$ is joined to the $2^{d-1}$ nearest points in $\Lambda_{0}$. Putting $A=\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \ldots\right)$ results in a Cayley sum graph having at least $2^{d-1}$ semiedges. A more attractive option is to put $A=\left(\frac{5}{4}, \frac{1}{4}, \frac{1}{4}, \ldots\right)$, which lies on no edge of $\widetilde{G}_{2}$. Provided that $\Lambda$ is a sublattice of $2 \Lambda_{\bullet}$, this results in a Cayley sum graph having no semiedges. When $d=3$, this construction gives a class of Cayley sum graphs having the local structure of diamond crystal. Such graphs satisfy $M=\{4,0,-2,-2\}$. Another attractive class is based on $D_{8}^{+}$, otherwise known as the $E_{8}$ lattice.
- The 24 -dimensional Leech lattice $\Lambda_{24}$ arises as the union of two cosets of a lattice $h \Lambda_{24}$ which is obtained from the binary Golay code (see [13, p. 124]). This yields a particularly attractive class of crystallographic Cayley sum graphs of high dimension.

We have constructed infinite families of Cayley sum graphs whose spectra have the form $M \cup L \cup$ $(-L)$, where $M$ is a fixed finite multiset. It would be interesting to find other natural examples of this phenomenon.

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## Appendix D

Highly arc-transitive digraphs Structure and counterexamples

# HIGHLY ARC-TRANSITIVE DIGRAPHS - STRUCTURE AND COUNTEREXAMPLES 

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Two problems of Cameron, Praeger, and Wormald [Infinite highly arc transitive digraphs and universal covering digraphs, Combinatorica (1993)] are resolved. First, locally finite highly arc-transitive digraphs with universal reachability relation are presented. Second, constructions of two-ended highly arc-transitive digraphs are provided, where each 'building block' is a finite bipartite digraph that is not a disjoint union of complete bipartite digraphs. Both of these were conjectured impossible in the above-mentioned paper. We also describe the structure of two-ended highly arc-transitive digraphs in more generality, heading towards a characterization of such digraphs. However, the complete characterization remains elusive.

## 1. Introduction

A digraph $D$ consists of a set of vertices $V(D)$ and $\operatorname{arcs}$ (also termed edges) $E(D) \subseteq V(D) \times V(D)$. We consider digraphs without loops and rely on standard terminology and notation as in [2] or [4]. In particular, an edge $(u, v) \in E(D)$ is shortly written as $u v$ and interpreted as the edge from $u$ to $v$.

[^5]An $s$-arc in a digraph is an $(s+1)$-tuple of vertices $\left(v_{0}, v_{1}, \ldots, v_{s}\right)$ such that $v_{i-1} v_{i}$ is an edge for each $i=1, \ldots, s$. A digraph $D$ is $s$-arc transitive if for every two $s$-arcs $\left(v_{i}\right)_{i=0}^{s},\left(v_{i}^{\prime}\right)_{i=0}^{s}$, there is an automorphism $f$ of $D$ such that $f\left(v_{i}\right)=v_{i}^{\prime}$ for each $i$. To exclude trivialities, it is also assumed that $D$ has no isolated vertices and that every arc of $D$ lies on some $s$-arc.

The notion of $s$-arc transitive digraphs parallels that of $s$-arc transitive undirected graphs. For those, an $s$-arc corresponds to a nonretracting walk of length $s$. Celebrated result of Tutte [13] states that a finite 3-regular graph can be $s$-arc transitive only if $s \leq 5$. Weiss [14] extended this (using the classification of finite simple groups) to finite $r$-regular graphs ( $r>2$ ); these can be $s$-arc transitive only if $s \leq 8$. (Somewhat trivially, cycles are $s$-arc transitive for every $s$.)

A digraph is highly arc-transitive if it is $s$-arc transitive for every $s \geq 0$. As one may expect, this is very demanding definition. Indeed, the only connected finite highly arc-transitive digraphs are the directed cycles (including cycles of length 1 and 2). Among infinite digraphs, the number of highly arctransitive ones is much larger. Still, they are rather restricted, which makes the constructions nontrivial, and one may hope to characterize all such digraphs, at least to some extent.

The motivation to study highly arc-transitive digraphs does not come solely from combinatorics. There is an intimate connection to totally disconnected locally compact groups that is presented in Möller [9], see also Malnič et al. [6].

An obvious infinite highly arc-transitive digraph is the two-way-infinite directed path, which we shall denote by $Z$. Another immediate example is obtained when we replace each vertex of $Z$ by an independent set of size $k$ and every arc by a (directed) complete bipartite graph $\vec{K}_{k, k}$ - formally this is the lexicographic product $Z\left[\bar{K}_{k}\right]$ with $\bar{K}_{k}$ denoting the graph with $k$ vertices and no edges. Confirm also Lemma 4.3 and Theorem 4.5 in [3] for more on products and high arc-transitivity.

The question of what other highly arc-transitive digraphs exist has started a substantial amount of research. The question was originally considered by Cameron, Praeger, and Wormald [3]. They presented some nontrivial constructions (details can be found in Section 3) and worked on ways to describe all highly arc-transitive digraphs. One approach to this involves the reachability relation.

Given a digraph $D$, an alternating walk is a sequence $\left(v_{0}, v_{1}, \ldots, v_{s}\right)$ of vertices such that $v_{i} v_{i+1}$ and $v_{i} v_{i-1}$ are arcs of $D$ either for all even $i$ or for all odd $i$; informally, when visiting the vertices $v_{0}, v_{1}, \ldots, v_{s}$, we use the arcs of $D$ alternately in the forward and backward direction. When $e, e^{\prime}$
are two arcs of $D$, we say that $e^{\prime}$ is reachable from $e$, in symbols $e \sim e^{\prime}$, if there is an alternating walk which has $e$ as the first arc and $e^{\prime}$ as the last one. One can easily see that this is an equivalence relation. Moreover, this relation is preserved by any digraph automorphism. Thus, whenever $D$ is 1 -arc transitive, then the digraphs induced by the equivalence classes are isomorphic to a fixed digraph, which will be denoted by $R(D)$ ( $R$ stands for reachability).

It is shown in [3] that if the reachability relation has more than one class, then $R(D)$ is bipartite and a construction is presented that, for an arbitrary directed bipartite digraph $R$, gives a highly arc-transitive digraph $D$ with $R(D) \simeq R$. In fact, a universal cover for all such digraphs is constructed. Thus a question arises, whether there are highly arc-transitive digraphs for which the reachability relation is universal (by which it is meant that there is just one equivalence class), as this approach to classify highly arc-transitive digraphs would not work for them. Actually, such digraphs are rather easy to construct if we allow infinite degree. One example would be the digraph $Q$ whose vertex set are all rational numbers, $V(Q)=\mathbb{Q}$, and two vertices $u, v$ are adjacent if $u<v$. So, the following question was asked in [3].

Question 1.1. Is there a locally finite highly arc-transitive digraph with universal reachability relation?

In Section 2 we present a construction of such digraphs - showing, in effect, that highly arc-transitive digraphs form a richer class of digraphs than one might expect.

Many highly arc-transitive digraphs possess a homomorphism onto $Z$. That is a mapping $f: V \rightarrow \mathbb{Z}$ such that for every edge $u v$ we have $f(v)=$ $f(u)+1$. This is called property $Z$ in [3], and the authors ask, whether all locally finite highly arc-transitive digraphs have this property. The first examples of locally finite highly arc-transitive digraphs without property Z were constructed by Malnič et al. in [7]. Our digraphs with universal reachability relation provide further examples, as a digraph with property Z has infinitely many reachability classes.

Another approach to classify highly arc-transitive digraphs is to use the number of ends. (See [4] for the definition of an end of a graph.)

It is well known that every infinite vertex-transitive graph, and hence also every highly arc-transitive digraph, has 1,2 , or infinitely many ends. An example with two ends is $Z$, with infinitely many ends a tree (where the in-degree of all vertices is some constant $d^{-}$and the out-degree of all vertices is some constant $d^{+}$). An example of a highly arc-transitive digraph with
just one end is $Q$. Locally finite examples are known, but they are harder to construct. In a few words, one can construct them as horocyclic products of trees, see [8] for details.

Let us focus on two-ended digraphs. This class includes the aforementioned basic examples $Z$ and $Z\left[\bar{K}_{k}\right]$, as well as a more complicated construction by McKay and Praeger [3, Remark 3.4] that is also discussed in our Section 3 as Construction 1. This construction was generalized in [3, Definition 4.6].

Based on their generalization and the lack of other examples, it was conjectured in [3] that for each connected highly arc-transitive digraph $D$ with two ends, the reachability digraph $R(D)$ is either infinite, or a complete bipartite digraph. We disprove this conjecture in Section 3, where we present several constructions that behave in a more complicated way. Independently from us, Christoph Neumann has constructed counterexample to Conjecture 3.1 using a different method.

Finally, in Section 4 we work towards characterizing all two-ended highly arc-transitive digraphs. We show, in particular, that every such digraph either admits a quotient by which we can reduce it to a simpler structure, or some lexicographic product $G\left[\bar{K}_{k}\right]$ (digraph $G$ with cloned vertices) can be constructed by a rather complicated Construction 4 described in Section 4. This construction uses a finite digraph with colored edges as a 'template'. While this construction provides many complicated new examples and is shown to be universal (upto cloning of vertices), we are lacking full understanding of when precisely it gives rise to a highly arc-transitive digraph.

## 2. Highly arc-transitive digraphs with universal reachability relation

The following result answers Question 1.1 in the affirmative.
Theorem 2.1. There is a locally finite highly arc-transitive digraph for which the reachability relation is universal. In fact, for every composite integer $d \geq 4$ there is such digraph with all in-degrees and all out-degrees equal to $d$.

Proof. Pick integers $a, b \geq 3$. We will construct a digraph $G_{a, b}$, in which every vertex has in- and out-degree equal to $(a-1)(b-1)$ and which satisfies the conditions of the theorem. Let $T=T_{a, b}$ be the infinite tree with vertex set $A \dot{\cup} B$, where every vertex in $A$ has $a$ neighbours in $B$, and every vertex in $B$ has $b$ neighbours in $A$. Next, we define the desired digraph with $V\left(G_{a, b}\right)=$


Figure 1. The digraph $G_{3,3}$ - a part of the digraph (with the underlying tree), without directions of edges. Vertices of the set $A$ are small circles, vertices of $B$ are squares.


Figure 2. The digraph $G_{3,3}$ - description of the direction of edges. Vertices of the set $A$ are small circles, vertices of $B$ are squares.
$E\left(T_{a, b}\right)$. For each $e=u v \in E\left(T_{a, b}\right)$, where $u \in A, v \in B$, we add an arc from each $e_{1} \neq e$ incident with $u$ to each $e_{2} \neq e$ that is incident with $v$. For each such pair $e_{1}, e_{2}$ we put $c\left(e_{1}, e_{2}\right):=e$. We let $G=G_{a, b}$ be the resulting digraph; in Fig. 1 and 2 we display part of $G_{3,3}$.

First we prove that $G$ is highly arc-transitive. Suppose $\mathbf{e}=\left(e_{0}, e_{1}, \ldots, e_{s}\right)$ is an $s$-arc in $G$, and let $P(\mathbf{e})$ be $e_{0}, c\left(e_{0}, e_{1}\right), e_{1}, \ldots, c\left(e_{s-1}, e_{s}\right), e_{s}$, the corresponding path in $T$. Now let $\mathbf{e}^{\prime}$ be another $s$-arc in $G$. Obviously $P(\mathbf{e})$ and $P\left(\mathbf{e}^{\prime}\right)$ are paths in $T$ of the same length, both starting at a vertex of $B$. Consequently, there is an automorphism $\varphi$ of $T$ that maps $P(\mathbf{e})$ to $P\left(\mathbf{e}^{\prime}\right)$. The mapping that $\varphi$ induces on $E(T)=V(G)$ is clearly an automorphism of $G$ that sends $\mathbf{e}$ to $\mathbf{e}^{\prime}$.

We still need to show that the reachability relation of $G$ is universal. Suppose $e, e^{\prime} \in V(G)$ are adjacent as edges in $T$, and that $h$ (resp. $h^{\prime}$ ) is an arc of $G$ starting at $e$ (resp. $e^{\prime}$ ). We will show that $h \sim h^{\prime}$; this is clearly sufficient. Assume first that $e$ and $e^{\prime}$ share a vertex of $A$. Let $h_{1}, h_{2}$ be arcs of $G$ as depicted in the left part of Fig. 3 (recall that $a \geq 3$ ). Obviously $h, h_{1}, h_{2}, h^{\prime}$ is an alternating walk, thus $h \sim h^{\prime}$. Secondly, assume $e$ and $e^{\prime}$ share a vertex of $B$. In this case pick arcs $h_{1}, h_{2}$ according to the right part of Fig. 3, utilizing that $b \geq 3$. Now $h \sim h_{1}$ and $h_{2} \sim h^{\prime}$ according to the first case. This shows that $h_{1} \sim h_{2}$ and completes the proof.


Figure 3. Two arcs of $G_{a, b}$ that start at adjacent edges of $T_{a, b}$ are equivalent.

Remark. It is known that highly arc-transitive digraphs with universal reachability relation do not exist if indegrees $d^{-}$and outdegrees $d^{+}$are not the same [11], and neither they exist if $d^{+}=d^{-}$is a prime [3]. However, whenever $d^{+}=d^{-}$is not a prime, it can be written as $(a-1)(b-1)$ for $a, b \geq 3$, so Theorem 2.1 provides an example of such a digraph.

Note that the structure of the digraph $G_{a, b}$ can also be described as follows. Consider a partition of the vertices of $K_{a, a(b-1)}$ into $a$ copies of a star, $K_{1, b-1}$. Let us denote these copies by $S_{1}, \ldots, S_{a}$. We let $H$ be $K_{a, a(b-1)}-\cup_{i} E\left(S_{i}\right)$. Then we take countably many copies of $H$ and glue them together (in a tree-like fashion) by identifying in pairs the sets corresponding to some of the $S_{i}$ 's. From this description it is immediate that $G_{a, b}$ has universal reachability relation.

## 3. Two-ended constructions

As mentioned in the introduction, a highly arc-transitive digraph can have 1,2 , or infinitely many ends; in the rest of this paper we concentrate on the case of two ends. It is not hard to show (see the proof of Proposition 4.1) that every two-ended 1-arc transitive digraph $D$ has the following structure: the vertices can be partitioned as $V(D)=\bigcup_{i=-\infty}^{\infty} V_{i}$ and all arcs go from some $V_{i}$ to $V_{i+1}$. Moreover, if $D$ is also vertex-transitive, then each of the induced digraphs $B_{i}=D\left[V_{i} \cup V_{i+1}\right]$ is isomorphic to a fixed bipartite 'tile' $B$. If $B$ is a complete bipartite digraph $\vec{K}_{k, k}$, we get the basic example $Z\left[\bar{K}_{k}\right]$. If $B$ is not the complete bipartite digraph, then $D$ is not determined just by $B$, as we need to specify how are consecutive copies $B_{i}$ and $B_{i+1}$ of $B$ 'glued' together at $V_{i+1}$. It is easy to see that all components of $B$ are isomorphic to the reachability digraph $R(D)$. The following was conjectured in [3].

Conjecture 3.1 (Cameron, Praeger, and Wormald [3]). If $D$ is a connected highly arc-transitive digraph such that there exists a homomorphism $f: D \rightarrow Z$ and $f^{-1}(0)$ is finite, then $R(D)$ is a complete bipartite digraph.

Next, we describe several constructions. We start with the one found by McKay and Praeger [3, Remark 3.4], that, while nontrivial, concurs with the above conjecture. Next, we shall present our construction (Construction 2 ), disproving the conjecture. Continuing, we shall provide some more complicated examples. In Section 4, we introduce a very general construction and provide some evidence that this construction essentially describes all two-ended highly arc symmetric digraphs.

We want to mention here that recently (and independently) Christoph Neumann has constructed [10] counterexample to Conjecture 3.1 using a different setting. His method (as well as ours) allows for many modifications and extensions, however his and ours smallest counterexamples are isomorphic.

Construction 1 (McKay and Praeger [3, Remark 3.4]). Let $S$ be a finite set, $n$ a positive integer, and let $V=\mathbb{Z} \times S^{n}$. The set $V$ is considered as the vertex-set of the digraph in which two vertices $a=\left(i, a_{1}, \ldots, a_{n}\right)$ and $b=\left(i+1, b_{1}, \ldots, b_{n}\right)$ are adjacent if $a_{j}=b_{j+1}$ for each $j=1, \ldots, n-1$; no other edges are present.

Here, the digraph $B$ is a disjoint union of complete bipartite digraphs (more precisely, $B$ consists of $|S|^{n-1}$ copies of $\vec{K}_{|S|,|S|}$ ), thus $R(D)$ is $\vec{K}_{|S|,|S|}$. The fact that this is a highly arc-transitive digraph is easy to show directly, but also follows from our next constructions.

Construction 2. Let $T$ be a "template" - an arc-transitive digraph that is bipartite with parts $A_{1}, A_{2}$, all arcs directed from $A_{1}$ to $A_{2}$. Let $D$ be the digraph with vertex-set $V=\mathbb{Z} \times A_{1} \times A_{2}$, in which two vertices $\left(i, a_{1}, a_{2}\right)$, $\left(i+1, b_{1}, b_{2}\right)$ are connected if $\left(a_{1}, b_{2}\right) \in E(T)$. We define $V_{i}=\{i\} \times A_{1} \times A_{2} \subset V$.

It is clear from the definition that the digraph joining $V_{i}$ and $V_{i+1}$ is isomorphic to the bipartite digraph $B$ which is obtained from $T$ by taking $\left|A_{2}\right|$ copies of each vertex in $A_{1}$ and $\left|A_{1}\right|$ copies of each vertex in $A_{2}$, and replacing each arc in $T$ by the complete bipartite digraph $\vec{K}_{\left|A_{2}\right|,\left|A_{1}\right|}$. If $T$ is connected, then $B$ is isomorphic to $R(D)$. As shown by Theorem 3.2, the resulting digraph is highly arc-transitive. Thus, by taking $T$ to be $\vec{K}_{3,3}$ minus a matching (alternately oriented 6 -cycle) we get a counterexample to Conjecture 3.1.

Construction 3. The next construction is a common generalization of Constructions 1 and 2. Let $T$ be a ( $t-1$ )-arc-transitive template digraph, with vertices in $t$ "levels", $A_{1}, \ldots, A_{t}$ and each arc leading from $A_{j}$ to $A_{j+1}$ for some $j$. We shall denote by $T_{i}$ the subgraph of $T$ induced by $A_{i} \cup A_{i+1}$. Suppose that each vertex $v \in V(T) \backslash A_{1}$ has in-degree at least 1 , and each vertex $v \in V(T) \backslash A_{t}$ has out-degree at least 1 . Now, define a digraph $D=D(T)$ with vertex-set $V=\mathbb{Z} \times A_{1} \times A_{2} \times \cdots \times A_{t}$, in which two vertices $a=\left(i, a_{1}, a_{2}, \ldots, a_{t}\right)$ and $b=\left(i+1, b_{1}, b_{2}, \ldots, b_{t}\right)$ are adjacent if $\left(a_{j}, b_{j+1}\right) \in E(T)$ for each $j=1, \ldots, t-1$, and no other edges are present in $D$. Clearly, for $t=2$ we get Construction 2. Construction 1 of McKay and Praeger is a special case of this one, with $T$ consisting of $|S|$ disjoint paths.

Theorem 3.2. If $T$ is as in Construction 3, then the digraph $D(T)$ is connected and highly arc-transitive. If all graphs $T_{i}$ are connected then $R(D(T))$ (equivalence class of the reachability relation) is isomorphic to the subgraph of $D(T)$ induced by vertices $\{0,1\} \times A_{1} \times \cdots \times A_{t}$.
Proof. As before, let $V_{i}=\{i\} \times A_{1} \times \cdots \times A_{t}$. For a vertex $a \in V_{i}$, we denote its $j$-th component by $a_{j}$, starting with $a_{0}=i$ and having $a_{j} \in A_{j}$ for $j=1, \ldots, t$. First we show that $D=D(T)$ is connected. It is easy to see that the following statement suffices for this: for every $a \in V_{0}$ and $b \in V_{t}$, there is a directed $(a, b)$-path. In order to prove this, observe that every vertex of $T$ is a part of at least one directed path with $t$ vertices. Let $P_{i}\left(Q_{i}\right.$, resp.) be such a path containing $a_{i}$ ( $b_{i}$, resp.). We let $P_{i, j}$ denote the $j$-th vertex on $P_{i}$, so that $P_{i, i}=a_{i}$ (and, similarly, $Q_{i, i}=b_{i}$ ). Now we define vertices $c_{0}, c_{1}, \ldots, c_{t}$ in $D$ forming a directed path. For $i=0, \ldots, t$ we set $c_{i, 0}=i$ and

$$
c_{i, j}= \begin{cases}Q_{j-i+t, j} & \text { if } 1 \leq j \leq i, \\ P_{j-i, j} & \text { if } i<j \leq t .\end{cases}
$$

Clearly, $c_{0, j}=P_{j, j}$ for $1 \leq j \leq t$, thus $c_{0}=a$. Similarly, $c_{t, j}=Q_{j, j}$ for $1 \leq j \leq t$, thus $c_{t}=b$. Comparing $c_{i}$ and $c_{i+1}(0 \leq i<t)$, we see that $c_{i, j}$ and $c_{i+1, j+1}$ are consecutive vertices of $T$ on the same path, $P_{j-i}$ or $Q_{j-i+t}$. This shows that $c_{i}$ and $c_{i+1}$ are adjacent in $D$, and shows that $D$ is connected.

Next, we study the reachability relation. Let $B=T\left[V_{0} \cup V_{1}\right]$. Obviously, no alternating walk can leave $B$; we only need to show, that any two edges in $B$ are connected by an alternating walk. Let $x y$ and $u v$ be two such edges. Each of the bipartite graphs $T_{i}$ is connected (by assumption), thus every two of its edges are connected by an alternating walk. We will use this for edges $x_{i} y_{i+1}$ and $u_{i} v_{i+1}$ and let $a_{i}(j) b_{i+1}(j)$ be the $j$-th edge of this walk (with $j=0$ corresponding to the starting edge). We may assume that all of these walks are of the same length and each of them starts by "fixing the head of the edge": that is, for every $i=1, \ldots, t-1$ we have $b_{i+1}(0)=b_{i+1}(1)=y_{i+1}$. Put $b_{1}(0)=b_{1}(1)=y_{1}, b_{1}(j)=v_{1}$ for $j>1$. Put $a_{t}(0)=x_{t}, a_{t}(j)=u_{t}$ for $j>0$. Finally, put $a_{0}(j)=0$ and $b_{0}(j)=1$ for all $j$. By construction, edges $a(j) b(j)$ form an alternating walk in $D$ connecting $x y$ and $u v$. It follows that $R(D)$ is isomorphic to $B$.

To prove that $D$ is highly arc-transitive, we describe some of its automorphisms. A trivial one is a shift in the first coordinate, $\tau: a \mapsto\left(a_{0}+1, a_{1}, \ldots, a_{t}\right)$. More interesting automorphisms are those that preserve the levels $V_{i}$. They come from the automorphisms of $T$. Let $\varphi \in \operatorname{Aut}(T)$. Let $\psi: V(D) \rightarrow V(D)$ be the mapping that applies $\varphi$ on the $j$-th coordinate in $V_{j}$ for $j=1, \ldots, t$ and is identity elsewhere. We shall show that $\psi$ is an automorphism of $D(T)$. Suppose $a b \in E(D)$, but $\psi(a) \psi(b) \notin E(D)$. Since $\psi$ preserves the sets $V_{i}$, from the construction of $D(T)$ (Construction 3) it follows, that there exists $j=1, \ldots, t-1$ such that $\psi(a)_{j} \psi(b)_{j+1} \notin E(T)$. By the definition of $\psi$, we conclude that $a \in V_{j}$ and $b \in V_{j+1}$. Moreover, $\psi(a)_{j}=\varphi\left(a_{j}\right)$ and $\psi(b)_{j+1}=\varphi\left(b_{j+1}\right)$. By assumption, $a b \in E(D)$, so $a_{j} b_{j+1}$ is an edge of $T$, and as $\varphi$ is an automorphism of $T, \varphi\left(a_{j}\right) \varphi\left(b_{j+1}\right)$ is an edge of $T$ as well. This contradicts our assumption and proves that $\psi$ is a homomorphism $D \rightarrow D$. Since $\psi$ is invertible (as $\psi^{-1}$ comes from the inverse automorphism $\varphi^{-1}$ of $T$ by the same construction as $\psi$ from $\varphi$ ), we conclude that $\psi$ is an automorphism of $D$.

Let $\Psi$ be the set of all automorphisms $\psi$ that are obtained from $\varphi \in$ $\operatorname{Aut}(T)$ in the way as described above. We claim that the group generated by $\tau$ and $\Psi$ acts transitively on the $s$-arcs in $D$ (for every $s$ ). Let $\left(v_{i}\right)_{i=0}^{s}$, $\left(v_{i}^{\prime}\right)_{i=0}^{s}$ be two $s$-arcs in $D(T)$. By applying $\tau$ or $\tau^{-1}$, we may assume that $v_{0} \in V_{0}$ and $v_{0}^{\prime} \in V_{0}$, and thus also $v_{i}, v_{i}^{\prime} \in V_{i}$ for each $i$. We imagine coordinates of the two arcs written in a grid: all coodinates of $v_{i}\left(v_{i}^{\prime}\right.$, resp.) in the $i$-th row. We are going to find an automorphism $\psi_{k}$ of $D$ such that $\psi_{k}(v)$ is closer
to $\left(v^{\prime}\right)$ than $(v)$. We shall do this by applying an automorphism $\psi \in \Psi$ on an appropriate diagonal (the first diagonal in which $(v)$ and $\left(v^{\prime}\right)$ differ). Now, we make this idea precise:

If $v_{i}=v_{i}^{\prime}$ for each $i$, then we are done; otherwise find $i$ and $j$ so that

$$
\begin{equation*}
\left(v_{i}\right)_{j} \neq\left(v_{i}^{\prime}\right)_{j} \quad \text { and } \quad k=i-j \text { is minimal. } \tag{*}
\end{equation*}
$$

We put $a_{\ell}=v_{\ell+k, \ell}$ if $0 \leq \ell+k \leq s$ and $1 \leq \ell \leq t$. After that, we pick $a_{\ell} \in A_{\ell}$ (for $\ell$ such that $1 \leq \ell \leq t$ but $\ell+k<0$ or $\ell+k>s$ ). The only condition now is that $a_{\ell} a_{\ell+1}$ is an arc for all $\ell=1, \ldots, t-1$, so that $\left(a_{\ell}\right)_{\ell=1}^{t}$ is a $(t-1)$-arc in $T$. Similarly, we define $a_{\ell}^{\prime}$ from $v^{\prime}$. Now $\left(a_{\ell}\right),\left(a_{\ell}^{\prime}\right)$ are two $(t-1)$-arcs in $T$, thus (by the symmetry assumptions on $T$ ) there is an automorphism $\varphi$ of $T$ such that $\varphi\left(a_{\ell}\right)=a_{\ell}^{\prime}$ for $\ell=1, \ldots, t$. Let $\psi$ be the automorphism of $D$ corresponding to $\varphi$, and let $\psi_{k}=\tau^{k} \psi \tau^{-k}$. The mapping $\psi_{k}$ permutes the $j$ th coordinate in $V_{k+j}$. Observe that $s$-arcs $\left(\psi_{k}\left(v_{i}\right)\right)_{i=0}^{s}$ and $\left(v_{i}^{\prime}\right)_{i=0}^{s}$ are closer (so that we get larger value of $k$ in $(*))$ than for $\left(v_{i}\right)$ and $\left(v_{i}^{\prime}\right)$. So, after repeating this procedure at most $s+t$ times we map one $s$-arc to the other.

As the requirements on the template $T$ are rather strong, let us describe a nice source of nontrivial templates. Consider a finite affine or projective space, $A G(n, q)$ or $P G(n, q)$. Let $A_{i}$ be the family of subspaces of dimension $i-1$. We let the arcs denote incidence, i.e., $(x, y)$ is an arc if and only if $x$ is a subspace of $y$ of codimension 1 . This gives a template with $t=n-1$. A $(t-1)$-arc corresponds to a flag (that is, a sequence of a subspaces one contained in another, one in each dimension). It is not hard to show that the geometric space is flag-transitive, which implies the following.

Claim 3.3. The template just described satisfies the conditions of Construction 3.

A natural question remains: does Construction 3 give some highly arctransitive digraphs that cannot be obtained by Construction 2? The answer is positive. To prove it, let us first define the notion of clones. Given a digraph, we call vertices $x, x^{\prime}$ right clones, if they have the same outneighbours ( $x y$ is an edge if and only if $x^{\prime} y$ is an edge); we call them left clones if they have the same inneighbours. It is not hard to show that in a highly arctransitive digraph, all vertices have the same number $c^{+}$of right clones and the same number $c^{-}$of left-clones. In Construction 2 we have $c^{+} \geq\left|A_{2}\right|$ and $c^{-} \geq\left|A_{1}\right|$, so $c^{+} c^{-} \geq\left|V_{0}\right|$. On the other hand, using Construction 3 with a template $T$ from finite geometries we have $c^{+}=\left|A_{t}\right|$ and $c^{-}=\left|A_{1}\right|$. In particular, when $t>2$, we have $c^{+} c^{-}<\left|V_{0}\right|$. This shows that these highly arc-transitive digraphs cannot be obtained by Construction 2 .

## 4. Structure in the two-ended case

The goal of this section is to prove a structural result concerning two-ended highly arc-transitive digraphs. Our structure theorem will show that every two-ended highly arc-transitive digraph either admits a quotient by which we can reduce it to a simpler structure, or up to vertex cloning, can be represented using a generalized construction which we describe next.
Construction 4. We define a coloured template to be a digraph $K$ equipped with a possibly improper colouring of the edges $\varphi: E(K) \rightarrow\{1, \ldots, t\}$ and also equipped with a distinguished partition of the vertices into sets $V_{0}, V_{1}, \ldots, V_{m}$ so that every edge goes from a point in $V_{i}$ to a point in $V_{i+1}$ for some $0 \leq$ $i<m$. Given such a template $K$, we define the digraph $\widehat{K}$ to have vertex set $\mathbb{Z} \times V_{0} \times V_{1} \times \cdots \times V_{m}$ and an edge from $\left(i, x_{0}, x_{1}, \ldots, x_{m}\right)$ to $\left(i+1, y_{0}, y_{1}, \ldots, y_{m}\right)$ whenever all of the $\operatorname{arcs}\left(x_{0}, y_{1}\right),\left(x_{1}, y_{2}\right), \ldots,\left(x_{m-1}, y_{m}\right)$ are present in $K$ and all have the same colour.

It is easy to see that Construction 4 generalizes Construction 3. However, the digraphs $\widehat{K}$ are not always highly arc-transitive. In this section we shall prove that all two-ended highly arc-transitive digraphs can be described by using Construction 4 combined with vertex-cloning operation. The proof of this will be built up slowly in a series of small lemmas.

Throughout this section, we shall always assume that $G$ is a highly arctransitive digraph ${ }^{1}$ such that the underlying undirected graph is connected and has two ends. For any partition $\mathcal{P}$ of the vertices, we let $G^{\mathcal{P}}$ denote the digraph obtained from $G$ by identifying the vertices in each block of $\mathcal{P}$ to a single new vertex and then deleting any parallel edges. We say that a system of imprimitivity $\mathcal{B}$ is a $\mathbb{Z}$-system if $G^{\mathcal{B}}$ is isomorphic to two-wayinfinite directed path. In this case the blocks of $\mathcal{B}$ can be enumerated $\left\{B_{i}\right\}_{i \in \mathbb{Z}}$ so that every edge has its tail in $B_{i}$ and its head in $B_{i+1}$ for some $i \in \mathbb{Z}$. Note that in this case, we have that for every $\varphi \in \operatorname{Aut}(G)$ there exists $j \in \mathbb{Z}$ so that $\varphi\left(B_{i}\right)=B_{i+j}$ for every $i \in \mathbb{Z}$.

Some of the results that follow, or parts of their proofs, can be found in [3] or in [8]. We include them for completeness.

Proposition 4.1. Every connected two-ended 2-arc transitive digraph has a unique $\mathbb{Z}$-system $\mathcal{B}$. Furthermore, $\mathcal{B}$ has finite blocks of imprimitivity, and every system of imprimitivity with finite blocks is a refinement of $\mathcal{B}$.

Proof. Every connected vertex-transitive two-ended digraph has a system of imprimitivity $\mathcal{B}$ with finite blocks and an (infinite) cyclic relation on $\mathcal{B}$

[^6]which is preserved by the automorphism group; this follows, for instance from Dunwoody's theorem [5] on cutting up graphs. Enumerate the blocks $\left\{B_{i}\right\}_{i \in \mathbb{Z}}$ so that this cyclic relation associates $B_{i}$ with $B_{i-1}$ and $B_{i+1}$ for every $i \in \mathbb{Z}$. Now, it follows from the assumption that the digraph $G$ is arctransitive that there exists a fixed integer $k$ so that every edge with one end in $B_{i}$ and one end in $B_{j}$ satisfies $|i-j|=k$. It then follows from the connectivity of the underlying graph that $k=1$. So, every edge has its ends in two consecutive blocks of $\left\{B_{i}\right\}_{i \in \mathbb{Z}}$.

Note that every vertex $x \in B_{i}$ must be adjacent in the underlying undirected graph to both a vertex in $B_{i-1}$ and in $B_{i+1}$ (otherwise every vertex would behave similarly, and the graph would be disconnected). Suppose (for a contradiction) that there exists a directed path $P$ of length two with vertex sequence $x_{0}, x_{1}, x_{2}$ so that both $x_{0}$ and $x_{2}$ are contained in the same block $B_{i}$. Choose a vertex $y$ which is adjacent to $x_{1}$ in the underlying undirected graph but is not in $B_{i}$. Now either $x_{0}, x_{1}, y$ or $y, x_{1}, x_{2}$ is the vertex sequence of a directed path of length two; we let $P^{\prime}$ denote this path. It follows immediately that no automorphism can map $P$ to $P^{\prime}$, and this contradicts the assumption of 2 -arc transitivity. Therefore, by possibly reversing our ordering, we may assume that every edge has its tail in some block $B_{i}$ and its head in $B_{i+1}$. Thus $\mathcal{B}$ is a $\mathbb{Z}$-system.

For the last part of the theorem, we let $\mathcal{C}$ be a system of imprimitivity with finite blocks, and suppose (for a contradiction) that $\mathcal{C}$ is not a refinement of $\mathcal{B}$. Choose a block $C$ of $\mathcal{C}$ and let $i \in \mathbb{Z}$ be the smallest integer with $B_{i} \cap C \neq \emptyset$ and let $j \in \mathbb{Z}$ be the largest integer with $B_{j} \cap C \neq \emptyset$ (and note that $i<j$ ). Now choose a vertex $u \in B_{i} \cap C$ and $v \in B_{j} \cap C$ and choose an automorphism $\varphi$ so that $\varphi(u)=v$. It now follows that $\varphi(C)=C$ and that $\varphi\left(B_{k}\right)=B_{k+j-i}$ for every $k \in \mathbb{Z}$, but this implies that $C$ is infinite, and thus we obtain a contradiction. Thus, $\mathcal{C}$ must be a refinement of $\mathcal{B}$. It follows immediately from this that the $\mathbb{Z}$-system $\mathcal{B}$ is unique.

In the sequel, we shall work extensively with group actions; our groups shall act on the left. For clarity, we shall always use upper case Greek letters for groups and lower case Greek letters for elements of groups. If $\Psi$ is a group and $\Lambda \leq \Psi$ we let $\Psi / \Lambda$ denote the set of left $\Lambda$-cosets in $\Psi$. Further, we let $G$ be a connected two-ended highly arc-transitive digraph and we let $\mathcal{B}=\left\{B_{i}\right\}_{i \in \mathbb{Z}}$ be its $\mathbb{Z}$-system.

Lemma 4.2. There exists a nontrivial automorphism of $G$ with only finitely many non-fixed points.

Proof. Let $\mathcal{B}=\left\{B_{i}\right\}_{i \in \mathbb{Z}}$ be the $\mathbb{Z}$-system, and suppose that every vertex has outdegree $d$ and that each block of $\mathcal{B}$ has size $k$. Next, choose an integer $n$
large enough so that $d^{n}>(k!)^{2}$ and consider a directed path $P$ of length $n$ with vertex sequence $x_{0}, x_{1}, \ldots, x_{n}$ with $x_{i} \in B_{i}$. Now, there are $d^{n}$ directed paths of length $n$ which start at the vertex $x_{0}$, and for each of them, we may choose an automorphism which maps $P$ to this path. Since $d^{n}>(k!)^{2}$ it follows that there must be two such automorphisms, say $\varphi_{1}$ and $\varphi_{2}$ which give exactly the same permutation of both $B_{0}$ and $B_{n}$. It follows that the automorphism $\psi=\varphi_{1} \varphi_{2}^{-1}$ is nontrivial, but gives the identity permutation on both $B_{0}$ and $B_{n}$. Now, we define a mapping $\psi^{\prime}: V(G) \rightarrow V(G)$ by the following rule

$$
\psi^{\prime}(x)= \begin{cases}\psi(x) & \text { if } x \in B_{1} \cup B_{2} \cup \cdots \cup B_{n-1} \\ x & \text { otherwise } .\end{cases}
$$

It is immediate that $\psi^{\prime}$ is a nontrivial automorphism which has only finitely many non-fixed points, as desired.

Based on the above lemma, there exists a smallest integer $\ell$ so that $G$ has a nontrivial automorphism which fixes all but $\ell+1$ blocks from the $\mathbb{Z}$ system pointwise. It is immediate that every such automorphism must give a non-identity permutation on $\ell+1$ consecutive blocks and the identity on all others. For every integer $i$, let $\Gamma_{i}$ denote the subgroup of automorphisms which pointwise fix all blocks of the $\mathbb{Z}$-system with the (possible) exception of $B_{i-\ell}, B_{i-\ell+1}, \ldots, B_{i}$. We let $\Gamma$ denote the subgroup of $\operatorname{Aut}(G)$ generated by $\cup_{i \in \mathbb{Z}} \Gamma_{i}$.
Lemma 4.3. The following statements hold:
(i) If $\alpha \in \Gamma_{i}$ and $\beta \in \Gamma_{j}$ with $i \neq j$, then $\alpha$ and $\beta$ commute.
(ii) If $\varphi \in \operatorname{Aut}(G)$ satisfies $\varphi\left(B_{0}\right)=B_{k}$ then $\varphi \Gamma_{j} \varphi^{-1}=\Gamma_{j+k}$ for every $j \in \mathbb{Z}$.
(iii) $\Gamma \triangleleft \operatorname{Aut}(G)$.

Proof. To prove claim (i), we consider the mapping $\gamma=\alpha \beta \alpha^{-1} \beta^{-1}$. Since $\alpha$ pointwise fixes all blocks but $B_{i-\ell}, B_{i-\ell+1}, \ldots, B_{i}$ and $\beta$ pointwise fixes all blocks but $B_{j-\ell}, B_{j-\ell+1}, \ldots, B_{j}$ the map $\gamma$ fixes pointwise any block, which is not in both of these lists. However, then $\gamma$ must pointwise fix all but fewer than $\ell+1$ blocks, so $\gamma$ is the identity.

For the second claim, we first note that $\varphi\left(B_{i}\right)=B_{i+k}$ for every $i \in \mathbb{Z}$. Now, for every $\alpha \in \Gamma_{j}$ we see that $\varphi \alpha \varphi^{-1}$ pointwise fixes all blocks except possibly $B_{j+k-\ell}, B_{j+k-\ell+1}, \ldots, B_{j+k}$ and it follows that $\varphi \alpha \varphi^{-1} \in \Gamma_{j+k}$ which proves the claim.

To prove claim (iii), let $\alpha \in \Gamma$ and express this element as $\alpha=\alpha_{1} \alpha_{2} \ldots \alpha_{m}$ where each $\alpha_{i}$ is in a subgroup of the form $\Gamma_{j}$. Now we have

$$
\varphi \alpha \varphi^{-1}=\left(\varphi \alpha_{1} \varphi^{-1}\right)\left(\varphi \alpha_{2} \varphi^{-1}\right) \ldots\left(\varphi \alpha_{m} \varphi^{-1}\right)
$$

so $\varphi \alpha \varphi^{-1}$ is also contained in $\Gamma$.
We call a two-way-infinite directed path a line. The following lemma may be proved with a straightforward compactness argument, and appears in Möller [8].

Lemma 4.4. Let $\mathbf{x}, \mathbf{y}$ be lines in $G$ with $x$ a vertex in $\mathbf{x}$ and $y$ a vertex in $\mathbf{y}$. Then there exists an automorphism $\varphi$ of $G$ which maps $\mathbf{x}$ to $\mathbf{y}$ and maps $x$ to $y$.

Lemma 4.5. Let $\Lambda \triangleleft \operatorname{Aut}(G)$ and let $\mathcal{C}$ be the partition of $V(G)$ given by the orbits under the action of $\Lambda$.
(i) $\mathcal{C}$ is a system of imprimitivity.
(ii) If $C, C^{\prime} \in \mathcal{C}$ and there is an edge from $C$ to $C^{\prime}$, then every vertex in $C$ has an outneighbour in $C^{\prime}$ and every vertex in $C^{\prime}$ has an inneighbour in $C$.
(iii) $G^{\mathcal{C}}$ is highly arc-transitive.
(iv) If $\mathbf{x}$ is a line in $G$, then the digraph $G_{\mathbf{x}}$ induced by the union of those blocks of $\mathcal{C}$ which contain a vertex in $\mathbf{x}$ is highly arc-transitive.
(v) If $\mathbf{x}$ and $\mathbf{y}$ are lines in $G$, then the digraphs $G_{\mathbf{x}}$ and $G_{\mathbf{y}}$ are isomorphic.

Proof. Part (i) is a standard fact about group actions. For the proof, let $u, v \in V(G)$ be in the same orbit of $\Lambda$, say $u=\alpha(v)$ for $\alpha \in \Lambda$, and let $\varphi$ be any automorphism. Now, $\varphi(u)=\varphi \alpha(v)=\varphi \alpha \varphi^{-1} \varphi(v)$. Since $\varphi \alpha \varphi^{-1} \in \Lambda$, $\varphi(u)$ and $\varphi(v)$ are also in the same orbit of $\Lambda$.

For part (ii), choose an edge $\left(u, u^{\prime}\right) \in E(G)$ with $u \in C$ and $u^{\prime} \in C^{\prime}$. Now, for every $v \in C$ there is an element in $\Lambda$ that maps $u$ to $v$. Since this element must fix $C^{\prime}$ setwise, it follows that $v$ has an outneighbour in $C^{\prime}$. A similar argument shows that every point in $C^{\prime}$ has an inneighbour in $C$.

To prove (iii), we let $C_{1}, C_{2}, \ldots, C_{k}$ and $C_{1}^{\prime}, C_{2}^{\prime}, \ldots, C_{k}^{\prime}$ be two sequences of blocks of $\mathcal{C}$ so that both form the vertex set of a directed path in the digraph $G^{\mathcal{C}}$. Using part 2 we may choose vertex sequences $x_{1}, \ldots, x_{k}$ and $x_{1}^{\prime}, \ldots, x_{k}^{\prime}$ in $G$ so that $x_{i} \in C_{i}$ and $x_{i}^{\prime} \in C_{i}^{\prime}$ for $1 \leq i \leq k$ and so that $\left(x_{i}, x_{i+1}\right),\left(x_{i}^{\prime}, x_{i+1}^{\prime}\right) \in$ $E(G)$ for $1 \leq i \leq k-1$. It follows from the high arc transitivity of $G$ that there is an automorphism $\varphi$ of $G$ so that $\varphi\left(x_{i}\right)=x_{i}^{\prime}$ for $1 \leq i \leq k$. Then $\varphi\left(C_{i}\right)=C_{i}^{\prime}$ for $1 \leq i \leq k$ so $\varphi$ induces an automorphism of $G^{\mathcal{C}}$ that maps $C_{1}, \ldots, C_{k}$ to $C_{1}^{\prime}, \ldots, C_{k}^{\prime}$. It follows that $G^{\mathcal{C}}$ is highly arc-transitive.

For the proof of (iv), set $X$ to be the union of those blocks of $\mathcal{C}$ which contain a point of $\mathbf{x}$, and set $G^{\prime}$ to be the digraph induced by $X$. Now we let $y_{1} y_{2} \ldots y_{k}$ and $y_{1}^{\prime} y_{2}^{\prime} \ldots y_{k}^{\prime}$ be two paths of length $k-1$ in $G^{\prime}$. It follows from part 2 that we may extend $y_{1} \ldots y_{k}$ and $y_{1}^{\prime} \ldots y_{k}^{\prime}$, respectively, to lines $\mathbf{y}$ and $\mathbf{y}^{\prime}$ in $G^{\prime}$. It now follows from the previous lemma that there is an
automorphism $\varphi$ of $G$ which maps $\mathbf{y}$ to $\mathbf{y}^{\prime}$ and further has $\varphi\left(y_{i}\right)=y_{i}^{\prime}$ for $1 \leq i \leq k$. It then follows that $\varphi(X)=X$ so $\varphi$ yields an automorphism of $G^{\prime}$ which sends $y_{1}, \ldots, y_{k}$ to $y_{1}^{\prime}, \ldots, y_{k}^{\prime}$. We conclude that $G^{\prime}$ is highly arctransitive.

Part (v) follows easily from Lemma 4.4.
We define $G$ to be essentially primitive if there does not exist $\Lambda \triangleleft \operatorname{Aut}(G)$ so that the orbits of $\Lambda$ on $V(G)$ generate a proper nontrivial system of imprimitivity with finite blocks which is not equal to the $\mathbb{Z}$-system. Parts 3-5 from the previous lemma show that any two-ended highly arc-transitive digraph which is not essentially primitive has a type of decomposition into a highly arc-transitive subgraph and a highly arc-transitive quotient. Although this decomposition does not seem to give us a construction, we will focus in the remainder of this section on understanding the structure of the essentially primitive digraphs. Note, however, that we do not know whether this is truly needed. The only examples of highly arc-transitive digraphs that are not essentially primitive that we are aware of are a disjoint union of two highly arc-transitive digraphs (rather trivial example) and digraphs obtained by a horocyclic product (see [1]): we have vertices ( $i, x, y$ ) for each pair of vertices $(i, x),(i, y)$ of the two factors, and vertex $(i, x, y)$ is connected to $\left(i+1, x^{\prime}, y^{\prime}\right)$ iff both $(i, x)\left(i+1, x^{\prime}\right)$ and $(i, y)\left(i+1, y^{\prime}\right)$ are arcs in the factors. However, such product of two highly arc-transitive digraphs obtained by our template construction can also be obtained by our construction using a more complicated template.

Continuing with our attempt for a structural characterization we describe orbits of the group $\Gamma$ (see the definition before Lemma 4.3).

Lemma 4.6. If $G$ is essentially primitive, then the orbits under the action of $\Gamma$ are the blocks $\left\{B_{i}: i \in \mathbb{Z}\right\}$ of the $\mathbb{Z}$-system of $G$.

Proof. This follows immediately from Lemma 4.3.
Next we shall introduce another useful subgroup of $\operatorname{Aut}(G)$. Let $\Gamma_{k}(k \in \mathbb{Z})$ and $\Gamma$ be the subgroups of $\operatorname{Aut}(G)$ introduced before Lemma 4.3. As before, let $\tau$ be an automorphism of $G$ so that $\tau\left(B_{0}\right)=B_{1}$ (so, more generally, $\left.\tau\left(B_{i}\right)=B_{i+1}\right)$, and let $\Phi$ be the subgroup of $\operatorname{Aut}(G)$ which is generated by $\tau$ and $\Gamma$. We will use $\Phi$ to describe our digraph, so let us record some key features of it. The listed properties follow easily from Lemma 4.3, and the details of the proof are left to the reader.

Lemma 4.7.
(i) $\tau^{-1} \Gamma_{k} \tau=\Gamma_{k-1}$ for every $k \in \mathbb{Z}$.
(ii) $\Gamma \triangleleft \Phi$.
(iii) $\langle\tau\rangle \cong \mathbb{Z}$.
(iv) $\Gamma \cap\langle\tau\rangle=\{1\}$.
(v) $\Phi$ is a semidirect product of $\langle\tau\rangle$ and $\Gamma$.

Next we introduce another family of subgroups of $\Phi$. For every $j \leq k$ we define $\bar{\Gamma}_{j . k}$ to be the subgroup of $\Gamma$ generated by $\left(\bigcup_{i<j} \Gamma_{i}\right) \cup\left(\bigcup_{i>k} \Gamma_{i}\right)$. Note that $\bar{\Gamma}_{0 . \ell}$ is precisely the subgroup of $\Gamma$ consisting of those automorphisms which act trivially on $B_{0}$.

## Lemma 4.8.

(i) Every coset of $\bar{\Gamma}_{j . k}$ in $\Phi$ has a unique representation as

$$
\tau^{m}\left(\prod_{i=j}^{k} \alpha_{i}\right) \bar{\Gamma}_{j . . k}
$$

where $\alpha_{i} \in \Gamma_{i}$ for every $j \leq i \leq k$ (henceforth we call this the standard form).
(ii) $\tau^{-1} \bar{\Gamma}_{j . k} \tau=\bar{\Gamma}_{j-1 . k-1}$
(iii) If $A \subseteq \tau \Gamma$ then $\bar{\Gamma}_{j . k} A=A \bar{\Gamma}_{j-1 . k-1}$.
(iv) $A$ set $A \subseteq \tau \Gamma$ satisfies $\bar{\Gamma}_{j . k} A \bar{\Gamma}_{j . k}=A$ if and only if $A \bar{\Gamma}_{j . k-1}=A$.

Proof. The first and the second property follow immediately from the previous lemma. For the third, choose $A^{\prime} \subseteq \Gamma$ so that $A=\tau A^{\prime}$ and observe that

$$
\bar{\Gamma}_{j . k} A=\bar{\Gamma}_{j . k} \tau A^{\prime}=\tau \bar{\Gamma}_{j-1 . . k-1} A^{\prime}=\tau A^{\prime} \bar{\Gamma}_{j-1 . . k-1}=A \bar{\Gamma}_{j-1 . . k-1} .
$$

To prove the last property it is enough to observe that for $A \subseteq \tau \Gamma$

$$
\bar{\Gamma}_{j . k} A \bar{\Gamma}_{j . k}=A \bar{\Gamma}_{j-1 . . k-1} \bar{\Gamma}_{j . . k}=A \bar{\Gamma}_{j . k-1} .
$$

The only additional ingredients required for our structure theorem are some standard properties of vertex-transitive digraphs. Let $\Psi$ be a group, $\Lambda$ a subgroup of $\Psi$, and suppose set $A \subseteq \Psi$ satisfies $\Lambda A \Lambda=A$. Then we define the Cayley coset digraph $\operatorname{Cay}(\Psi / \Lambda, A)$ to be the digraph whose vertexset are the left cosets $\Psi / \Lambda$, where there is an edge from $g \Lambda$ to $h \Lambda$ if and only if $\Lambda g^{-1} h \Lambda \subseteq A$. The group $\Psi$ has a natural action on the vertices by left multiplication, and this action preserves the edges, and is transitive. The following well-known result of Sabidussi [12] shows that every vertextransitive digraph is isomorphic to a Cayley coset digraph. Here, if $\Psi$ acts on a set $X$ and $u \in X$ we let $\Psi_{u}=\{\gamma \in \Psi: \gamma(u)=u\}$ denote the point stabilizer of $u$.

Proposition 4.9. Let $H$ be a digraph, let $u \in V(H)$ and let $\Phi \leq \operatorname{Aut}(H)$ act transitively on $V(H)$. Then there exists $A \subseteq \Phi$ so that $H \cong \operatorname{Cay}\left(\Phi / \Phi_{u}, A\right)$, and this isomorphism may be chosen so that the vertex $u$ corresponds to the trivial coset $\Phi_{u}$.

Let us recall that cloning a vertex in a digraph $G$ means the operation of adding a new vertex $v^{\prime}$ whose inneighbours (and outneighbours) are precisely the inneighbours (and the outneighbours) of $v$. Also, let us note that the digraph obtained from $G$ by cloning each vertex $k-1$ times is just the lexicographic product $G\left[\bar{K}_{k}\right]$ of $G$ with the empty graph on $k$ vertices.

Proposition 4.10. Let $G=\operatorname{Cayley}(\Phi / \Lambda, A)$ and let $\Lambda^{\prime} \leq \Lambda$ with $\left[\Lambda: \Lambda^{\prime}\right]=k$. Then $G^{\prime}=\operatorname{Cayley}\left(\Phi / \Lambda^{\prime}, A\right)$ is a Cayley coset digraph which is isomorphic to the digraph obtained from $G$ by cloning each vertex $k-1$ times.

Proof. (sketch) By definition, in the digraph $G^{\prime}$ there will be an edge from $Q \in \Phi / \Lambda^{\prime}$ to $R \in \Phi / \Lambda^{\prime}$ if $Q^{-1} R \subseteq A$. If $R$ and $R^{\prime}$ lie in the same $\Lambda$-coset then $Q^{-1} R \Lambda=Q^{-1} R^{\prime} \Lambda$. Since $A \Lambda=A$, it follows that there is an edge from $Q$ to $R$ if and only if there is an edge from $Q$ to $R^{\prime}$. So, two vertices which lie in the same $\Lambda$-coset will have the same inneighbours. A similar argument shows that they have the same outneighbours. Thus, $G^{\prime}$ is isomorphic to the digraph obtained from $G$ by cloning each vertex exactly $k-1$ times.
Theorem 4.11. If a two-ended highly arc-transitive digraph $G$ is essentially primitive, then there exists a digraph $G^{+}$obtained from $G$ by cloning each vertex the same (finite) number of times and a coloured template $K$ so that $G^{+} \cong \widehat{K}$.

Proof. It follows immediately from Lemma 4.6 that the group $\Phi$ generated by $\tau$ and $\Gamma$ acts transitively on $V(G)$. As before, let $B_{i}(i \in \mathbb{Z})$ be the blocks of the $\mathbb{Z}$-system on $G$. Choose a vertex $u \in B_{0}$ and apply Proposition 4.9 to obtain $A \subseteq \Phi$ so that $G \cong \operatorname{Cay}\left(\Phi / \Phi_{u}, A\right)$. Since $\Phi_{u}$ is the stabilizer of $u$ and $\bar{\Gamma}_{0 . \ell}$ is the subgroup of $\Phi$ which fixes every point in $B_{0}$ we have $\bar{\Gamma}_{0 . . \ell} \leq \Phi_{u} \leq \Phi$ (and note that this also implies that $\left[\Phi_{u}: \bar{\Gamma}_{0 . . \ell}\right]$ is finite). It now follows from Proposition 4.10 that $G^{+}=\operatorname{Cay}\left(\Phi / \bar{\Gamma}_{0 . \ell}, A\right)$ is obtained from $G$ by cloning each vertex the same number of times, so it shall suffice to prove that $G^{+}$ can be obtained from our construction.

By assumption, $A$ must satisfy $\bar{\Gamma}_{0 . . \ell} A \bar{\Gamma}_{0 . . \ell}=A$ and then it follows from Lemma 4.8 that $A \bar{\Gamma}_{0 . . \ell-1}=A$, so we may partition $A$ into cosets of $\bar{\Gamma}_{0 . . \ell-1}$ as $\left\{A_{1}, A_{2}, \ldots, A_{t}\right\}$. Now, each $A_{q}$ also satisfies $\bar{\Gamma}_{0 . . \ell} A_{q} \bar{\Gamma}_{0 . . \ell}=A_{q}$, so we may define a Cayley coset digraph $G_{q}^{+}=\operatorname{Cay}\left(\Phi / \bar{\Gamma}_{0 . . \ell}, A_{q}\right)$ and now $G^{+}$is the edgedisjoint union of the digraphs $G_{1}^{+}, \ldots, G_{t}^{+}$. We may now view each $q=1, \ldots, t$ as a colour and view $G^{+}$as having its edges coloured accordingly.

Fix $1 \leq q \leq t$ and consider the digraph $G_{q}^{+}$and let $A_{q}=\tau\left(\prod_{i=0}^{\ell-1} \gamma_{i}\right) \bar{\Gamma}_{0 . \ell-1}$ be represented in standard form. Let $v=\tau^{k}\left(\prod_{i=0}^{\ell} \alpha_{i}\right) \bar{\Gamma}_{0 . \ell}$ be a vertex of $G_{q}^{+}$in standard form. Within the digraph $G_{q}^{+}$, the vertex $v$ will have outneighbours consisting of exactly those cosets of $\bar{\Gamma}_{0 . \ell}$ that are contained in the set

$$
\begin{aligned}
v A_{q} & =\tau^{k}\left(\prod_{i=0}^{\ell} \alpha_{i}\right) \bar{\Gamma}_{0 . . \ell} \tau\left(\prod_{i=0}^{\ell-1} \gamma_{i}\right) \bar{\Gamma}_{0 . \ell-1} \\
& =\tau^{k+1}\left(\prod_{i=0}^{\ell} \tau^{-1} \alpha_{i} \tau\right) \tau^{-1} \bar{\Gamma}_{0 . \ell} \tau\left(\prod_{i=0}^{\ell-1} \gamma_{i}\right) \bar{\Gamma}_{0 . \ell-1} \\
& =\tau^{k+1}\left(\prod_{i=1}^{\ell} \tau^{-1} \alpha_{i} \tau\right)\left(\prod_{i=0}^{\ell-1} \gamma_{i}\right) \bar{\Gamma}_{0 . \ell-1} \\
& =\tau^{k+1}\left(\prod_{i=1}^{\ell} \tau^{-1} \alpha_{i} \tau \gamma_{i-1}\right) \bar{\Gamma}_{0 . \ell-1 .}
\end{aligned}
$$

In other words, a vertex $w$ is an outneighbour of $v$ if and only if in standard form $w=\tau^{k+1}\left(\prod_{i=0}^{\ell} \beta_{i}\right) \bar{\Gamma}_{0 . \ell}$ where $\beta_{i-1}=\tau^{-1} \alpha_{i} \tau \gamma_{i-1}$ for every $1 \leq i \leq \ell$ (and there is no restriction on $\beta_{\ell}$ ). Next we shall define a template $K_{q}$ with ordered vertex partition $\left(\Gamma_{\ell}, \Gamma_{\ell-1}, \ldots, \Gamma_{0}\right)$ and an edge from $\delta \in \Gamma_{i}$ to $\epsilon \in \Gamma_{i-1}$ if and only if $\epsilon=\tau^{-1} \delta \tau \gamma_{i-1}$. It now follows that $(v, w)$ is an edge of $G_{q}^{+}$if and only if (using standard form) $v=\tau^{i} \alpha_{0} \alpha_{1} \ldots \alpha_{\ell} \bar{\Gamma}_{0 . \ell \ell}$ and $w=\tau^{j} \beta_{0} \beta_{1} \ldots \beta_{\ell} \bar{\Gamma}_{0 . \ell}$ satisfy $j=i+1$ and $\left(\alpha_{i}, \beta_{i-1}\right)$ is an edge of $K_{q}$ for every $1 \leq i \leq \ell$. It follows from this that $G_{q}^{+} \cong \widehat{K_{q}}$ by way of the isomorphism which maps a vertex $v=\tau^{i} \alpha_{0} \alpha_{1} \ldots \alpha_{\ell} \bar{\Gamma}_{0 . \ell}$ of $G_{q}^{+}$to the vertex $\left(i, \alpha_{\ell}, \alpha_{\ell-1}, \ldots, \alpha_{0}\right)$ of $\widehat{K}_{q}$.

We now define $K$ to be a coloured template with vertex set $\Gamma_{1} \cup \Gamma_{2} \cup \cdots \cup \Gamma_{\ell}$, vertex partition $\left\{\Gamma_{1}, \Gamma_{2}, \ldots, \Gamma_{\ell}\right\}$, and an edge from $\delta \in \Gamma_{i}$ to $\epsilon \in \Gamma_{i+1}$ of colour $q$ if and only if this edge exists in the template $K_{q}$. It now follows that $G^{+} \cong \widehat{K}$ which completes the proof.

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## Appendix E

## Sabidussi Versus Hedetniemi for Three Variations of the Chromatic Number

# SABIDUSSI VERSUS HEDETNIEMI <br> FOR THREE VARIATIONS OF THE CHROMATIC NUMBER 

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#### Abstract

We investigate vector chromatic number $\left(\chi_{v e c}\right)$, Lovász $\vartheta$-function of the complement $(\bar{\vartheta})$, and quantum chromatic number $\left(\chi_{q}\right)$ from the perspective of graph homomorphisms. We prove an analog of Sabidussi's theorem for each of these parameters, i.e., that for each of the parameters, the value on the Cartesian product of graphs is equal to the maximum of the values on the factors. Interestingly, as a consequence of this result for $\bar{\vartheta}$, we obtain analog of Hedetniemi's conjecture, i.e., that the value of $\bar{\vartheta}$ on the categorical product of graphs is equal to the minimum of its values on the factors. We conjecture that the analogous results hold for vector and quantum chromatic number, and we prove that this is the case for some special classes of graphs.


## 1. Introduction

The chromatic number is a well known graph parameter which can be defined in terms of homomorphisms. A graph homomorphism from $G$ to $H$ is a function $\varphi: V(G) \rightarrow V(H)$ such that $\varphi(u)$ is adjacent to $\varphi(v)$ whenever $u$ is adjacent to $v$. In this terminology, a graph $G$ is $n$-colorable if and only if there exists a homomorphism from $G$ to $K_{n}$. There are many interesting variants of chromatic number which can also be defined via homomorphisms. We are concerned with the following:

[^7]- Vector chromatic number $\left(\chi_{v e c}\right)$
- Strict vector chromatic number $(\bar{\vartheta})$
- Quantum chromatic number $\left(\chi_{q}\right)$

As the notation suggests, the strict vector chromatic number is equal to the Lovász $\vartheta$-function of the complement, i.e., $\bar{\vartheta}(G)=\vartheta(\bar{G})$ where $\bar{G}$ denotes the complement of graph $G$. We, however, do not define it in this way, rather we approach both $\bar{\vartheta}$ and $\chi_{\text {vec }}$ in terms of homomorphisms. From this viewpoint they can both be seen as relaxations of chromatic number defined in terms of assigning unit vectors to vertices such that vectors assigned to adjacent vertices have some specified inner product. Quantum chromatic number can also be viewed in terms of homomorphisms, however, for this parameter we assign to vertices tuples of orthogonal projectors which must satisfy certain constraints.

In this paper we are concerned with how these parameters behave on certain graph products. We are in particular focused on the Cartesian and categorical products, denoted by $G \square H$ and $G \times H$ respectively. A well known theorem of Sabidussi [23] states that the chromatic number of the Cartesian product of two graphs is equal to the maximum of the chromatic numbers of its factors. An equally, if not more, well known conjecture of Hedetniemi proposes that the chromatic number of the categorical product of two graphs is equal to the minimum of the chromatic numbers of the factors. Our aim is to prove or make steps towards proving analogs of these two statements for the three parameters above. Interestingly, for $\bar{\vartheta}$ the analog of Hedetniemi's conjecture follows as a consequence of the analog of Sabidussi's theorem. The rest of the paper is outlined as follows.

In Section 2 we define the basic concepts and notation used throughout the paper. We give the background needed for our results on vector and strict vector colorings in Section 3. This is followed by Section 4, in which we show that analogs of Sabidussi's theorem hold for $\bar{\vartheta}$ and $\chi_{v e c}$, and that the $\bar{\vartheta}$ version of Hedetniemi's conjecture is true. In Section 5, we investigate a class of graphs called 1-homogeneous graphs, which include edge transitive graphs. We give an explicit formula for $\bar{\vartheta}$ and $\chi_{v e c}$ for these graphs in terms of their largest and smallest eigenvalues. As a consequence, we see that these two parameters coincide for this class of graphs, and thus the $\chi_{\text {vec }}$ version of Hedetniemi's conjecture holds for 1-homogeneous graphs. In Section 6, we introduce quantum homomorphisms and give the background needed for our results on quantum chromatic number. Then in Section 7, we prove the quantum analog of Sabidussi's theorem, and show that quantum Hedetniemi's conjecture holds for a family of graphs which initiated the study of quantum chromatic number.

## 2. Preliminaries

Here we give the background on the basic tools such as homomorphisms and graph products that we use throughout the paper. For a more detailed introduction we refer the reader to [13,15] for homomorphisms, and [14] for graph products.

Let $G$ and $H$ be graphs (by which we mean undirected simple finite graphs). We denote the existence of a homomorphism from $G$ to $H$ by writing $G \rightarrow H$. It is easy to see that homomorphisms compose, so $G \rightarrow H \rightarrow K$ implies $G \rightarrow K$. In fact, graphs with homomorphisms form a category. More relevant for graph theory is that many graph theoretic notions can be simply expressed in terms of homomorphisms. In particular, a graph $G$ is $n$-colorable if and only if $G \rightarrow K_{n}$.

A graph parameter $f$ is called homomorphism-monotone if we have $f(G) \leq f(H)$ whenever $G \rightarrow H$. Examples of homomorphism-monotone pa${ }_{\bar{\sigma}}$ rameters include $\chi, \chi_{f}, \omega$, etc. We will see that the three parameters, $\chi_{v e c}$, $\bar{\vartheta}$, and $\chi_{q}$ are homomorphism-monotone as well, and they are even quantum homomorphism-monotone. (Quantum homomorphisms will be defined in Section 6).

Given graphs $G$ and $H$, we define four graphs with vertex set $V(G) \times$ $V(H)$. In the categorical product $G \times H$ (also called direct, or tensor product), tuples $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$ are adjacent if and only if

$$
u_{1} \sim u_{2} \text { and } v_{1} \sim v_{2} .
$$

In the Cartesian product $G \square H$, tuples $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$ are adjacent if and only if

$$
\left(u_{1} \sim u_{2} \text { and } v_{1}=v_{2}\right) \text { or }\left(u_{1}=u_{2} \text { and } v_{1} \sim v_{2}\right) .
$$

The strong product $G \boxtimes H$ is defined as the edge union of $G \times H$ and $G \square H$. In the disjunctive product $G * H$ (also referred to as the conormal product) the tuples $\left(u_{1}, v_{1}\right)$ and $\left(u_{2}, v_{2}\right)$ are adjacent if $u_{1} \sim u_{2}$ or $v_{1} \sim v_{2}$. It is trivial to see that $G \boxtimes H$ is a subgraph of $G * H$, and a little thought reveals that $\overline{G * H}=\bar{G} \boxtimes \bar{H}$, where $\bar{G}$ denotes the complement of $G$.

It is easy to see that $G \rightarrow G \square H$ (and also $H \rightarrow G \square H$ ), indeed, $G \square H$ contains copies of both $G$ and $H$. By projecting onto each coordinate, we see that $G \times H \rightarrow G$ and $G \times H \rightarrow H$. (This is indeed true in any category and $G \times H$ is called the categorical product because it is, in fact, a product in the sense of category theory.) Consequently, we have the following lemma:

Lemma 2.1. If $f$ is a homomorphism-monotone graph parameter and $G$ and $H$ are graphs, then

$$
f(G \times H) \leq \min \{f(G), f(H)\} \quad \text { and } \quad f(G \square H) \geq \max \{f(G), f(H)\} .
$$

Proof. This follows immediately from the fact that $G \times H \rightarrow G, H$ and $G, H \rightarrow G \square H$.

This lemma allows us to easily establish that $\chi_{v e c}, \bar{\vartheta}$, and $\chi_{q}$ must all satisfy the above inequalities. We can then ask if/when equality holds. Indeed, that is the main focus of this paper.

Much attention has specifically been given to the value of the chromatic number on the Cartesian and categorical products, and this is of course part of the motivation for our work. Applying the above lemma to the chromatic number, which is homomorphism-monotone, we obtain the following:

$$
\begin{equation*}
\chi(G \times H) \leq \min \{\chi(G), \chi(H)\} \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\chi(G \square H) \geq \max \{\chi(G), \chi(H)\} . \tag{2}
\end{equation*}
$$

As mentioned above, a well known theorem of Sabidussi [23] states that (2) holds with equality, and we provide a proof here for the reader's convenience and comparison with Theorems 4.2 and 7.2.

Theorem 2.2 (Sabidussi 1957). For graphs $G$ and $H$,

$$
\chi(G \square H)=\max \{\chi(G), \chi(H)\} .
$$

Proof. Let $m=\max \{\chi(G), \chi(H)\}$. Clearly, we need at least $m$ colors to color $G \square H$. So it suffices to show that $G \square H$ can be $m$-colored. There are colorings $g$ of $G$ and $h$ of $H$ using $m$ colors, which we may assume are the integers modulo $m$. It is easy to check that assigning $(g(u)+f(v)) \bmod m$ to vertex ( $u, v$ ) gives an $m$-coloring of $G \square H$.

Determining whether (1) holds with equality turns out to be much more difficult, and the following conjecture remains open to this day:

Conjecture 2.3 (Hedetniemi 1966). For all graphs $G$ and $H$,

$$
\chi(G \times H)=\min \{\chi(G), \chi(H)\} .
$$

It is worth noting that many different versions of this statement have been either conjectured or proven since its inception. Perhaps most significantly, Zhu has recently proved in [26] that

$$
\chi_{f}(G \times H)=\min \left\{\chi_{f}(G), \chi_{f}(H)\right\}
$$

where $\chi_{f}$ denotes fractional chromatic number.
Zhu's proof makes use of the fact that fractional chromatic number can be written as a linear program and thus suffers from strong duality. As we will see below, the strong duality property of the semidefinite programs for $\bar{\vartheta}$ and $\chi_{v e c}$ are crucial for our proofs as well. This suggests that the lack of strong duality for chromatic number is one reason for the difficulty in attempting to prove Hedetniemi's conjecture.

## 3. Vector and Strict Vector Colorings

Vector and strict vector colorings were first introduced in [16], in which Karger, Motwani, and Sudan also defined the vector chromatic number. However, a parameter equal to $\chi_{v e c}$ of the complement was actually introduced a few decades earlier in [20] and [25], but this seems to have gone unnoticed by many. We will first define strict vector colorings:

Definition 3.1. Let $\mathcal{S}^{d}$ denote the unit sphere in $\mathbb{R}^{d+1}$. For a graph $G$, a $\operatorname{map} \varphi: V(G) \rightarrow \mathcal{S}^{d}$ is called a strict vector $k$-coloring if whenever $u \sim v$,

$$
\varphi(u)^{T} \varphi(v)=-\frac{1}{k-1}
$$

So a strict vector $k$-coloring can be viewed as a homomorphism to the infinite graph whose vertices are unit vectors in $\mathbb{R}^{d}$ such that vectors $u$ and $v$ are adjacent whenever $u^{T} v=-1 /(k-1)$. The strict vector chromatic number of $G$ is the infimum of real numbers $k$ such that $k>1$ and $G$ admits a strict vector $k$-coloring (for all nonempty graphs this infimum can be obtained and thus is just the minimum). It has been shown [16] that the strict vector chromatic number of $G$ is equal to $\vartheta(\bar{G})$, and thus we use $\bar{\vartheta}$ to denote this parameter. The Lovász $\vartheta$-function has been well studied and it possesses many interesting properties, some of which we will present below. For a more detailed look at this graph parameter we refer the reader to [18] and [17].

If we relax the definition above to only require that adjacent vertices be assigned unit vectors which have inner product at most $-1 /(k-1)$, then we obtain what is known as a vector $k$-coloring [16]. The smallest $k$ for which $G$ admits a vector $k$-coloring is the vector chromatic number of $G$,
and we denote this by $\chi_{v e c}(G)$. Since any strict vector $k$-coloring is clearly a vector $k$-coloring as well, we have that $\chi_{v e c}(G) \leq \bar{\vartheta}(G)$ for any graph $G$. The basic motivation behind these definitions is that mapping the vertices of the complete graph $K_{n}$ to the vertices of the ( $n-1$ )-dimensional simplex gives a (strict) vector $n$-coloring, and therefore, any $n$-colorable graph is also (strict) vector $n$-colorable. This of course implies that $\chi_{v e c}(G) \leq \bar{\vartheta}(G) \leq \chi(G)$ for all graphs $G$. On the other hand, [20] and [25] established (though in other terms) that $\omega(G) \leq \chi_{v e c}(G) \leq \bar{\vartheta}(G)$.

Defining these parameters in terms of homomorphisms as above allows us to easily see that both $\chi_{v e c}$ and $\bar{\vartheta}$ are homomorphism-monotone. Therefore, by Lemma 2.1, we have that

$$
\begin{aligned}
\chi_{v e c}(G \times H) & \leq \min \left\{\chi_{v e c}(G), \chi_{v e c}(H)\right\} \\
\bar{\vartheta}(G \times H) & \leq \min \{\bar{\vartheta}(G), \bar{\vartheta}(H)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\chi_{v e c}(G \square H) & \geq \max \left\{\chi_{v e c}(G), \chi_{v e c}(H)\right\} \\
\bar{\vartheta}(G \square H) & \geq \max \{\bar{\vartheta}(G), \bar{\vartheta}(H)\}
\end{aligned}
$$

It turns out that both $\bar{\vartheta}$ and $\chi_{v e c}$ can be written as semidefinite programs (SDPs). The practical advantage of this is that one can compute them to arbitrary precision in polynomial time. The other advantage is that we can use duality to assist in proving theorems. In general, strong duality does not hold for all SDPs, however, one can show that it holds for the SDPs defining $\bar{\vartheta}$ and $\chi_{v e c}$ using Slater's condition. Below we give both the primal and dual SDPs for $\bar{\vartheta}$ and $\chi_{v e c}$. Here, $P \succeq 0$ means that the matrix $P$ is positive semidefinite, while $P \geq 0$ means that the entries of $P$ are nonnegative. We use $J$ to denote the all ones matrix, and o to denote Schur product. The $\underline{m}$ matrix $A$ in the SDP constraints refers to the adjacency matrix of $G$, and $\bar{A}:=J-I-A$ is the adjacency matrix of $\bar{G}$.

\[

\]

To see that the SDPs for $\chi_{v e c}$ are equivalent to the vector coloring definition of this parameter, one can use the fact that the positive semidefinite matrix $M$ in the primal SDP is a Gram matrix of a set of vectors. Assigning these (normalized) vectors to the vertices of the graph gives a valid vector coloring of the appropriate value. The reverse procedure converts a vector coloring to a feasible solution to the primal. The same technique works for $\bar{\vartheta}$ as well $[18,17,25]$.

## 4. Strict Vector Chromatic Number

To prove the strict vector chromatic number version of Sabidussi's theorem, we need the following lemma which shows that any graph $G$ which can be strict vector $k$-colored, can also be strict vector $k^{\prime}$-colored for any $k^{\prime} \geq k$. For chromatic number, as well as vector chromatic number, this is trivial, since any $k$-coloring can be viewed as a $k^{\prime}$-coloring for any $k^{\prime} \geq k$.

Lemma 4.1. Suppose $G$ is a graph such that $\bar{\vartheta}(G)=k$. Then for every real $k^{\prime} \geq k$, there is a strict vector $k^{\prime}$-coloring of $G$.

Proof. Let $\varphi: V(G) \rightarrow \mathbb{R}^{d}$ be a strict vector $k$-coloring of $G$. Let $t=-1 /(k-1)$, and $t^{\prime}=-1 /\left(k^{\prime}-1\right)$. As $k^{\prime} \geq k>1$, we have that $t \leq t^{\prime}<0$. Consequently, there exists an $\alpha \in[0,1]$ such that $\alpha^{2} t+\left(1-\alpha^{2}\right)=t^{\prime}$. Define the mapping $\varphi^{\prime}=\left(\alpha \varphi, \sqrt{1-\alpha^{2}}\right)$. It is easy to check that $\varphi^{\prime}$ is a strict vector $k^{\prime}$-coloring of $G$.

We are now able to prove that Sabidussi's theorem holds for $\bar{\vartheta}$.
Theorem 4.2. For graphs $G$ and $H$,

$$
\bar{\vartheta}(G \square H)=\max \{\bar{\vartheta}(G), \bar{\vartheta}(H)\} .
$$

Proof. As we have already seen in Section 3,

$$
\bar{\vartheta}(G \square H) \geq \max \{\bar{\vartheta}(G), \bar{\vartheta}(H)\} .
$$

Thus, we only need to show the reverse inequality. Let $k=\max \{\bar{\vartheta}(G), \bar{\vartheta}(H)\}$. By Lemma 4.1, there exist strict vector $k$-colorings $g: V(G) \rightarrow \mathbb{R}^{d_{1}}$ and $h: V(H) \rightarrow \mathbb{R}^{d_{2}}$. We will consider the tensor product $g \otimes h: V(G \square H) \rightarrow \mathbb{R}^{d_{1} d_{2}}$. Explicitly, we put $(g \otimes h)(u, v)=g(u) \otimes h(v)$, where $u \in V(G)$ and $v \in V(H)$.

Now consider an edge of the form $(u, v)\left(u^{\prime}, v\right)$ in $G \square H$. Let $t=-1 /(k-1)$ as in Definition 3.1. Using standard properties of the tensor product we get

$$
(g(u) \otimes h(v))^{T}\left(g\left(u^{\prime}\right) \otimes h(v)\right)=\left(g(u)^{T} g\left(u^{\prime}\right)\right)\left(h(v)^{T} h(v)\right)=t \cdot 1=t .
$$

By symmetry, we get the same condition for edges of the form $(u, v)\left(u, v^{\prime}\right)$. Consequently, $g \otimes h$ is a strict vector $k$-coloring of $G \square H$, as required.

We also have the following:
Lemma 4.3. For graphs $G$ and $H$,

$$
\chi_{v e c}(G \square H)=\max \left\{\chi_{v e c}(G), \chi_{v e c}(H)\right\} .
$$

Proof. Same as in Theorem 4.2, without the need for Lemma 4.1 since vector $k$-colorings only require that adjacent vertices have inner product at most $-1 /(k-1)$.

We will use Lemma 4.2 to prove the $\bar{\vartheta}$ version of Hedetniemi's conjecture, but we will also need some basic facts about how $\bar{\vartheta}$ behaves on the strong and disjunctive products, as well as the edge union of two graphs.

In [18] it was shown that $\vartheta(G \boxtimes H)=\vartheta(G) \vartheta(H)$. A slight modification of the same proof shows that $\vartheta(G * H)=\vartheta(G) \vartheta(H)$, and in fact this is proven in [17]. Translating these two facts into terms of $\bar{\vartheta}$, we obtain the following lemma.

Lemma 4.4. For graphs $G$ and $H$,

$$
\bar{\vartheta}(G \boxtimes H)=\bar{\vartheta}(G) \bar{\vartheta}(H)=\bar{\vartheta}(G * H) .
$$

Proof. Since $\overline{G * H}=\bar{G} \boxtimes \bar{H}$ (and equivalently $\overline{G \boxtimes H}=\bar{G} * \bar{H}$ ), we have that

$$
\bar{\vartheta}(G * H)=\vartheta(\bar{G} \boxtimes \bar{H})=\vartheta(\bar{G}) \vartheta(\bar{H})=\bar{\vartheta}(G) \bar{\vartheta}(H)
$$

and

$$
\bar{\vartheta}(G \boxtimes H)=\vartheta(\bar{G} * \bar{H})=\vartheta(\bar{G}) \vartheta(\bar{H})=\bar{\vartheta}(G) \bar{\vartheta}(H) .
$$

From this lemma we can easily obtain the following corollary which is analogous to a well known upper bound on the chromatic number of the union of two graphs. Given two graphs $G$ and $H$ on the same vertex set $V$, we use $G \cup H$ to denote the graph with vertex set $V$ and edge set $E(G) \cup E(H)$.

Corollary 4.5. If $G$ and $H$ are graphs on the same vertex set $V$, then

$$
\bar{\vartheta}(G \cup H) \leq \bar{\vartheta}(G) \bar{\vartheta}(H) .
$$

Proof. The vertices of the form $(v, v)$ for $v \in V$, induce a subgraph of $G * H$ isomorphic to $G \cup H$ and thus

$$
\bar{\vartheta}(G \cup H) \leq \bar{\vartheta}(G * H)=\bar{\vartheta}(G) \bar{\vartheta}(H)
$$

by Lemma 4.4.
With these tools in hand, we are able to give a simple and elegant proof of Hedetniemi's conjecture for $\bar{\vartheta}$.

Theorem 4.6. For graphs $G$ and $H$,

$$
\bar{\vartheta}(G \times H)=\min \{\bar{\vartheta}(G), \bar{\vartheta}(H)\} .
$$

Proof. We have already seen that

$$
\bar{\vartheta}(G \times H) \leq \min \{\bar{\vartheta}(G), \bar{\vartheta}(H)\} .
$$

So we only need to show the reverse inequality. For this we observe that $G \boxtimes H=(G \square H) \cup(G \times H)$. Using Corollary 4.5 for $G \square H$ and $G \times H$, as well as Lemma 4.4, we obtain

$$
\bar{\vartheta}(G) \bar{\vartheta}(H)=\bar{\vartheta}(G \boxtimes H) \leq \bar{\vartheta}(G \square H) \bar{\vartheta}(G \times H) .
$$

Combining this with Theorem 4.2 finishes the proof.

## 5. 1-Homogeneous Graphs

A graph $G$ is said to be 1-homogeneous if it satisfies the following two conditions:

1. The number of closed walks of length $k$ in $G$ that begin at a vertex $u$ is independent of $u$ for all $k \in \mathbb{Z}$.
2. The number of walks of length $k$ in $G$ that begin at vertex $u$ and end at adjacent vertex $v$ is independent of the edge $u v$.
The first condition can be viewed as a type of combinatorial relaxation of vertex transitivity. Indeed, it is easy to see that any vertex transitive graph has this property. The second condition can similarly be viewed as a combinatorial relaxation of edge transitivity, and again, any edge transitive graph trivially has this property. So any graph which is both edge and vertex transitive is 1 -homogeneous. Note that letting $k=2$ in the first condition guarantees that any such graph is regular.

Although 1-homogeneous graphs are not a well known class of graphs, they include several well known classes of graphs. In particular, distance regular (and thus strongly regular) graphs are 1-homogeneous. More generally, any graph which is a single class in an association scheme is 1-homogeneous. We will also see that any edge-transitive graph is either 1-homogeneous or bipartite, thus the results of this section apply to all of these classes of graphs.

If $A$ is the adjacency matrix of a graph $G$, then the $u v$-entry of $A^{k}$ is the number of walks of length $k$ in $G$ starting at $u$ and ending at $v$. From this
it is easy to see that $G$ being 1-homogeneous is equivalent to the existence of constants $b_{k}$ and $c_{k}$ for all $k \in \mathbb{N}$ such that

$$
\begin{equation*}
A^{k} \circ I=b_{k} I \quad \& \quad A^{k} \circ A=c_{k} A . \tag{3}
\end{equation*}
$$

In this section, we will present an explicit formula for the vector chromatic number of a 1-homogeneous graph in terms of its largest and smallest eigenvalues. Furthermore, we will show that $\bar{\vartheta}$ and $\chi_{\text {vec }}$ are equal in this case. As a result, we will see that the vector chromatic number version of Hedetniemi's conjecture holds for all 1-homogeneous graphs. The results of this section rely heavily on the SDP formulations of $\chi_{v e c}$ and $\bar{\vartheta}$ given in Section 3.

Before we give our results on 1-homogeneous graphs, we prove a general lower bound on vector chromatic number. The following two lemmas are from [11], which is unpublished, so we include their proofs here.

Lemma 5.1 ([11]). Let $G$ be a graph with $n$ vertices, e edges, and least eigenvalue $\tau$. Then

$$
\chi_{v e c}(G) \geq 1-\frac{2 e / n}{\tau}
$$

Proof. Let $A$ be the adjacency matrix of $G$. Then $A-\tau I \succeq 0$ and

$$
(A-\tau I) \circ \bar{A}=0 .
$$

Since $\tau<0$, we have that

$$
A-\tau I \geq 0
$$

Furthermore, since

$$
\operatorname{Tr}(A-\tau I)=-n \tau,
$$

the matrix $\frac{1}{-n \tau}(A-\tau I)$ is a feasible solution to the dual formulation of $\chi_{v e c}$ with objective value $1-\frac{2 e / n}{\tau}$. This gives the lower bound and proves the lemma.

Note that $2 e / n$ is the average degree of the graph, and is thus simply the degree for regular graphs. The next lemma states that the above bound is tight for 1-homogeneous graphs.

Lemma 5.2 ([11]). If $G$ is 1-homogeneous with degree $k$ and least eigenvalue $\tau$, then

$$
\chi_{v e c}(G)=\bar{\vartheta}(G)=1-\frac{k}{\tau} .
$$

Proof. We make use of the identity $\operatorname{Tr}\left(A^{T} B\right)=\operatorname{sum}(A \circ B)$ where $\operatorname{sum}(M)$ denotes the sum of all the entries of the matrix $M$. From the previous lemma, we have that $\chi_{v e c}(G) \geq 1-\frac{2 e / n}{\tau}=1-\frac{k}{\tau}$. As we saw in Section 3, $\chi_{v e c}(G) \leq \bar{\vartheta}(G)$, and thus we only need to show that $\bar{\vartheta}(G) \leq 1-\frac{k}{\tau}$. To do this we will find a suitable solution to the primal SDP formulation of $\bar{\vartheta}$. Let $A$ be the adjacency matrix of $G$ and let $E_{\tau}$ denote the projection onto the $\tau$-eigenspace of $A$. Since $G$ is 1-homogeneous and $E_{\tau}$ is a polynomial in $A$, by (3) we have that

$$
E_{\tau} \circ I=b I \quad \text { and } \quad E_{\tau} \circ A=c A
$$

for some constants $b$ and $c$. Let $r$ be the rank of $E_{\tau}$. Since $E_{\tau}$ is a projection, $\operatorname{Tr}\left(E_{\tau}\right)=r$, and so $b=r / n$. Now

$$
\operatorname{sum}\left(A \circ E_{\tau}\right)=\operatorname{Tr}\left(A E_{\tau}\right)=\operatorname{Tr}\left(\tau E_{\tau}\right)=r \tau
$$

and also

$$
\operatorname{sum}\left(A \circ E_{\tau}\right)=c \cdot \operatorname{sum}(A)=c n k,
$$

whence $c=r \tau / n k$. If we define

$$
M:=-\frac{n k}{r \tau} E_{\tau},
$$

then

$$
M \circ I=-\frac{n k}{r \tau} b I=-\frac{k}{\tau} I=((1-k / \tau)-1) I
$$

and

$$
M \circ A=-\frac{n k}{r \tau} c A=-A .
$$

Since $E_{\tau}$ is a projection, $M$ is positive semidefinite, and is thus a feasible solution to the primal SDP with objective value $1-\frac{k}{\tau}$. Therefore, $\bar{\vartheta}(X) \leq 1-\frac{k}{\tau}$ and the lemma is proven.

Applying the above to edge transitive graphs, we obtain the following corollary

Corollary 5.3. If $G$ is edge transitive with greatest and least eigenvalues $\lambda$ and $\tau$, respectively, then

$$
\chi_{v e c}(G)=\bar{\vartheta}(G)=1-\frac{\lambda}{\tau} .
$$

Proof. First note that we can assume that $G$ has no isolated vertices since removing them does not change any of $\lambda, \tau, \chi_{v e c}(G)$, or $\bar{\vartheta}(G)$. If $G$ is also vertex transitive, then it is 1-homogeneous and $\lambda$ is the degree, $k$, of $G$. Thus the result holds by the above. If $G$ is not vertex transitive, then by Lemma 3.2.1 from [12], it is nonempty and bipartite and thus $\chi_{\text {vec }}(G)=$ $2=\bar{\vartheta}(G)$. However, the spectrum of any bipartite graph is symmetric about zero, and so $\tau=-\lambda$, and therefore, $1-\frac{\lambda}{\tau}=2$.

The following lemma shows that the class of 1-homogeneous graphs is closed under categorical products.

Lemma 5.4 ([10]). If $G$ and $H$ are 1-homogeneous graphs, then the graph $G \times H$ is 1-homogeneous.

As a consequence of the above lemmas, we obtain a proof of the $\chi_{v e c}$ version of Hedetniemi's conjecture for 1-homogeneous graphs.

Theorem 5.5. If $G$ and $H$ are 1-homogeneous, then

$$
\chi_{v e c}(G \times H)=\min \left\{\chi_{v e c}(G), \chi_{v e c}(H)\right\} .
$$

Proof. Since $G$ and $H$ are 1-homogeneous, and by Lemma 5.4 the product $G \times H$ is as well, we have that

$$
\chi_{v e c}(G \times H)=\bar{\vartheta}(G \times H)=\min \{\bar{\vartheta}(G), \bar{\vartheta}(H)\}=\min \left\{\chi_{v e c}(G), \chi_{v e c}(H)\right\},
$$

where the second equality follows from Theorem 4.6.
Note that one can also prove the above theorem without the aid of Theorem 4.6 by writing the degree and smallest eigenvalue of $G \times H$ in terms of the same parameters for $G$ and $H$.

As a corollary, we get that Hedetniemi's conjecture for $\chi_{v e c}$ also holds for all edge transitive graphs.

Corollary 5.6. If $G$ and $H$ are edge transitive, then

$$
\chi_{v e c}(G \times H)=\min \left\{\chi_{v e c}(G), \chi_{v e c}(H)\right\} .
$$

Proof. As in Corollary 5.3, we can assume that neither $G$ nor $H$ contains any isolated vertices. If both graphs are vertex transitive, then they are 1homogeneous and the result holds by Theorem 5.5. Otherwise, at least one of them is bipartite and therefore, their product is bipartite and the conjecture holds trivially in this case.

## 6. Quantum Colorings

As a result of the continuing attempt to isolate the differences between quantum and classical mechanics, a large literature has developed which is devoted to the study of communication protocols based on the use of quantum resources, such as shared physical systems. In order to approach this problem with quantitative techniques and from a combinatorial angle, quantum colorings and the quantum chromatic number were introduced in $[9,6]$ and [2], respectively. These concepts were further investigated in $[5,8$, 24,19].

A seminal result in the study of quantum colorings was the discovery of a family of graphs $\left\{\Omega_{4 n}: n \in \mathbb{N}\right\}$ which exhibit an exponential separation between $\chi\left(\Omega_{4 n}\right)$ and $\chi_{q}\left(\Omega_{4 n}\right)[4,3,9]$. Here, $\Omega_{n}$ is the graph with vertex set $\{ \pm 1\}^{n}$ such that orthogonal vectors are adjacent. In [7] it was shown that when $n$ is a multiple of four, the graph $\Omega_{n}$ has chromatic number exponential in $n$. In contrast, it was shown in $[4,3,9]$ that when $n$ is a power of two, $\chi_{q}\left(\Omega_{n}\right) \leq n$. This result was extended to all $n$ divisible by four in [2].

In [22], Mančinska and Roberson introduce the notion of quantum homomorphisms, which generalize quantum colorings in the same way that homomorphisms generalize colorings. It is this framework we will use to study quantum colorings and quantum chromatic number.

For a more detailed look at quantum colorings and quantum homomorphisms we refer the reader to [5] and [22].

Though quantum homomorphisms were originally defined via a game played between two players and a referee, by the results of $[5,22]$, one can equivalently define them using homomorphisms. To do this, we require the following definition which comes from [22]:

Definition 6.1. For a graph $G$ and integer $d$, let $M(G, d)$ be the following graph. The vertices of $M(G, d)$ are the tuples $\mathbf{E}=\left(E_{v}\right)_{v \in V(G)}$ such that $E_{v} \in \mathbb{C}^{d \times d}$ is an orthogonal projector for all $v \in V(G)$ and

$$
\begin{equation*}
\sum_{v \in V(G)} E_{v}=I \tag{4}
\end{equation*}
$$

Two vertices $\mathbf{E}=\left(E_{v}\right)_{v \in V(G)}$ and $\mathbf{E}^{\prime}=\left(E_{v}^{\prime}\right)_{v \in V(G)}$ are adjacent if whenever $v \nsim v^{\prime}$,

$$
E_{v} E_{v^{\prime}}^{\prime}=0 .
$$

We refer to the graph $M(G, d)$ as the measurement graph of $G$ in dimension $d$.

Note that in the above we do not consider a vertex to be adjacent to itself and thus $v \nsim v$ for all $v \in V(G)$. Furthermore, note that condition (4) implies that for distinct vertices $v, v^{\prime} \in V(G)$, we have that $E_{v} E_{v^{\prime}}=0$.

The reasoning behind the name of the measurement graph is that its vertices are what are known as "projective quantum measurements". In general, a quantum measurement can consist of any positive semidefinite operators which sum to identity, but if each of the operators is a projection, then it is referred to as a projective measurement.

We say that $G$ has a quantum homomorphism to $H$, and write $G \xrightarrow{q} H$, if $G \rightarrow M(H, d)$ for some $d \in \mathbb{N}$. We will also refer to a homomorphism from $G$ to $M(H, d)$ as a quantum homomorphism from $G$ to $H$. Note that if $\varphi$ is a homomorphism from $G$ to $H$, then the map which takes $u \in V(G)$ to the tuple whose $\varphi(u)$ coordinate is $I$ and all other coordinates are 0 is a quantum homomorphism from $G$ to $H$. Therefore, $G \rightarrow H \Rightarrow G \xrightarrow{q} H$.

Now that we have defined quantum homomorphisms, we can define quantum colorings and quantum chromatic number in the obvious way: a quantum $n$-coloring of a graph $G$ is simply a quantum homomorphism from $G$ to $K_{n}$, and the quantum chromatic number of $G$, denoted $\chi_{q}(G)$, is the minimum $n$ such that $G \xrightarrow{q} K_{n}$. Note that since $G \rightarrow H \Rightarrow G \xrightarrow{q} H$, for all $G$ and $H$, we have that $\chi_{q}(G) \leq \chi(G)$ for all graphs $G$.

The definition of quantum homomorphism may seem a bit arbitrary, but it arises from the following physical considerations.

For graphs $G$ and $H$, the $(G, H)$-homomorphism game consists of two players, Alice and Bob, trying to convince a referee that they have a homomorphism from $G$ to $H$. More precisely, the referee sends Alice and Bob vertices $u_{A}, u_{B} \in V(G)$, respectively, and they respond with vertices $v_{A}, v_{B} \in V(H)$ accordingly. To win, the following conditions must be satisfied:

$$
\begin{aligned}
& \text { if } u_{A}=u_{B}, \text { then } v_{A}=v_{B} ; \\
& \text { if } u_{A} \sim u_{B} \text {, then } v_{A} \sim v_{B} .
\end{aligned}
$$

Players can decide upon a strategy beforehand, but cannot communicate once play has commenced. The game is played for only one round, but we require a "winning" strategy to win with probability 1. It is not too difficult to see that classical players (who can use probabilistic strategies and have access to shared randomness) can win the ( $G, H$ )-homomorphism game with certainty if and only if there exists a homomorphism from $G$ to $H$. However, if players are allowed to perform quantum measurements on a shared entangled state, then it is sometimes possible for them to win the $(G, H)$-homomorphism game even when $G \nrightarrow H$. A general introduction
to the theory of quantum entanglement can be found in [21]. In Chapter 10 of [1] the interested reader may find a short elementary analysis of a different communication game, which exhibits an analogous difference between the classical and the quantum version.

In [5] it was proven that for $H=K_{n}$, the $(G, H)$-homomorphism game can be won by quantum players if and only if $G \rightarrow M(H, d)$ for some $d \in \mathbb{N}$ (though it was not phrased in this way). In [22] they note that the same proof works for any graph $H$ and they introduce the measurement graph.

The general idea behind the correspondence between winning quantum strategies for the ( $G, H$ )-homomorphism game and homomorphisms from $G$ to $M(H, d)$ is as follows: Let $\varphi$ be a homomorphism from $G$ to $M(H, d)$. If Alice and Bob receive $u_{A}, u_{B} \in V(G)$, respectively, then Alice and Bob can perform measurements $\varphi\left(u_{A}\right)$ and $\varphi\left(u_{B}\right)^{T}$ on what is known as a "maximally entangled state" to win the game. Here, $\varphi\left(u_{B}\right)^{T}$ corresponds to taking the transpose of each coordinate of $\varphi\left(u_{B}\right)$. The adjacency condition for $M(H, d)$ will correspond to the probability of outputting an incorrect response being zero.

In many ways quantum homomorphisms behave similarly to homomorphisms. In $[22]$ it was shown that they are transitive, i.e., if $G \xrightarrow{q} H \xrightarrow{q} K$, then $G \xrightarrow{q} K$. This means that $\chi_{q}$ is quantum homomorphism-monotone, i.e., that $G \xrightarrow{q} H \Rightarrow \chi_{q}(G) \leq \chi_{q}(H)$. Note that in general $G \xrightarrow{q} H$ does not imply that $\chi(G) \leq \chi(H)$. Similarly, many other graph parameters defined via homomorphisms are not quantum homomorphism-monotone. However, in [22] it was shown that both $\chi_{v e c}$ and $\bar{\vartheta}$ are quantum homomorphism-monotone, i.e., $G \xrightarrow{q} H$ implies that

$$
\chi_{v e c}(G) \leq \chi_{v e c}(H) \text { and } \bar{\vartheta}(G) \leq \bar{\vartheta}(H) .
$$

Since $\bar{\vartheta}\left(K_{n}\right)=n$, it follows that strict vector chromatic number lower bounds quantum chromatic number. From this we obtain the following lemma:
Lemma 6.1. For any graph $G$, we have

$$
\chi_{v e c}(G) \leq \bar{\vartheta}(G) \leq \chi_{q}(G) .
$$

We mentioned above that $\chi_{q}$ is quantum homomorphism-monotone. This is in fact a stronger condition than being homomorphism-monotone. Indeed, $G \rightarrow H \Rightarrow G \xrightarrow{q} H \Rightarrow f(G) \leq f(H)$ for any quantum homomorphismmonotone parameter $f$. Therefore, $\chi_{q}$ is homomorphism-monotone, and thus by Lemma 2.1 we have

$$
\chi_{q}(G \square H) \geq \max \left\{\chi_{q}(G), \chi_{q}(H)\right\},
$$

and

$$
\chi_{q}(G \times H) \leq \min \left\{\chi_{q}(G), \chi_{q}(H)\right\} .
$$

So we have seen that the easy directions of Sabidussi's theorem and Hedetniemi's conjecture hold for all three of the parameters we are investigating.

## 7. Quantum Chromatic Number

Here we will prove the quantum analog of Sabidussi's theorem, and use Theorem 4.6 to show that the quantum analog of Hedetniemi's conjecture holds in certain cases. First, we need the following lemma. We denote by $G[H]$ the lexicographic product of $G$ with $H$, for a definition see [14].

Lemma 7.1. Suppose that $G, H, F, K$ are graphs such that $G \xrightarrow{q} F$ and $H \xrightarrow{q} K$. Then the following hold

1. $G \square H \xrightarrow{q} F \square K$;
2. $G \times H \xrightarrow{q} F \times K$;
3. $G \boxtimes H \xrightarrow{q} F \boxtimes K$;
4. $G * H \xrightarrow{q} F * K$;
5. $G[H] \xrightarrow{q} F[K]$.

Proof. We only give the proof for item (1), but it is obvious that a similar proof works for the others. For a function $f$ from vertices to tuples, we will use $f_{u}(v)$ to denote the $u$ coordinate of $f(v)$. Suppose that $\varphi^{1}$ and $\varphi^{2}$ are homomorphisms from $G$ to $M\left(F, d_{1}\right)$ and from $H$ to $M\left(K, d_{2}\right)$ respectively. Define $\varphi: V(G \square H) \rightarrow V\left(M\left(F \square K, d_{1} d_{2}\right)\right)$ as follows:

$$
\varphi_{(w, z)}(u, v)=\varphi_{w}^{1}(u) \otimes \varphi_{z}^{2}(v)
$$

for all $(u, v) \in V(G \square H)$ and $(w, z) \in V(F \square K)$. First, we must show that $\varphi$ is indeed a map to the vertices of $M\left(F \square K, d_{1} d_{2}\right)$. Since $\varphi_{w}^{1}(u)$ and $\varphi_{z}^{2}(v)$ are orthogonal projectors in dimensions $d_{1}$ and $d_{2}$ respectively, their tensor product is an orthogonal projector in dimension $d_{1} d_{2}$. Furthermore, since

$$
\sum_{w \in V(F)} \varphi_{w}^{1}(u)=I \text { for all } u \in V(G)
$$

and

$$
\sum_{z \in V(K)} \varphi_{z}^{2}(v)=I \text { for all } v \in V(H)
$$

we have that

$$
\begin{aligned}
\sum_{(w, z) \in V(F \square K)} \varphi_{(w, z)}(u, v) & =\sum_{w \in V(F), z \in V(K)} \varphi_{w}^{1}(u) \otimes \varphi_{z}^{2}(v) \\
& =\left(\sum_{w \in V(F)} \varphi_{w}^{1}(u)\right) \otimes\left(\sum_{z \in V(K)} \varphi_{z}^{2}(v)\right) \\
& =I \otimes I=I .
\end{aligned}
$$

Now recall from the definition of $M\left(F, d_{1}\right)$ that for $u \sim u^{\prime} \in V(G)$, we have that $\varphi_{w}^{1}(u) \varphi_{w^{\prime}}^{1}\left(u^{\prime}\right)=0$ whenever $w \nsim w^{\prime}$. We also have that $\varphi_{w}^{1}(u) \varphi_{w^{\prime}}^{1}(u)=0$ for distinct $w, w^{\prime} \in V(F)$, and the analogous conditions for $\varphi^{2}$.

To show that $\varphi$ is a homomorphism, we must show that for $(u, v) \sim\left(u^{\prime}, v^{\prime}\right)$, we have $\varphi_{(w, z)}(u, v) \varphi_{\left(w^{\prime}, z^{\prime}\right)}\left(u^{\prime}, v^{\prime}\right)=0$ whenever $(w, z) \nsim\left(w^{\prime} z^{\prime}\right)$. Since

$$
\varphi_{(w, z)}(u, v) \varphi_{\left(w^{\prime}, z^{\prime}\right)}\left(u^{\prime}, v^{\prime}\right)=\varphi_{w}^{1}(u) \varphi_{w^{\prime}}^{1}\left(u^{\prime}\right) \otimes \varphi_{z}^{2}(v) \varphi_{z^{\prime}}^{2}\left(v^{\prime}\right)
$$

it suffices to show that either $\varphi_{w}^{1}(u) \varphi_{w^{\prime}}^{1}\left(u^{\prime}\right)=0$ or $\varphi_{z}^{2}(v) \varphi_{z^{\prime}}^{2}\left(v^{\prime}\right)=0$.
Since $(u, v) \sim\left(u^{\prime}, v^{\prime}\right)$, without loss of generality we have that $u \sim u^{\prime}$ and $v=v^{\prime}$. The latter implies that $\varphi_{z}^{2}(v) \varphi_{z^{\prime}}^{2}\left(v^{\prime}\right)=0$ unless $z=z^{\prime}$. However, if $z=z^{\prime}$ and $(w, z) \nsim\left(w^{\prime}, z^{\prime}\right)$, then we must have that $w \nsim w^{\prime}$ and thus $\varphi_{w}^{1}(u) \varphi_{w^{\prime}}^{1}\left(u^{\prime}\right)=0$. Therefore, we have shown that $\varphi$ is a homomorphism from $G \square H$ to $M\left(F \square K, d_{1} d_{2}\right)$, and thus $G \square H \xrightarrow{q} F \square K$.

We will in fact only need item (1) from the above lemma. We state the others simply because they follow from an essentially identical proof. Recall from Section 6 that quantum homomorphisms are transitive, and that $G \rightarrow H \Rightarrow G \xrightarrow{q} H$ for any graphs $G$ and $H$. With these facts and the above lemma, we can easily prove the quantum version of Sabidussi's theorem.
Theorem 7.2. For graphs $G$ and $H$,

$$
\chi_{q}(G \square H)=\max \left\{\chi_{q}(G), \chi_{q}(H)\right\}
$$

Proof. We saw in Section 6 that $\chi_{q}(G \square H) \geq \max \left\{\chi_{q}(G), \chi_{q}(H)\right\}$, so we only need to show the other inequality. Let $n=\max \left\{\chi_{q}(G), \chi_{q}(H)\right\}$. Then we have that $G \xrightarrow{q} K_{n}$ and $H \xrightarrow{q} K_{n}$. Therefore, by Lemma 7.1 and the original Sabidussi's theorem, we have

$$
G \square H \xrightarrow{q} K_{n} \square K_{n} \rightarrow K_{n}
$$

and thus

$$
G \square H \xrightarrow{q} K_{n} .
$$

Therefore, $\chi_{q}(G \square H) \leq n$, and we are done.

Although we are not able to prove the general quantum version of Hedetniemi's conjecture, we can use the $\bar{\vartheta}$ version of Hedetniemi's conjecture to prove a special case.

Theorem 7.3. Suppose that graphs $G$ and $H$ are such that $\chi_{q}(G)=\bar{\vartheta}(G)$ and $\chi_{q}(H)=\bar{\vartheta}(H)$. Then

$$
\chi_{q}(G \times H)=\min \left\{\chi_{q}(G), \chi_{q}(H)\right\} .
$$

Proof. In Section 6 we saw that

$$
\chi_{q}(G \times H) \leq \min \left\{\chi_{q}(G), \chi_{q}(H)\right\} .
$$

Therefore, we only need to show the reverse inequality. Suppose that $G$ and $H$ satisfy the conditions above. Recall from Lemma 6.1 that $\bar{\vartheta}(K) \leq \chi_{q}(K)$ for any graph $K$. Thus

$$
\chi_{q}(G \times H) \geq \bar{\vartheta}(G \times H)=\min \{\bar{\vartheta}(G), \bar{\vartheta}(H)\}=\min \left\{\chi_{q}(G), \chi_{q}(H)\right\},
$$

by Theorem 4.6.
Recall that $\Omega_{n}$ is the graph with vertex set $\{ \pm 1\}^{n}$ such that orthogonal vectors are adjacent. In Section 6 , we saw that these graphs exhibit exponential separation between $\chi_{q}$ and $\chi$ for $n$ a multiple of 4 , and they have been central to the investigation of quantum chromatic number since its beginnings.

For $n$ odd, $\Omega_{n}$ is empty and thus $\chi_{q}\left(\Omega_{n}\right)=1=\bar{\vartheta}\left(\Omega_{n}\right)$. For $n \equiv 2 \bmod 4$, $\Omega_{n}$ is nonempty and bipartite, and thus $\chi_{q}\left(\Omega_{n}\right)=2=\bar{\vartheta}\left(\Omega_{n}\right)$. For $n$ a multiple of 4, combining results from [2] and [22] shows that $\chi_{q}\left(\Omega_{n}\right)=n=\bar{\vartheta}\left(\Omega_{n}\right)$. Therefore, $\chi_{q}\left(\Omega_{n}\right)=\bar{\vartheta}\left(\Omega_{n}\right)$ for all $n$ and thus we have the following corollary.

Corollary 7.4. For any $m, n \in \mathbb{N}$,

$$
\chi_{q}\left(\Omega_{m} \times \Omega_{n}\right)=\min \left\{\chi_{q}\left(\Omega_{m}\right), \chi_{q}\left(\Omega_{n}\right)\right\} .
$$

## 8. Concluding Remarks

We have shown that the $\chi_{v e c}, \bar{\vartheta}$, and $\chi_{q}$ versions of Sabidussi's theorem hold. We have also shown that the $\bar{\vartheta}$ version of Hedetniemi's conjecture holds, the $\chi_{v e c}$ version holds for 1-homogeneous graphs, and the $\chi_{q}$ version holds for graphs with strict vector chromatic number equal to quantum chromatic number. It is not surprising that we were more succesful with the analogs of Sabidussi's theorem, as this seems to be the easier of the two
problems in general. However, we conjecture that the $\chi_{v e c}$ and $\chi_{q}$ versions of Hedetniemi's conjecture hold in general.

With the similarity between $\chi_{v e c}$ and $\bar{\vartheta}$, it is worthwhile considering why the proof of Theorem 4.6 cannot be used to prove a version of Hedetniemi's conjecture for $\chi_{v e c}$. The proof of Theorem 4.6 relies on the following three properties of $\bar{\vartheta}$ :

1. $\bar{\vartheta}(G \square H)=\max \{\bar{\vartheta}(G), \bar{\vartheta}(H)\}$
2. $\bar{\vartheta}(G \boxtimes H) \geq \bar{\vartheta}(G) \bar{\vartheta}(H)$
3. $\bar{\vartheta}(G \cup H) \leq \bar{\vartheta}(G) \bar{\vartheta}(H)$

Combining the last two gives that

$$
\bar{\vartheta}(G) \bar{\vartheta}(H) \leq \bar{\vartheta}(G \boxtimes H) \leq \bar{\vartheta}(G \square H) \bar{\vartheta}(G \times H),
$$

which along with the first proves the theorem. We noted after Theorem 4.2 that (1) also holds for $\chi_{v e c}$, and it can be shown (using essentially the same proof as for $\bar{\vartheta}$ ) that (2) holds for $\chi_{v e c}$ as well. However, (3) is false for $\chi_{v e c}$, as already shown by Schrijver in [25] (his $\theta^{\prime}$ is equal to $\chi_{v e c}$ of the complement). Of course this does not mean that a version of Hedetniemi's conjecture for $\chi_{\text {vec }}$ cannot be proved, but a different approach is needed.

We can consider the same analysis for $\chi_{q}$. Theorem 7.2 shows that (1) holds for $\chi_{q}$. Item (4) of Lemma 7.1 concerning the disjunctive product shows that $\chi_{q}(G * H) \leq \chi_{q}(G) \chi_{q}(H)$, and then the same trick used to prove Corollary 4.5 shows that (3) holds for $\chi_{q}$. This leaves (2), but it is not hard to see $\chi\left(C_{5} \boxtimes C_{5}\right)=5$ and thus

$$
\chi_{q}\left(C_{5} \boxtimes C_{5}\right) \leq \chi\left(C_{5} \boxtimes C_{5}\right)=5<9=\chi_{q}\left(C_{5}\right)^{2} .
$$

Note that $\chi_{q}\left(C_{5}\right)=3$ follows from the fact that $\chi_{q}(G)=2$ if and only if $\chi(G)=2$, which was proven in [5].

Of $\chi_{v e c}$ and $\chi_{q}$, it seems that proving the analog of Hedetniemi's conjecture for the former should be more tractable. This is because one can use strong duality when working with $\chi_{v e c}$, whereas $\chi_{q}$ is not known to have this property. On the other hand, finding a counterexample to the conjecture (if one exists) is also likely easier for $\chi_{v e c}$ since it can be computed efficiently, and $\chi_{q}$ is not even known to be computable.

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## Appendix F

## Fractional covering by cuts \& Semidefinite programming

# Cubical coloring - fractional covering by cuts and semidefinite programming* 

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#### Abstract

We introduce a new graph invariant that measures fractional covering of a graph by cuts. Besides being interesting in its own right, it is useful for study of homomorphisms and tension-continuous mappings. We study the relations with chromatic number, bipartite density, and other graph parameters.

We find the value of our parameter for a family of graphs based on hypercubes. These graphs play for our parameter the role that circular cliques play for the circular chromatic number. The fact that the defined parameter attains on these graphs the 'correct' value suggests that the definition is a natural one. In the proof we use the eigenvalue bound for maximum cut and a recent result of Engström, Färnqvist, Jonsson, and Thapper.

We also provide a polynomial time approximation algorithm based on semidefinite programming and in particular on vector chromatic number (defined by Karger, Motwani and Sudan [Approximate graph coloring by semidefinite programming, J. ACM 45 (1998), no. 2, 246-265]).


## 1 Introduction

All graphs we consider are undirected and loopless; to avoid trivialities we do not consider edgeless graphs. For a set $W \subseteq V(G)$ we let $\delta(W)$ denote the set of edges leaving $W$ and we call any set of form $\delta(W)$ a cut. Other terminology we shall be using is standard, and can be found in, e.g., [6].

Let us call a (cut) $n / k$-cover of $G$ an $n$-tuple $\left(X_{1}, \ldots, X_{n}\right)$ of cuts in $G$ such that every edge of $G$ is covered by at least $k$ of them. We define two closely related parameters of $G$. We let

$$
x(G)=\inf \left\{\left.\frac{n}{k} \right\rvert\, \text { exists } n / k \text {-cover of } G\right\}
$$

and call $x(G)$ the fractional cut-covering number of $G$. Its 'rescaling'

$$
\chi_{q}(G)=\frac{2}{2-x(G)}
$$

[^8]will be called the cubical chromatic number of $G$. This terminology is motivated by analogy with the circular chromatic number, see the discussion following Equation (1). The rescaling function $2 /(2-x)$ serves the purpose of aligning the value with other variants of chromatic number, namely of attaining the right value for complete graphs. However, the rescaling function is far from arbitrary, as the values for other graphs are also modified in a proper way, see Theorem 5.2.

If $k=1$, i.e., if we want to cover every edge at least once, then we need at least $\left\lceil\log _{2} \chi(G)\right\rceil$ of them (see, e.g., [5]). Here we consider a fractional version. In this context we may find it surprising that $x(G)<2$ for every $G$ (Corollary 2.3).

From another perspective, $x(G)$ is the fractional chromatic number of a certain hypergraph: it has $E(G)$ as points and odd cycles of $G$ as hyperedges. This suggests that $x(G)$ is a solution of a linear program, see Equations (2) and (3).

The parameter $x(G)$ has found surprising use in computer science. Färnqvist, Jonsson, and Thapper [10] study the approximability of MAXCUT and its generalizations (so-called $H$-COLORING) using a suitably defined metric space. The function used to define the metric is in [7] recognized as a natural generalization of fractional covering by cuts. See the concluding remarks for further discussion.

As another point of view we note that $x(G)$ is a certain type of chromatic number, but instead of complete graphs (or Kneser graphs or circulants) which are used to define chromatic number (or fractional or circular chromatic number) it uses another graph scale. Let $Q_{n / k}$ denote a graph with $\{0,1\}^{n}$ as the set of vertices, where $x y$ forms an edge iff $d(x, y) \geq k$ (here $d(x, y)$ is the Hamming distance of $x$ and $y$ ).

Observation 1.1 A graph has $n / k$-cover if and only if it is homomorphic to $Q_{n / k}$.
Proof: If $\left(X_{1}, \ldots, X_{n}\right)$ is a cut $n / k$-cover of a graph $G$ then we can define homomorphism $f: V(G) \rightarrow V\left(Q_{n / k}\right)$ as follows: for each $i$ we write $X_{i}$ as $\delta\left(W_{i}\right)$; we put $f(v)=1$ if $v \in W_{i}$ and $f(v)=0$ otherwise. Now $f=\left(f_{1}, \ldots, f_{n}\right)$ is a homomorphism. If, on the other hand, we are given a homomorphism $f: V(G) \rightarrow V\left(Q_{n / k}\right)$ then we can put $W_{i}=\left\{v \in V(G): f_{i}(v)=1\right\}$ and observe that $\left(\delta\left(W_{1}\right), \ldots, \delta\left(W_{n}\right)\right)$ is a cut $n / k$-cover.

The above observation implies that an alternative definition of $x(G)$ is

$$
\begin{equation*}
x(G)=\inf \left\{\left.\frac{n}{k} \right\rvert\, G \xrightarrow{h o m} Q_{n / k}\right\} . \tag{1}
\end{equation*}
$$

An immediate corollary is that $x(G)$ is a homomorphism invariant, that is if $G \xrightarrow{\text { hom }} H$ then $x(G) \leq x(H)$. This will be strengthened in Lemma 1.2.

For a graph $H$ let $H^{\geq k}$ denote the graph with vertices $V(H)$ and edges $u v$ for any $u, v \in V(H)$ with distance in $H$ at least $k$. Further let $Q_{n}$ denote the $n$-dimensional cube. Then $Q_{n / k}=Q_{n}^{\geq}$. This corresponds to the definition of circular chromatic number, where the target graph is $C_{n}^{\geq k}$. This observation inspires the term cubical chromatic number. However, as we will see later (in Corollary 2.3), a rescaling of $x(G)$ is in order to make it behave like a version of chromatic number, thus the definition of $\chi_{q}$.

The original motivation for defining $x(G)$ was the study $[22,19]$ of cut-continuous mappings (defined in [5]). Given graphs $G, H$ we call a mapping $f: E(G) \rightarrow E(H)$ cut-continuous, if for every cut $U \subseteq E(H)$, the preimage $f^{-1}(U)$ is a cut in $G$. The following lemma is straightforward, but useful.

Lemma 1.2 Let $G, H$ be graphs. Then if there is a cut-continuous mapping from $G$ to $H$ (in particular, if there is a homomorphism $G \xrightarrow{\text { hom }} H$ ), then $x(G) \leq x(H)$ and (equivalently) $\chi_{q}(G) \leq \chi_{q}(H)$.

Proof: It suffices to show that whenever $H$ has an $n / k$-cover, $G$ has it as well. So let $f$ be some cut-continuous mapping from $G$ to $H$, let $X_{1}, \ldots, X_{n}$ be an $n / k$-cover and consider $X_{i}^{\prime}$-a preimage of the cut $X_{i}$ under $f$. By definition, $X_{i}^{\prime}$ is also a cut. If $e$ is an edge of $G, f(e)$ is an edge of $H$, hence it is covered by at least $k$ of the cuts $X_{i}$. Thus $e$ is covered by at least $k$ of the cuts $X_{i}^{\prime}$. For the homomorphism part, one may observe that the mapping induced on edges by a homomorphism is cut-continuous [5], or just use the alternative definition in Equation (1).

As each graph $Q_{n / k}$ is a Cayley graph on $\mathbb{Z}_{2}^{n}$, it follows (see [22]) that for every graph $G$ the existence of a homomorphism from $G$ to $Q_{n / k}$ is equivalent to the existence of a cut-continuous mapping from $G$ to $Q_{n / k}$. Consequently, we may as well use cut-continuous mapping to $Q_{n / k}$ in Equation (1). This also provides an indirect proof of Lemma 1.2.

It is a standard exercise to show that $x(G)$ is the solution of the following linear program ( $\mathcal{C}$ denotes the family of all cuts in $G$ )

$$
\begin{equation*}
\text { minimize } \sum_{X \in \mathcal{C}} w(X) \text { subject to: for every edge } e, \sum_{X, e \in X \in \mathcal{C}} w(X) \geq 1 \text {. } \tag{2}
\end{equation*}
$$

We conclude that we can replace inf by min in the definition of $x(G)$-the infimum is always attained. We can also consider the dual program

$$
\begin{equation*}
\text { maximize } \sum_{e \in E(G)} y(e) \text { subject to: for every cut } X, \sum_{e, e \in X} y(e) \leq 1 . \tag{3}
\end{equation*}
$$

This program is useful for computation of $x(G)$ for some $G$. (Färnqvist, Jonsson, and Thapper [10] used a modification of this program. There is an optimal solution $y^{*}$ of the above program, that respects symmetries of $G$ : if there is an automorphism of $G$ that maps edge $e$ to edge $f$, then $y^{*}(e)=y^{*}(f)$. This decreases the size of the linear program for graphs with nontrivial automorphism group.) Moreover, in the final section we use this dual program to discuss yet another definition of $x(G)$ in terms of the bipartite subgraph polytope.

There is another possibility to dualize the notion of fractional cut covering, namely fractional cycle covering. Bermond, Jackson and Jaeger [1] proved that every bridgeless graph has a cycle $7 / 4$-cover (i.e., a collection of 7 cycles, that cover every edge at least 4 times), and Fan [8] proved that it has a 10/6-cover. An equivalent formulation of the Berge-Fulkerson conjecture claims that every cubic bridgeless graph has a $6 / 4$-cover. On the other hand, Edmonds characterization of the matching polytope implies that every cubic bridgeless graph has a cycle $3 k / 2 k$-cover (for some $k$ ). It is open, whether for some fixed $k$ every cubic bridgeless graph has a cycle $3 k / 2 k$-cover.

## 2 Basic properties

We let $\operatorname{MAXCUT}(G)$ denote the number of edges in the largest cut in $G$ and write $b(G)=\operatorname{MAXCUT}(G) /|E(G)|$ for the bipartite density of $G$.

Lemma 2.1 For any graph $G$ it holds $x(G) \geq 1 / b(G)$. If $G$ is edge-transitive, then equality holds.

Proof: Suppose $x(G)=n / k$ and let $X_{1}, \ldots, X_{n}$ be an $n / k$-cover. Then $\sum_{i=1}^{n}\left|X_{i}\right| \leq$ $n \cdot b(G)|E(G)|$, on the other hand this sum is at least $k \cdot|E(G)|$, as every edge is counted at least $k$ times. This proves the first part of the lemma. To prove the second part, let $\mathcal{X}=\left\{X_{1}, \ldots, X_{n}\right\}$ be all cuts of the maximal size (i.e., $\left|X_{i}\right|=b(G)|E(G)|$ ). From the edge-transitivity follows that every edge is covered by the same number (say $k$ ) of elements of $\mathcal{X}$. Now $k \cdot|E(G)|=\sum_{i=1}^{n}\left|X_{i}\right|=n \cdot b(G)|E(G)|$, which finishes the proof.

Corollary 2.2 Let Pt denote the Petersen graph.

$$
\begin{array}{ll}
x\left(K_{2 n}\right)=x\left(K_{2 n-1}\right)=2-1 / n & \chi_{q}\left(K_{2 n}\right)=\chi_{q}\left(K_{2 n-1}\right)=2 n \\
x\left(C_{2 k+1}\right)=1+1 /(2 k) & \chi_{q}\left(C_{2 k+1}\right)=2+2 /(2 k-1) \\
x(\mathrm{Pt})=5 / 4 & \chi_{q}(\mathrm{Pt})=8 / 3
\end{array}
$$

In the following result, $g_{o}(G)$ denotes the length of a shortest odd cycle in $G$.
Corollary 2.3 For any graph $G$,

$$
2+\frac{2}{g_{o}(G)-2} \leq \chi_{q}(G) \leq 2\left\lceil\frac{\chi(G)}{2}\right\rceil
$$

Equivalently, $1+\frac{1}{g_{o}(G)-1} \leq x(G) \leq 2-\frac{1}{\lceil\chi(G) / 2\rceil}$.
In particular, $x(G) \in[1,2)$ and $\chi_{q}(G) \geq 2$.
Proof: Let $l=g_{o}(G)$, i.e., $C_{l}$ is the shortest odd cycle that is a subgraph of $G$. Put $n=\chi(G)$. Then there are homomorphisms $C_{l} \rightarrow G \rightarrow K_{n}$, so it remains to use Lemma 1.2 and Corollary 2.2.

By combining Lemma 1.2 and Corollary 2.2 we get that there is no cut-continuous mapping from $K_{n+2}$ to $K_{n}$. As there is obviously a cut-continuous mapping (indeed, even a homomorphism) in the other direction, we conclude that the even cliques $K_{2 n}$ form a strictly ascending chain in the poset defined by cut-continuous mappings. This application was the original point in defining $x(G)$, the result is not as straightforward as it appears (for example, there is a cut-continuous mapping $K_{4} \rightarrow K_{3}$ ).

Next, we will study how good are the bounds of Corollary 2.3. While they obviously are tight for $G$ equal to a complete graph, resp. odd cycle, they can be arbitrarily far off, as documented by Corollary 2.5 and Theorem 2.6. Before we get to that we need to look at $\chi_{f}(G)$-the fractional chromatic number of $G$. This may be defined by $\chi_{f}(G)=\inf \{n / k \mid G \xrightarrow{\text { hom }} K(n, k)\}$, where $K(n, k)$ is the Kneser graph.

Lemma 2.4 Let $k, n$ be integers such that $0<2 k \leq n$. Then

1. $b(K(n, k)) \geq 2 k / n$.
2. $x(K(n, k)) \leq n /(2 k)$.

Consequently, for any graph $G$ we have $x(G) \leq \frac{1}{2} \chi_{f}(G)$.
(Note that the bound is only useful if $k>n / 4$.)
Proof: For the first part we let $U=\{S \subseteq[n] \mid 1 \in S\}$ and observe that $\delta(U)$ contains $\binom{n-1}{k-1}\binom{n-k}{k}$ edges. As Kneser graphs are edge-transitive, the second part follows by Lemma 2.1. The rest follows by Lemma 1.2 and the definition of fractional chromatic number.

Corollary 2.5 For every $\varepsilon>0$ and every integer $b$ there is a graph $G$ such that

$$
\chi_{q}(G)<2+\varepsilon \quad \text { and } \quad \chi(G)>b
$$

Proof: Let $G=K(n, k)$, for $n=2 k+t, k=t^{2}$ and $t$ large enough. Then by Corollary 2.4 we have $x(G) \leq n / 2 k=1+t /\left(2 t^{2}\right)$, thus (for $t$ large enough) $\chi_{q}(G) \leq 2+\varepsilon$. On the other hand, it is known [17] that $\chi(G)=n-2 k+2=t+2$. Cf. also Corollary 5.4, where a stronger result is proved using semidefinite approximation.

By Corollary 2.3, we can view Corollary 2.5 as a strengthening of the well-known fact that there are graphs with no short odd cycle and with a large chromatic number. It also shows that the converse of Lemma 1.2 is far from being true: just take $G$ from the Corollary 2.5 and let $H=K_{b / 2}$ (for $b$ large). Then $\chi_{q}(G)$ is close to 2 and $\chi_{q}(H)$ is at least $b / 2$, still by an application of Proposition 6.7 of [5] there is no cut-continuous mapping from $G$ to $H$.

It is interesting to find how various graph properties affect $\chi_{q}(G)$. From the values in Corollary 2.2 we might think that $\chi_{q}(G)$ is always larger than the fractional chromatic number $\chi_{f}(G)$. However, this is very far from the truth, as shown in Corollary 5.4. We saw already that small $\chi(G)$ makes $\chi_{q}(G)$ small (Corollary 2.3), while large $\chi(G)$ does not force $\chi_{q}$ to be large (Corollary 2.5). Also small $g_{o}(G)$ makes $\chi_{q}(G)$ large (Corollary 2.3 again). Complementing Corollary 2.5 we show that large $g_{o}(G)$ does not make $\chi_{q}(G)$ small (but cf. Question 2.7).

Theorem 2.6 For any integers $k$, $l$ there is a graph $G$ such that $\chi_{q}(G)>k$ and $G$ contains no circuit of length at most $l$.

Proof: We modify the famous Erdős' proof of existence of high-girth graphs of high chromatic number.

Let $p=n^{\alpha-1}$ (where $\alpha \in(0,1 / l)$ ) and consider the random graph $G(n, p)$. The expected number of circuits of length at most $l$ is $O\left((p n)^{l}\right)=o(n)$, therefore by Markov inequality with probability $1-o(1)$ the graph $G(n, p)$ contains at most $n$ circuits of length at most $l$.

Using Lemma 3.1, and in particular its Claim 1, where we put $\delta=n^{-\alpha / 3}$ we get that a.a.s. $b(G(n, p)) \leq \frac{1}{2}\left(1+O\left(n^{-\alpha / 3}\right)\right)$ and $|E(G(n, p))|>n^{1+\alpha} / 3$.

We take a graph $G^{\prime}$ satisfying all these three requirements. Then we delete one edge from each of the at most $n$ short circuits and let $G$ be the resulting graph.

Clearly $G$ contains no short cycles. To show $\chi_{q}(G)$ is large it is enough to show that $x(G)$ can be arbitrary close to 2 , or (using Lemma 2.1) to show that $b(G)$ can be arbitrary close to $1 / 2$.

As $\left|E\left(G^{\prime}\right)\right|=\Omega\left(n^{1+\alpha}\right)$, and as we delete at most $n$ edges of $G^{\prime}$ to get $G$, we have $|E(G)| \geq\left|E\left(G^{\prime}\right)\right|(1-o(1))$. Obviously, MAXCUT in $G$ cannot be larger than in $G^{\prime}$, thus $b(G) \leq b\left(G^{\prime}\right)(1+o(1))=\frac{1}{2}(1+o(1))$, which finishes the proof.

In the previous result it was crucial that the graphs had large degrees. For graphs of small degree the situation differs:

Question 2.7 Let $G$ be a cubic graph with no cycle of length $\leq c$. How large can $\chi_{q}(G)($ resp. $x(G)) b e$ ?

For $c=3$, it follows from Brooks' theorem that $x(G) \leq x\left(K_{3}\right)=3 / 2\left(\chi_{q}(G) \leq\right.$ 4). For $c=17$, it is known [4] that $G$ has a cut-continuous mapping to $C_{5}$, hence $x(G) \leq x\left(C_{5}\right)=5 / 4\left(\chi_{q}(G) \leq 8 / 3\right)$. On the other hand, there is $\varepsilon>0$ such that cubic graphs $G$ of arbitrary high girth exist with $b(G)<1-\varepsilon$ (an unpublished result of McKay, see also [22]), hence with $x(G)>1+\varepsilon$ and so $\chi_{q}(G)>2+2 \varepsilon$.

We conclude this section by a simple lemma that shows that $\chi_{q}$ and $x$ enjoy some of the properties of other chromatic numbers. (Here $G_{1} \square G_{2}$ denotes the Cartesian product of graphs, $G_{1} \times G_{2}$ the categorical one (also called tensor product), see [14].)

Lemma 2.8 1. $x(G)=\max \left\{x\left(G^{\prime}\right) \mid G^{\prime}\right.$ is a component of $\left.G\right\}$
2. $x(G)=\max \left\{x\left(G^{\prime}\right) \mid G^{\prime}\right.$ is a 2-connected block of $\left.G\right\}$ for a connected graph $G$.
3. $x\left(G_{1} \square G_{2}\right)=\max \left\{x\left(G_{1}\right), x\left(G_{2}\right)\right\}$
4. $x\left(G_{1} \times G_{2}\right) \leq \min \left\{x\left(G_{1}\right), x\left(G_{2}\right)\right\}$

The same formulas are true for $\chi_{q}$ in place of $x$.

Proof: We will prove that if $G^{\prime}, G^{\prime \prime}$ are graphs that share at most one vertex, then $x\left(G^{\prime} \cup G^{\prime \prime}\right)=\max \left\{x\left(G^{\prime}\right), x\left(G^{\prime \prime}\right)\right\}$. Clearly, this proves 1 and 2 . Let $x\left(G^{\prime}\right)=$ $n / k$, and $x\left(G^{\prime \prime}\right)=m / l$ (by discussion after Equation (2) the infimum is attained) and suppose $X_{1}^{\prime}, \ldots, X_{n}^{\prime}$ is an $n / k$-cover of $G^{\prime}$, while $X_{1}^{\prime \prime}, \ldots, X_{m}^{\prime \prime}$ is an $m / l$-cover of $G^{\prime \prime}$. Consider the collection of $m n$ cuts $\left\{X_{i}^{\prime} \cup X_{j}^{\prime \prime}\right\}$ (these are cuts, indeed, as $G^{\prime}$ and $G^{\prime \prime}$ share at most one vertex). An edge of $G^{\prime}$ is covered at least $m k$ times, an edge of $G^{\prime \prime}$ at least $n l$ times. Hence $x(G) \leq \frac{m n}{\min \{m k, n l\}}=\max \left\{\frac{n}{k}, \frac{m}{l}\right\}=\max \left\{x\left(G^{\prime}\right), x\left(G^{\prime \prime}\right)\right\}$. On the other hand, both $G^{\prime}$ and $G^{\prime \prime}$ are subgraphs of $G$, hence by Lemma 1.2 the other inequality follows.

Part 3 follows from Lemma 1.2, as between $G_{1} \square G_{2}$ and $G_{1} \cup G_{2}$ exists a cutcontinuous mapping in both directions.

Part 4 follows from Lemma 1.2 as there are homomorphisms (and therefore $T T$ mappings) $G_{1} \times G_{2} \rightarrow G_{i}$ (for $i=1,2$ ).

As $\chi_{q}=2 /(2-x)$ (which is an increasing function for the values that $x$ can attain), the results for $\chi_{q}$ follow immediately.

## 3 Cubical chromatic number of random graphs

In this section we consider the value of cubical chromatic number of random graphs. After a short technical lemma (that is also used in the proof of Theorem 2.6) we bound $\chi_{q}$ of a random graph $G(n, 1 / 2)$ using a simple self-contained proof. We complement this by a result that provides the correct order of magnitude using results from Section 5.

Lemma 3.1 Let $p, \delta$ be functions of $n$ such that $p, \delta \in[0,1]$ and $\delta^{2} p \geq 7 \log n / n$. Then $b(G(n, p)) \leq \frac{1}{2}(1+O(1 / n)+O(\delta))$ a.a.s. In particular, we have

$$
b(G(n, p)) \leq \frac{1}{2}+O\left(\sqrt{\frac{\log n}{p n}}\right) \quad \text { a.a.s. }
$$

Proof: We will prove that almost all graphs have "many edges but no huge cut".
Claim 1. $|E(G(n, p))|>(1-\delta) p\binom{n}{2}$ a.a.s.
To prove this we use Chernoff inequality (as stated in Corollary 2.3 of [15]) for random variable $X=|E(G(n, p))|$. It claims $\operatorname{Pr}[|X-\mathbb{E} x| \geq \delta \mathbb{E} X] \leq 2 e^{-\frac{\delta^{2}}{3} \mathbb{E} X}$ for $\delta \leq 3 / 2$ and as $\mathbb{E} X=p\binom{n}{2}$, Claim 1 follows.

Claim 2. $\operatorname{MAXCUT}(G(n, p))<(1+\delta) p \frac{n^{2}}{4}$ a.a.s.
For a set $A \subseteq V(G(n, p))$ we let $X_{A}$ be the random variable that counts the edges leaving $A$, and put $a=|A| \leq n / 2$. By Chernoff inequality for $X_{A}$ we easily get

$$
\operatorname{Pr}\left[X_{A} \geq(1+\delta) p n^{2} / 4\right] \leq 2 e^{-\frac{\delta^{2}}{3} p a(n-a)} \leq 2 e^{-\frac{\delta^{2} p a n}{6}}
$$

It remains to estimate the total probability of a large cut:

$$
\operatorname{Pr}\left[(\exists A) X_{A} \geq(1+\delta) p n^{2} / 4\right] \leq \sum_{a=1}^{n / 2}\binom{n}{a} 2 e^{-\frac{\delta^{2} p a n}{6}} \leq 2\left(\left(1+e^{-\frac{\delta^{2} p n}{6}}\right)^{n}-1\right)
$$

For $\delta^{2} p \geq 7 \log n / n$ the last expression tends to zero, which finishes the proof of Claim 2. The rest of the proof of the lemma is a simple calculation.

## Theorem 3.2

$$
\Omega(\sqrt{n / \log n}) \leq \chi_{q}(G(n, 1 / 2)) \leq O(n / \log n) \quad \text { a.a.s. }
$$

Proof: The lower bound follows by Lemma 3.1, the upper one by an application of Corollary 2.3 and the well-known fact that $\chi(G(n, 1 / 2))=O(n / \log n)$.

Theorem $3.3 \chi_{q}(G(n, p))=\Theta(\sqrt{p n}) \quad$ a.a.s.
Proof: The result follows directly using Theorem 5.1 and Theorem 5.2.

## 4 Measuring the scale

In this section we will discuss the 'invariance property' of cubical chromatic number. In analogy with $\chi\left(K_{n}\right)=n, \chi_{c}\left(C_{n}^{\geq k}\right)=n / k, \chi_{f}(K(n, k))=n / k$, and 'dimension of product of $n$ complete graphs is $n$ ' we would like to prove that $x\left(Q_{n / k}\right)=n / k$. The following lemma shows, that the situation is not that simple for $x$.

Lemma 4.1 Let $1 \leq k \leq n$ be integers. Then we have $x\left(Q_{n / k}\right) \leq \frac{n}{k}$. If $k$ is odd, then $x\left(Q_{n / k}\right) \leq \frac{n+1}{k+1}$.

Proof: For the first part, it suffices to consider the identical homomorphism $Q_{n / k} \xrightarrow{\text { hom }}$ $Q_{n / k}$. For the second part, mapping $V\left(Q_{n / k}\right) \rightarrow V\left(Q_{\frac{n+1}{k+1}}\right)$ given by $\left(x_{1}, \ldots, x_{n}\right) \mapsto$ $\left(x_{1}, \ldots, x_{n}, x_{1}+\cdots+x_{n} \bmod 2\right)$ is a homomorphism whenever $k$ is odd.

Another complication is that by Corollary 2.3 we have $x(G)<2$ for any graph $G$. However, with this exception, the bounds in Lemma 4.1 are optimal:

Theorem 4.2 Let $k$, $n$ be integers such that $k \leq n \leq 2 k$. Then

1. if $k$ is even and $n<2 k$ then $x\left(Q_{n / k}\right)=\frac{n}{k}$; and
2. if $k$ is odd then $x\left(Q_{n / k}\right)=\frac{n+1}{k+1}$.

This theorem was announced as a conjecture in the author's thesis [22], together with a part of a possible proof. The proof was finished by Engström, Färnqvist, Jonsson, and Thapper [7, Lemma 4.4], who did prove the inequality in Lemma 4.4.

We'll use the following result (see Lemma 13.7.4 and 13.1.2 of [11]).
Lemma 4.3 Let $G$ be an r-regular graph with $n$ vertices, let $\lambda_{\text {min }}$ be the smallest eigenvalue of $G$. Then $b(G) \leq \frac{1}{2}\left(1-\frac{\lambda_{\text {min }}}{r}\right)$.

The following lemma was proved (using a clever induction) by Engström, Färnqvist, Jonsson, and Thapper [7, Lemma 4.4], resolving thus a question from the author's thesis [22].

Lemma 4.4 Let $k, n$ be integers such that $k \leq n<2 k$ and $k$ is even, let $x$ be an integer such that $1 \leq x \leq n$. Then

$$
\sum_{o d d t}\binom{x}{t}\binom{n-x}{k-t} \leq\binom{ n-1}{k-1}
$$

Proof: (of Theorem 4.2) Lemma 4.1 provides the upper bound, we will establish the lower bound now. Suppose first that $k$ is even. We shall use a spanning subgraph of $Q_{n / k}=Q_{n}^{\geq k}$, that contains only edges of length precisely $k$; we shall use $Q_{n}^{=k}$ to denote this subgraph.

By Lemma 1.2 and 2.1 we have that $x\left(Q_{n / k}\right) \geq x\left(Q_{n}^{=k}\right)=1 / b\left(Q_{n}^{=k}\right)$. By Lemma 4.3 it is enough to determine the smallest eigenvalue $\lambda_{\min }$ of $Q_{n}^{=k}$. As $Q_{n}^{=k}$ is $\binom{n}{k}$-regular, we have

$$
\frac{1}{b\left(Q_{n}^{=k}\right)} \geq \frac{2}{1-\lambda_{\min } /\binom{n}{k}}
$$

It is standard (see, e.g., Problem 11.8 in [18] or the theory of Association Schemes in Chapter 30 of [23]) that the eigenvalues of $Q_{n}^{=k}$ are

$$
\sum_{t=0}^{k}(-1)^{t}\binom{x}{t}\binom{n-x}{k-t}
$$

By using Vandermonde's identity and Lemma 4.4, we get that the above sum is at least $\binom{n}{k}(1-2 k / n)$, which is equal to the sum for $x=1$. Thus the smallest eigenvalue $\lambda_{\text {min }}$ equals $\binom{n}{k}(1-2 k / n)$, and we obtain $x\left(Q_{n / k}\right) \geq n / k$ as desired.

For odd values of $k$ we cannot use the same method, as then $Q_{n}^{=k}$ is bipartite, hence $b\left(Q_{n}^{=k}\right)=1$. However, observe that $Q_{\frac{n+1}{k+1}} \xrightarrow{\text { hom }} Q_{n / k}$, hence by Lemma 1.2 and the result for (even) $k+1$ we have

$$
x\left(Q_{n / k}\right) \geq x\left(Q_{\frac{n+1}{k+1}}\right) \geq \frac{n+1}{k+1} .
$$

Corollary 4.5 The set $\{x(G): G$ is a graph $\}$ equals $\mathbb{Q} \cap[1,2]$. Consequently, the set $\left\{\chi_{q}(G): G\right.$ is a graph $\}$ equals $\mathbb{Q} \cap[2, \infty)$.

## 5 Semidefinite approximation

In this section we show how to approximate $\chi_{q}$ in polynomial time up to a factor of $\pi / 2$. Key to this approximation is the vector coloring, introduced by [16] based on the Lovász' $\vartheta$ function. The concept of vertex coloring is extended by using highdimensional unit vectors as colors, and requiring adjacent vertices to be assigned distant vectors. Precisely: given a graph $G$ and real $t<0$ consider a mapping $f: V(G) \rightarrow$ $\mathbb{R}^{n}$ (where $n=|V(G)|$ ), so that

- $\|f(v)\|_{2}=1$ for every vertex $v$ and
- $\langle f(u), f(v)\rangle \leq t$ for every edge $u v$.

We let $t(G)$ denote the minimum $t$ such that function $f$ with the above properties exists. The vector chromatic number of $G$ is defined as $\chi_{v}(G)=1-\frac{1}{t(G)}$.

As these conditions for $t(G)$ can be formulated as a semidefinite program, the minimum indeed exists; more importantly, $t(G)$ can be approximated with an absolute error $\varepsilon$ in time polynomial in $n$ and $\log \frac{1}{\varepsilon}$. Indeed, Karger, Motwani and Sudan [16, Lemma 3.2] prove that if a graph $G$ has $\chi_{v}(G)=k$ then it is possible to find a vector $(k+\varepsilon)$-coloring in time polynomial in $n$ and $\log 1 / \varepsilon$ - in particular, one finds approximation to $\chi_{v}$ up to an absolute error $\varepsilon$.

It is easy to see that $\chi_{v}(G) \leq \chi(G)$ - given a proper $k$-coloring, we may map all vertices of one color to one vertex of a simplex with $k$ vertices. This will lead to $t=-\frac{1}{k-1}$, and so indeed $\chi_{v}(G) \leq k$. However, the fraction $\chi(G) / \chi_{v}(G)$ can be arbitrarily large [9], in fact as large as $n / \operatorname{polylog}(n)$ (where $n=|V(G)|$ ); this contrasts sharply with Theorem 5.2.

For further properties of $\chi_{v}$ see [16] and [3]. In the latter the following is shown.
Theorem 5.1 ([3]) $c_{1} \sqrt{n p} \leq \chi_{v}\left(G_{n, p}\right) \leq c_{2} \sqrt{n p}$ with probability $1-o(1)$.
Now we proceed to show to connection between $\chi_{q}$ and $\chi_{v}$.
Theorem 5.2 For every graph $G$ we have

$$
\chi_{v}(G) \leq \chi_{q}(G) \leq \frac{\pi}{2} \chi_{v}(G)
$$

Proof: We prove the lower bound first. Recall that $\chi_{q}(G)=\frac{2}{2-x(G)}$ and $x(G)=n / k$, for some $n, k$ where there is a $k$-cover of $G$ by $n$ cuts. (The fact that the infimum in the definition of $x(G)$ is attained follows from the linear-programming reformulation, see Equation (2).) Equivalently, there is a mapping $g: V(G) \rightarrow\{ \pm 1\}^{n}$ (the $i$-th coordinate encodes the $i$-th cut so that for every edge $u v$ the vectors $g(u)$ and $g(v)$ differ in $\geq k$ coordinates. Put $f(v)=g(v) / \sqrt{n}$. Obviously, each $f(v)$ is a unit vector, while for every edge $u v$ we have

$$
\langle f(u), f(v)\rangle=1-\frac{2 d_{H}(g(u), g(v))}{n} \leq 1-\frac{2 k}{n}=1-\frac{2}{x(G)} .
$$

Therefore, for this $f$ we get $t \leq 1-2 / x(G)$. Consequently,

$$
\chi_{v}(G) \leq 1-\frac{1}{t} \leq 1-\frac{x(G)}{x(G)-2}=\frac{2}{2-x(G)}=\chi_{q}(G)
$$

For the upper bound we use probabilistic approach, motivated by the algorithm for approximating MAXCUT by Goemans and Williamson [12]. Consider a mapping $f$ as above, the scalar products are at most $t$ with $\chi_{v}(G)=1-1 / t$. For a large $N$, we choose $N$ uniformly random hyperplanes in $\mathbb{R}^{n}$ through the origin. With probability 1 none of them contains any of the points $f(v)$ for $v \in V(G)$, therefore each hyperplane defines a cut. We shell prove that with probability $1-o(1)$ this cut covering gives us the desired bound.

To this end, consider an edge $u v \in E(G)$, let $\alpha$ be the angle between the unit vectors $f(u)$ and $f(v)$. The following elementary observation (used also in [12]) is crucial for the calculation:

A random hyperplane through origin separates $f(u)$ and $f(v)$ with probability $\frac{\alpha}{\pi}$.
For an edge $e=u v$ let $X_{e}$ be the random variable that counts how many of the $N$ hyperplanes separate the end-vertices of $e$. Obviously, $X_{e}$ follows a binomial distribution $\operatorname{Bin}(N, p)$ with $p=\frac{\alpha}{\pi}$. We have $\cos \alpha=\langle f(u), f(v)\rangle \leq t$, so $p \geq \frac{\arccos t}{\pi}$. By the Chernoff inequality we have $\operatorname{Pr}\left[X_{e}<p N-s\right]<e^{-\frac{s^{2}}{2 N_{p}}}$. Putting $s=\left\lceil N^{2 / 3}\right\rceil$ we obtain

$$
\operatorname{Pr}\left[X_{e}<p N-\left\lceil N^{2 / 3}\right\rceil\right]<e^{-\frac{N^{1 / 3}}{2 p}}=o(1)
$$

(the $o(1)$ is with respect to $N$ growing to infinity). Thus, with probability $1-\binom{n}{2} o(1)=$ $1-o(1)$ we have $X_{e} \geq p N-\left\lceil N^{2 / 3}\right\rceil$ for every edge $e$. So for every large enough $N$ there is a cut covering achieving this and from the definition of $x(G)$, we get that

$$
x(G) \leq \frac{N}{p N-\left\lceil N^{2 / 3}\right\rceil}=\frac{1}{p}(1+o(1)) .
$$

As we may choose arbitrarily large $N$, we get from here that $x(G) \leq \frac{1}{p}=\frac{\pi}{\arccos t}$. Now from the definition we obtain

$$
\frac{\chi_{q}(G)}{\chi_{v}(G)}=\frac{\frac{2}{2-x(G)}}{1-\frac{1}{t}} \leq \frac{\frac{2}{2-\frac{\pi}{\arccos t}}}{1-\frac{1}{t}}=\frac{t \arccos t}{(\arccos t-\pi / 2)(t-1)}
$$

Putting $t=\cos \alpha$ and $\beta=\alpha-\pi / 2$ (so that $t=-\sin \beta$ ), the last expression equals

$$
\frac{\sin \beta}{\beta} \frac{\beta+\frac{\pi}{2}}{\sin \beta+1} \leq 1 \cdot \frac{\pi}{2}
$$

(we used the elementary estimate $\frac{2}{\pi} \beta \leq \sin \beta \leq \beta$ valid for $\beta \in[0, \pi / 2]$ ).
We note that the above proof also yields bound $\chi_{q}(G) \leq 1 /\left(1-\frac{\pi}{2 \arccos \frac{1}{1-\chi_{v}(G)}}\right)$, which is, for small values of $\chi_{v}(G)$, slightly better than the above theorem.

Corollary 5.3 There is a polynomial-time algorithm that approximates $\chi_{q}(G)$ with approximation factor almost $\frac{\pi}{2}$. More precisely: to get an approximation factor at most $\frac{\pi}{2}(1+\varepsilon)$ we need an algorithm polynomial in $|V(G)|$ and $\log 1 / \varepsilon$.

Corollary 5.4 For every graph $G$ we have

$$
\chi_{q}(G) \leq \frac{\pi}{2} \chi_{f}(G)
$$

Moreover, there is a sequence of graphs for which $\chi_{q}(G)$ is bounded, while $\chi_{f}(G)$ is unbounded.

Proof: For the first part it is enough to use Theorem 5.2, the bound $\chi_{v}(G) \leq \vartheta(\bar{G})$ (Theorem 8.2 of [16]) and the well-known bound $\vartheta(\bar{G}) \leq \chi_{f}(G)$. We use Theorem 1.2 of [9]: There are infinitely many graphs $G$ that are vector 3-colorable and satisfy $\alpha(G) \leq n^{0.843}$ (where $n$ is the number of vertices of $G$ ). Each such graph $G$ satisfies $\chi_{q}(G) \leq 3 \pi / 2<5$, and $\chi_{f}(G) \geq n / n^{0.843}=n^{0.157}$.

## 6 Concluding Remarks

Bipartite subgraph polytope For a bipartite subgraph $B \subseteq G$, let $c_{B}$ be the characteristic vector of $E(B)$. Bipartite subgraph polytope $P_{B}(G)$ is the convex hull of points $c_{B}$, for all bipartite graphs $B \subseteq G$. The study of this polytope was motivated by the MAXCUT problem: to look for a weighted maximum cut of $G$ simply means to solve a linear program over $P_{B}(G)$. Thus, for graphs where $P_{B}(G)$ has simple description, we can have polynomial-time algorithm for MAXCUT; this in particular happens for weakly bipartite graphs (which include planar graphs), see [13]. We apply $P_{B}$ to yield yet another definition of $x$.

Theorem 6.1 $x(G)=\max \left\{\sum_{e \in E(G)} y_{e} \mid y \cdot c \leq 1\right.$ defines a facet of $\left.P_{B}(G)\right\}$

Proof: By LP duality $x(G)$ is a solution to the program (3). This means, that we are maximizing over such $y$, that for each cut $X$ satisfy $y \cdot c_{X} \leq 1$. As the convex hull of vectors $c_{X}$ is $P_{B}$, we are maximizing the sum of coordinates of an element of the dual polytope $P_{B}^{*}$. This maximum is attained for some vertex of $P_{B}^{*}$, that is for $y$ such that $y \cdot c \leq 1$ defines a facet of $P_{B}$.
'Natural' facets of $P_{B}(G)$ are defined by $\sum_{e \in E(H)} y_{e} \leq \operatorname{MAXCUT}(H)$ for some $H \subseteq G$. (This inequality is satisfied for every graph $H$, but it doesn't always define a face of maximal dimension.) This proves the following observation (we add a direct proof, too).

Lemma 6.2 $x(G) \geq 1 /\left(\min _{H \subseteq G} b(H)\right)$

Proof: Let $H \subseteq G$. Then $H \xrightarrow{T T_{2}} G$, which by Lemma 1.2 and 2.1 implies $1 / b(H) \leq$ $x(G)$.

Let us return to Lemma 2.1 for a while. In general $x(G)$ and $1 / b(G)$ can be as distant as possible: Let $G$ be a disjoint union of a $K_{n}$ and $K_{N, N}$. Now $x(G)$ is close to 2 (because $G$ is homomorphically equivalent to $K_{n}$, hence $x(G)=x\left(K_{n}\right)$ ) and $b(G)$ is close to 1 (provided $N$ is sufficiently large). This motivates Lemma 6.2, which improves the original bound. A natural question is whether this improvement gives the correct size of $x$. It turns out it does not (contrary to a conjecture in the author's thesis). In [7] it is shown, that the circular clique $K_{11 / 4}$ is a counterexample.

A failed attempt The proof of Theorem 4.2 could be attempted by another way: First, observe that the Kneser graph $K(n, r)$ is a subgraph of $Q_{n / 2 r}$. By Lemma 1.2 and 2.1 we have $x\left(Q_{n / 2 r}\right) \geq x(K(n, r)) \geq \frac{1}{b(K(n, r))}$. Thus, if we knew the value of $b(K(n, r))$ (and it turned out to be $2 r / n$ for the range of $r$ we are interested in), we would be done.

In [20] it is claimed that if $2 r \leq n \leq 3 r$ then, indeed, $b(K(n, r))=2 r / n$. This would imply the conjecture for even $k$ less than $3 / 2 \cdot n$; unfortunately the proof in [20] is incomplete (as already observed by [2]). Thus, the true value of MAXCUT for Kneser graphs remains open.

Generalizations and future work As already mentioned in the introduction, the metric that is used in [10, 7] to study approximability of MAX- $H$-COLORING can be computed from a generalization of fractional covering by cuts. One only needs to consider more general edge sets in place of cuts, namely edge sets of graphs that are homomorphic to $H$. Then the cube $Q_{n / k}$ in Equation (1) is replaced by appropriately defined power of $H$. One may also use this motivation to define $H$-continuous mappings as follows. We call a subset $X \subseteq E(G)$ an $H$-cut in $G$ whenever there is a mapping $g: V(G) \rightarrow V(H)$ for which $g^{-1}(E(H))=X$. We say a mapping $f: E\left(G_{1}\right) \rightarrow E\left(G_{2}\right)$ is $H$-continuous whenever a preimage of each $H$-cut is an $H$-cut. This notion deserves further attention.

Number of cuts required By definition, if $x(G)=t$ then there is a cut $n / k$-cover for some $n$, $k$ satisfying $t=\frac{n}{k}$. It would be nice to know how large $n$ is required. To be precise, define $n(G)$ to be the smallest $n$ as above. Then we let

$$
f(v)=\max \{n(G): G \text { is a graph with } v \text { vertices }\}
$$

This maximum clearly exists (as there are only finitely many graphs on $v$ vertices).
Question 6.3 How fast doest $f(v)$ grow? Is $f(v) \leq 2^{v}$ ?
The estimate by $2^{v}$ seems natural, as there is only $2^{v-1}$ different cuts in a graph on $v$ vertices. However, one may be forced to take some cuts repeatedly.

Complexity In view of the complexity of computing other variants of chromatic number, the following conjecture is natural. Note, however, that in contrast with chromatic or fractional chromatic number, cubical chromatic number can be approximated up to a constant factor.

Conjecture 6.4 For any $s>2$ determining if an input graph $G$ satisfies $\chi_{q}(G) \leq s$ is NP-complete.

Cubic graphs For the reader's convenience we restate here Question 2.7. For known partial results we refer the reader to Section 2.

Question 6.5 Let $G$ be a cubic graph with no cycle of length $\leq c$. How large can $x(G)$ $\left(\right.$ resp. $\left.\chi_{q}(G)\right) b e$ ?

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## Appendix G

## Flow-continuous mappings - The influence of the group

# Flow-continuous mappings-The influence of the group ${ }^{\text {* }}$ 

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#### Abstract

Many conjectures at the core of graph theory can be formulated as questions about certain group-valued flows: examples include the cycle double-cover conjecture, the Berge-Fulkerson conjecture, and Tutte's 3-flow, 4-flow, and 5-flow conjectures. As an approach to these problems, Jaeger, and DeVos, Nešetřil, and Raspaud define a notion of graph morphisms continuous with respect to groupvalued flows. We discuss the influence of the group on these maps. In particular, we prove that the number of flow-continuous mappings between two graphs does not depend on the group, but only on the largest order of an element of the group (i.e., on the exponent of the group). Further, there is a nice algebraic structure describing for which groups a mapping is flow-continuous.

On the combinatorial side, we show that for cubic graphs the only relevant groups are $\mathbb{Z}_{2}, \mathbb{Z}_{3}$, and $\mathbb{Z}$.


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## 1. Introduction

Throughout this paper $G$ and $H$ will be digraphs (finite multidigraphs with loops and parallel edges allowed), $f: E(G) \rightarrow E(H)$ a mapping, and $M, N$ abelian groups.

Recall that a mapping $\varphi: E(G) \rightarrow M$ is a flow (an $M$-flow when we want to emphasize $M$ ) when it satisfies Kirchhoff's law at every vertex, that is, for every $v \in V(G)$ we have

$$
\sum_{e \in E(G): e \text { leaves } v} \varphi(e)=\sum_{e \in E(G): e \text { enters } v} \varphi(e)
$$

[^9]The theory of flows on (di)graphs is a very rich one, but also full of longstanding conjectures (the cycle double cover conjecture, the Berge-Fulkerson conjecture, Tutte's 3 -flow, 4 -flow, and 5 -flow conjectures, etc.); see [7,2] or [9] for a detailed treatment of this area.

In this paper we are going to study a notion introduced by Jaeger [3] and by DeVos, Nešetřil, and Raspaud [1] as an approach to these problems.

We say that a mapping $f: E(G) \rightarrow E(H)$ is $M$-flow-continuous if "the preimage of every $M$-flow is an $M$-flow". More precisely, for every $M$-flow $\varphi$ on $H$, the composition $\varphi f$ (applying first $f$ then $\varphi$ ) is an $M$-flow on $G$. For short, we will call $M$-flow-continuous mappings just $F F_{M}$; in the important case $M=\mathbb{Z}_{n}$ we use the typographically nicer $F F_{n}$ instead of $F F_{\mathbb{Z}_{n}}$. We will write $G \xrightarrow{F F F_{M}} H$ to denote that there exists some $F F_{M}$ mapping from $G$ to $H$.

The main reason for introducing this notion is Jaeger's conjecture [3] that every bridgeless cubic graph $G$ has a $\mathbb{Z}_{2}$-flow-continuous mapping to the Petersen graph. If true, this conjecture would imply the cycle double cover conjecture, and many others. In this paper we will study the notion of $M$-flow-continuous mappings per se, with the aim of making clear what the role of the group $M$ is; this question has not been addressed in previous treatments. For $M=\mathbb{Z}_{2}$ we do not need to consider the orientation of edges; thus this part of the theory is relevant for undirected graphs. As our emphasis is on general abelian groups, we will mostly deal with digraphs.

In some of our proofs we will use an alternative characterization of $F F$-mappings; to state it we need to briefly introduce two notions. Given $\tau: E(G) \rightarrow M$ and $f: E(G) \rightarrow E(H)$, we denote by $\tau_{f}$ the algebraic image of $\tau$, i.e., the mapping $\tau_{f}: E(H) \rightarrow M$ defined by

$$
\tau_{f}(e)=\sum_{e^{\prime} \in E(G) ; f\left(e^{\prime}\right)=e} \tau\left(e^{\prime}\right) .
$$

A mapping $t: E(G) \rightarrow M$ is called an $M$-tension if for every circuit $C$ the sum of $t$ over all clockwise edges is the same as the sum over all counterclockwise edges. It is not hard to see that $M$-tensions in a plane digraph $G$ correspond to $M$-flows in the dual $G^{*}$. More relevant for us is that for every digraph the vector spaces of all $M$-flows and of all $M$-tensions are orthogonal complements. (For this we need $M$ to be a ring. As our consideration will be restricted to finitely generated abelian groups, i.e., to groups in the form (1), this will not limit our use of the following lemma.) This allowed DeVos, Nešetřil, and Raspaud [1, Theorem 3.1] to prove the following useful result.

Lemma 1.1. Let $f: E(G) \rightarrow E(H)$ be a mapping; let $M$ be a ring. Mapping $f$ is $F F_{M}$ if and only if for every $M$-tension $\tau$ on $G$, its algebraic image $\tau_{f}$ is an $M$-tension on $H$.

Moreover, it is sufficient to verify the condition for all tensions that are nonzero only on a neighborhood of a single vertex.

As an easy corollary of this lemma, we observe that $F F_{2}$-mappings between cubic bridgeless graphs map a 3-edge-cut to a 3-edge-cut. In particular, if the target graph is cyclically 4-edge-connected, then the image of an elementary cut (all edges around a vertex) is an elementary cut.

## 2. The influence of the group

In this section we study how the notion of $M$-flow-continuous mapping depends on the group $M$. Although the existence of $M$-flow-continuous mappings seems to be strongly dependent on the choice of $M$ we prove here (in Theorem 2.4) that this dependence relates only to the largest order of an element of $M$.

As we are interested only in finite digraphs, we can restrict our attention to finitely generated groups-clearly $f$ is $M$-flow-continuous if and only if it is $N$-flow-continuous for every finitely generated subgroup $N$ of $M$. Hence, there are integers $\alpha, k, \beta_{i}, n_{i}(i=1, \ldots, k)$ such that

$$
\begin{equation*}
M \simeq \mathbb{Z}^{\alpha} \times \prod_{i=1}^{k} \mathbb{Z}_{n_{i}}^{\beta_{i}} \tag{1}
\end{equation*}
$$

Note that each such group has a canonical ring structure; thus we will be able to use Lemma 1.1.

For a group $M$ in the form (1), let $n(M)=\infty$ if $\alpha>0$; otherwise let $n(M)$ be the least common multiple of $\left\{n_{1}, \ldots, n_{k}\right\}$. When $n(M)$ is finite, it is called the exponent of the group $M$. An alternative definition is that $n(M)$ is the largest order of an element of $M$ (here order of $a \in M$ is the smallest $n>0$ such that $n \cdot a=a+a+\cdots+a=0)$.

As a first step towards a complete characterization we consider a specialized question: given an $F F_{M}$ mapping, when can we conclude that it is $F F_{N}$ as well?

Lemma 2.1. 1. If $f$ is $F F_{\mathbb{Z}}$ then it is $F F_{M}$ for any abelian group $M$.
2. Let $M$ be a subgroup of abelian group $N$. If $f$ is $F F_{N}$ then it is $F F_{M}$.

Proof. 1. This appears as Theorem 4.4 in [1].
2. Let $\varphi$ be an $M$-flow on $H$. As $M \leq N$, we may regard $\varphi$ as an $N$-flow; hence $\varphi f$ is an $N$-flow on $G$. As it attains only values in the range of $\varphi$, and hence in $M$, it is an $M$-flow, too.

Lemma 2.2. Let $M_{1}, M_{2}$ be two abelian groups. Mapping $f$ is $F F_{M_{1}}$ and $F F_{M_{2}}$ if and only if it is $F F_{M_{1} \times M_{2}}$.
Proof. As $M_{1}, M_{2}$ are isomorphic to subgroups of $M_{1} \times M_{2}$, one implication follows from the second part of Lemma 2.1. For the other implication let $\varphi$ be an $\left(M_{1} \times M_{2}\right)$-flow on $H$. Write $\varphi=\left(\varphi_{1}, \varphi_{2}\right)$, where $\varphi_{i}$ is an $M_{i}$-flow on $H$. By assumption, $\varphi_{i} f$ is an $M_{i}$ flow on $G$; consequently $\varphi f=\left(\varphi_{1} f, \varphi_{2} f\right)$ is a flow too.

The following (somewhat surprising) lemma shows that we can restrict our attention to cyclic groups only.

Lemma 2.3. 1. If $n(M)=\infty$ then $f$ is $F F_{M}$ if and only if it is $F F_{\mathbb{Z}}$.
2. Otherwise $f$ is $F F_{M}$ if and only if it is $F F_{n(M)}$.

Proof. In the first part, each implication follows from one part of Lemma 2.1. In the second part: If $f$ is $F F_{M}$, we use the fact that $\mathbb{Z}_{n(M)}$ is isomorphic to a subgroup of $M$; thus the second part of Lemma 2.1 implies $f$ is $F F_{n(M)}$. For the other implication, suppose that $f$ is $F F_{n(M)}$. Note that whenever $\mathbb{Z}_{n_{i}}$ occurs in the expression (1) for $M$, then $\mathbb{Z}_{n_{i}}$ is a subgroup of $\mathbb{Z}_{n(M)}$. Consequently (Lemma 2.1, second part) $f$ is $F F_{n_{i}}$. Repeated application of Lemma 2.2 implies that $f$ is $F F_{M}$ as well.

By a theorem of Tutte [8], for a finite abelian group $M$, the number of nowhere-zero $M$-flows on a given (di)graph only depends on the order of $M$ (see also [2, Chapter 6]). Before proceeding in the main direction of this section, let us note a consequence of Lemma 2.3, which is an analogue of Tutte's result.

Theorem 2.4. Given digraphs $G, H$, the number of $F F_{M}$ mappings from $G$ to $H$ depends only on $n(M)$.
Lemma 2.3 suggests defining for two digraphs the set

$$
F F(G, H)=\left\{n \geq 1 \mid \text { there is } f: E(G) \rightarrow E(H) \text { such that } f \text { is } F F_{n}\right\}
$$

and for a particular $f: E(G) \rightarrow E(H)$

$$
F F(f, G, H)=\left\{n \geq 1 \mid f \text { is } F F_{n}\right\}
$$

We remark that most of these sets contain $1: \mathbb{Z}_{1}$ is a trivial group, hence any mapping is $F F_{1}$. Therefore $1 \in F F(f, G, H)$ for every $f: E(G) \rightarrow E(H)$, while $1 \in F F(G, H)$ if and only if there exists a mapping $E(G) \rightarrow E(H)$. This happens always, unless $E(H)$ is empty and $E(G)$ nonempty.

Although we are working with finite digraphs throughout the paper, in the following results we stress this-contrary to most of the other results, these are not true for infinite digraphs.

Lemma 2.5. Let $G$ be a finite digraph. Either $F F(f, G, H)$ is finite or $F F(f, G, H)=\mathbb{N}$. In the latter case $f$ is $F F_{\mathbb{Z}}$.

Proof. It is enough to prove that $f$ is $F F_{\mathbb{Z}}$ if it is $F F_{n}$ for infinitely many integers $n$. To this end, take a $\mathbb{Z}$-flow $\varphi$ on $H$. As $\varphi_{n}: e \mapsto \varphi(e) \bmod n$ is a $\mathbb{Z}_{n}$-flow, $\varphi_{n} f=\varphi f \bmod n$ is a $\mathbb{Z}_{n}$-flow whenever $f$ is $F F_{n}$. To show that $\varphi f$ is a $\mathbb{Z}$-flow consider a vertex $v$ of $G$ and let $s$ be the " $\pm$-sum" (in $\mathbb{Z}$ ) around $v$ :

$$
s=\sum_{e \text { leaves } v}(\varphi f)(e)-\sum_{e \text { enters } v}(\varphi f)(e) .
$$

As $s \bmod n=0$ for infinitely many values of $n$, we have $s=0$.
Any $f$ induced by a local isomorphism is $F F_{\mathbb{Z}}$, thus providing an example where $F F(f, G, H)$ is the whole of $\mathbb{N}$. For finite sets the situation is more interesting. By the next theorem, the sets $F F(f, G, H)$ are precisely the ideals in the divisibility lattice.

Theorem 2.6. Let $S$ be a finite subset of $\mathbb{N}$. Then the following are equivalent.

1. There are $G, H, f$ such that $S=F F(f, G, H)$.
2. There is $n \in \mathbb{N}$ such that $S$ is the set of all divisors of $n$.

Proof. First we show that 1 implies 2 . The set $S=F F(f, G, H)$ has the following properties:
(i) If $a \in S$ and $b \mid a$ then $b \in S$. (We use the second part of Lemma 2.1: if $b$ divides $a$, then $\mathbb{Z}_{b} \leq \mathbb{Z}_{a}$.)
(ii) If $a, b \in S$ then the least common multiple of $a, b$ is in $S$. (We use Lemmas 2.1 and 2.2: if $l=\operatorname{lcm}(a, b)$ then $\mathbb{Z}_{l} \leq \mathbb{Z}_{a} \times \mathbb{Z}_{b}$.)

Let $n$ be the maximum of $S$. By ( i ), all divisors of $n$ are in $S$. If there is a $k \in S$ that does not divide $n$ then $\operatorname{lcm}(k, n)$ is an element of $S$ larger than $n$, a contradiction.

For the other implication, let $\vec{D}_{n}$ be a graph with two vertices and $n$ parallel edges in the same direction, and let $L$ be a loop (a digraph with a single vertex and one edge). Let $f$ be the only mapping from $\vec{D}_{n}$ to $L$. Then $F F\left(f, \vec{D}_{n}, L\right)=S$ : mapping $f$ is $F F_{k}$ if and only if for any $a \in \mathbb{Z}_{k}$ the constant mapping $E\left(\vec{D}_{n}\right) \mapsto a$ is a $\mathbb{Z}_{k}$-flow; this occurs precisely when $k$ divides $n$.

Let us turn to describing the sets $F F(G, H)$.
Lemma 2.7. Let $G, H$ be finite digraphs. Either $\operatorname{FF}(G, H)$ is finite or $F F(G, H)=\mathbb{N}$. In the latter case $G \xrightarrow{\mathrm{FF}_{\mathrm{Z}}} H$.

Proof. As in the proof of Lemma 2.5, the only difficult step is to show that if $G \xrightarrow{F F_{n}} H$ for infinitely many values of $n$, then $G \xrightarrow{F F_{Z}} H$. As $G$ and $H$ are finite, there are only a finite number of possible mappings between their edge sets. Hence, there is one of them, say $f: E(G) \rightarrow E(H)$, that is $F F_{n}$ for infinitely many values of $n$. By Lemma 2.5 we have $f: G \xrightarrow{\text { FF }} H$.

When characterizing the sets $F F(G, H)$ we first remark that an analogue of Lemma 2.2 does not hold: there is an $F F_{M}$ mapping from $\vec{D}_{9}$ to $\vec{D}_{7}$ for $M=\mathbb{Z}_{2}$ (a mapping that identifies three edges and is $1-1$ on the other ones) and for $M=\mathbb{Z}_{3}$ (e.g., a constant mapping), but not the same mapping for both; hence there is no $F F_{\mathbb{Z}_{2} \times \mathbb{Z}_{3}}$ mapping. We will see that the sets $F F(G, H)$ are precisely the down-sets in the divisibility poset. First, we prove a lemma that will help us to construct pairs of digraphs $G, H$ with a given $F F(G, H)$. The integer cone of a set $\left\{s_{1}, \ldots, s_{t}\right\} \subseteq \mathbb{N}$ is the set $\left\{\sum_{i=1}^{t} a_{i} s_{i} \mid a_{i} \in \mathbb{Z}, a_{i} \geq 0\right\}$.

Lemma 2.8. Let $A$, $B$ be nonempty subsets of $\mathbb{N}, n \in \mathbb{N}$; define $G=\bigcup_{a \in A} \vec{D}_{a}$, and $H=\bigcup_{b \in B} \vec{D}_{b}$. Then there is an $F F_{n}$ mapping from G to H if and only if
$A$ is a subset of the integer cone of $B \cup\{n\}$.
Proof. We use Lemma 1.1. Consider a tension $\tau_{a}$ taking the value 1 on $\vec{D}_{a}$ and 0 elsewhere. The algebraic image of this tension is a tension; hence it is (modulo $n$ ) a sum of several tensions on the digons $\vec{D}_{b}$, implying that $a$ is in integer cone of $B \cup\{n\}$. On the other hand if $a=\sum_{i} b_{i}+c n$ (with
$b_{i} \in B$ ) then we can map any $c n$ edges of $\vec{D}_{a}$ to one (arbitrary) edge of $H$, and for each $i$ any ("unused") $b_{i}$ edges bijectively to $\vec{D}_{b_{i}}$. After we have done this for each $a \in A$ we will have constructed a $\mathbb{Z}_{n}$-flowcontinuous mapping from $G$ to $H$.

Theorem 2.9. Let $S$ be a finite subset of $\mathbb{N}$. Then the following are equivalent.

1. There are $G, H$ such that $S=F F(G, H)$.
2. There is a finite set $T \subset \mathbb{N}$ such that

$$
S=\{s \in \mathbb{N} ;(\exists t \in T) s \mid t\} .
$$

Proof. If $S$ is empty, we take $T$ empty in part 2. In part 1, we just consider digraphs such that $E(H)$ is empty and $E(G)$ is not. Next, we suppose that $S$ is nonempty.

By the same reasoning as in the proof of Theorem 2.6 we see that if $a \in F F(G, H)$ and $b \mid a$ then $b \in F F(G, H)$. Hence, 1 implies 2, as we can take $T=S$ (or, to make $T$ smaller, let $T$ consist of the maximal elements of $S$ in the divisibility relation).

For the other implication let $p>4 \max T$ be a prime, and let $p^{\prime} \in(1.25 p, 1.5 p)$ be an integer. Let $A=\left\{p, p^{\prime}\right\}$ and

$$
B=\{p-t ; t \in T\} \cup\left\{p^{\prime}-t ; t \in T\right\} ;
$$

note that every element of $B$ is larger than $\frac{3}{4} p$. As in Lemma 2.8 we define $G=\bigcup_{a \in A} \vec{D}_{a}, H=$ $\bigcup_{b \in B} \vec{D}_{b}$. We claim that $F F(G, H)=S$. By Lemma 2.8 it is immediate that $F F(G, H) \supseteq S$. For the other direction take $n \in F F(G, H)$. By Lemma 2.8 again, we can express $p$ and $p^{\prime}$ in the form

$$
\begin{equation*}
\sum_{i=1}^{k} b_{i}+c n \tag{2}
\end{equation*}
$$

for integers $c, k \geq 0$, and $b_{i} \in B$.

- If $k \geq 2$ then the sum in (2) is at least $1.5 p$; hence neither $p$ nor $p^{\prime}$ can be expressed with $k \geq 2$.
- If $k=1$ then we distinguish two cases.
- $p=(p-t)+c n$; hence $n$ divides $t$ and thus $n \in S$.
- $p=\left(p^{\prime}-t\right)+c n$; hence $p^{\prime}-p \leq t$. But $p^{\prime}-p>0.25 p>t$, a contradiction.

Considering $p^{\prime}$ we find that either $n \in S$ or $p^{\prime}=(p-t)+c n$.

- Finally, consider $k=0$. If $p=c n$ then either $n=1$ (so $n \in S$ ) or $n=p$. (We do not claim anything about $p^{\prime}$.)
To summarize, if $n \in F F(G, H) \backslash S$ then necessarily $n=p$. For $p^{\prime}$ we have only two possible expressions: $p^{\prime}=c n($ for $k=0)$ and $p^{\prime}=(p-t)+c n$ (for $k=1$ ). We easily check that both of them lead to a contradiction. The first one contradicts $1.25 p<p^{\prime}<1.5 p$. In the second expression $c=0$ implies $p^{\prime}<p$ while $c \geq 1$ implies $p^{\prime} \geq 2 p-t \geq 1.75 p$, again a contradiction.

Remark 2.10. This paper concentrates on $F F$ mappings. We remark, however, that analogous proofs describe the role of the group for mappings where preimages of tensions are tensions, or preimages of tensions are flows (or preimages of flows are tensions). For a discussion of the relevance of these types of mappings the reader may consult the series [5,4] by the authors and the second author's Ph.D. Thesis [6].

## 3. Cubic graphs

In the previous section we studied how the group $M$ influences the notion of $F F_{M}$-mappings; it turned out that there is an algebraic structure behind this. In this section we look at the combinatorially more relevant case of cubic graphs. (The degree of each vertex is 3 ; the orientation is arbitrary.) Indeed, many longstanding conjectures in the area have been reduced to the case of cubic graphs. There it turns out that we only need to consider three groups: $\mathbb{Z}_{2}, \mathbb{Z}_{3}$, and $\mathbb{Z}$.

Theorem 3.1. Let $n>3$ be an integer; suppose that $G$, $H$ are digraphs with maximum degree less than $n$. Then $G \xrightarrow{F F_{n}} H$ is equivalent to $G \xrightarrow{F F_{Z}} H$.
Proof. One direction follows from Lemma 2.1. For the other one, consider a mapping $f: E(G) \rightarrow E(H)$. We will show that if it is $F F_{n}$, it is $F F_{\mathbb{Z}}$ as well. Taking a $\mathbb{Z}$-flow $\varphi$ on $H$, we will show that $\varphi f$ is a $\mathbb{Z}$-flow on $G$. We only need to test this on elementary flows (those taking only values $\pm 1$ around a circuit), as these form a basis for $\mathbb{Z}$-flows. So suppose that $\varphi$ is one of these; notice that it is both a $\mathbb{Z}$-flow and a $\mathbb{Z}_{n}$-flow. Thus, $\varphi f$ is a $\mathbb{Z}_{n}$-flow on $G$. Consider a vertex $v \in V(G)$ of degree $d<n$ and let $e_{1}, e_{2}, \ldots, e_{d}$ be the edges incident with it; further, let $a_{i}=\varphi\left(f\left(e_{i}\right)\right)$. As $\varphi f$ is a $\mathbb{Z}_{n}$-flow, we have that $s= \pm a_{1} \pm a_{2} \pm \cdots \pm a_{d} \equiv 0(\bmod n)$ (the signs are chosen based on the orientation of the edges). Now $|s| \leq d<n$; thus $s=0$. It follows that $\varphi f$ also satisfies Kirchhoff's law at $v$ in $\mathbb{Z}$; thus $\varphi f$ is also a flow over $\mathbb{Z}$.

Corollary 3.2. Let $G, H$ be digraphs of maximum degree 3; let $n>3$ be an integer. Then $G \xrightarrow{F F_{n}} H$ is equivalent to $G \xrightarrow{\mathrm{FF} \mathrm{Z}_{\mathrm{Z}}} H$.

Together with Lemma 2.3, the above corollary implies that for subcubic digraphs we only need to consider $\mathbb{Z}_{2}{ }^{-}, \mathbb{Z}_{3^{-}}$, and $\mathbb{Z}$-flow-continuous mappings.

By Lemma 2.1 a $\mathbb{Z}$-flow-continuous mapping is also $\mathbb{Z}_{2}$ - and $\mathbb{Z}_{3}$-flow-continuous. In the following examples we show that the existence of $F F_{2}$ mappings and the existence of $F F_{3}$ mappings are independent, even for subcubic digraphs. Let $f$ be any bijection from $E\left(\vec{D}_{3}\right)$ to $E\left(\vec{C}_{3}\right)$. Mapping $f$ is $F F_{n}$ only if $n$ is a multiple of 3; thus it is $F F_{3}$ but not $F F_{2}$ or $F F_{\mathbb{Z}}$. On the other hand, consider an edge 3-coloring for $\vec{K}_{4}$ (a $K_{4}$ with an arbitrary orientation of edges) as a mapping $g: \vec{K}_{4} \rightarrow \vec{D}_{3}$. This mapping is $F F_{2}$ (as a 4-cycle in $K_{4}$ is also a cut). However, $g$ is not $F F_{3}$ : consider a $\mathbb{Z}_{3}$-flow $\varphi$ in $\vec{D}_{3}$ that equals 1 on all three edges of $\vec{D}_{3}$. Clearly the composition $\varphi f$ is not a $\mathbb{Z}_{3}$-flow on $\vec{K}_{4}$.

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## Appendix H

Cycle-continuous mappings - order structure

# Cycle-continuous mappings - order structure 

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#### Abstract

Given two graphs, a mapping between their edge-sets is cycle-continuous, if the preimage of every cycle is a cycle. The motivation for this definition is Jaeger's conjecture that for every bridgeless graph there is a cycle-continuous mapping to the Petersen graph, which, if solved positively, would imply several other important conjectures (e.g., the Cycle double cover conjecture). Answering a question of DeVos, Nešetriil, and Raspaud, we prove that there exists an infinite set of graphs with no cycle-continuous mapping between them. Further extending this result, we show that every countable poset can be represented by graphs and the existence of cycle-continuous mappings between them.


## 1 Introduction

Many questions at the core of graph theory can be formulated as questions about cycles or more generally about flows on graphs. Examples are the Cycle double cover conjecture, the Berge-Fulkerson conjecture, and Tutte's 3-Flow, 4-Flow, and 5-Flow conjectures. For a detailed treatment of this area the reader may refer to [16], [11], [18] or [19].

As an approach to these problems Jaeger [10] and DeVos, Nešetřil, and Raspaud [6] defined a notion of graph morphism continuous with respect to group-valued flows. In this paper we restrict ourselves to the case of $\mathbb{Z}_{2}$-flows, that is to cycles. Thus, the following is the principal notion we study in this paper:

Given graphs (parallel edges or loops allowed) $G$ and $H$, a mapping $f: E(G) \rightarrow$ $E(H)$ is called cycle-continuous, if for every cycle $C \subseteq E(H)$, the preimage $f^{-1}(C)$ is a cycle in $G$. We emphasize, that by a cycle we understand (as is common in this area) a set of edges such that every vertex is adjacent with an even number of them. (So a cycle is an edge-disjoint union of circuits - connected 2-regular graphs.) For shortness we sometimes call cycle-continuous mappings just $c c$ mappings.

The fact that $f$ is a $c c$ mapping from $G$ to $H$ is denoted by $f: G \xrightarrow{c c} H$. If we just need to say that there exists a $c c$ mapping from $G$ to $H$, we write $G \xrightarrow{c c} H$; inspired by the notation common for graph homomorphisms. Note, that the identity is $c c$ and that $c c$ mappings compose, so they are truly morphisms in the sense of category theory.

With the definition covered, we mention the main conjecture describing the properties of $c c$ mappings.

[^11]Conjecture 1.1 (Jaeger [10]) For every bridgeless graph $G$ we have $G \xrightarrow{c c} \mathrm{Pt}$, where Pt denotes the Petersen graph.

If true, this would imply many conjectures in the area. To illustrate this, suppose we want to find a 5 -tuple of cycles in a graph $G$ covering each of its edges twice (this is conjectured to exist by the 5-Cycle double cover conjecture [15, 17, 3]). Further, suppose $f: G \xrightarrow{c c} \mathrm{Pt}$. We can use $C_{1}, \ldots, C_{5}-\mathrm{a} 5$-tuple of cycles in the Petersen graph double-covering its edges - and then it is easy to check that $f^{-1}\left(C_{1}\right), \ldots, f^{-1}\left(C_{5}\right)$ has the same property in $G$.

DeVos et al. [6] study $c c$ mappings further and ask the following question about their structure. We say that graphs $G, G^{\prime}$ are $c c$-incomparable if there is no $c c$ mapping between them, that is $G \stackrel{c q}{\longrightarrow} G^{\prime}$ and $G^{\prime} \xrightarrow{c q} G$.

Question 1.2 ([6], Problem 5.8) Is there an infinite set $\mathcal{G}$ of bridgeless graphs such that every two of them are cc-incomparable?

A negative answer to this would suggest a way to attack Conjecture 1.1. DeVos et al. [6] prove in their Theorem 2.9 that if there is no infinite set as in the above question, and no infinite chain $G_{1} \xrightarrow{c c} G_{2} \xrightarrow{c c} G_{3} \xrightarrow{c c} \cdots$ (such that $G_{n+1} \xrightarrow{c q} G_{n}$ for all $n$ ), then there is a single graph $H$ such that for every other bridgeless graph $G$ we have $G \xrightarrow{c c} H$.

DeVos et al. [6] also show that arbitrarily large sets of $c c$-incomparable graphs exist. Their proof is based on the notion of critical snarks and on Lemma 3.1; these will be crucial also for our proof.

We will show that the answer to Question 1.2 is positive. Thus, the following is the first main result of this paper. (A graph is said to be cubic if it is 3-regular.)

Theorem 1.3 There is an infinite set $\mathcal{G}$ of cubic bridgeless graphs such that every two of them are cc-incomparable.

While this definitely shouldn't be interpreted as an indication that Conjecture 1.1 is false, it eliminates some easy paths towards the possible proof of it. As a further indication of the complexity of the structure of $c c$ mappings, we study the order that $c c$ mappings induce on graphs.

When given a set of objects and morphisms between them, it is standard to consider a poset in which $x \leq y$ iff there is a mapping from $x$ to $y$. (Note that [6] puts $x \geq y$ in this situation, i.e., their poset is the opposite of ours.) In this sense, we can speak about the poset of $c c$ mappings and ask what subposets it contains. The above theorem can be restated: this poset contains infinite antichains (posets with no relation). We prove that this poset contains every countable poset, a surprising outcome of Question 1.2.

Theorem 1.4 Every countable (finite or infinite) poset can be represented by a set of cubic graphs and the existence of cycle-continuous mappings between them.

By a cut we mean a set of edges of the form $\delta(U)$ - all edges leaving some set $U$ of vertices. Such a set may be empty (if $U$ is empty, or a union of connected components), but if it is not, removing it increases the number of components. Note, however, that not all edge-sets whose removal increases the number of components are cuts in our sense! The set $\delta(\{v\})$ will be called the elementary edge-cut determined by the vertex $v$. For a plane graph $G$, a cycle in $G$ corresponds to a cut in $G^{*}$, while a cut in $G$ corresponds to a cycle in $G^{*}$ (Theorem 10.16 of [1].) We recall the cut-cycle duality: When we look
at subsets of $E(G)$ as a vector space over $G F(2)$ (with symmetric difference as a sum), then the set of cycles and the set of cuts are orthogonal complements (Theorem 1.9.4 of [7]).

To further illustrate the topic, we briefly mention the related concept of cut-continuous mappings. A mapping $f: E(G) \rightarrow E(H)$ is cut-continuous if the preimage of every cut is a cut. Cut-continuous mappings behave in many contexts similarly to homomorphisms (see [13, 12]), in particular Question 1.2 would be trivial for cutcontinuous mappings. The cycle-continuous mappings, on the other hand, have been hard to tame so far, perhaps because of their connection with so many longstanding conjectures. The main contribution of this paper is an approach to study the behaviour of $c c$ mappings - although only for graphs with a special structure.

## 2 Properties of cycle-continuous mappings

### 2.1 Basics

Before we describe our construction, we introduce basic properties of cycle-continuous mappings. Many of them are folklore, we still provide some proofs as a warm-up for the proofs in the rest of the paper.

By a graph we mean a multigraph with loops and parallel edges allowed. We start with a dual definition of cycle-continuity in terms of cuts. To prepare for it, we define for a set $X \subseteq E(G)$ and a mapping $f: E(G) \rightarrow E(H)$ a "parity image" of $X$ under $f$, where two edges with the same image cancel out: we put

$$
f^{\text {odd }}(X):=\left\{e \in E(H):\left|f^{-1}(e) \cap X\right| \text { is odd }\right\}
$$

The following alternative characterization of $c c$ mappings basically appears as Theorem 3.1 of [6], the proof is a simple use of the cut-cycle duality.

Lemma 2.1 ([6]) Let $f: E(G) \rightarrow E(H)$ be a mapping. The following are equivalent:
(1) $f$ is cycle-continuous;
(2) the set $f^{\text {odd }}(X)$ is a cut in $H$ for every cut $X$ in $G$;
(3) the set $f^{\circ \text { odd }}(X)$ is a cut in $H$ for every elementary cut $X$ in $G$, that is for every $X$ of the form $\delta(\{v\})$.

Proof: The equivalence of (1) and (2) is proved in [6] in greater generality: instead of speaking of cycles and cuts, they discuss $K$-flows and $K$-tensions; for $K=\mathbb{Z}_{2}$ the support of a $K$-flow is a cycle, the support of a $K$-tension a cut. Thus, their $\mathbb{Z}_{2}{ }^{-}$ flow-continuous mappings are exactly our cc mappings. Obviously (2) implies (3), it remains to show the converse. To this end consider the vector space (over $G F(2)$ ) of subsets of $E(G)$ with symmetric difference as a sum. The set of all cuts is a vector space generated by the elementary cuts (Proposition 1.9.2 of [7]). It is easy to verify that $f^{\text {odd }}(X \Delta Y)=f^{\text {odd }}(X) \Delta f^{\text {odd }}(Y)$, which finishes the proof.

Corollary 2.2 Suppose $f: G \xrightarrow{c c} H$ and $H$ is bridgeless. Then for every 3-edge-cut $\left\{e_{1}, e_{2}, e_{3}\right\}$ the set $\left\{f\left(e_{1}\right), f\left(e_{2}\right), f\left(e_{3}\right)\right\}$ is a 3-edge-cut.

Proof: Note that $\left|f^{\circ o d d}(X)\right|$ has the same parity as $|X|$ (for any $f$ and $X$ ). It follows that for $X=\left\{e_{1}, e_{2}, e_{3}\right\}$, the set $f^{\text {odd }}(X)$ is either a single edge, or it is the set
$\left\{f\left(e_{1}\right), f\left(e_{2}\right), f\left(e_{3}\right)\right\}$ of size 3. Moreover, $f^{\text {odd }}(X)$ is a cut (Lemma 2.1), and $H$ is bridgeless thus the former case is impossible.

We let $K_{2}^{3}$ denote the graph with two vertices connected by three parallel edges. It will play a major role in our paper, because of the following observation.

Lemma 2.3 The following are equivalent properties of a cubic graph $G$ :

- $G \xrightarrow{c c} K_{2}^{3}$
- G admits a proper 3-edge-coloring

Proof: While it is easy to give a direct elementary proof, we will use Lemma 2.1 to illustrate this useful technique. Let $f: E(G) \rightarrow E\left(K_{2}^{3}\right)$ be any mapping. For any $v \in V(G)$ the set $f^{\text {odd }}(\delta(\{v\}))$ contains an odd number of edges (as $\operatorname{deg} v=3$ ). There is only one cut in $K_{2}^{3}$ with an odd number of edges, namely the whole edge-set. It follows that $f^{\text {odd }}(\delta(\{v\}))$ is a cut iff $f$ is a bijection between $\delta(\{v\})$ and $E\left(K_{2}^{3}\right)$.

To sum it up, $f$ is a $c c$ mapping if and only if it is a proper 3-edge-coloring (using edges of $K_{2}^{3}$ as colors), which finishes the proof.

We remark that for any graph $G$ (not necessarily cubic) the statement $G \xrightarrow{c c} K_{2}^{3}$ is equivalent with $G$ having a nowhere-zero 4 -flow.

Snarks. A graph is called a snark if it is

- connected,
- cubic,
- bridgeless, and
- not 3-edge-colorable.

In view of Lemma 2.3, we may replace the last condition by $G \stackrel{c q}{\nrightarrow} K_{2}^{3}$. The notion of snark is crucial to this area of graph theory (say, for discussing the Cycle double cover conjecture, the Berge-Fulkerson conjecture, or Tutte's Flow conjectures). The main theme is that if a graph is not a snark, then solving these conjectures is easy. As a consequence, some authors restrict the notion of snark to add further conditions of being "nontrivial". For instance, in [6] a snark is required to be cyclically 4-edgeconnected: for any set $X$ of $<4$ edges, at most one component of $G-X$ contains a cycle. (For cubic graphs it is equivalent to say, that no set of $<4$ edges disconnects the graph, except for sets of the form $\delta(\{v\})$.) Following [18], we do not include such extra conditions; instead we add the condition to those theorems about snarks, where it is needed.

The following result, while easy to prove, provides a way to construct $c c$ mappings in a homomorphism-like fashion.

Corollary 2.4 Let $f: E(G) \rightarrow E(H)$ be a mapping such that for each $v \in V(G)$ there is $w \in V(H)$ such that a restriction of $f$ to $\delta(\{v\})$ is a bijection to $\delta(\{w\})$. Then $f$ is cycle-continuous.

Proof: This is a direct consequence of Lemma 2.1: if $f$ has the described properties, then for every $v \in V(G)$, the set $f^{\text {odd }}(\delta(\{v\}))$ is $\delta(\{w\})$ for some $w \in V(H)$.

A simple corollary of this is that every isomorphism induces a $c c$ mapping: if $h: V(G) \rightarrow V(H)$ is an isomorphism then the mapping $h^{\sharp}: E(G) \rightarrow E(H)$ defined by $h^{\sharp}(u v)=h(u) h(v)$ is cycle-continuous.

We note that if $G, H$ are cubic graphs and $H$ is cyclically 4-edge-connected, then every $c c$ mapping from $G$ to $H$ is of the type described in Corollary 2.4. This explains a frequently mentioned version of Conjecture 1.1: every cubic bridgeless graph $G$ has a mapping $f: E(G) \rightarrow E(P t)$ such that adjacent edges are mapped to adjacent edges.

TODO: put to the end Corollary 2.4 shows an interesting relation with homomorphisms of line graphs. If $f: L(G) \rightarrow L(H)$ is a homomorphism and $G, H$ are cubic triangle-free graphs, then $f$ is also a $c c$ mapping from $G$ to $H$. As for some instances of $G, H$ this describes all $c c$ mappings between $G$ and $H$, we can use our results to prove facts about homomorphisms of line graphs: In [8] universality of homomorphism order of line-graphs is proved. A special case of their result (for line-graphs of cubic graphs) follows from our Theorem 4.2.
$\left.G\right|_{x=y}$ is the graph obtained by identifying vertices $x$ and $y$ of $G$, keeping possible loops and multiple edges. The following easy observation will be used a lot.

Lemma 2.5 $\left.G\right|_{x=y} \xrightarrow{c c} G$ for every graph $G$ and its vertices $x, y$.
Proof: We map every edge of $\left.G\right|_{x=y}$ to the corresponding edge of $G$. To verify that this is a $c c$ mapping, we consider a cycle $C$ in $G$. We need to observe that after identifying $x$ with $y$ we get a cycle $C^{\prime}$ in $\left.G\right|_{x=y}$.

Lemma 2.6 Let $G^{\prime}$ be obtained from $G$ by removing a loop. Then $G^{\prime} \xrightarrow{c c} G$.

Proof: The identity mapping is $c c$, as preimage of any cycle $C$ is either $C$ itself or $C$ with a loop removed, thus still a cycle.

We shall call a natural inclusion the above $c c$ mapping from $\left.G\right|_{x=y}$ (or any graph obtained from it by removing loops) to $G$. Obviously, a composition of $c c$ mappings is a $c c$ mapping. Thus, in particular, for any $F \subseteq E(G)$ we have a natural inclusion $G / F \xrightarrow{c c} G$.

Lemma 2.7 Suppose $f: G \xrightarrow{c c} H$ is a cc mapping, let $H^{\prime}$ be the subgraph of $H$ consisting of the edges in the range of $f$. Then $f: G \rightarrow H^{\prime}$ is also cycle-continuous.

Proof: Every cycle in $H^{\prime}$ is also a cycle in $H$, thus its preimage is a cycle in $G$.

### 2.2 Properties of a 2-join

In this and the next section we will describe two common constructions of snarks. While the constructions are known (see, e.g., [18]), the relation to cycle-continuous mappings has not been investigated elsewhere, and is crucial to our result. Before we start, we must mention that the constructed graphs have 2- or 3-edge cuts that are not elementary, so they are not snarks according to some authors' definition. It will be convenient for our purpose though, to consider them snarks, following the definition in [18].

The first construction can be informally described as adding a "gadget" on an edge of a graph. Formally, let $G_{1}, G_{2}$ be graphs, and let $e_{i}=x_{i} y_{i}$ be an edge of $G_{i}$. We delete edge $e_{i}$ from $G_{i}$ (for $i=1,2$ ), and connect the two graphs by adding two new edges $x_{1} x_{2}$ and $y_{1} y_{2}$. The resulting cubic graph will be called the 2 -join of the graphs $G_{1}, G_{2}$ (some authors call this a 2 -cut construction); it will be denoted by $G_{1} \asymp G_{2}$.

We note that the resulting graph depends on our choice of the edges $x_{i} y_{i}$, but for our purposes this coarse description will suffice. By connecting edges of $G_{1} \asymp G_{2}$ we mean the edges $x_{1} x_{2}$ and $y_{1} y_{2}$. Its vertices and edges arising from edges of $G_{1}-e_{1}$ will be called the left vertices/edges, those coming from $G_{2}-e_{2}$ the right vertices/edges.


Figure 1: Illustration of the 2-join contstruction.

Lemma 2.8 For every $G_{1}, G_{2}$ we have $G_{i} \xrightarrow{c c} G_{1} \asymp G_{2}$ for $i \in\{1,2\}$.

Proof: Assume $i=1$. We can get $G_{1}$ from $G_{1} \asymp G_{2}$ by identifying all right vertices, removing all resulting loops, and finally contracting $y_{1} y_{2}$. Thus using Lemma 2.5 and 2.6 finishes the proof.

As in Section 2.1, we call the mapping $G_{i} \rightarrow G_{1} \asymp G_{2}$ a natural inclusion.
Lemma 2.9 Let $G_{1}, G_{2}$ be any graphs. Let $K$ be an edge-transitive graph. Then $G_{1} \asymp G_{2} \xrightarrow{c c} K$ if and only if $G_{1} \xrightarrow{c c} K$ and $G_{2} \xrightarrow{c c} K$.

Proof: For the forward implication it is enough to use Lemma 2.8 and the fact that $c c$ mappings compose. For the other direction, consider cycle-continuous mappings $f_{i}: E\left(G_{i}\right) \rightarrow E(K)$, let $e_{i}=x_{i} y_{i}$ be the edges on which the 2-join operation is performed. As $K$ is edge-transitive, we may assume that $f_{1}\left(e_{1}\right)=f_{2}\left(e_{2}\right)$. Thus, we may define $f: E\left(G_{1} \asymp G_{2}\right) \rightarrow E(K)$ in a natural way: $f\left(x_{1} x_{2}\right)=f\left(y_{1} y_{2}\right)=$ $f_{1}\left(e_{1}\right)$ (which equals $f_{2}\left(e_{2}\right)$ ), and $f(e)=f_{i}(e)$ whenever $e \neq e_{i}$ is an edge of $G_{i}$. Corollary 2.4 implies easily that $f$ is cycle-continuous.

As an immediate corollary we get the following classical result about snarks and 2-joins.

Corollary 2.10 ([18]) Let $G_{1}, G_{2}$ be connected bridgeless cubic graphs. Then $G_{1} \asymp G_{2}$ is a snark if and only if at least one of $G_{1}, G_{2}$ is a snark.

Proof: As $G_{1}, G_{2}$, and thus also $G_{1} \asymp G_{2}$ are connected, cubic, and bridgeless, we only need to verify the nonexistence of 3-edge-coloring, or, equivalently, of $c c$ mapping to $K_{2}^{3}$. This is easy by Lemma 2.9 , as $K_{2}^{3}$ is edge-transitive.

Another easy corollary of Lemma 2.9 is that a minimal counterexample (if it exists) to Conjecture 1.1 does not contain a nontrivial 2-edge-cut.

Corollary 2.11 Let $G_{1}, G_{2}$ be cubic bridgeless graphs. If $G_{1} \asymp G_{2} \stackrel{\text { cq }}{\rightarrow} \mathrm{Pt}$ then $G_{i} \xrightarrow{c q}$ Pt for some $i \in\{1,2\}$.

### 2.3 Properties of a 3-join

A 3-join (also called a 3-cut construction) is another method to create new snarks ones that contain nontrivial 3-edge cuts. One way to view this is that we replace a vertex in a graph by a "gadget" created from another graph.

To be more precise, we consider graphs $G_{1}$ and $G_{2}$, delete a vertex $v_{i}$ of each $G_{i}$, and add a matching between the neighbors of former vertices $v_{1}$ and $v_{2}$.

The resulting cubic graph in general depends on our choice of $v_{i}$ 's, and of the matching, but in our applications it either will not matter, or will be discussed in advance, so we do not introduce any special notation for this. We use $G_{1} \equiv G_{2}$ to denote (any of) the resulting graph(s); we call it the 3-join of $G_{1}$ and $G_{2}$. Connecting edges of the 3-join are the three edges we added to connect $G_{1}$ and $G_{2}$. The vertices/edges of $G_{1} \equiv G_{2}$ arising from edges of $G_{1}-v_{1}$ will be called the left vertices/edges, those coming from $G_{2}-v_{2}$ the right vertices/edges.


Figure 2: Illustration of the 3-join contstruction.
We collect several easy properties of the 3-join operation.
Lemma 2.12 For any graphs $G_{1}, G_{2}$ we have $G_{i} \xrightarrow{c c} G_{1} \equiv G_{2}$ for $i=1,2$.

Proof: Assume $i=1$. We can get $G_{1}$ by identifying all right vertices of $G_{1} \equiv G_{2}$ and removing the resulting loops. Thus using Lemma 2.5 and 2.6 suffices again.

Again, we shall call the cycle-continuous mapping from $G_{i}$ to $G_{1} \equiv G_{2}$ that is used in the above lemma a natural inclusion.

We remind the reader of the definition of 2-transitivity. A triple $\left(s_{0}, s_{1}, s_{2}\right)$ of vertices of $G$ is called a $2-\operatorname{arc}$, if $s_{0} s_{1}$ and $s_{1} s_{2}$ are edges and $s_{0} \neq s_{2}$. A graph $G$ is called 2 -transitive if it contains some 2 -arc and for every two 2 -arcs $\left(s_{0}, s_{1}, s_{2}\right)$, $\left(t_{0}, t_{1}, t_{2}\right)$ there is an automorphism $\varphi$ of $G$ such that $\varphi\left(s_{i}\right)=t_{i}$. Note that for cubic graph $G$ 2-transitivity is equivalent with the following symmetry condition:

Whenever $u_{i}(i=1,2)$ is a vertex and $x_{i, 1}, x_{i, 2}, x_{i, 3}$ is an ordering of $N\left(u_{i}\right)$, there is an automorphism $\varphi$ of $G$ such that $\varphi\left(u_{1}\right)=u_{2}$ and $\varphi\left(x_{1, j}\right)=x_{2, j}$ for $j=1,2,3$.

Lemma 2.13 Let $G_{1}, G_{2}$ be any graphs. Let $K$ be a cyclically 4-edge-connected cubic graph that is 2-transitive.

Then $G_{1} \equiv G_{2} \xrightarrow{c c} K$ if and only if $G_{1} \xrightarrow{c c} K$ and $G_{2} \xrightarrow{c c} K$.

Proof: The 'only if' part follows from Lemma 2.12. For the other direction, consider any $f_{i}: G_{i} \xrightarrow{c c} K(i=1,2)$. Also let $v_{i}$ be the vertex of $G_{i}$ deleted in the 3-join operation, and let $a_{i}, b_{i}, c_{i}$ be the edges incident to $v_{i}$, labeled in an order compatible with the matching chosen in the 3 -join operation.

As $\delta\left(\left\{v_{i}\right\}\right)=\left\{a_{i}, b_{i}, c_{i}\right\}$, Corollary 2.2 implies that $S_{i}=\left\{f_{i}\left(a_{i}\right), f_{i}\left(b_{i}\right), f_{i}\left(c_{i}\right)\right\}$ is a 3-edge-cut in $K$.

Graph $K$ is cyclically 4-edge-connected and cubic, so its only 3-edge-cuts are the elementary cuts. Thus $S_{i}$ is a cut of the form $\delta\left(\left\{u_{i}\right\}\right)$ for some $u_{i} \in V(K)$. As $K$ is 2transitive and as each isomorphism induces a cc mapping, we can assume that $S_{1}=S_{2}$, and even $f_{1}\left(a_{1}\right)=f_{2}\left(a_{2}\right), f_{1}\left(b_{1}\right)=f_{2}\left(b_{2}\right)$, and $f_{1}\left(c_{1}\right)=f_{2}\left(c_{2}\right)$. Consequently, we may define a mapping $f: G_{1} \equiv G_{2} \xrightarrow{c c} K$ in a straightforward way: if $e$ is an edge of $G_{i}$, we let $f(e)=f_{i}(e)$. Because of the above assumption, the connecting edges are mapped consistently. To verify that $f$ is cycle-continuous, we use Corollary 2.4.

As an immediate corollary, we get the following classical result about snarks and 3joins. (Recall that, unlike some other authors, we do not require snarks to be cyclically 4-edge-connected, otherwise $G_{1} \equiv G_{2}$ would not be a snark.)

Corollary 2.14 ([18]) Let $G_{1}, G_{2}$ be cubic bridgeless graphs. Then $G_{1} \equiv G_{2}$ is a snark, iff at least one of $G_{1}, G_{2}$ is a snark.

Proof: Apply Lemma 2.13 for $K=K_{2}^{3}$.
As another easy application, we observe that minimal counterexample (if it exists) to Conjecture 1.1 does not contain a nontrivial 3-edge-cut.
Corollary 2.15 Let $G_{1}, G_{2}$ be cubic bridgeless graphs. If $G_{1} \equiv G_{2} \xrightarrow{c q} \mathrm{Pt}$ then $G_{i} \xrightarrow{c q}$ Pt for some $i \in\{1,2\}$.

The above notwithstanding, we proceed to study the structure of cycle-continuous mappings in graphs with 3-edge-cuts, for two reasons: first we believe, it provides insights that might be useful in further progress towards solving Conjecture 1.1; second, we find it has an independent interest.

We close this section with two lemmas that will be the key to the construction in the next section.

Lemma 2.16 Let $G_{1}, G_{2}$ be cc-incomparable snarks. Then
$G_{1} \equiv G_{2} \xrightarrow{c ¢} G_{i}$ for each $i \in\{1,2\}$.
Proof: Immediate from Lemma 2.12.
In contrast with Lemma 2.16, $G \equiv G \xrightarrow{c c} G$ holds for every graph $G$, taking the identity mapping on both copies of $G$. For this mapping to be $c c$ it suffices if we choose the connecting vertex and the order of their neighbors in the same way in both copies of $G$. We will use this in the proof of Theorem 4.2.

The following lemma provides a partial converse to Lemma 2.12.
Lemma 2.17 Let $G_{1}, G_{2}$ be connected cubic graphs, consider a cc mapping $f: F \xrightarrow{c c}$ $G_{1} \equiv G_{2}$. Suppose for every edge $e \in E(F)$ the image $f(e)$ is a left or a connecting edge of $G_{1} \equiv G_{2}$. We let $f^{\prime}$ be the same mapping as $f$ but considered as a mapping $E(F) \rightarrow E\left(G_{1}\right)$. Then $f^{\prime}$ is also cycle-continuous.

Proof: Let $C$ be a cycle in $G_{1}$, we need to check that $f^{\prime-1}(C)$ is a cycle in $F$. If $C$ does not contain $v_{1}$ (the vertex used in the construction of $G_{1} \equiv G_{2}$ ), then $C$ is also a cycle in $G_{1} \equiv G_{2}$. In this case $f^{\prime-1}(C)=f^{-1}(C)$, which is a cycle in $F$.

Suppose next, that $C$ contains $v_{1}$ and two of its neighbors. As $G_{2}$ is connected, we can find a cycle $C^{\prime}$ in $G_{1} \equiv G_{2}$ such that $C^{\prime} \supseteq C$ and $C^{\prime} \backslash C$ only contains right edges of $G_{1} \equiv G_{2}$. Consequently, $f^{\prime-1}(C)=f^{-1}\left(C^{\prime}\right)$, which is a cycle by our assumptions.

## 3 The proof

### 3.1 Critical snarks

For our construction we will need the following notion of criticality of snarks. It appears in DeVos et al. [6], and also in [4], where these graphs are called flow-critical snarks.

Recall that we call a graph $G$ a snark if $G$ is cubic, connected, bridgeless and $G \stackrel{c c}{\longrightarrow} K_{2}^{3}$ (where $K_{2}^{3}$ is a graph formed by two vertices and three parallel edges). We say that $G$ is a critical snark if it is a snark and for every edge $e$ of $G$ we have $G-e \xrightarrow{c c} K_{2}^{3}$. (Equivalently [4], $G / e \xrightarrow{c c} K_{2}^{3}$.)

The following lemma (appearing as Proposition 5.9 in [6]) is the basis of our control over cycle-continuous mappings between graphs in our construction. (We recall that a snark, or 3 -snark in [6] is, according to our definitions, a cyclically 4-edge-connected snark.) As the 'Moreover' part is not formulated in [6], we present the short proof for the reader's convenience.
Lemma 3.1 ([6]) Let $G$, $H$ be cyclically 4-edge-connected cubic graphs, both of which are critical snarks, suppose that $|E(G)|=|E(H)|$. Then $G \xrightarrow{c c} H$ iff $G \cong H$.

Moreover, every cycle-continuous mapping $G \xrightarrow{c c} H$ is a bijection of the edge-sets that is induced by an isomorphism of $G$ and $H$.

Proof: As $G, H$ are snarks, neither of them contains $K_{2}^{3}$. If $G$ and $H$ are isomorphic, Corollary 2.4 implies that $G \xrightarrow{c c} H$. For the other implication it is enough to prove the 'Moreover' part. Consider a mapping $g: E(G) \rightarrow E(H)$. We prove that if $g$ is cycle-continuous, then $g$ is induced by an isomorphism $f: V(G) \rightarrow V(H)$ : for every edge $u v$ of $G$ we have $g(u v)=f(u) f(v)$.

Consider $v \in V(G)$. Corollary 2.2 implies that $g(\delta(\{v\}))$ is a 3-edge-cut; as $H$ is cyclically 4-edge-connected, there is a vertex $w \in V(H)$ such that $g\left(\delta_{G}(\{v\})\right)=$ $\delta_{H}(\{w\})$. We put $f(v)=w$ (clearly, $w$ is uniquely determined) and show, that this defines an isomorphism.

First we show that $g$ is a mapping onto $E(H)$ : if there is an edge $e \in E(H)$ that is not in the range of $g$, then $g$ as a mapping $G \rightarrow H-e$ is also $c c$ (Lemma 2.7). As $H$ is a critical snark, we have $G \xrightarrow{c c} H-e \xrightarrow{c c} K_{2}^{3}$, thus $G$ is not a snark, a contradiction.

Consequently, $g$ is onto, and thus injective. It follows that $f$ is injective. Further, for $u v \in E(G)$ we observe that $\delta_{G}(\{u\})$ a $\delta_{G}(\{v\})$ share exactly one edge, namely $u v$. As $g$ is injective, there images, sets $\delta_{H}(\{f(u)\})$ a $\delta_{H}(\{f(v)\})$ also share exactly one edge. It follows that $f(u) f(v)$ is an edge of $H$, thus $f$ is a homomorphism. As $G$ and $H$ are cubic and of the same size, the rest follows.

DeVos et al. [6] claim that if $G$ is critical then the dot product of $G$ and the Petersen graph is critical as well (see [18] for the definition of dot product). This allows (by different ways of taking the dot product) to create arbitrarily large sets of nonisomorphic critical snarks with the same number of vertices. However, these claims are not proved there (the first one is proved in the preprint version [5]), thus we will only use the following two graphs that will suffice for our purposes. It is well-known (Proposition B.1.13 of [19]) that there are two nonisomorphic snarks on 18 vertices, called Blanuša snarks, let us use $B_{i}(i=1,2)$ to denote them.

Lemma 3.2 The Blanuša snarks $B_{1}, B_{2}$ are critical and nonisomorphic. Moreover, with the notation of Figure 3, there is no isomorphism of $B_{2}$ mapping s to b. Furthermore, the only automorphism of $B_{2}$ that fixes $s$ is the identity.


Figure 3: Blanuša snarks (see Lemma 3.2).

Proof: Both of the Blanuša snarks can be obtained from two copies of Petersen graph by dot product. This is the easiest way to prove that they are not 3-edge-colorable, for details see [18], Section 3.7.3 or [19], Section B.1.3.

We will prove criticality using the well-known fact ([2] or [19], Section B.1.5), that there is no cyclically 4 -edge connected snark on 16 vertices, and the only such graph on $\leq 10$ vertices is the Petersen graph. (*)

Suppose $B_{i}(i=1$ or 2$)$ is not critical, let $e$ be an edge for which $B_{i}-e \xrightarrow{c q} K_{2}^{3}$. The graph $B_{i}-e$ is a subdivision of a cubic graph $G$ on 16 vertices. Lemma 2.5 and 2.6 imply that $G \xrightarrow{c q} K_{2}^{3}$.

Thus, from (*) we see that $G$ is not cyclically 4-edge connected. Since $B_{i}$ is, $e$ is an edge of some 4 -edge cut in $B_{i}$, that separates cycles. This gives us (up to symmetry) four possibilities for the choice of $i$ and $e$, which would be easy to go over and provide 3-edge coloring of each of them. An easier way though is to observe, that for each such choice, $G$ can be written as $G_{1} \equiv G_{2}$ with $G_{1}, G_{2}$ cubic graphs, $\left|V\left(G_{1}\right)\right|=8,\left|V\left(G_{2}\right)\right|=10$. Therefore, $G_{1}$ is 3-edge-colorable (*) and so $G_{2}$ is a snark (Corollary 2.14).

In Figure 4 we depict all eight possibilities for $G_{2}$ : among the dashed lines one is deleted and the vertex of degree 2 supressed.


Figure 4: Graph $G_{2}$ from the proof that Blanuša snarks are critical. On the left is the case $i=1$, on the right the case $i=2$.

If $G_{2}$ is a snark, it must be the Petersen graph (*). It isn't, however, as the Petersen graph has girth 5 , which is not the case for $G_{2}$ : All of the graphs depicted on the left contain a 4 -cycle with one black and one white vertex. The graphs displayed on the right either contain the 3 -cycle through the white vertices or the 4 -cycle through the black vertices.

To show that $B_{1}$ and $B_{2}$ are not isomorphic, we observe that $B_{1}$ has has exactly one 4-edge cut separating two cycles, while $B_{2}$ has two such cuts.

It is easy to check (and we will do it shortly) that $\operatorname{Aut}\left(B_{2}\right) \cong \mathbb{Z}_{2}^{2}$, with the obvious four automorphisms being generated by the horizontal and vertical symmetry of
the drawing, $\varphi_{h}, \varphi_{v}$. This implies (by checking these four automorphisms) that no automorphism of $B_{2}$ maps $s$ to $b$ and that the only automorphism that fixes $s$ is the identity.

Finally, we verify the automorphism group of $B_{2}$. Call an edge special if it is a part of a 4-edge cut separating two cycles (there are six of them in $B_{2}$ ). Every automorphism of $B_{2}$ either fixes or switches $x$ and $y$ - they are the only two vertices incident with two special edges. Consider $f \in \operatorname{Aut}\left(B_{2}\right)$ and suppose without loss of generality that $f(x)=x$ and $f(y)=y$ (otherwise we consider $\varphi_{v} \circ f$ instead of $f$ ). With this assumption, the vertices $u$ and $v$ are either switched or both fixed by $f$, suppose the latter (otherwise we consider $\varphi_{h} \circ f$ ). We will show that with these assumptions, $f$ is the identity; this will finish the proof. The unique path of length three from $u$ to $y$ is fixed, as well as the unique path of length three from $v$ to $y$. Now we repeatedly use the observation, that if a vertex $t$ and two of its neighbors are fixed by $f$, the remaining neighbor of $t$ is fixed by $f$, too.

### 3.2 Tree of snarks

Let $\mathcal{G}=\left\{G_{1}, \ldots, G_{\ell}\right\}$ be a family of graphs such that

- each $G_{i}$ is a cyclically 4-edge-connected graph
- each $G_{i}$ is a critical snark
- all graphs in $\mathcal{G}$ are of the same size
- for $i \neq j$, the graphs $G_{i}$ and $G_{j}$ are not isomorphic

Observe, that Lemma 3.1 implies, that we may replace the last condition with

- $G_{i} \xrightarrow{c c} G_{j}$ implies $i=j$

Let $T$ be a tree with a vertex coloring (not necessarily proper) $c: V(T) \rightarrow[\ell]$. We denote by $T(\mathcal{G})$ a family of graphs that can be obtained by replacing each $v \in V(T)$ by a copy of $G_{c(v)}$ and performing a 3-join for each edge; see Figure 5 for an illustration. There are in general many graphs that can be constructed in this way, depending on our choices.


Figure 5: An illustration of the "tree-snark" construction.
More precisely, for each $v \in V(T)$ we fix a bijection $r_{v}$ from $N_{T}(v)$ to an independent set $A_{v}$ in $G_{c(v)}$, we also specify an ordering of edges going out of vertices of $A_{v}$. (If $G_{c(v)}$ does not have large enough independent set, then $T(\mathcal{G})$ is empty.) Next, we split each vertex $w$ in $A_{v}$ into three degree 1 vertices; these will be denoted by $w_{1}, w_{2}$, $w_{3}$. (This graph will be denoted by $G_{c(v)}^{\prime}$.) For each edge $u v$ of $T$ we identify vertices $r_{u}(v)_{i}$ with $r_{v}(u)_{i}$ for $i=1,2,3$. Finally, we suppress all vertices of degree 2 . Note, that the construction can be also described as a repeated application of 3-join.

If $H$ is a graph in $T(\mathcal{G})$ and $v$ a vertex of $T$, we let $H_{v}$ denote the part of $H$ constructed from $v$ : the isomorphic copy of $G_{c(v)}^{\prime}$. Further, we let $\iota_{v}$ denote the natural inclusion (see Section 2.3) of $G_{c(v)}$ into $H$, which is a bijection from $E\left(G_{c(v)}\right)$
to $E\left(H_{v}\right)$. (Note that $\iota_{v}$ is a $c c$ mapping, not a homomorphism.) Finally, for an edge $u v$ of $T$, we will let $H_{u, v}$ denote the three edges in the intersection of $H_{u}$ and $H_{v}$.

The following lemma and theorem are crucial for getting control over $c c$ mappings on our graphs. We note here, that while the $T(\mathcal{G})$ construction could be easily generalized for the case when $T$ is not a tree, Lemma 3.3 would be false in $T$ were not a tree. Recall that $c c$ mappings act on edges and that $f^{\sharp}$ denotes the mapping induced on edges by a homomorphism $f$.

Lemma 3.3 Let $\mathcal{G}$ be as above. Take $H \in T(\mathcal{G})$ and $K \in \mathcal{G}$. Then $K \xrightarrow{c c} H$ iff $K \cong G_{i}$ for some $G_{i} \in \mathcal{G}$ such that color $i$ is used on $T$. Moreover, all mappings $K \xrightarrow{c c} H$ can be written as $\iota_{v} \circ f^{\sharp}$ where $v$ is a vertex of $T$ with $c(v)=i$ and $f$ an automorphism of $K$.

Proof: If $K \cong G_{i}$ and $c(v)=i$ for some $v \in V(T)$ then $K \xrightarrow{c c} G_{i}$ (repeated application of Lemma 2.12). For the other implication it is enough to show the 'Moreover' part.

Consider a cycle-continuous mapping $f: E(K) \rightarrow E(H)$, let $R$ be the set of edges in the range of $f$. Suppose first, that $R$ is exactly the edge set of one of the graphs $H_{v}$. As $f$ only uses edges of $H_{v}$, Lemma 2.17 (applied repeatedly) implies that $K \xrightarrow{c c} G_{c(v)}$. The rest follows by assumptions on $\mathcal{G}$ and Lemma 3.1.

Suppose next, that for every $v$, some edge of $H_{v}$ is not in $R$; let $H_{v}^{\prime}$ be the subgraph of $G_{c(v)}$ with edges $E\left(H_{v}\right) \cap R$ (we are identifying here edges of $H_{v}$ and $G_{c(v)}$ ). As each graph of $\mathcal{G}$ is critical, each graph $H_{v}^{\prime}$ has a $c c$ mapping to $K_{2}^{3}$. The graph $H^{\prime}$ subgraph of $H$ induced by $R$ - is produced from graphs $H_{v}^{\prime}$ (for $v \in V(T)$ ) by 2-join and 3-join operations, which implies (Lemma 2.9 and 2.13) that $H^{\prime} \xrightarrow{c c} K_{2}^{3}$. On the other hand, $g$ is a $c c$ mapping $K \xrightarrow{c c} H^{\prime}$ (Lemma 2.7). By composition, $K \xrightarrow{c c} K_{2}^{3}$, but this is a contradiction as $K$ is a snark.

The next theorem shows that every $c c$ mapping between a graph in $T_{1}(\mathcal{G})$ and a graph in $T_{2}(\mathcal{G})$ (for trees $T_{1}, T_{2}$ and $\mathcal{G}$ as above) is guided by a homomorphism $g: T_{1} \rightarrow T_{2}$ of reflexive colored graphs.

Theorem 3.4 Let $T_{1}, T_{2}$ be trees and let $c_{i}: V\left(T_{i}\right) \rightarrow[\ell](i=1,2)$ be arbitrary colorings. Let $\mathcal{G}$ be as above.

Consider $H^{i} \in T_{i}(\mathcal{G})$ for $i=1,2$. For every cc mapping $h: H^{1} \xrightarrow{c c} H^{2}$ there is a mapping $g: V\left(T_{1}\right) \rightarrow V\left(T_{2}\right)$ such that

- $c_{2}(g(v))=c_{1}(v)$ ( $g$ preserves colors), and
- if uv is an edge of $T_{1}$, then $g(u)=g(v)$ or $g(u) g(v)$ is an edge of $T_{2}$. In the latter case, $h$ maps $H_{u, v}^{1}$ to $H_{g(u), g(v)}^{2}$.
Moreover, for every $v \in V\left(T_{1}\right)$ the mapping $h_{v}: E\left(G_{c_{1}(v)}\right) \rightarrow E\left(G_{c_{2}(g(v))}\right)$ given by $h_{v}=\iota_{g(v)}^{-1} \circ h \circ \iota_{v}$ is cycle-continuous.

Proof: For a vertex $v$ of $T_{1}$, consider the composition of $h$ with $\iota_{v}$. It is a cyclecontinuous mapping from $G_{c_{1}(v)}$ to $H^{2}$. By Lemma 3.3 this mapping is onto some $H_{v^{\prime}}^{2}$ for which $c_{2}\left(v^{\prime}\right)=c_{1}(v)$. We put $g(v)=v^{\prime}$ (obviously, $v^{\prime}$ is unique). Next, for an edge $u v$ of $T_{1}$ we observe that $H_{u, v}^{1}$ is a part of both $H_{u}^{1}$ and $H_{v}^{1}$, thus $H_{g(u)}^{2}$ and $H_{g(v)}^{2}$ must have common edges. If follows that either $g(u)=g(v)$ or $g(u) g(v)$ is an edge of $T_{2}$. The rest follows easily.

The 'Moreover' part follows from Lemma 2.17 as $h \circ \iota_{v}$ maps $G_{c_{1}(v)}$ to $H_{g(v)}^{2}$.

## 4 Applications

In this section we provide two applications of the construction from the previous section. The second theorem is a strengthening of the first one. We include both, however, as the first one, that already answers Question 1.2, is easier to prove and self-contained.


Figure 6: Construction of an infinite set of $c c$-incomparable graphs.

Theorem 4.1 There is an infinite set of cc-incomparable graphs.

Proof: Let $T_{n}$ be a path with vertices $\{0,1, \ldots, n\}$ colored as $1(2)^{n-1} 1$. We let $\mathcal{G}=\left\{G_{1}, G_{2}\right\}$, where $G_{i} \cong B_{i}$ are the Blanuša snarks from Lemma 3.2. For every vertex $v \in V\left(T_{n}\right)$ of degree 2 we define $r_{v}$ so, that $r_{v}(v-1)=s$ and $r_{v}(v+1)=b$. We specify neither $A_{0}$ nor $A_{n}$, nor the order of the edges adjacent to $s$ or $b$. We pick $H^{n}$ arbitrarily from $T_{n}(\mathcal{G})$ for every integer $n \geq 2$.

We will show that $\left\{H^{n}, n \geq 2\right\}$ has the required properties. To this end, consider $H^{m}$ and $H^{n}$ and suppose that $h: H^{m} \xrightarrow{c c} H^{n}$ is a $c c$ mapping. We will show that necessarily $m=n$.

Let $g: V\left(T_{m}\right) \rightarrow V\left(T_{n}\right)$ be the mapping guaranteed by Theorem 3.4. As $g$ preserves colors, we have $\{g(0), g(m)\} \subseteq\{0, n\}$, also $0<i<m$ implies $0<g(i)<n$. Suppose first that $g(0)=n$. Then $g(1)=n-1$ and $h\left(H_{0,1}^{m}\right)=H_{n, n-1}^{n}$. It follows that $h_{1}=\iota_{n-1}^{-1} \circ h \circ \iota_{1}$ is a mapping $B_{2} \xrightarrow{c c} B_{2}$ that maps $\delta(\{s\})$ to $\delta(\{b\})$, a contradiction (Lemma 3.1 and 3.2). Thus $g(0)=0$, consequently $g(1)=1$ and $h\left(H_{0,1}^{m}\right)=H_{0,1}^{n}$.

We will prove by induction that $g(i)=i$ and $h\left(H_{i-1, i}^{m}\right)=H_{i-1, i}^{n}$. For $i=1$ we already know this, we will prove the induction step. From the assumption we know that the mapping $h_{i}: B_{2} \xrightarrow{c c} B_{2}$ maps $\delta(\{s\})$ to $\delta(\{s\})$. It follows (Lemma 3.1 and 3.2) that $h_{i}$ maps $\delta(\{b\})$ to $\delta(\{b\})$, thus $h\left(H_{i, i+1}^{m}\right)=H_{i, i+1}^{n}$. If $g(i+1)=i+1$, we are done, so assume not. Then $g(i+1)=i$ (as $H_{g(i)}^{n}$ contains $H_{i, i+1}^{n}$ ). In this case the mapping $h_{i+1}: B_{2} \xrightarrow{c c} B_{2}$ maps $\delta(\{s\})$ to $\delta(\{b\})$, a contradiction. It follows that $g(i)=i$ for every $i$, thus $m=n$.

Question 1.2 should be understood as a question about how complicated is the structure of $c c$ mappings. Next, we provide even further indication, that the structure is complicated indeed.

Theorem 4.2 Every countable (finite or infinite) poset can be represented by a set of cubic graphs and the existence of cc mappings between them.

Proof: We use the result of Hubička and Nešetřil [9], claiming that arbitrary countable posets can be represented by finite directed paths and the existence of homomorphisms between them. We may assume that only paths with at least one edge are used, as we may first modify our poset by adding a new element as a least element, if path of a single vertex is used, it may be only for this new element.


Figure 7: Construction used for representation of arbitrary posets by $c c$ mappings.

Thus, we only need to find an injective mapping $m$ that assigns cubic bridgeless graphs to directed paths, so that $P \xrightarrow{\text { hom }} P^{\prime}$ iff $m(P) \xrightarrow{c c} m\left(P^{\prime}\right)$. To do this, we use the construction depicted in Fig. 7.

Informally, we replace each directed edge by a copy of $B_{2}$ "from $a$ to $b$ " and perform a 3-join operation in-between each pair of adjacent edges. Further, we join a copy of $B_{1}$ to each copy of $B_{2}$ by a 3 -join operation. We make the 3 -joins carefully, so that any homomorphism $P \rightarrow P^{\prime}$ will correspond to a 'folding' of $m(P)$ to $m\left(P^{\prime}\right)$ this mapping will be locally an isomorphism, thus also a $c c$ mapping (Corollary 2.4). Basically, we are using the fact that $G \equiv G \xrightarrow{c c} G$ that was mentioned earlier, in the discussion following Lemma 2.16. Next, we use the properties of our tree-of-snarks construction to find that $m(P) \xrightarrow{c c} m\left(P^{\prime}\right)$ implies $P \rightarrow P^{\prime}$.

Formally, let $P$ be a path with vertices $v_{0}, \ldots, v_{k}$. The edge $e_{i}$ is either $\left(v_{i-1}, v_{i}\right)$ (a forward edge) or ( $v_{i}, v_{i-1}$ ) (a backward edge). We will use the construction from Section 3.2. To construct our tree $T$ we start with a path with vertices $e_{1}, \ldots, e_{k}$ (an undirected line-graph of $P$ ), then we join a new vertex $f_{i}$ by an edge to $e_{i}$ (for $i=1, \ldots, k)$. All vertices $e_{i}$ are colored by 2 , all $f_{i}$ 's by 1 . As before, our set of snarks will consist of the two Blanuša snarks, i.e., $\mathcal{G}=\left\{G_{1}, G_{2}\right\}$, where $G_{1} \cong B_{1}$ and $G_{2} \cong B_{2}$.

We remind the reader of the notation in Figure 3, we let $z$ be any vertex in $B_{1}$. We define $r_{e_{i}}\left(f_{i}\right)=s$, we let $r_{f_{i}}\left(e_{i}\right)=z$. If $e_{i}$ is a forward edge, we put $r_{e_{i}}\left(e_{i-1}\right)=a$, and $r_{e_{i}}\left(e_{i+1}\right)=b$; if it is a backward edge, we put $r_{e_{i}}\left(e_{i-1}\right)=b$, and $r_{e_{i}}\left(e_{i+1}\right)=a$ (in case $i \in\{0, k\}$ we use only one of the two formulas, as $e_{0}, e_{k+1}$ are not defined). We choose an ordering of the edges going out of $a, b, x$ and $z$; we keep this fixed for all vertices of all paths. Then we let $m(P)$ be the graph in $T(\mathcal{G})$ determined by the above described choices.

We further remind the reader of the notation $m(P)_{v}$ from the construction of the tree of snarks ( $v$ is a vertex of the tree, i.e. $v=e_{i}$ or $v=f_{i}$ ). We define $V_{i}$ to be the connecting edges between $m(P)_{e_{i}}$ and $m(P)_{e_{i+1}}$. We extend this definition for $i \in\{0, k\}$ in the natural way: $V_{0}$ are the three edges of $m(P)_{e_{1}}$ corresponding to $\delta(\{a\})$ (if $e_{1}$ is a forward edge) or to $\delta(\{b\})$ (if $e_{1}$ is a backward edge), similarly
for $V_{k}$. Finally, we put $E_{i}=m(P)_{e_{i}} \cup m(P)_{f_{i}}$.
With the construction in place, we need to show that for any directed paths $P$ and $P^{\prime}$, we have $P \xrightarrow{h o m} P^{\prime}$ if and only if $m(P) \xrightarrow{c c} m\left(P^{\prime}\right)$. Shortly, the proof of the 'only if' part is a direct consequence of the construction, the 'if' part uses Theorem 3.4 and Lemma 3.2.

In detail, for the forward implication consider a homomorphism $f: P \xrightarrow{\text { hom }} P^{\prime}$. As above, the vertices and edges of $P$ are $v_{i}, e_{i}(i \leq k)$; the vertices and edges of $P^{\prime}$ will be denoted $v_{j}^{\prime}, e_{j}^{\prime}(j \leq l)$.

Homomorphism $f$ induces a mapping $g: E(P) \rightarrow E\left(P^{\prime}\right)$ : we put $g((u, v))=$ $(f(u), f(v))$. We define $h: m(P) \rightarrow m\left(P^{\prime}\right)$ separately on each $E_{i}:$ whenever $g\left(e_{i}\right)=$ $e_{j}^{\prime}$, we let (the restriction of) $h$ be an identity between isomorphic graphs $E_{i}$ and $E_{j}^{\prime}$. Corollary 2.4 implies that $h$ is $c c$, but we need to verify that $h$ is well-defined, as we are defining the value of $h$ on the connecting edges twice. It is enough to verify that when $f\left(v_{i}\right)=v_{j}^{\prime}$ then we have defined $h$ to map $V_{i}$ to $V_{j}^{\prime}$ bijectively in the predetermined order both on $E_{i}$ and $E_{i-1}$ (except when $i \in\{0, k\}$, when we only defined $h$ on $V_{i}$ once). To do this we only observe that if $e$ has its tail (head, resp.) at $v_{i}$ then (by definition) $g(e)$ has its tail (head, resp.) at $f\left(v_{i}\right)$.

For the backward implication: suppose we have $h: m(P) \xrightarrow{c c} m\left(P^{\prime}\right)$, we want to prove that there is a homomorphism $f: P \rightarrow P^{\prime}$. There is a natural way to define $f$ : if $h$ maps $V_{i}$ to $V_{j}^{\prime}$ than we define $f\left(v_{i}\right)=v_{j}^{\prime}$. We need to show that $f$ is well-defined and that it is, indeed, a homomorphism.

We use Theorem 3.4 to find a mapping $g: V(T) \rightarrow V\left(T^{\prime}\right)$. As $E(P) \subseteq V(T)$ and as $g$ respects colors, a restriction of $g$ is a mapping $g^{\prime}: E(P) \rightarrow E\left(P^{\prime}\right)$. We will show that $g^{\prime}$ is induced by the above-defined homomorphism $f$.

Using Theorem 3.4, for every $i$ there is a $j$ such that $g\left(e_{i}\right)=e_{j}^{\prime}$ and $g\left(f_{i}\right)=f_{j}^{\prime}$. This implies that $h$ maps the three edges shared by $m(P)_{e_{i}}$ and $m(P)_{f_{i}}$ to those of $m\left(P^{\prime}\right)_{e_{j}^{\prime}} \cap m\left(P^{\prime}\right)_{f_{j}^{\prime}}$. Thus, a mapping $h_{e_{i}}=\iota_{e_{j}^{\prime}}^{-1} \circ h \circ \iota_{e_{i}}: B_{2} \xrightarrow{c c} B_{2}$ fixes the three edges in $\delta(\{s\})$. Using Lemma 3.2, this mapping is the identity, therefore it also fixes the three edges in $\delta(\{a\})$ and those in $\delta(\{b\})$. Consequently, if $e_{i}=(u, v)$ then $f(u)$ and $f(v)$ are well-defined and $g\left(e_{i}\right)=(f(u), f(v))$. Repeating this for all edges $e_{i}$ we find that $f$ is, indeed, a homomorphism (and that $g$ is induced by $f$ ).

We remark that the construction would also work without the vertices $f_{i}$, but the proof is easier with them.

To close this section, we describe an interesting application of Theorem 4.2 to a problem solved in [8]. They study the homomorphism order defined on undirected graphs by $G \leq H$ iff $G \rightarrow H$ (i.e., iff there is a homomorphism from $G$ to $H$ ). They prove that this order is universal (it contains every countable poset as a subposet), even if restricted to graphs that are line-graphs of graphs of given maximal degree. Note that the graphs we utilize in Theorem 4.2 are all 3-regular and triangle-free. So if $G, H$ are two of our graphs (graphs of form $m(P)$ for some directed path $P$ ), then $G \xrightarrow{c c} H$ is equivalent with $L(G) \rightarrow L(H)$. It follows that a special case of their result (namely for line-graphs of 3-regular graphs) follows from our result.

## 5 Concluding remarks

While being a resolution to Question 1.2, none of the family of examples we gave does violate Conjecture 1.1:

Theorem 5.1 If $H \in T(\mathcal{G})$ and if every $G \in \mathcal{G}$ satisfies $G \xrightarrow{c c} \mathrm{Pt}$ then $H \xrightarrow{c c} \mathrm{Pt}$.

Proof: It suffices to repeatedly use Corollary 2.15.
Still, the presented results illustrate the complexity of $c c$ mappings. To better understand the structure of these mappings, we suggest the following questions:

Question 5.2 Does the poset of cubic cyclically 4-edge-connected graphs and cc mappings between them have infinite antichains? Does it contain every countable poset as a subposet? How about cyclically 5-edge-connected graphs?

For the next question, recall that in a poset $(X, \leq)$ an interval $(a, b)$ is the set $\{x \in X: a<x<b\}$ (we must have $a<b$ for this definition to make sense, otherwise we call ( $a, b$ ) degenerate interval).

Question 5.3 In the poset of graphs and cc mappings between them, is every nondegenerate interval nonempty? Does every non-degenerate interval contain an infinite antichain? Does every non-degenerate interval contain every countable poset?

Note, that if Conjecture 1.1 is true, then $\left(P t, K_{2}\right)$ is an empty but non-degenerate interval. Is there some other?

We also briefly note the more general definition of flow-continuous mappings, that extends the notion of cycle-continuous mappings: a mapping $f: E(G) \rightarrow E(H)$ is called $M$-flow-continuous (for an abelian group $M$ ) if for every $M$-flow $\varphi$ on $H$, the composition $\varphi \circ f$ is an $M$-flow on $G$. For detailed discussion, see [6] or [14]. We only mention here, that cycle-continuous mappings are exactly $\mathbb{Z}_{2}$-flow-continuous ones, and that our main results, Theorem 4.1 and 4.2 extend trivially to $\mathbb{Z}$-flow-continuous mappings.

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[^3]:    ${ }^{6}$ Still, we use $v v$ to denote a semiedge at a vertex $v$.

[^4]:    7 Note that in this context, no graph with semiedge is bipartite.

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[^6]:    ${ }^{1}$ Let us note that some of our lemmas hold more generally.

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